VALUES OF MINORS OF AN INFINITE FAMILY OF
D-OPTIMAL DESIGNS AND THEIR APPLICATION TO
THE GROWTH PROBLEM: II∗

C. KOUKOUVINOS†, M. MITROULI‡, AND JENNIFER SEBERRY§

Abstract. We obtain explicit formulae for the values of the $2v − j$ minors, $j = 0, 1, 2$, of $D$-optimal designs of order $2v = x^2 + y^2$, $v$ odd, where the design is constructed using two circulant or type 1 incidence matrices of $2 − \{x^2 + x + 1; \frac{s(s−1)}{2}, \frac{s(s+1)}{2}, \frac{s(s−1)}{2}\}$ supplementary difference sets (SDS). This allows us to obtain information on the growth problem for families of matrices which have moderately large growth. Some of our theoretical formulae suggest that growth greater than $2v$ may occur, but experimentation has not yet supported this result. An open problem remains to establish whether the $(1, −1)$ completely pivoted (CP) incidence matrices of $2 − \{x^2 + x + 1; \frac{s(s−1)}{2}, \frac{s(s+1)}{2}, \frac{s(s−1)}{2}\}$ SDS, which yield $D$-optimal designs, can have growth greater than $2e$.

Key words. $D$-optimal designs, supplementary difference sets, symmetric designs, Gaussian elimination, growth, complete pivoting

AMS subject classifications. 05B20, 15A15, 65F05, 65G05

1. Introduction. In this paper, we use several concepts from orthogonal design theory (e.g., see [5]), but here we will formulate those concepts in matrix notation.

Let $A = [a_{ij}] ∈ \mathbb{R}^{n×n}$. We reduce $A$ to upper triangular form by using Gaussian elimination with complete pivoting (GECP) [15]. Let $A^{(k)} = [a^{(k)}_{ij}]$, $k = 1, 2, ..., n$, denote the matrix obtained after the first $k$ pivoting operations, so $A^{(n)}$ is the final upper triangular matrix. A diagonal entry of that final matrix will be called a pivot. Matrices with the property that no exchanges are actually needed during GECP are called completely pivoted (CP). Let

$$g(n, A) = \max_{i, j, k} \frac{|a^{(k)}_{ij}|}{\max_{i, j} |a_{ij}|}$$

denote the growth associated with GECP on $A$ and

$$g(n) = \sup \{ g(n, A)/A ∈ \mathbb{R}^{n×n} \}.$$ 

The problem of determining $g(n)$ for various values of $n$ is called the growth problem [6].

The values of $g(n)$ are usually less than $n$. One of the curious frustrations of the growth problem is that it is difficult but possible to construct any examples of $n × n$ matrices $A$ for which $g(n, A)$ is greater than or equal to $n$ [6].

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A Hadamard matrix of order $n$ is an $n \times n$ matrix of 1's and −1's with $HH^T = H^T H = nI_n$. Hadamard matrices were first studied by Sylvester in 1867. In 1893 Hadamard discovered that if $X = (x_{ij})$ is a matrix of order $n$, then

$$|\det X|^2 \leq \prod_{i=1}^{n} \sum_{j=1}^{n} |x_{ij}|^2.$$  

Hadamard showed that matrices satisfying the equality and with entries in the unit disc (i.e., $|x_{ij}| \leq 1$ ) have order 1, 2, or $\equiv 0 \pmod{4}$ and entries $\{1, -1\}. \text{ He produced examples for orders up to 20. Subsequently, matrices which satisfy the equality of Hadamard’s inequality came to be known as Hadamard matrices. We refer the interested reader to [5] for more details. Two Hadamard matrices $H_1$ and $H_2$ are called equivalent (or Hadamard equivalent, or H-equivalent) if one can be obtained from the other by a sequence of row negations, row permutations, column negations, and column permutations. Equivalent Hadamard matrices give different pivot structures when GECP is performed on them. When GECP is done on an $n \times n$ Hadamard matrix $H$, the last pivot has magnitude $n$. This was proved by Cryer in [2] because it is the reciprocal of an entry from $H^{-1}$ and that equals $(\frac{1}{n})^T A^T$. Thus $g(n, H) \geq n$. Cryer [2] also evaluated the two pivots preceding the last which take the value of $\frac{n}{2}$, and he remarked that it is unlikely any earlier pivot under GECP could exceed $n$.

In [3] it was proved that the last six pivots cannot exceed $n$ when GECP is done on a Hadamard matrix. The equality $g(n, H) = n$ has been proved for the equivalence class of $n \times n$ Hadamard matrices containing the Sylvester–Hadamard matrix $H$. This evidence supports Cryer’s hunch that $g(n, H) = n$ for any Hadamard matrix $H$.

A matrix $W$ with entries $\{0, \pm 1\}$ satisfying $WW^T = kI_n$, $k \in \{1, 2, \ldots, n\}$, is called a weighing matrix of order $n$ and weight $k$. For more details and construction methods concerning Hadamard and weighing matrices, see [5]. It has also been observed that weighing matrices of order $n$ can give $g(n, W) = n - 1$ [11].

