System theoretic based characterisation and computation of the least common multiple of a set of polynomials

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Received 16 March 2002; accepted 4 November 2003

Submitted by V. Mehrmann

Abstract

The paper provides a system theoretic characterisation of the least common multiple (LCM) \( m(s) \) of a given set of polynomials \( \mathcal{P} \) which leads to an efficient numerical procedure for the computation of LCM that avoids root finding and use of greatest common divisor (GCD) procedures. The procedure that is presented also leads to the computation of the associated set of multipliers of \( \mathcal{P} \) with respect to LCM. The basis of the new characterisation and computational procedure are the controllability properties of a natural realization \( S(A,b,C) \) associated with the set \( \mathcal{P} \). It is shown, that the coefficients of the LCM are defined by the properties of the controllable subspace of the pair \( (A, b) \), which also leads to the characterisation of associated multipliers. An algorithmic procedure that exploits the companion structure of \( A \) is formulated and its numerical properties are investigated. The new method provides a robust procedure for the computation of LCM and enables the computation of approximate values, when the original data are inaccurate.

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Keywords: Algebraic computations; Least common multiple; Polynomials; System theory

1. Introduction

Two of the key problems of algebraic computations [3,6,10] are the computation of the greatest common divisor (GCD) and the computation of the least
common multiple (LCM) of a set of polynomials. From the applications in Control Theory viewpoint the GCD is linked with the computation of zeros of representations whereas LCM is connected with the derivation of minimal fractional representations of rational models, which are essential for the study of a variety of algebraic design problems [10].

We may always consider a set of polynomials in a degree ordered representation i.e. \( \mathcal{P} = \{ p_i(s), i \in k : p_i(s) \in \mathbb{R}[s], \deg(p_i(s)) = d_i, d_1 = n \geq d_2 = m \geq d_i, i = 3, \ldots, k \} \). Such sets, with fixed the first two maximal degrees \((n, m)\) may be represented by their coefficient vectors \( p_i \), where \( p_1 \in \mathbb{R}^{n+1}, p_i \in \mathbb{R}^{m+1}, i = 2, 3, \ldots, k \) and the set \( \mathcal{P} \) by the vector \( p = [p_1^T, p_2^T, \ldots, p_k^T]^T \in \mathbb{R}^\sigma, \sigma = (n + 1) + (m + 1)(k - 1) \), or by a point \( P \) in the projective space \( \mathbb{P}^{\sigma-1}(\mathbb{R}) \). Such a representation is linked to the generalised resultant [1] and allows the discussion of properties of sets \( \mathcal{P} \), when there is uncertainty in their coefficients.

The GCD and LCM problems are naturally interlinked [9], but they are of a different nature. For \( k \)-sets of polynomials ordered by the two maximal degrees \((n, m)\), those who have a nontrivial GCD (different than 1), belong to a proper variety of \( \mathbb{P}^{\sigma-1}(\mathbb{R}) \). Such variety is defined by the set of minors of the generalised resultant associated with \( \mathcal{P} \). In this sense, the existence of a nontrivial GCD is a nongeneric property. However, for every \( \mathcal{P} \) set the corresponding LCM always exist. It is such properties which make the GCD and LCM computational processes different. In fact, the difference in nature of GCD and LCM computations is expressed by the fact that uncertainty in the coefficients of the set of polynomials results in coprimeness and thus we have a trivial GCD equal to 1. This is due to that the existence of nontrivial GCD is a nongeneric property [5,13,17]. However, the same problem for LCM results always in a nontrivial solution with error in the coefficients and the degree.

For the case of two polynomials \( t_1(s), t_2(s) \) with LCM \( p(s) \) and GCD \( z(s) \), we have the standard identity that \( t_1(s)t_2(s) = z(s)p(s) \), which indicates the natural linking of the two problems. For randomly selected polynomials, the existence of a nontrivial GCD is a “nongeneric” property [9,17], but the corresponding LCM always exists. This suggests that there are fundamental differences between the two computational problems; such problems acquire a special dimension for engineering models, where uncertainty about the true value of the parameters on one hand and (rounding off) computational errors on the other, makes the computation of nongeneric values of invariants a very difficult task. This creates the need for the development of procedures, which lead to meaningful “approximate” solutions. Such computations are referred to as “approximate”, or robust algebraic computations [9]. The case of GCD computation has taken a lot of attention [8,9,12] (and references there in) and algorithms dealing with the evaluation of exact and approximate solutions have been derived.

This paper deals with the corresponding problems of LCM computation, which has taken much less attention so far. Existing procedures for the computation of LCM
rely on the standard factorisation of polynomials, computation of a minimal basis of a special polynomial matrix \([2]\) and use of algebraic identities, GCD algorithms and numerical factorisation of polynomials \([7]\). This paper presents an alternative approach to the computation of LCM, which is based on standard system theory concepts and avoids root finding, as well as use of the algebraic procedure and GCD computation. The new characterisation of LCM leads to an efficient computational procedure based on properties of controllability of a linear system, associated with the given set of polynomials, and also provides a procedure for the computation of the associated set of polynomial multipliers linked to LCM.

For a given set of polynomials \(P\), a natural realization \(S(A, b, C)\) is defined by inspection of the elements of the set \(P\). It is shown that the degree \(r\) of LCM is equal to the dimension of the controllable subspace of the pair \((A, b)\), whereas the coefficients of LCM express the relation of \(A'\vec{b}\) with respect to the basis of the controllable space. The companion form structure of \(A\) simplifies the computation of controllability properties and leads to a simple procedure for defining the associated set of polynomial multipliers of \(P\) with respect to LCM. A special feature of the algorithmic procedure is that a number of possibly difficult steps are substituted by simple closed formulae derived from the special structure of the system. An overall algorithmic procedure is formulated for computing LCM and multipliers, which is based on standard numerical linear algebra procedures. The numerical aspects of the algorithm are investigated and demonstrated by a number of examples. The developed results provide a robust procedure for the computation of LCM and enable the computation of approximate values, when the original data have some numerical inaccuracies. In such cases the method computes an approximate LCM with degree smaller than the "generic" degree (see Example 4.3 of Section 4 of the paper). In fact, a generic set of polynomials is coprime and thus their LCM is their product. The existence of an LCM with degree different than that of the product of polynomials occurs only when the given set of polynomials is not coprime. In fact, an LCM with degree less than the degree of the product of all polynomials implies that at least two polynomials in the set have a nontrivial GCD and this is a "nongeneric" property. Therefore, this degree with value less than the degree of the product is a "nongeneric" property for the set of all \(k\)-number polynomials (ordered by the \((n, m)\) degrees.

Throughout the paper if a property is said to be true for \(i \in k, k \in \mathbb{Z}^+\), this means it is true for all \(1 \leq i \leq k\). The proof of the results are given in Appendix A.

2. Problem statement, definitions and preliminary results

Consider the set of polynomials \(\mathcal{P} = \{p_i(s), i \in k : p_i(s) \in \mathbb{R}[s], \partial{p_i(s)} = d_i\}\), where \(\mathbb{R}[s]\) denotes the ring of polynomials over \(\mathbb{R}\) and \(\partial{\cdot}\) denotes the degree function. By LCM \(\{\mathcal{P}\} \triangleq p(s)\) and GCD \(\{\mathcal{P}\} \triangleq z(s)\) we shall denote the LCM and GCD.
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of the set $\mathcal{P}$ respectively. Without loss of generality, we may assume the polynomials $p_i(s)$ to be monic, i.e.

$$p_i(s) = s^{d_i} + a_1^i s^{d_i-1} + \cdots + a_{d_i-1}^i s + a_{d_i}^i, \quad i \in k,$$  

(2.1)

where $a_j^i$ denotes the coefficient $a_j$ of the $i$th polynomial of the set. We define the rational vector associated with $\mathcal{P}$

$$g_{\mathcal{P}}(s) = \begin{bmatrix} p_1(s)^{-1} \\ p_2(s)^{-1} \\ \vdots \\ p_k(s)^{-1} \end{bmatrix} \in \mathbb{R}^k(s)$$  

(2.2)

as the associated rational representative (ARR) of $\mathcal{P}$. Clearly if the $p_i(s)$ are not monic, then we can always write

$$g_{\mathcal{P}}(s) = \text{diag}\{c_1, \ldots, c_k\} \cdot \tilde{g}_{\mathcal{P}}(s),$$  

(2.3)

where $\tilde{g}_{\mathcal{P}}(s)$ corresponds to a monic set. The problem we consider here is the computation of the LCM of $\mathcal{P}$ using the system based properties of the minimal linear system associated with $g_{\mathcal{P}}(s)$ [3,6]. This is aimed as an alternative to the algebraic procedures [7] based on algebraic properties and use of GCD, or alternative algebraic procedures [2].