Following Kharaghani [7] a matrix $B$ of order $n$ is a $D$-optimal matrix or $D$-optimal design if the determinant of $B$ is the maximal determinant among all matrices with entries $\pm 1$ (a $\pm 1$ matrix) of order $n$. Let $d_n$ denote the maximum absolute value of determinant of all $n \times n$ matrices with elements $\pm 1$. It follows from Hadamard’s inequality that $d_n \leq n^{\frac{n}{2}}$, and it is easily shown that equality can only hold if $n = 1$ or 2 or if $n \equiv 0 \pmod{4}$, as described above. If $n \equiv 0 \pmod{4}$ and a Hadamard matrix $H$ of order $n$ exists, then $H$ has absolute value of determinant $n^{\frac{n}{2}}$, and thus it is a $D$-optimal matrix. It still remains open if a Hadamard matrix of order $n$ exists for every $n \equiv 0 \pmod{4}$. The smallest value of $n$ is 428 for which a Hadamard matrix of order $n$ and consequently a $D$-optimal design of the same order is not yet known.

A $D$-optimal design $A$ of order $n$ is said to be constructible from two circulant matrices if it can be written in the form $A = [A_1 \ A_2]$, where $A_1$, $A_2$ are circulant matrices of order $\frac{n}{2}$.

Let $X$ be an $n \times n$ matrix of the form $aI + bJ$, where $J$ is an $n \times n$ matrix, every entry of which is $1$. The eigenvalues of this matrix are $a$ with multiplicity $(n - 1)$ and $a + bn$, thus $\det(X) = (a + bn)a^{n-1}$. This paper studies $(+1, -1)$ matrices $C$ of size $(2v) \times (2v)$, where $v$ is odd, and they satisfy $CCT = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$, where $a = 2v - 2$ and $b = 2$.  
Thus $\det(C) = \det(X) = (4v - 2)(2v - 2)^{v-1}$. We shall here be concerned with the case $n \equiv 2 \pmod{4}$, $n \neq 2$, and this will be implicitly assumed in what follows. Ehlich [4] has proved the following theorem.

**Theorem 1.** We have

$$d_n \leq (2n - 2)(n - 2)^{\frac{n}{2} - 1}$$

and equality can hold only if $2n - 2 = x^2 + y^2$, where $x$ and $y$ are integers.

Thus, the above matrices $A$ are $D$-optimal by Ehlich’s theorem.

Since Hadamard matrices of order $n$ are $D$-optimal designs for $n \equiv 0 \pmod{4}$, when they exist, and they have large growth, it is natural to inquire how big growth could be for other $D$-optimal matrices. This is examined in the present paper for an infinite family of $D$-optimal matrices.

**Notation 1.** Write $A$ for a matrix of order $n$ whose initial pivots $p_i$, $i = 1, 2, \ldots$, are derived from matrices with CP structure. Write $A(j)$ for the absolute value of the determinant of the $j \times j$ principal submatrix in the upper left-hand corner of the matrix $A$. Throughout this paper, we find all possible values of the $n - j$ minors, $j = 1, 2, \ldots, n$. Hence, if any minor is CP, it must have one of these values. It can be proved [2] that

$$g(n, A) = \max \left\{ 1, \max_{1 \leq k \leq n-1} \frac{A(k+1)}{A(k)} \right\}.$$ 

Thus, the magnitude of the pivots appearing after the application of Gaussian elimination operations on a CP matrix $A$ is given by

$$p_j = \frac{A(j)}{A(j-1)}, \quad j = 1, 2, \ldots, n, \quad A(0) = 1.$$ 

(1)

**2. $D$-optimal designs of order $2v \equiv 2 \pmod{4}$ from symmetric balanced incomplete block designs.** For the purpose of this paper we will define a symmetric balanced incomplete block design (SBIBD) $(v, k, \lambda)$ to be a $v \times v$ matrix, $B$, with entries 0 or 1, which has exactly $k$ entries +1 and $v - k$ entries 0 in each row and column and for which the inner product of any distinct pairs of rows and columns is $\lambda$. The $(1, -1)$ incidence matrix of $B$ is obtained by letting $A = 2B - J$, where $J$ is the $v \times v$ matrix with entries all +1. We write $I$ for the identity matrix of order $v$. Then we have

$$BB^T = (k - \lambda)I + \lambda J \quad \text{(2)}$$

and

$$AA^T = 4(k - \lambda)I + (v - 4(k - \lambda))J \quad \text{(3)}$$

It can be easily shown that

$$\det B = (k - \lambda)^{\frac{v-2}{2}} \sqrt{k + (v - 1)\lambda},$$

and since $\lambda(v - 1) = k^2 - k$,

$$\det A = 2^{v-1}(k - \lambda)^{\frac{v-1}{2}} |v - 2k|.$$ 

(4)
In this paper we evaluate the $2v - j$, $j = 0, 1, 2$, minors for $(1, -1)$ incidence matrices of certain SBIBDs which yield $D$-optimal designs.

For the purpose of this paper we will define two supplementary difference sets $2 - \{v; k_1, k_2; \lambda\}$, abbreviated as SDS, to be two circulant (or type 1) $v \times v$ matrices $B_1$ and $B_2$, with entries 0 or 1, which have exactly $k_i$ entries +1 and $v - k_i$ entries 0, $i = 1, 2$, respectively, in each row and column and for which the inner product of any pair of rows of $[B_1 \ B_2]$ is $\lambda$, where $\lambda = (k_1 + k_2) - \frac{0 - 1}{2}$. We note that circulant matrices commute, and that the transpose of a circulant matrix is also a circulant matrix. The $(1, -1)$ incidence matrices of $B_i$ are obtained by letting $A_i = 2B_i - J$, $i = 1, 2$.