For the transfer function matrix $g_{\mathcal{P}}(s)$ we may always define a left, right coprime matrix fraction description (MFD) [6]

$$g_{\mathcal{P}}(s) = D_\ell(s)^{-1} n_\ell(s) = n_r(s) D_r(s)^{-1},$$  

(2.4)

where $D_\ell(s) \in \mathbb{R}^{k \times k}[s], D_r(s) \in \mathbb{R}[s]$. For the system represented by $g_{\mathcal{P}}(s)$ we have the following properties:

**Proposition 2.1.** The ARR $g_{\mathcal{P}}(s)$ of $\mathcal{P}$ has the following properties:

(i) $g_{\mathcal{P}}(s)$ has no finite zeros and it is strictly proper.

(ii) If $v = \min\{d_i : d_i = \partial\{p_i(s)\}, \forall i \in k\}$, then $g_{\mathcal{P}}(s)$ has an infinite zero of order $v$.

(iii) If $p(s)$ is an LCM of $\mathcal{P}$, then $p(s)$ is defined (modulo $c \in \mathbb{R}, c \neq 0$) by

$$p(s) = D_r(s) = |D_r(s)|.$$  

(2.5)

(iv) If $S(A', b', C')$ is any realization of $g_{\mathcal{P}}(s)$, then it is minimal, if and only if the following conditions hold true

$$p(s) = |sI - A'| = D_r(s) = |D_r(s)|.$$  

(2.6)

The above result provides a procedure for computing LCM as the characteristic polynomial of a minimal realization. An efficient procedure for such a computation is summarised by the following remark.
Remark 2.1. If $S(A^*, b^*, C^*)$ is a balanced minimal realization of $g_P(s)$ [11], then $p(s) = |sI - A^*|$ provides an efficient, realization based computational procedure for LCM. Although a balanced realization provides a stable numerical procedure, it has the disadvantage that it transforms the original data and this may lead to creation of additional numerical instabilities.

In the following, we shall work with a realization that is not minimal, which however is based on the original data and thus avoids introduction of additional numerical errors. Such a realization is defined below:

**Proposition 2.2.** Let $g_P(s) \in \mathbb{R}^{k(s)}$ be the ARR of $\mathcal{P}$. We may define a state space realization of $g_P(s)$ as follows:

(a) For every $p_i(s)$ as in (2.1), we define the phase variable realization of $p_i(s)^{-1}$ as:

$$p_i(s)^{-1} = c_i^T (sI - A_i)^{-1} b_i,$$

where

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_i^d & -a_i^{d_i-1} & \cdots & -a_i^2 & -a_i^1 \end{bmatrix} \in \mathbb{R}^{d_i \times d_i},$$

(2.7b)

$$c_i^T = [1, 0, \ldots, 0, 0] \in \mathbb{R}^{1 \times d_i}, \quad b_i = [0, 0, \ldots, 0, 1] \in \mathbb{R}^{d_i}.$$ (2.8a)

(b) Then $g_\mathcal{P}(s)$ has the following phase variable realization

$$g_\mathcal{P}(s) = C_\mathcal{P}(sI - A_\mathcal{P})^{-1} b_\mathcal{P},$$

where $C_\mathcal{P} \in \mathbb{R}^{k \times \sigma}$, $A_\mathcal{P} \in \mathbb{R}^{\sigma \times \sigma}$, $b_\mathcal{P} \in \mathbb{R}^{\sigma}$, $\sigma = \sum_{i=1}^{k} d_i$ and

$$C_\mathcal{P} = \text{bl-diag} \{ \ldots, c_i^T, \ldots \}, \quad A_\mathcal{P} = \text{bl-diag} \{ \ldots, A_i, \ldots \},$$

$$b_\mathcal{P} = \begin{bmatrix} b_1 \\ \vdots \\ b_\sigma \end{bmatrix}.$$ (2.8b)

The above realization is the phase variable realization [6] of $g_\mathcal{P}(s)$ and $S(A_\mathcal{P}, b_\mathcal{P}, C_\mathcal{P})$ will be called the normal system associated with $\mathcal{P}$ and shall be denoted in short by $S_\mathcal{P}$. The advantage of such realization is that it is defined by
inspection of the set $\mathcal{P}$ without any numerical operations. A key property of this realization is defined below:

**Proposition 2.3.** The normal system $S(A, b, C)$ associated with the set $\mathcal{P}$ is always observable, but not always controllable [6]. For the case where $\mathcal{P}$ has polynomials which are pairwise coprime, then $(A, b)$ is also controllable.

The above results provide the means for the computation of LCM based on the properties of the natural system $S$. The study of these properties lead to a new system theoretic characterisation of LCM and is considered next.

3. **LCM characterisation in terms of properties of the natural system**

The controllability properties of $S(A, b, C)$ are investigated next and are linked to the characterisation of LCM.

**Theorem 3.1.** Consider the normal system $S(A, b, C)$ of $\mathcal{P}$ and let $p(s)$ be the LCM of $\mathcal{P}$. The following properties hold true:

(i) If $r$ is the dimension of the controllable subspace of the pair $(A, b)$, then $r = \partial\{p(s)\}$.

(ii) Let $\tilde{x} = Px$ be a coordinate transformation that reduces $(A, b, C)$ triple to the controllability normal form $(\tilde{A}, \tilde{b}, \tilde{C})$

$$\tilde{A} = PA^{-1} = \begin{bmatrix} \tilde{A}_{12} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ O & \cdots & \cdots & \cdots \end{bmatrix}, \quad \tilde{b} = Pb = \begin{bmatrix} \tilde{b}_1 \\ \cdots \\ 0 \end{bmatrix},$$

$$\tilde{C} = C\tilde{A}^{-1}, \quad (3.1)$$

where $(\tilde{A}, \tilde{b})$ is controllable. Then,

(a) The monic LCM of $\mathcal{P}$ is defined by $p(s) = |sI - \tilde{A}|$.

(b) A right coprime MFD of $g_\mathcal{P}(s)$ is defined by

$$g_\mathcal{P}(s) = \tilde{C}\text{adj}(sI - \tilde{A})\tilde{b}|sI - \tilde{A}| = n(s) \cdot p(s)^{-1}.$$  

The above result establishes the link of LCM to the properties of the controllable space of $(A, b)$ pair. This space $\mathcal{Q}$ is defined as the column space of the controllability matrix [6].

$$Q(A, b) = [b, A\tilde{b}, \ldots, A\tilde{b}^{-1}]$$  

(3.3)
and the number \( r = \partial[p(s)] \) is defined by
\[
    r = \operatorname{rank} Q(A_{\varphi}, b_{\varphi}) = \dim \mathcal{Z}_{\varphi}.
\] (3.4)

**Lemma 3.1.** The rank \( r \) of \( Q(A_{\varphi}, b_{\varphi}) \) is the minimal number for which the set \( \{Q_i\} = \{b_{\varphi}, A_{\varphi}b_{\varphi}, \ldots, A_{\varphi}^{r-1}b_{\varphi}\} \) is independent whereas \( A_{\varphi}^{r-1}b_{\varphi} \) is dependent on the set \( \{Q_i\} \).