Then it is true that

$$A_1A_1^T + A_2A_2^T = (2v - 2)I + 2J$$

when the matrix $A = [A_1 \ A_2; -A_2^T]$ is constructible from two $(1, -1)$ circulants as just described. Thus when such a $A$ exists, it has determinant $(4v - 2)(2v - 2)^{v-1}$; hence by Ehlich’s theorem, it is a $D$-optimal design.

Only two infinite families of $D$-optimal designs are known:

1. The first, which uses $2 - \{s^2 + s + 1; s(s-1)/2, s(s+1)/2\}$ SDS, is based on the family of $SBIBD(s^2 + s + 1, s + 1, 1)$ for $s$ a prime power, found by Singer [12] and used extensively by Spence [13]. Koukouvinos, Kounias, and Seberry [8] showed how to use these SDS to form an infinite family of $D$-optimal designs, constructible from two circulant matrices, for $n = 2(s^2 + s + 1)$, where $s = 2, 4, 6, 8$ or $s$ is an odd prime power. This family is called the Koukouvinos–Kounias–Seberry–Singer–Spence (KKSSS) family. If the $D$-optimal design $A$ is constructed from the above SDS, then

$$\text{det } A = (4v - 2)(2v - 2)^{v-1}. \quad (6)$$

2. The second family is based on Brouwer’s [1] family of $2 - \{2s^2 + 2s + 1; s, s^2, s(s-1)\}$ SDS, where the two SDS are in fact identical. Whiteman [14] showed how to form these SDS into an infinite family of $D$-optimal designs, constructible from two circulant matrices, for $n = 2(2s^2 + 2s + 1)$, where $s$ is an odd prime power.

In [9] the pivot structure of $(1, -1)$ incidence matrices of $SBIBD(v, k, \lambda)$ was studied. In [10] values for the pivots of $2 - \{2s^2 + 2s + 1; s^2, s^2, s(s-1)\}$ SDS were evaluated (as previously noted, the two SDS are in fact identical for this case). In the present paper we obtain values for the pivots of $2 - \{s^2 + s + 1; s(s-1)/2, s(s+1)/2, s(s-1)/2\}$ SDS and $D$-optimal designs made from them. Our calculations here and in [10] have given moderately large values of growth for the $D$-optimal matrices of both KKSSS and Brouwer types, but it is not known yet whether there exist any $(+1, -1) n \times n$ $D$-optimal matrices with growth greater than $n$.

2.1. Minors of size $(2v - 1)$. We denote by $A = \Delta(h, i, j, k, m)$ the following matrix of order $2v$:
\[ A = \Delta(h, i, j, k, m) = \]
\[
\begin{bmatrix}
\begin{array}{cccc}
  h & 1 & \cdots & 1 \\
  1 & m & \cdots & 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & 1 & \cdots & m \\
  3 & 3 & \cdots & 3 \\
  3 & 3 & \cdots & 3 \\
  \vdots & \vdots & \ddots & \vdots \\
  3 & 3 & \cdots & 3 \\
  - & - & \cdots & - \\
  - & - & \cdots & - \\
  1 & 1 & \cdots & m \\
  1 & 1 & \cdots & 1 \\
  1 & 1 & \cdots & 1 \\
  1 & 1 & \cdots & 1 \\
  3 & 3 & \cdots & 3 \\
  3 & 3 & \cdots & 3 \\
  \vdots & \vdots & \ddots & \vdots \\
  3 & 3 & \cdots & 3 \\
\end{array}
\end{bmatrix}
\]

where \( m = 2v = h + i + j + k \). Then by the determinant simplification theorem [10],

\[
\det \Delta(h, i, j, k, m) = (m-1)^{m-4} \begin{vmatrix}
  m - 1 + h & 3h & -h & h \\
  3i & m - 1 + i & i & -i \\
  -j & j & m - 1 + j & 3j \\
  k & -k & 3k & m - 1 + k
\end{vmatrix}
\]

and

\[
\det \Delta(h, i, j, k, m) = (m-1)^{m-4} \left((m-1)^4 + (m-1)^3(i + j + h + k) - 8(m-1)^2(jk + ih) - 16(m-1)(jk(i + h) + ih(j + k))\right).
\]

The \((2v - 1) \times (2v - 1)\) minors are obtained by removing a row and column from \( A \) to get \( D \). We note that this means the number of rows becomes \( m - 1 \) instead of \( m \). The number of columns being reduced by one means we have one of \( h - 1, i, j, k \), or \( h, i - 1, j, k \), or \( h, i, j - 1, k \), or \( h, i, j, k - 1 \), as the number of columns of each type. Thus \( \det DD^T \) is \( \det \Delta(h - 1, i, j, k, m - 1) \) or \( \det \Delta(h, i, j, k, m - 1) \) or \( \det \Delta(h, i - 1, k, m - 1) \) or \( \det \Delta(h, i, j - 1, k, m - 1) \).

**Notation 2.** We use the notation \( M_j \) to denote a \( j \times j \) minor of \( A \). We use \( \sim \) to denote \( \sim^{-1} \) throughout this paper.

**Notation 3.** In the work that follows we simplify the typesetting by defining two expressions \( T \) and \( \mathcal{P} \):

\[
T = 2s^{2+s+1}s^{2+s}(s+1)^{s^{2+s}} = 2(2v - 2)^{v-1},
\]
\[
\mathcal{P} = 2s^{2} + 2s + 1 = 2v - 1.
\]

**Lemma 1.** The \((2v - 1) \times (2v - 1)\) minors of the \( D \)-optimal designs of the KKSSS series are

\[
s \frac{s}{s + 1} T, \quad \frac{s + 1}{s} T, \quad \frac{s^2 + s + 1}{s(s + 1)} T, \quad T,
\]

where \( T = 2s^{2+s+1}s^{2+s}(s+1)^{s^{2+s}} \).
Proof. Here we use the $(1, -1)$ incidence matrices of the $2 - \{s^2 + s + 1; s(s - 1), s(s + 1); s(s - 2)\}$ SDS. By the reasoning above, with $v = s^2 + s + 1$, $h = \frac{s(s + 1)(s + 2)}{2}$, $i = \frac{s(s - 1)}{2}$, $j = \frac{s + 2}{2}$, $m = 2s^2 + 2s + 2$ substituted into $\det \Delta(h - 1, i, j, k, m - 1)$, $\det \Delta(h, i, j - 1, k, m - 1)$, and $\det \Delta(h, i, j, k - 1, m - 1)$ we obtain the result.

Specifically the $(2v - 1) \times (2v - 1)$ minor is the square root of the determinant and is given by one of the following:

1. $\det \Delta(h - 1, i, j, k, m - 1) = 2s^2 + s + 1 \left(\frac{s(s - 1)}{2}\right)^2 + (s + 1)(s^2 + s + 1)$.
2. $\det \Delta(h, i - 1, j, k, m - 1) = 2s^2 + s + 1 \left(\frac{s(s + 1)}{2}\right)^2 - (s + 1)(s^2 + s + 1)$.
3. $\det \Delta(h, i, j - 1, k, m - 1) = 2s^2 + s + 1 \left(\frac{s + 1}{2}\right)^2 - (s^2 + s + 1)$.

2.2. Minors of size $(2v - 2)$. Now remove two rows and two columns of $A$.

We have not included the generic matrix in expanded form, except for two cases, but moved straight to the determinant after it has been simplified using the determinant simplification theorem [10]. Thus the determinant of a submatrix of $A$ obtained by removing two rows and two columns is $(2v - 2)^{v - 5} \sqrt{\det D}$, where

$$D = \begin{bmatrix}
2v - 2 & 2u_2 & 2u_3 & 4u_4 & -2u_5 & 0 & 0 & 2u_6 \\
2u_1 & 2v - 2 & 4u_3 & 2u_4 & 0 & -2u_6 & 2u_7 & 0 \\
2u_1 & 4u_2 & 2v - 2 & 2u_4 & 0 & 2u_6 & -2u_7 & 0 \\
4u_1 & 2u_2 & 2u_3 & 2v - 2 & 2u_5 & 0 & 0 & -2u_8 \\
-2u_1 & 0 & 0 & 2u_4 & 2v - 2 & 2u_6 & 2u_7 & 4u_8 \\
0 & -2u_2 & 2u_3 & 0 & 2u_5 & 2v - 2 & 4u_7 & 2u_8 \\
0 & 2u_2 & -2u_3 & 0 & 2u_5 & 4u_6 & 2v - 2 & 2u_8 \\
2u_1 & 0 & 0 & -2u_4 & 4u_5 & 2u_6 & 2u_7 & 2v - 2 \\
\end{bmatrix}.$$ 

Diagrammatically, we have used the matrix form

$$\begin{bmatrix}
A_1 & A_2 \\
A_2^T & -A_1^T
\end{bmatrix} = \begin{bmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{bmatrix}.$$ 

For Case I both rows and columns are removed from $A_1$; for Case II one row is from $A_1$ and one from $A_3$, but both columns are from $A_1$; for Case III one row is from $A_1$ and one from $A_3$, and one column is from $A_1$ and one column is from $A_2$.

To calculate the minors of size $(2v - 2)$ we distinguish three major cases. This leads to the following seven subcases:

Case Ia. $[\begin{array}{c} x \\ y
\end{array}]$, where the $(1,1)$ and the $(2,1)$ elements have the same sign, the $(1,2)$ element and the $(2,2)$ element have opposite signs, and the inner product of row one and two with each other is 2.

Case Ib. $[\begin{array}{c} x \\ y
\end{array}]$, where the $(1,1)$ and the $(2,1)$ elements have the same sign, the $(1,2)$ element and the $(2,2)$ element have the same signs, and the inner product of rows one and two with each other is 2.

Case Ic. $[\begin{array}{c} x \\ y
\end{array}]$, where the $(1,1)$ and the $(2,1)$ elements have opposite sign, the $(1,2)$ element and the $(2,2)$ element have opposite signs, and the inner product of row one and two with each other is 2.

Case IIa. $[\begin{array}{c} x \\ y
\end{array}]$, where the $(1,1)$ element and the $(2,1)$ element have the same signs, the $(1,2)$ element and the $(2,2)$ element have different signs, and the inner product of rows one and two with each other is zero.
### Table 1

<table>
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<th>(w_3)</th>
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<td>(\lambda - \lambda_1)</td>
<td>(k_2 - \lambda + \lambda_1)</td>
<td>(v_2 + \lambda - \lambda_1)</td>
<td>(k_2 - \lambda + \lambda_1)</td>
</tr>
<tr>
<td>– –</td>
<td>(\lambda_1)</td>
<td>(k_1 - \lambda_1)</td>
<td>(k_1 - \lambda_1 - 1)</td>
<td>(v_1 + \lambda)</td>
<td>(\lambda - \lambda_1)</td>
<td>(k_2 - \lambda + \lambda_1)</td>
<td>(v_2 + \lambda - \lambda_1)</td>
<td>(k_2 - \lambda + \lambda_1)</td>
</tr>
</tbody>
</table>