From the above it follows that determining a basis of \( \mathcal{Z}_{\varphi} \) does not require computation of the whole controllability matrix; such a basis is determined as follows:

**Remark 3.1.** A basis for \( \mathcal{Z}_{\varphi} \) is constructed by a searching procedure that tests the rank of matrices \( r_i = \operatorname{rank} Q_i(A_{\varphi}, b_{\varphi}) \), where
\[
    Q_i(A_{\varphi}, b_{\varphi}) = [b_{\varphi}, A_{\varphi}b_{\varphi}, \ldots, A_{\varphi}^{i-1}b_{\varphi}], \quad i = 0, 1, 2, \ldots, \sigma - 1.
\] (3.5)
The smallest index \( j \) for which \( r_{j-2} < r_{j-1} = r_j = \cdots \) defines the rank of \( Q(A_{\varphi}, b_{\varphi}) \) and thus \( Q_{r-1}(A_{\varphi}, b_{\varphi}) \) is a basis of \( \mathcal{Z}_{\varphi} \).

**Note.** The test for computing the possible increase of the rank at each step may become numerical. This may provide the means to introduce the LCM with a certain accuracy defined from the data set.

The basis matrix for \( \mathcal{Z}_{\varphi} \) defined by the columns of \( Q_{r-1}(A_{\varphi}, b_{\varphi}) \) is referred to as natural basis of \( \mathcal{Z}_{\varphi} \) and its significance for the computation of LCM is shown below:

**Theorem 3.2.** Let \( \mathcal{Z}_{\varphi} \) be the controllable subspace of \( S_{\varphi} \). The monic LCM of \( \mathcal{P} \) is linked to \( \mathcal{Z}_{\varphi} \) as shown below:

(i) \( p(s) \) is the characteristic polynomial of the restriction \( \tilde{A}_{\varphi} \) of the map \( A_{\varphi} \) on the \( A_{\varphi} \)-invariant subspace \( \mathcal{Z}_{\varphi} \).

(ii) The monic LCM of \( \mathcal{P} \) is the \( r \)-degree polynomial \( p(s) \), \( r = \dim \mathcal{Z}_{\varphi} \), \( p(s) = s^r + c_1s^{r-1} + \cdots + c_2s + c_1 \) the coefficients of which express the dependence of \( A_{\varphi}^{r}b_{\varphi} \) with respect to the natural basis \( \{Q_{r-1}(A_{\varphi}, b_{\varphi})\} \) of \( \mathcal{Z}_{\varphi} \) i.e.
\[
    A_{\varphi}^{r}b_{\varphi} = -c_1b_{\varphi} - c_2A_{\varphi}b_{\varphi} - \cdots - c_rA_{\varphi}^{r-1}b_{\varphi}.
\] (3.6)

The above results provide a system theoretic characterisation of LCM and a simple algorithmic procedure for its computation. A problem that is closely linked to LCM computation of the set \( \mathcal{P} = \{p_i(s), i \in k\} \) is the computation of the set of multipliers \( \mathcal{R} = \{r_i(s), i \in k\} \) of \( \mathcal{P} \) with respect to the LCM. These are defined by the factorisation problems
\[
    p(s) = p_i(s)r_i(s), \quad r_i(s) \in \mathbb{R}[s].
\] (3.7)
Remark 3.2. The factorisation problem defined by (3.7) may be reduced to a standard numerical linear algebra problem and an efficient procedure for its computation has been suggested in [9] as an alternative to the Euclidean division.

The computation of LCM, as described by (3.6) leads also to the following characterisation of LCM:

Corollary 3.1. Let \( \bar{Q} = Q_r(A, b) = [b, Ab, \ldots, A^{r-1}b, A'b] \), where \( r = \text{rank } Q(A, b) \). The row echelon form of \( \bar{Q} \) is given by

\[
\bar{Q} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & : & -c_1 \\
0 & 1 & 0 & \cdots & 0 & : & -c_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 & : & -c_r \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\
& & & & & & O \\
\end{bmatrix},
\]

(3.8)

where \( c_i \) define the LCM as \( p(s) = s^r + c_1 s^{r-1} + \cdots + c_2 s + c_1 \).

Remark 3.3. If \( \tilde{\bar{Q}} = [b, Ab, \ldots, A^{r-1}b] \) is the reduced controllability matrix of 
\((A, b)\) and \( \hat{Q} \) denotes any left inverse of \( \tilde{\bar{Q}} \), then the coefficient vector of the monic LCM \( p(s) = s^r + c_1 s^{r-1} + \cdots + c_2 s + c_1 \), i.e. \( \xi = [c_1, c_2, \ldots, c_r]^{\top} \) is defined by the first \( r \)-coordinates of the vector

\[
\begin{bmatrix}
\xi \\
\vdots \\
0
\end{bmatrix} = -\tilde{\bar{Q}} A' b.
\]

(3.9)

We consider now an alternative, direct approach for the characterisation and computation of the set of multiplies \( \mathcal{R} = \{r_i(s), i \in \mathbb{R}\} \). This alternative approach is based on the factorization of \( g_{\pm}(s) \) and is considered below. Consider the right MFD of \( g_{\pm}(s) \) i.e.

\[
g_{\pm}(s) = [\ldots, p_i(s)^{-1}, \ldots] = [\ldots, r_i(s), \ldots]^{\top} p(s)^{-1}.
\]

(3.10)

It is clear from the coprimeness property that the numerator polynomials are the multipliers of \( \mathcal{R} \) in \( p(s) \). Using the notation introduced above we have the following result:

Corollary 3.2. For the set \( \mathcal{R} \) with LCM \( p(s) \) and \( S(A, b, C) \) associated normal system, a right coprime MFD for \( g_{\pm}(s) \) is given by

\[
g_{\pm}(s) = [C \bar{Q} \text{adj}(sI - \bar{A}) \hat{Q}b]/[sI - \bar{A}]]^{-1} = \bar{n}_{\pm}(s)p(s)^{-1}.
\]

(3.11)
The vector \( r(s) = [r_1(s), \ldots, r_k(s)]^T \) defines the set of multipliers of \( P \) with respect to \( p(s) \).

The results derived here provide a system theoretic characterisation of the LCM and corresponding multipliers and provide the basis for a numerical linear algebra procedure for their computation. The general theoretical algorithm is summarised below:

3.1. Algorithm for LCM and multipliers

Given the set \( \mathcal{P} = \{ p_i(s) \in \mathbb{R}[s], i \in k \} \) we construct by inspection the normal system \( S(A, b, C) \), where \( A \in \mathbb{R}^{\sigma \times \sigma} \), \( \sigma = \sum_{i=1}^{k} d_i, d_i = \partial[p_i(s)], i \in k \). The LCM \( p(s) \) and the corresponding set of multipliers \( \mathcal{R} = \{ r_i(s), i \in k : p(s) = r_i(s)p_i(s) \} \) are defined as follows:

**Step 1:** Compute the matrices \( Q_i = [b, Ab, \ldots, A^{r-1}b] \) for \( i = 1, 2, \ldots, \sigma - 1 \) and determine the smallest integer \( r \) such that \( Q_{r-1} \) has full rank, but \( Q_r \) is rank deficient.

**Step 2:** Determine the dependency relationship amongst the vectors \( \{b, Ab, \ldots, A^{r-1}b, \ldots\} \) i.e.

\[
A' b = -c_1 b - c_2 Ab - \cdots - c_r A^{r-1}b.
\]

**Step 3:** Define the LCM \( p(s) \) as

\[
p(s) = s^r + c_r s^{r-1} + \cdots + c_2 s + c_1.
\]

**Step 4:** Define the reduced controllability matrix \( \tilde{Q} \),

\[
\tilde{Q} = [b, Ab, \ldots, A^{r-1}b]
\]

and define a left inverse of \( \tilde{Q} \), \( \tilde{Q} \in \mathbb{R}^{r \times \sigma} \).