### Table 4

<table>
<thead>
<tr>
<th>2 × 2 submatrix</th>
<th>(w_1)</th>
<th>(w_2)</th>
<th>(w_3)</th>
<th>(w_4)</th>
<th>(w_5)</th>
<th>(w_6)</th>
<th>(w_7)</th>
<th>(w_8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1</td>
<td>(\lambda_1 - 1)</td>
<td>(k_1 - \lambda_1)</td>
<td>(k_2 - \lambda_1 - 1)</td>
<td>(v_1 + \lambda)</td>
<td>(\lambda - \lambda_1)</td>
<td>(k_2 - \lambda + \lambda_1)</td>
<td>(v_2 + \lambda - \lambda_1)</td>
<td>(k_2 - \lambda + \lambda_1)</td>
</tr>
<tr>
<td>– –</td>
<td>(\lambda_1)</td>
<td>(k_1 - \lambda_1)</td>
<td>(k_1 - \lambda_1 - 1)</td>
<td>(v_1 + \lambda)</td>
<td>(\lambda - \lambda_1)</td>
<td>(k_2 - \lambda + \lambda_1)</td>
<td>(v_2 + \lambda - \lambda_1)</td>
<td>(k_2 - \lambda + \lambda_1)</td>
</tr>
<tr>
<td>1 –</td>
<td>(\lambda_1)</td>
<td>(k_1 - \lambda_1 - 1)</td>
<td>(k_1 - \lambda_1 - 1)</td>
<td>(v_1 + \lambda)</td>
<td>(\lambda - \lambda_1)</td>
<td>(k_2 - \lambda + \lambda_1)</td>
<td>(v_2 + \lambda - \lambda_1)</td>
<td>(k_2 - \lambda + \lambda_1)</td>
</tr>
<tr>
<td>– –</td>
<td>(\lambda_1)</td>
<td>(k_1 - \lambda_1)</td>
<td>(k_1 - \lambda_1 - 1)</td>
<td>(v_1 + \lambda)</td>
<td>(\lambda - \lambda_1)</td>
<td>(k_2 - \lambda + \lambda_1)</td>
<td>(v_2 + \lambda - \lambda_1)</td>
<td>(k_2 - \lambda + \lambda_1)</td>
</tr>
</tbody>
</table>

### Case IIb

\[ \begin{bmatrix} x & y \\ x & y \end{bmatrix} \]

where the (1,1) element and the (2,1) element have the same signs, the (1,2) element and the (2,2) element also have the same sign, and the inner product of row one and two with each other is zero.

### Case IIIa

\[ \begin{bmatrix} x & y \\ x & y \end{bmatrix} \]

where one of the columns in the submatrix has two identical elements and the other has two different elements.

### Case IIIb

\[ \begin{bmatrix} x & y \\ x & y \end{bmatrix} \]

where both columns in the submatrix have identical elements.

In [10] we analyzed which 2 × 2 submatrices gave independent values for the distribution of rows in the minors of order 2v – 2. These are summarized for the KKSSS family in Tables 1, 2, 3, 4. Case III is covered by Tables 5 and 6. Set \(v_1 = v - 2k_1\), \(v_2 = v - 2k_2\) in Tables 1–4.
Table 5

<table>
<thead>
<tr>
<th>2 x 2 subquadrant</th>
<th>n₁</th>
<th>n₂</th>
<th>n₃</th>
<th>n₄</th>
<th>n₅</th>
<th>n₆</th>
<th>n₇</th>
<th>n₈</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 1</td>
<td>ρ - 1</td>
<td>k₁ - ρ</td>
<td>k₂ - ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₂ - ρ</td>
<td>ρ - 1</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₁ - ρ</td>
</tr>
<tr>
<td>1 - 2</td>
<td>ρ</td>
<td>k₁ - ρ - 1</td>
<td>k₂ - ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₂ - ρ - 1</td>
<td>ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₁ - ρ</td>
</tr>
<tr>
<td>2 - 1</td>
<td>ρ</td>
<td>k₁ - ρ - 1</td>
<td>k₂ - ρ - 1</td>
<td>v - k₁ - k₂ + ρ - 1</td>
<td>k₂ - ρ - 1</td>
<td>ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₁ - ρ</td>
</tr>
<tr>
<td>1 - -</td>
<td>ρ</td>
<td>k₁ - ρ - 1</td>
<td>k₂ - ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₂ - ρ</td>
<td>ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₁ - ρ</td>
</tr>
<tr>
<td>2 - -</td>
<td>ρ</td>
<td>k₁ - ρ - 1</td>
<td>k₂ - ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₂ - ρ</td>
<td>ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₁ - ρ</td>
</tr>
</tbody>
</table>