**Step 5:** From the computed \( p(s) = s^r + c_r s^{r-1} + \cdots + c_2 s + c_1 \) define the associated companion matrix

\[
\tilde{A} = \begin{bmatrix}
0 & 0 & \cdots & 0 & -c_1 \\
1 & 0 & \cdots & 0 & -c_2 \\
0 & 1 & \cdots & 0 & -c_3 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -c_r
\end{bmatrix}
\]

and compute \( \text{adj}(sI - \tilde{A}) \).
Step 6: The vector of multipliers is then defined as

\[
\begin{bmatrix}
    r_1(s) \\
    r_2(s) \\
    \vdots \\
    r_k(s)
\end{bmatrix}
= C \tilde{Q} \text{adj}(sI - \bar{A}) \hat{Q} b.
\] (3.14)

The theoretical algorithm presented above involves a number of key computational issues, which will be considered in the following section. We close this section by demonstrating the theoretical algorithm in terms of an example.

Example 3.4. Consider the set \( P = \{p_1(s) = s^2 + 3s + 2, p_2(s) = s^2 + 4s + 3\} \).

The normal system is defined by inspection as

\[
A = \begin{bmatrix}
    0 & 1 & \cdots & O \\
    -2 & -3 & \cdots & \cdots \\
    \vdots & \vdots & \ddots & \vdots \\
    O & \cdots & -3 & -4
\end{bmatrix}, \quad
\hat{b} = \begin{bmatrix}
    0 \\
    1 \\
    \vdots \\
    -3
\end{bmatrix}, \quad
C = \begin{bmatrix}
    1 & 0 & : & 0 & 0 \\
    0 & 0 & : & 1 & 0
\end{bmatrix}.
\]

Step 1: Compute the vectors \( \hat{b}, A\hat{b}, A^2\hat{b} \) since the LCM is of degree at least two i.e.

\[
\hat{b} = \begin{bmatrix}
    0 \\
    1 \\
    \vdots \\
    1
\end{bmatrix}, \quad
A\hat{b} = \begin{bmatrix}
    1 \\
    -3 \\
    \vdots \\
    -4
\end{bmatrix}, \quad
A^2\hat{b} = \begin{bmatrix}
    -3 \\
    7 \\
    \vdots \\
    13
\end{bmatrix}.
\]

The matrix \([\hat{b}, A\hat{b}, A^2\hat{b}]\) has rank 3 and thus we compute next \( A^3\hat{b} \) and check the rank of \([\hat{b}, A\hat{b}, A^2\hat{b}, A^3\hat{b}]\)

\[
A^3\hat{b} = \begin{bmatrix}
    7 \\
    -15 \\
    \cdots \\
    -40
\end{bmatrix} \quad \text{and} \quad Q_3 = \begin{bmatrix}
    0 & 1 & -3 & 7 \\
    1 & -3 & 7 & -15 \\
    0 & 1 & -4 & 13 \\
    1 & -4 & 13 & -40
\end{bmatrix}.
\]

Since rank\( (Q_3) = 3 \) we have that \( r = 3 \).

Step 2: Use elementary row operations to reduce \( Q_3 \) to a row echelon form \( \tilde{Q}_3 \) and then solve the equation \( \tilde{Q}_3 \xi = 0 \), i.e.

\[
\tilde{Q}_3 = \begin{bmatrix}
    0 & 1 & -3 & 7 \\
    1 & -3 & 7 & -15 \\
    0 & 1 & -4 & 13 \\
    1 & -4 & 13 & -40
\end{bmatrix} \quad \tilde{Q}_3 = \begin{bmatrix}
    1 & 0 & 0 & -6 \\
    0 & 1 & 0 & -11 \\
    0 & 0 & 1 & -6 \\
    0 & 0 & 0 & 0
\end{bmatrix}.
\]
from the essential part of $\tilde{Q}_3$ (nonzero part) we have that
\[
\begin{bmatrix}
1 & 0 & 0 & -6 \\
0 & 1 & 0 & -11 \\
0 & 0 & 1 & -6
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4
\end{bmatrix} = 0 \rightarrow \zeta = \begin{bmatrix}
6 \\
11 \\
6 \\
1
\end{bmatrix}
\]
and thus the LCM of $\mathcal{P}$ is defined by $p(s) = s^3 + 6s^2 + 11s + 6$.

**Step 3:** Having found the LCM we may define $\tilde{A}$ as well as $\tilde{Q}_1$ as:
\[
\tilde{A} = \begin{bmatrix}
0 & 0 & -6 \\
1 & 0 & -11 \\
0 & 1 & -6
\end{bmatrix}
\quad \text{and} \quad
\tilde{Q} = \begin{bmatrix}
0 & 1 & -3 \\
1 & -3 & 7 \\
0 & 1 & -4 \\
1 & -4 & 13
\end{bmatrix}.
\]

For this $\tilde{A}$ we have
\[
\text{adj}(sI - \tilde{A}) = \begin{bmatrix}
s^2 + 6s + 11 & -6 & -6s \\
s + 6 & s^2 + 6s & -11s - 6 \\
1 & s & s^2
\end{bmatrix}
\]
and
\[
C \tilde{Q} = \begin{bmatrix}
0 & 1 & -3 \\
0 & 1 & -4
\end{bmatrix}.
\]
\[
\hat{Q} = \begin{bmatrix}
5 & 1 & -2 & 0 \\
4 & 0 & -3 & 0 \\
1 & 0 & -1 & 0
\end{bmatrix}.
\]
Thus
\[
C \hat{Q} \text{adj}(sI - \tilde{A}) \tilde{Q}_b = \begin{bmatrix}
r_1(s) \\
r_2(s)
\end{bmatrix} = \begin{bmatrix}
s + 3 \\
s + 2
\end{bmatrix},
\]
which verifies the fact that $p(s) = p_1(s)r_1(s) = p_2(s)r_2(s)$.

4. Numerical aspects of theoretical algorithm

The numerical implementation of the theoretical algorithm involves a number of particular computational problems, which are considered below:

4.1. Computational problems (CP)

*(CP.a)* Computation of $Q_i(A, b) = [b, Ab, \ldots, A^i b], i = 1, 2, \ldots, r$.
*(CP.b)* Test of rank of $Q_i(A, b), i = 1, 2, \ldots, r, r \leq \sigma$.
*(CP.c)* Computation of solution of $Q_r(A, b)x = 0$.
*(CP.d)* Computation of left inverse of $\tilde{Q} = Q_{r-1}(A, b)$.
*(CP.e)* Computation of adj$(sI - \tilde{A})$ and determination of the multipliers.
Next we analyze these problems.

(CP.a): Given the set \( \mathcal{P} \), the system \( S(A, b, C) \) is constructed by inspection and \( A \) contains the original data of \( \mathcal{P} \). The numerical aspects of computation of a controllability matrix for general pair \( (A, B) \) has been considered in [14] and in certain cases may be an unstable process. Here, however, the special structure of the pair \( (A, b) \) suggests that such computations are reduced to the simpler computation of the SISO phase variable form, where

\[
A_0 = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_\mu & -a_{\mu-1} & -a_{\mu-2} & \cdots & -a_1
\end{bmatrix}, ~ b_0 = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}, ~ a_0 = \begin{bmatrix}
-a_\mu \\
-a_{\mu-1} \\
\vdots \\
-a_1
\end{bmatrix}.
\]

(4.1)

In fact, for \( i = 1, 2, \ldots, \rho, \rho \geq \mu \), the block-diagonal structure of (2.7b), (2.8b) implies that

\[
A_i b_0 = \begin{bmatrix}
x_1^i \\
x_2^i \\
\vdots \\
x_\mu^i
\end{bmatrix}, \quad A_{i+1} b_0 = \begin{bmatrix}
x_1^{i+1} \\
x_2^{i+1} \\
\vdots \\
x_\mu^{i+1}
\end{bmatrix},
\]

where for all \( k > 0 \) we have

\[
x_1^{i+1} = x_2^i, \quad x_2^{i+1} = x_3^i, \ldots, x_{\mu-1}^{i+1} = x_\mu^i, \quad x_\mu^{i+1} = a_0^i A_0^i b_0
\]

and thus every step involves the computation of one scalar product.