Table 6

<table>
<thead>
<tr>
<th>2 x 2 subquadrant</th>
<th>n₁</th>
<th>n₂</th>
<th>n₃</th>
<th>n₄</th>
<th>n₅</th>
<th>n₆</th>
<th>n₇</th>
<th>n₈</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 1</td>
<td>ρ - 1</td>
<td>k₁ - ρ</td>
<td>k₂ - ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₂ - ρ - 1</td>
<td>ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₁ - ρ</td>
</tr>
<tr>
<td>1 - 2</td>
<td>ρ</td>
<td>k₁ - ρ - 1</td>
<td>k₂ - ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₂ - ρ - 1</td>
<td>ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₁ - ρ</td>
</tr>
<tr>
<td>2 - 1</td>
<td>ρ</td>
<td>k₁ - ρ - 1</td>
<td>k₂ - ρ - 1</td>
<td>v - k₁ - k₂ + ρ - 1</td>
<td>k₂ - ρ - 1</td>
<td>ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₁ - ρ</td>
</tr>
<tr>
<td>1 - -</td>
<td>ρ</td>
<td>k₁ - ρ - 1</td>
<td>k₂ - ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₂ - ρ</td>
<td>ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₁ - ρ</td>
</tr>
<tr>
<td>2 - -</td>
<td>ρ</td>
<td>k₁ - ρ - 1</td>
<td>k₂ - ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₂ - ρ</td>
<td>ρ</td>
<td>v - k₁ - k₂ + ρ</td>
<td>k₁ - ρ</td>
</tr>
</tbody>
</table>

Case Ia. To illustrate the derivation of the tables such as Table 1 we give Case Ia as an example.

<table>
<thead>
<tr>
<th>1</th>
<th>y</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>y</td>
<td>-</td>
<td>y</td>
</tr>
<tr>
<td>1</td>
<td>y</td>
<td>-</td>
<td>y</td>
</tr>
</tbody>
</table>

Inner Product of rows is 2

```
1 1
1 1
```

```
λ₁ - 1
1 1
```

```
λ₁
```

```
λ₁ - 1
```

```
λ₁ - 1
```

```
v - 2k₁ + λ₁
```

```
v - 2k₁ + λ₁ - 1
```

```
λ₂
```

```
λ₂
```

```
λ = λ₁ + λ₂
```

```
v rows
```

```
λ = λ₁ + λ₂
```

```
v rows
```

```
k₂ - λ₂
```

```
k₂ - λ₂
```

```
k₂ - λ₂
```

```
k₂ - λ₂
```

```
v - 2k₂ + λ₂
```

```
v - 2k₂ + λ₂
```
Case III. To help understand Case III we recall that in this case one column removed comes from the columns with \(k_1 + k_2\) ones per column and the other from the columns with \(v - k_2 + k_1\) ones per column in the original design. This means the generic form of these two columns is

\[
\begin{array}{cccccccc}
1 & 1 & \vdots & \rho & \vdots & k_1 - \rho & \vdots & v - k_1 - k_2 + \rho \\
1 & 1 & & & & & & \\
1 & k_1 & & & & & & \\
\vdots & \vdots & \vdots & k_2 & \vdots & k_2 - \rho & \vdots & v - k_2 \\
1 & & & & & & & \\
\vdots & \vdots & k_1 - \rho & \vdots & & & \vdots & \\
1 & & & & & & & \\
\end{array}
\]

Note that they have inner product zero.

The results given are quite general for the minors of size \(2v - 2\) constructed from any \(2 - \{v; k_1, k_2; \lambda\}\) SDS. We now apply these results to the special case of the \(2 - \{s^2 + s + 1; \frac{s(s-1)}{2}, \frac{s(s+1)}{2}; \frac{s(s-1)}{2}\}\) SDS.

**Lemma 2.** The \((2v - 2) \times (2v - 2)\) minors of the D-optimal design of the KKSSS series are

\[
\begin{align*}
0, & \quad \frac{1}{s^2 T}, & \quad \frac{1}{(s+1)^2 T}, & \quad \frac{1}{s(s+1) T}, & \quad \frac{1}{s^2(s+1)^2 T}, & \quad \frac{1}{s(s+1)^2 T}, \\
\frac{1}{s^2(s+1)^2 T}, & \quad \frac{2s + 1}{s^2(s+1)^2 T}, & \quad \frac{s^2 + 1}{s^2(s+1)^2 T}, & \quad \frac{s^2 + s + 1}{s^2(s+1)^2 T}, & \quad \frac{s^2 + 2s + 2}{s^2(s+1)^2 T},
\end{align*}
\]

where \(T = 2^{s^2 + s + 1}s^2 + s(s+1)^2s^{2+s} \).

**Proof.** Here \(\lambda = \frac{1}{2}s(s-1), \quad k_1 = \frac{1}{2}s(s-1), \quad k_2 = \frac{1}{2}s(s+1), \quad v = s^2 + s + 1\). The expressions for \(u_i, \ i = 1, \ldots, 8\), were calculated in each case. Maple was then used to evaluate the determinant for \(D\) giving the required result. Case Ia gives the values \(2^{10} s^4(s+1)^8, \ 2^{10} s^8(s+1)^4, \) and \(2^{10} s^4(2s+1)^2(s+1)^4\). Case IIa gives the values \(2^{10} s^4(s+1)^8, \ 2^{10} s^8(s+1)^8, \) and \(2^{10} s^4(s^2 + s + 1)^2(s+1)^4\) and \(2^{10} s^8(s+1)^4\).

Case Ib gives the value zero for the determinant. Case IIb gives the value \(2^{10} s^8(s+1)^4\) and the value zero for the determinant.
Case IIIa gives the values $2^{10}s^4(s^2 + 2s + 2)^2(s + 1)^4$, $2^{10}s^6(s + 1)^6$, $2^{10}s^8(s + 1)^4$, $2^{10}s^4(s^2 + 1)^2(s + 1)^4$, and $2^{10}s^4(s + 1)^8$, whereas Case IIIb gives the values $2^{10}s^4(s + 1)^4$, $2^{10}s^6(s + 1)^6$, and $2^{10}s^8(s + 1)^4$.