Error analysis: If we perform floating point arithmetic with unit round off \( u_1 \), then [16]

\[
\lfl(a_0^i A_0^i) = a_0^i A_0^i b_0 + \epsilon_0,
\]

where

\[
|\epsilon_0| \leq (\mu + 1)u_1 |a_0^i| \cdot |A_0^i b_0|.
\]

The relative error (Rel) satisfies the relation

\[
\text{Rel} \leq (\mu + 1)u_1 \frac{|a_0^i| \cdot |A_0^i b_0|}{|a_0^i| \cdot |A_0^i b_0|}
\]

The relative error in the computed \( Q_i (A, b) \) may not be small if \( |a_0^i| \cdot |A_0^i b_0| \gg |a_0^i| \cdot |A_0^i b_0| \) [4].

If we proceed in analyzing further the required inner product computation it holds that:

\[
x_\mu^{i+1} = -a_{i+1} - a_i x_\mu^i - \sum_{j=-1}^{1} a_j x_{\mu+j}^{i+1-j}, \quad i = 0, 1, 2, \ldots
\]

(4.3)
We see that

For $i = 0$, $x_1 = -a_1$, and thus no rounding error is introduced in this computation.

For $i = 1$, $x_2 = -a_2 + a_1^2$. There is danger for rounding errors if $a_2 \approx a_1^2$ since we will have the problem of subtracting approximate equal numbers. In this case $|a_1^2| \cdot |A_0 b_0| \gg |a_1^2| \cdot A_0 b_0|$. If $a_2 \approx a_2$ and $a_3 \approx a_1^3$ we will have again the problem of subtracting approximate equal numbers. Again for this case will hold $|a_1^2| \cdot |A_2 b_0| \gg |a_1^2| \cdot A_2 b_0|$. For $i = 2$, $x_3 = -a_3 + a_1a_2 + (-a_2 + a_1^2)a_1$. If $a_2 \approx a_2$ and $a_3 \approx a_1^3$ we will have again the problem of subtracting approximate equal numbers. As proceeding further, since in the expression (4.3) the signs are interchanged, the difference between $|a_1^2| \cdot |A_2 b_0|$ and $|a_1^2| \cdot A_2 b_0|$ will be eliminated. Thus, in the final computation of the LCM the introduced rounding error will be caused from the initial subtractions of approximate equal numbers, which will appear only if the coefficients of the given polynomials belong to special subvarieties, and will be characterised from the loss of accuracy that will appear in the final result, according to course to the available significant digits provided from the mantissa (see Example 4.4).

**Computational complexity:** If we have $k$ polynomials of degrees $d_1, d_2, \ldots, d_k$ the formulation of $Q_i(A, b)$ requires $\sum_{j=1}^{k} (i - 1)d_j$ flops. Thus we will have flops of order $O((i - 1)\sigma)$.

(CP.b)–(CP.c): In the application of the algorithm we use the notion of numerical-$\epsilon$ rank of a matrix $A$ [14] i.e. the number of singular values of matrix $A$ that are greater than a specified accuracy $\epsilon$. According to this accuracy we will determine the smallest integer $r$ such that $Q_{r-1}$ has full numerical-$\epsilon$ rank and $Q_r$ is numerically-$\epsilon$ rank deficient. The computation of the singular values of $Q_i(A, b)$ is a stable process. When we have uncertainty in the coefficients we have to think how we select the accuracy $\epsilon$, since this affects the estimated order $r$ of the LCM. If we plot the numerical-$\epsilon$ ranks, $\rho_{i, \epsilon}(A)$, as a function of $i$ we would expect to get a nondecreasing function, which after some number of steps should reach a steady state value. The minimum $i$ required for this is the degree $r_i$. According to the used accuracy $\epsilon_i$ the rank increases or remains the same. Fig. 1 demonstrates this situation.

![Fig. 1. Rank behaviour.](image-url)
The determination of the dependency relationship amongst the vectors \([b, Ab, \ldots, A^{r-1}b, A^r b]\) is performed in a numerically stable way if we compute, using the singular value decomposition, the one dimensional numerical-\(\epsilon\) right null space \((N_{r, \epsilon}(Q_r), \epsilon(Q_r))\) of matrix \(Q_r = [b, Ab, \ldots, A^{r-1}b, A^r b]\). For this index \(r\) we will have that \(N_{r, \epsilon}(Q_1) = 0, N_{r, \epsilon}(Q_{r+1}) = 0, \ldots, N_{r, \epsilon}(Q_r) = 0, N_{r, \epsilon}(Q_{r+1}) \neq 0\). Extra care is needed for the appropriate estimation of the accuracy \(\epsilon\) according to which \(N_{r, \epsilon}(Q_{r+1}) = 0\). The singular vector corresponding to the smallest singular value is used as the best approximation. If there is uncertainty about the value of \(r\), according to the specified accuracy, we can determine exact and approximate solutions.

(CP.d): The computation of the left inverse \(\hat{Q}\) of \(\tilde{Q}\) can be performed in a stable way using the singular value decomposition of \(\tilde{Q}\). If \(\tilde{Q} = U \cdot \Sigma \cdot V^T\) is the SVD of \(\tilde{Q}\), then \(\hat{Q}\) can be expressed as: \(\hat{Q} = V \cdot \Sigma^{-1} \cdot U^T\).

(CP.e): The computation of multipliers is based on the following result.

**Proposition 4.1.** Consider the polynomial \(p(s) \in \mathbb{R}[s]\), where

\[
p(s) = s^r + c_r s^{r-1} + \cdots + c_2 s + c_1 \tag{4.4}
\]

with an associated companion matrix \(\tilde{A}\) and pencil \(sI - \tilde{A}\)

\[
sI - \tilde{A} = \begin{bmatrix}
s & 0 & \cdots & \cdots & 0 & c_1 \\
-1 & s & \cdots & \cdots & 0 & c_2 \\
0 & -1 & \cdots & \cdots & 0 & c_3 \\
\vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & -1 & s + c_r
\end{bmatrix} \tag{4.5}
\]

The adjoint of \(sI - \tilde{A}\) is a polynomial matrix \(P(s) \in \mathbb{R}^{r \times r}[s]\) with special structure. The polynomial entries \(p_{ij}(s)\) of \(P(s)\) are defined as follows:

(a) The main diagonal elements are polynomials derived from \(p(s)\) and they all have degree \(r - 1\). This set is

\[
p_{ii}(s) = \sum_{j=r-1}^{i-1} c_{j+2} s^j, \quad i = 1, 2, \ldots, r. \tag{4.6}
\]

(b) For a given row \(i\) the \(p_{ii}(s)\) element defines the rest of the elements of this row as follows:

(i) The elements to the right of \(p_{ii}(s)\) are

\[
p_{i,i+j}(s) = \sum_{k=i}^{1} -c_k s^{k-1}, \quad i < r,
\]

\[
p_{i,i+j}(s) = s^{i-1} p_{i,i+1}(s), \quad j = 2, \ldots, r - i. \tag{4.7}
\]
(ii) The elements to the left of \( p_{i}(s) \) are

\[
p_{i,j}(s) = \sum_{k=r-1}^{i-1} c_k s^{k-j}, \quad j = 1, 2, \ldots, i - 1.
\]

The above result is readily established by induction. An interesting implication of formulas (4.5)–(4.7) is that as long as we know the polynomial \( p(s) \), then \( \text{adj}\left((sI - \bar{A})\right) \) is defined by using these formulas that does not involve any additional calculations and thus no additional errors. This calculation can be directly done symbolically. In the sequel, the computation of the vector of multipliers \( r_k(s) \in \mathbb{R}^k[s] \) can be implemented symbolically as well by computing directly the product \( C \cdot \bar{Q} \cdot P(s) \cdot \bar{Q} \cdot b \).