Taking the square root and multiplying by $(2s^2 + 2s)^{2+s-4}$ gives the required result. \[ \]

**Remark 1.** The values \( \frac{1}{s(s+1)^2}T, \frac{1}{s}T, \frac{1}{s+1}T \) all arise from a $2 \times 2$ corner block

\[
\begin{array}{cc}
x & y \\
x & y
\end{array}
\]

which cannot occur as the leading $2 \times 2$ block when GECP is done here. Also the value $\frac{2s+1}{s(s+1)^2}T$ arises from a $2 \times 2$ corner block

\[
\begin{array}{cc}
x & \bar{y} \\
\bar{x} & y
\end{array}
\]

which cannot occur as the leading $2 \times 2$ block when GECP is done here. \[ \]

### 3. Pivot structure for the KKSSS family of $D$-optimal designs.

**Conjecture (growth conjecture for the KKSSS family).** Let $A$ be a $2v \times 2v$ CP $D$-optimal design of the KKSSS family which is constructed from $2 - \{s^2 + s + 1; \frac{s(s-1)}{2}; \frac{s(s+1)}{2}; \frac{s(s-1)}{2} \}$ SDS. Reduce $A$ by GECP and recall that $\mathcal{P} = 2s^2 + 2s + 1$. Then we conjecture the following:

- (i) $g(v, A) = \frac{s+1}{s} \mathcal{P}$, or $\frac{s}{s+1} \mathcal{P}$, or $\frac{s(s+1)}{s(s+1+1)} \mathcal{P}$, or $\mathcal{P}$;
- (ii) the last pivot is equal to $\frac{s+1}{s+1} \mathcal{P}$, or $\frac{s}{s+1} \mathcal{P}$, or $\frac{s(s+1)}{s(s+1)} \mathcal{P}$, or $\mathcal{P}$;
- (iii) the second-to-last pivot can take the values given in Table 8;
- (iv) every pivot before the last has magnitude at most $2v$;
- (v) the first four pivots are equal to 1, 2, 2, 4;
- (vi) the fifth pivot may be 2 or 3.

We prove (ii) and (iii) in this paper. (v) and (vi) were proved for Brouwer’s $SBIBD(2s^2 + 2s + 1, s^2, 2(s - 1))$ in [9] and we also show they hold for the KKSSS family.

We recall that for any CP matrix $A$ of $SBIBD(v, k, \lambda)$, the two last pivots $p_v$ and $p_{v-1}$ are given from the formulae

\[
p_v = \frac{A(v)}{A(v-1)}, \quad p_{v-1} = \frac{A(v-1)}{A(v-2)}.
\]

**Theorem 2.** Let $A$ be the $2v \times 2v$ $D$-optimal design of the KKSSS family. Reduce $A$ by GECP. Then the last pivot, $p_{2v}$, is $\frac{s+1}{s} \mathcal{P}$, or $\frac{s}{s+1} \mathcal{P}$, or $\frac{s(s+1)}{s(s+1+1)} \mathcal{P}$, or $\mathcal{P}$. The only possible values of the second-to-last pivot, $p_{2v-1}$, are those given in Table 8.

**Proof.** From (4), (6), and Lemma 1 we have for the $D$-optimal design made using $2 - \{s^2 + s + 1; \frac{s(s-1)}{2}; \frac{s(s+1)}{2}; \frac{s(s-1)}{2} \}$ SDS the results given in Table 7, where the first row gives the values of $M_{2v}$, the first column gives the values of $M_{2v-1}$, and the entries are $p_{2v} = \frac{M_{2v}}{M_{2v-1}}$.

From (4) and Lemmas 1 and 2 we have for the $D$-optimal design made using $2 - \{s^2 + s + 1; \frac{s(s-1)}{2}; \frac{s(s+1)}{2}; \frac{s(s-1)}{2} \}$ SDS the results given in Table 8, where the first row gives the values of $M_{2v-1}$, the first column gives the values of $M_{2v-2}$, and the other entries are the only possible values of $p_{2v-1} = \frac{M_{2v-1}}{M_{2v-2}}$.

**Remark 2.** The entries marked $\ast$ in Tables 7 and 8 are those obtained in experiments. It is not known whether all the values shown in Tables 7 and 8 can actually
Table 7
The only possible values of $p_{2v}$.  

<table>
<thead>
<tr>
<th>$M_{2v-1}$</th>
<th>$\frac{s+1}{s+1} T$</th>
<th>$\frac{1}{s} T$</th>
<th>$\frac{s}{s+1} T$</th>
<th>$\frac{s}{s+1} T$</th>
<th>$\frac{1}{s} T$</th>
<th>$\frac{1}{s^2+2s+1} T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{2v}$</td>
<td>$\frac{s}{s+1} T$</td>
<td>$\frac{1}{s} T$</td>
<td>$\frac{s}{s+1} T$</td>
<td>$\frac{s}{s+1} T$</td>
<td>$\frac{1}{s} T$</td>
<td>$\frac{1}{s^2+2s+1} T$</td>
</tr>
</tbody>
</table>

Table 8
The only possible values of $p_{2v-1}$.  