4.2. Numerical results

The proposed algorithm was programmed in the Matlab environment and tested on a Pentium machine over several sets of polynomials \( \mathcal{P} \) characterised by various properties. The symbolic math toolbox of Matlab was used for the symbolic computations of (CPe). The accuracy \( \epsilon \) specifies the numerical \( \epsilon \) rank used for the determination of the degree of the LCM.

In order to measure the accuracy of the computed results we estimate the relative error (rel) between the expected theoretical value of the LCM and the computed value of the LCM following the proposed algorithm. Since the LCM is given from a polynomial of the form \( p(s) = s^r + c_r s^{r-1} + \cdots + c_2 s + c_1 \) we associate with it the vector \( p = [1 \quad c_r \quad c_{r-1} \quad \cdots \quad c_1]^T \). We say that the 2-norm of \( p(s) \) is equal to the 2-norm of its associated vector \( \| p \|_2 \). Thus, we will estimate the relative error using the 2-norm of the vector expressing the difference of the theoretical LCM and the computed LCM over the 2-norm of the vector expressing the theoretical LCM. Throughout the following examples, we will always use three significant digits for the estimation of rel.

Example 4.1. The following polynomial set contains real polynomials.

\[
\begin{align*}
p_1(s) &= s^3 + 4.6s^2 + 9.85s + 7.8 = (s + 1.5)(s^2 + 3.1s + 5.2), \\
p_2(s) &= s^2 + 5.2s + 5.55 = (s + 1.5)(s + 3.7).
\end{align*}
\]

For \( \epsilon = 10^{-16} \) we get the following results:

- computed LCM = \( s^4 + 8.299999999999993s^3 + 26.8699999999945s^2 
+ 44.24449999999855s + 28.8599999999773 \).

The relative error concerning the LCM computation is: \( \text{rel} = 4.59 \times 10^{-14} \).
Using seven significant digits, we get the following set of multipliers:

\[
\begin{bmatrix}
    r_1(s) \\
    r_2(s)
\end{bmatrix} = \begin{bmatrix}
    0.999999s + 3.699998 \\
    0.999991s^2 + 3.099998s + 5.199998
\end{bmatrix}.
\]

The relative error concerning the computation of multipliers \(r_1(s)\) and \(r_2(s)\) is:

\[
\text{rel}(r_1(s)) = 5.83 \times 10^{-7},
\]

\[
\text{rel}(r_2(s)) = 3.64 \times 10^{-7}.
\]

The relative error can be further reduced if we perform the symbolic computation using more significant digits.

**Example 4.2.** The following polynomial set contains integer polynomials of rather high degree

\[
p_1(s) = s^{10} + 2s^9 + 3s^8 + 4s^7 + 6s^6 + 12s^5 + 8s^4 + 4s^3 + 9s^2 + 6s + 1,
\]

\[
= (s^6 + 2s^5 + 3s^4 + 2s^3 + s^2 + 4s + 1)(s^4 + 2s + 1)
\]

\[
p_2(s) = s^8 + 2s^7 + 4s^6 + 4s^5 + 4s^4 + 6s^3 + 2s^2 + 4s + 1,
\]

\[
= (s^6 + 2s^5 + 3s^4 + 2s^3 + s^2 + 4s + 1)(s^2 + 1)
\]

\[
p_3(s) = s^8 + 2s^7 + 5s^6 + 6s^5 + 7s^4 + 8s^3 + 3s^2 + 8s + 2,
\]

\[
= (s^6 + 2s^5 + 3s^4 + 2s^3 + s^2 + 4s + 1)(s^2 + 1)
\]

\[
p_4(s) = s^7 + 5s^6 + 9s^5 + 11s^4 + 7s^3 + 7s^2 + 13s + 3,
\]

\[
= (s^6 + 2s^5 + 3s^4 + 2s^3 + s^2 + 4s + 1)(s + 3)
\]

For \(\epsilon = 10^{-16}\) and using 10 significant digits we get the following results:

computed LCM = \(s^{15} + 4.999999997s^{14} + 11.99999998s^{13}
\]

\[+ 27.9999997s^{12} + 46.99999992s^{11} + 78.9999998s^{10}
\]

\[+ 115.9999998s^9 + 143.9999997s^8 + 188.9999996s^7
\]

\[+ 176.9999995s^6 + 169.9999995s^5 + 157.9999996s^4
\]

\[+ 98.9999992s^3 + 74.9999998s^2 + 37.9999998s
\]

\[+ 5.99999943.
\]

The relative error concerning the LCM computation is: \(\text{rel} = 2.46 \times 10^{-9}\).

Using eight significant digits, we get the following set of multipliers:

\[
\begin{bmatrix}
    r_1(s) \\
    r_2(s) \\
    r_3(s) \\
    r_4(s)
\end{bmatrix} = \begin{bmatrix}
    s^5 + 3s^4 + 3s^3 + 9s^2 + 2s + 6,
    s^7 + 3s^6 + 2s^5 + 8s^4 + 7s^3 + 7s^2 + 14s + 6,
    s^7 + 3s^6 + s^5 + 5s^4 + 7s^3 + 5s^2 + 7s + 3,
    s^8 + 2.9999960s^6 + 2.0000025s^5 + 2.9999979s^4 + 6s^3
\end{bmatrix}
\]

\[+ 2.9999949s^2 + 4.0000024s + 1.99999.
\]

The relative error concerning the computation of the multiplier \(r_4(s)\) is: \(\text{rel} = 1.23 \times 10^{-6}\).
Example 4.3. The following polynomial set contains polynomials with varying coefficients.

\[ t_1(s) = s^2 - 3s + 2 = (s - 1)(s - 2), \]
\[ t_2(s) = s^2 - (3 - \varepsilon_1)s + (2 - \varepsilon_2). \]

For the case of exact coefficients (\( \varepsilon_1 = \varepsilon_2 = 0 \)), we have \( t_1(s) = t_2(s) \). Thus the exact GCD of the polynomials and the LCM is \( s^2 - 3s + 2 \). However if the coefficients of the polynomials become inexact (\( \varepsilon_1, \varepsilon_2 \neq 0 \)) we have a lot of LCMs whose degrees are 3 or 2. These depend strongly on the selection of the accuracy \( \epsilon \) that will define the numerical-\( \epsilon \) rank. More precisely, if

(i) \( \varepsilon_1 = \varepsilon_2 = -0.0001 \)

For \( \epsilon = 10^{-16} \):

approximate LCM = \(-4.00019999992399 + 8.00029999988601s - 5.00009999996201s^2 + s^3\).

Thus we compute an approximate LCM that equals to \( s^3 - 5.0001s^2 + 8.0003s - 4.0002 \) (keeping five significant digits).

Almost the same approximate LCM was computed in [7].

(ii) \( \varepsilon_1 = \varepsilon_2 = 2 \times 10^{-4} \)

\( \epsilon = \frac{1}{2} \times 10^{-4} \): approximate LCM = \( 2.000005003839484 - 3.00005001303614s + s^2 \).

Thus we compute an approximate LCM that equals to \( s^2 - 3.00005s + 2.00005 \) (keeping six significant digits).