<table>
<thead>
<tr>
<th>$M_{2v-1}$</th>
<th>$\frac{s}{s+1} T$</th>
<th>$\frac{s+1}{s} T$</th>
<th>$\frac{s^2+1}{s^2+2s+1} T$</th>
<th>$\frac{s}{s+1} T$</th>
<th>$\frac{s}{s+1} T$</th>
<th>$\frac{1}{s} T$</th>
<th>$\frac{1}{s^2+2s+1} T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{2v}$</td>
<td>$\frac{s}{s+1} T$</td>
<td>$\frac{s+1}{s} T$</td>
<td>$\frac{s^2+1}{s^2+2s+1} T$</td>
<td>$\frac{s}{s+1} T$</td>
<td>$\frac{s}{s+1} T$</td>
<td>$\frac{1}{s} T$</td>
<td>$\frac{1}{s^2+2s+1} T$</td>
</tr>
</tbody>
</table>

Table 9
Numerical values of $p_{2v}$.  

<table>
<thead>
<tr>
<th>$2v$</th>
<th>$s$</th>
<th>$\frac{s+1}{s} T$</th>
<th>$\frac{s}{s+1} T$</th>
<th>$\frac{s}{s+1} T$</th>
<th>$\frac{1}{s} T$</th>
<th>$\frac{1}{s^2+2s+1} T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>2</td>
<td>19.5</td>
<td>26</td>
<td>78</td>
<td>13</td>
<td>56</td>
</tr>
<tr>
<td>26</td>
<td>3</td>
<td>100</td>
<td>78</td>
<td>126</td>
<td>25</td>
<td>244</td>
</tr>
<tr>
<td>42</td>
<td>4</td>
<td>54.41</td>
<td>44.41</td>
<td>20.41</td>
<td>241</td>
<td></td>
</tr>
</tbody>
</table>

occur as $p_{2v}$ and $p_{2v-1}$ when GECP is done to a matrix of KKSSS type. In particular, notice that the first value listed for $p_{2v}$, $\frac{s+1}{s} T$, is greater than $2v$, but in experiments using GECP on such matrices we never saw it arise.

Remark 3. We experimented with $2v = 14$ by testing 100,000 equivalent transformations. The theoretical values for $M_{2v-1}$ are $2^{14} \cdot 3^5$, $2^{12} \cdot 3^7$, $2^{12} \cdot 3^5 \cdot 7$, and $2^{13} \cdot 3^6$. In our calculations we always found $p_{2v} = \frac{26}{s}$ and $\frac{78}{s}$. This leaves as an open problem the existence of a $14 \times 14$ matrix having growth equal to 19.5.

The next result is easy to prove using a counting argument and noting that the inner product of every pair of rows is +1 to see that the design always contains a $4 \times 4$ Hadamard matrix.
Table 10
Numerical values of $p_{2v-1}$.

<table>
<thead>
<tr>
<th>$2v$</th>
<th>$s$</th>
<th>$p_{2v-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(s+1)^2$</td>
<td>$\frac{(s+1)^3}{2s^2+3s+2}$</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>26</td>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>42</td>
<td>4</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 11
Growth factors and pivots patterns for small CP KKSSS designs.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$2v$</th>
<th>Growth</th>
<th>Pivot pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>14</td>
<td>$\frac{25}{7}$</td>
<td>$1, 2, 2, 4, 3, 10, 18, \ldots, 6, \frac{25}{7}$</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>$\frac{25}{7}$</td>
<td>$1, 2, 2, 4, 3, 10, 18, \ldots, 5.4, \frac{25}{7}$</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>$\frac{25}{7}$</td>
<td>$1, 2, 2, 4, 3, 10, 18, \ldots, 6, \frac{25}{7}$</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>$\frac{25}{7}$</td>
<td>$1, 2, 2, 4, 3, 10, 18, \ldots, 6, \frac{25}{7}$</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>$\frac{25}{7}$</td>
<td>$1, 2, 2, 4, 3, 10, 18, \ldots, 9, \frac{25}{7}$</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>$\frac{25}{7}$</td>
<td>$1, 2, 2, 4, 3, 10, 18, \ldots, 5.4, \frac{25}{7}$</td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>$\frac{75}{7}$</td>
<td>$1, 2, 2, 4, 3, 10, 18, \ldots, 12, \frac{75}{7}$</td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>$\frac{75}{7}$</td>
<td>$1, 2, 2, 4, 3, 10, 18, \ldots, \frac{34^3}{17}, \frac{75}{7}$</td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>$\frac{12.25}{13}$</td>
<td>$1, 2, 2, 4, 3, 10, 18, \ldots, 12, \frac{12.25}{13}$</td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>$\frac{72}{7}$</td>
<td>$1, 2, 2, 4, 3, 10, 18, \ldots, \frac{34^3}{17}, \frac{72}{7}$</td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>$\frac{12.25}{13}$</td>
<td>$1, 2, 2, 4, 3, 10, 18, \ldots, 16, \frac{12.25}{13}$</td>
</tr>
</tbody>
</table>

**Proposition 1** (see [10]). Let $A$ be the $2v \times 2v (1, -1)$ incidence matrix of an SBJBD of the KKSSS family. Reduce $A$ by GECP. Then the magnitudes of the first four pivots are 1, 2, 2, and 4; the magnitude of $|a_{55}^{(4)}|$ is 2 or 3.

The values presented in Table 11 are those we saw in experiments when we used GECP on some small KKSSS matrices. The first seven pivots and the last two are presented. All the other intermediate pivots take a variety of values. At least 54 different pivot structures were detected for $2v = 14$ and over 20,000 for $2v = 26$.

**Remark 4.** We note that experimentally, for $s = 1$, we always found the unique pivot structure $(1, 2, 2, 4, 3, \frac{10}{7})$.

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**REFERENCES**