Example 4.4. The coefficients of the following polynomial set belong to special subvarieties.

\[ p_1(s) = s^3 + 0.7495626s^2 + 0.56184312s + 0.42113924, \]
\[ p_2(s) = s + 1. \]

For this set we have that \( \alpha_2 \approx \alpha_1^2 \) and \( \alpha_3 \approx \alpha_1^3 \). The existence of this property in the coefficients of the polynomials is not a “generic” phenomenon. The theoretical LCM is: \( s^4 + 1.74956263s^3 + 1.31140575s^2 + 0.98298236s + 0.42113924 \).

We start the computation using a mantissa that can contain five significant decimal digits. We notice that:

For \( i = 1 \), \( |a_0^1| \cdot |A_1^1| = 1.12368726 \gg |a_0^1 \cdot A_1^1| = 1.01629251 \times 10^{-6} \).

For \( i = 2 \), \( |a_0^1| \cdot |A_1^1| = 0.84227661 \gg |a_0^1 \cdot A_1^1| = 3.39510029 \times 10^{-6} \).
For values of $i$ $\geq$ 3 the above quantities are approximate equal.
The computed LCM equals to:

$$s^4 + 1.74957120005612s^3 + 1.31141420638632s^2 + 0.98299073114751s + 0.42113924.$$

The relative error is equal to $0.6438 \times 10^{-6}$.

**Remark 4.5.** If we compare the proposed method with the balanced minimal realization method we notice that the balanced minimal realization method requires much more flops since many transformations of the original data are needed. More specifically in the Example (4.1) the computation of the LCM with the proposed method requires 5052 flops whereas the same computation with the balanced minimal realization method requires 48,444 flops. Furthermore the balanced minimal realization method cannot define various approximate LCM’s according to the specified accuracy. Thus in the Example (4.3) the proposed method, according to the selected accuracy, computed two approximate LCM’s of degrees 3 and 2 whereas the balanced minimal realization method computed only the LCM of degree 3 with of course much more flops (7059 flops were needed instead of 2703 that were required using the proposed method).

5. Conclusions

A new characterisation of the LCM of a set of polynomials has been given based on the properties of the controllable space of a linear system that is associated with the given set of polynomials. The distinguishing property of the new procedure is that the system is defined from the original data, without involving transformations and that it also provides the set of associated multipliers. The essential part of the numerical procedure is the determination of the successive ranks of parts of the controllability matrix, which due to the special structure of the system may be computed involving numerically stable procedures. The procedure has the advantage that it provides a meaningful way for determining the LCM in a numerically robust way and also indicates that approximate solutions to LCM are equivalent to determining the controllable space, with given accuracy of the associated natural realization.

Appendix A. Proof of results

**Proof of Proposition 2.1.** (i) If $s = z$ is a zero of $g_p(s)$, then $g_p(z) = 0$ and thus $p_i(z)^{-1} = 0, \forall i \in \mathbb{Z}$, which contradicts the fact that $p_i^{-1}(s)$ has no finite zero.
(ii) If $v$ is defined as above, then we can use the valuation representation [15] i.e.

$$\frac{1}{p_i(s)} = \frac{1}{s^{d_i}} u_i(s), \quad \forall \ i \in k,$$

where $u_i(s)$ is an appropriate unit of $R_{pr}(s)$. Clearly, then from the definition of valuations and infinite zeros [15] we have that

$$v = \min\{d_i, \forall \ i \in k\}.$$

(iii), (iv) These parts are standard from linear systems [3,6]. □

**Proof of Proposition 2.2.** By defining the realization of $p_i(s)^{-1} = c^i_I(sI - A_i)^{-1} b_i$ then

$$g_{\beta}(s) = \begin{bmatrix} c^i_I(sI - A_i)^{-1} b_i \\ \vdots \\ c^n_I(sI - A_n)^{-1} b_n \end{bmatrix} = \text{bl-diag}[\ldots, c^i_I, \ldots]\text{bl-diag}[\ldots, sI - A_i, \ldots]^{-1} b_i$$

and this completes the proof of the result. □

**Proof of Proposition 2.3.** The block-diagonal structure of $(A_\beta, C_\beta)$ pair suggests that observability is equivalent to the fact that each pair $(A_i, c^i_I)$ is observable. However, for SISO systems the phase variable realization is both observable and controllable and this establishes the first part of the result. Note that if the polynomials of $\beta$ are pairwise coprime, then LCM is defined as $p(s) = \prod_{i=1}^k p_i(s)$ and $\partial\{p(s)\} = \sigma = \sum_{i=1}^k \sigma_i$ is the McMillan degree of $g_{\beta}(s)$ which coincides with the dimension of the realization. This completes the proof. □

**Proof of Theorem 3.1.** We note that $S_{\beta}(A_{\beta}, b_{\beta}, C_{\beta})$ is observable but not necessarily controllable. Let $\bar{x} = P_{\tilde{A}}$ be the coordinate transformation that reduces to the controllable normal form, then $r$ is the dimension of the controllable space of $(A_{\beta}, b_{\beta})$. For the new description we have:

$$g_{\beta} = C_{\beta}(sI - A_{\beta})^{-1} b_{\beta} = \bar{C}_{\beta}(sI - \tilde{A}_{\beta})^{-1} \bar{b}_{\beta}$$

$$= [\bar{C}_{\beta}, \bar{C}_{\beta}'] \begin{bmatrix} (sI - \tilde{A}_{\beta})^{-1} & \cdots & X \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (sI - \tilde{A}_{\beta})^{-1} \end{bmatrix} \begin{bmatrix} \bar{b}_{\beta} \\ \vdots \\ 0 \end{bmatrix}$$

Clearly, the triple $(\tilde{A}_{\beta}, \bar{b}_{\beta}, \bar{C}_{\beta})$ is controllable and observable (the original system is observable) and thus it is a minimal realization of $g_{\beta}(s)$. Condition (2.10) is then readily established. □
Proof of Lemma 3.1. Assume that \( r = \text{rank}(Q(A_P, b_P)) \) and let us denote by \( q_i = A_i P b_P \). Assume now that

\[
q_{r+1} = c_{r-1} q_{r-1} + \cdots + c_1 q_1 + c_0 q_0,
\]

then

\[
q_{r+1} = A q_{r+1} = c_{r-1} A q_{r-1} + \cdots + c_1 A q_1 + c_0 A q_0
= c_{r-1} q_{r-1} + \cdots + c_1 q_1 + c_0 q_0
\]

\[
= e_{r-1} \left\{ \sum_{j=0}^{r-1} c_j q_j \right\} + e_{r-2} q_{r-1} + \cdots + c_1 q_2 + c_0 q_0
\]

and thus \( q_{r+1} \) is dependent on \( \{Q_r\} \); by induction we may prove that for all \( j > r + 1 \) this property also holds true. The above establishes the dependence of all \( q_j \), for \( j \geq r \) on \( \{Q_r\} \). Note now that the set \( \{Q_r\} \) has to be independent, since otherwise \( Q(A_P, b_P) \) will have rank less than \( r \) and this completes the proof. \( \square \)

Proof of Theorem 3.2. It is known that the controllable subspace \( \mathcal{Z}_P \) is \( A_P \)-invariant [17]. The construction of the coordinate transformation that leads to the derivation of the controllable form in (8) is \( \tilde{x} = P \bar{x} \), where [3]

\[
P^{-1} \triangleq Q = [q_1, q_2, \ldots, q_r, q_{r+1}, \ldots, q_{\sigma}]
\]

(A.1)

with the first columns \( [q_1, q_2, \ldots, q_r] \) defined by the vectors in \( Q_{r-1}(A_P, b_P) \) and the remaining \( \sigma - r \) vectors entirely arbitrary, so long as the matrix \( Q \) is nonsingular.

The transformation defined above reduces the \( (A_P, b_P) \) pair to the \( (\tilde{A}_P, \tilde{b}_P) \) description in (3.1) and clearly, \( \tilde{A}_P \) is the restriction of \( A_P \) on \( \mathcal{Z}_P \).

To compute the restriction map \( \tilde{A}_P \) we consider the basis \( [q_1, q_2, \ldots, q_r] \) for \( \mathcal{Z}_P \).

Then

\[
A_P[q_1, q_2, \ldots, q_r] = [AQ_1, AQ_2, \ldots, AQ_{r-1}, AQ_r]
= [q_2, q_3, \ldots, q_r, A' b_P].
\]

(A.2)

If \( A' P b_P = q_{r+1} \) is expressed as in (11b) then (A.2) leads to

\[
A_P[q_1, q_2, \ldots, q_r] = [q_2, \ldots, q_r, -c_1 \ldots -c_1]

= [q_1, q_2, \ldots, q_r]
\]

(A.3)
and thus with respect to the basis \([q_1, \ldots, q_r]\), the restriction map is
\[
\bar{A}_\rho = \begin{bmatrix}
0 & 0 & \cdots & 0 & -c_1 \\
1 & 0 & \cdots & 0 & -c_2 \\
0 & 1 & \cdots & 0 & -c_3 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -c_r \\
\end{bmatrix}
\] (A.4)

and this completes the proof. \(\square\)

Proof of Corollary 3.1. By (3.6) the coefficients of LCM are defined by
\[
[b, Ab, \ldots, A^{-1}b, A'b] = \begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_r \\
1 \\
\end{bmatrix} = 0.
\] (A.5a)

Given that \([b, Ab, \ldots, A^{-1}b]\) has rank \(r\), there exists \(T \in \mathbb{R}^{\sigma \times \sigma}, |T| \neq 0\) such that
\[
T[b, Ab, \ldots, A^{-1}b] = \begin{bmatrix}
I_r \\
\vdots \\
0 \\
\end{bmatrix}.
\] (A.5b)

Given that \(A'b\) is dependent on \([b, Ab, \ldots, A^{-1}b]\), it follows that for the transformation \(T\) defined above we also have
\[
T[b, Ab, \ldots, A^{-1}b, A'b] = \begin{bmatrix}
I_r & \vdots & d \\
\vdots & \ddots & \vdots \\
0 & \vdots & \tilde{d} \\
\end{bmatrix}.
\] (A.5c)

From the dependence of \(A'b\) on \([b, Ab, \ldots, A^{-1}b]\), it follows that \(\tilde{d} = 0\), since otherwise \(\text{rank}([b, Ab, \ldots, A^{-1}b, A'b]) > r\). If we now denote by \(d = [d_1, d_2, \ldots, d_r]\), then by multiplying (A.5a) on the left by \(T\) and using (A.5c), it is readily shown that
\[
d_i = -c_i, \quad \forall \ i \in r
\] (A.5d)

and this completes the proof. \(\square\)

Proof of Corollary 3.2. Consider the normal system \(S(A, b, C)\) of \(\dot{\tilde{x}} = A\tilde{x} + b\tilde{u}, \ y = C\tilde{x}\) and carry out the coordinate transformation \(\tilde{x} = P\bar{x}, \ \bar{x} = P^{-1}Q\) such that
\[
P^{-1} = Q = [\bar{b}, \ Ab, \ldots, A^{-1}b; q_{r+1}, \ldots, q_r] = [Q, \bar{Q}].
\] (A.6a)
where $Q_1$ is the reduced controllability matrix and $Q_2$ provides a completion of a basis to $\mathbb{R}^\sigma$. The transformed system is

$$
S(\tilde{A}, \tilde{b}, \tilde{C}) : \begin{cases}
\tilde{A} = PAP^{-1} = Q^{-1}AQ, \\
\tilde{b} = Pb = Q^{-1}b, \tilde{C} = CP^{-1} = CQ.
\end{cases}
$$

(A.6b)

where also

$$
\tilde{A} = Q^{-1}AQ = \begin{bmatrix}
\hat{A} & \cdots & \hat{A}_{12} \\
\vdots & \ddots & \vdots \\
0 & \cdots & \hat{A}'
\end{bmatrix}.
$$

(A.6c)

The resolvent $(sI - \tilde{A})^{-1}$ may be computed by using

$$
\begin{bmatrix}
sI - \tilde{A} & \cdots & -\tilde{A}_{12} \\
\vdots & \ddots & \vdots \\
0 & \cdots & sI - \tilde{A}'
\end{bmatrix}
\begin{bmatrix}
X_1(s) & \cdots & X_2(s) \\
\vdots & \ddots & \vdots \\
X_3(s) & \cdots & X_4(s)
\end{bmatrix} = \begin{bmatrix}
I_r & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & I_{\sigma - r}
\end{bmatrix}
$$

from which it follows that

$$
(sI - \tilde{A})X_1(s) - \tilde{A}_{12}X_3(s) = I, \quad (sI - \tilde{A})X_2(s) = 0,
$$

$$
(sI - \tilde{A})X_3(s) - \tilde{A}_{12}X_4(s) = 0, \quad (sI - \tilde{A})X_4(s) = I
$$

and from the above we readily have that

$$
\begin{bmatrix}
sI - \tilde{A} & \cdots & -\tilde{A}_{12} \\
\vdots & \ddots & \vdots \\
0 & \cdots & sI - \tilde{A}'
\end{bmatrix}^{-1} = \begin{bmatrix}
(sI - \tilde{A})^{-1} & \cdots & (sI - \tilde{A})^{-1}\tilde{A}_{12} (sI - \tilde{A}')^{-1} \\
\vdots & \ddots & \vdots \\
0 & \cdots & (sI - \tilde{A}')^{-1}
\end{bmatrix}.
$$

(A.6d)

If we set $Q^{-1} = \hat{Q}$, then

$$
\hat{Q} = \begin{bmatrix}
\hat{Q}_1 & \cdots & \hat{Q}_2 \\
\hat{Q}_1 & \cdots & \hat{Q}_2
\end{bmatrix}
$$

and

$$
\tilde{b} = Pb = Q^{-1}b = \begin{bmatrix}
\hat{b} \\
0
\end{bmatrix}.
$$

(A.6e)

Note that $\hat{Q}_2b = 0$, since the coordinate transformation reduces $(A, b)$ to the controllable form (see Theorem (3.1)). Similarly, if

$$
Q = \begin{bmatrix}
\hat{Q}_1 & \cdots & \hat{Q}_2
\end{bmatrix}
$$

then $\tilde{C} = C[Q_1, Q_2] = [Q_1, CQ_2].$

(A.6f)

By the definition $\hat{Q}$ we have

$$
\begin{bmatrix}
\hat{Q}_1 & \hat{Q}_2
\end{bmatrix}
\begin{bmatrix}
I_r & 0 \\
0 & I_{\sigma - r}
\end{bmatrix} = \hat{Q}_1Q_1 = I_r
$$

(A.6g)
and thus $\tilde{Q}_1$ is a left inverse of $Q_1$. The overall transfer function may thus be expressed as:

$$P_\rho(s) = [CQ_1; CQ_2] \begin{bmatrix} (sI - \bar{A})^{-1} & \cdots & (sI - \bar{A})^{-1}\bar{A}_{12} (sI - \bar{A}')^{-1} \\ \vdots & \ddots & \vdots \\ O & \cdots & (sI - \bar{A}')^{-1} \end{bmatrix} \begin{bmatrix} \tilde{Q}_1b \\ \vdots \\ 0 \end{bmatrix} = CQ_1(sI - \bar{A})^{-1}\tilde{Q}_1b = (CQ_1 \text{adj}(sI - \bar{A})\tilde{Q}_1b)[|sI - \bar{A}|]^{-1}. \quad (A.6h)$$

Clearly, since the pole polynomial $|sI - \bar{A}|$ is the LCM, the factorisation is minimal and thus the multipliers may be readily found from the numerator

$$[r_1(s), \ldots, r_k(s)]^T = CQ_1 \text{adj}(sI - \bar{A})\tilde{Q}_1b. \quad \square \quad (A.6i)$$

References