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ON A CLASS OF SOME SPECIAL SETS ON THE k-SKELETON OF A CONVEX COMPACT SET

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ABSTRACT

In this paper, generalizing the notion of a path we define a k-area to be the set $D = \{g(t) : t \in J\}$ on the k-skeleton of a convex compact set K in a Hilbert space, where g is a continuous injection map from the k-dimensional convex compact set J to the k-skeleton of K. We also define an E^k-area on K, where E^k is a k-dimensional subspace, to be a k-area with the property $\pi(g(t)) = t$, $t \in \pi(K)$, where π is the orthogonal projection on E^k. This definition generalizes the notion of an increasing path on the 1-skeleton of K. The existence of such sets is studied when K is a subset of a Euclidean space or of a Hilbert space. Finally some conjectures are quoted for the number of such sets in some special cases.

1. Introduction

Let K be a convex compact set in a Hilbert space $\mathscr{H}$ and let E^k be a k-dimensional subspace of $\mathscr{H}$. Then the orthogonal complement of E^k is a subspace

$$(E^k)^\perp = \{l_1(x) = l_2(x) = \cdots = l_k(x) = 0\},$$

where $l_1, l_2, \ldots, l_k$ are linearly independent continuous linear functionals. Let π be the orthogonal projection on E^k parallel to $(E^k)^\perp$. We quote now the following definitions.

Definition 1.1. A subset D of K is defined to be a *k-area on K*, $k = 1, 2, \ldots$ iff there exists a k-dimensional compact convex subset J of $\mathscr{H}$ and a continuous injection map $g : J \to \mathrm{skel}_k K$ with $D = g(J)$.

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DEFINITION 1.2. A subset D of K is defined to be an *E^k-area of K* iff D is a k-area of K and $\pi(g(t)) = t$, $t \in \pi(K)$.

Note that for $k = 1$ a k-area on K is a path on the $\mathrm{skel}_1 K$ and an E^k-area is an l_1-strictly increasing path on the $\mathrm{skel}_1 K$.

The existence and the number of l_1-increasing paths on $\mathrm{skel}_1 K$ in a Euclidean space E^d was studied in [8], [4] and [5], while the same problem in a normed space E of infinite dimension was studied in [7], [1] and [2].

If K is a convex body in E^d then it is an E^d-area.

In this paper we study the existence of a k-area and of E^k-area in Euclidean and Hilbert spaces as well as several related problems.

2. Existence of a k-area of K in E^d

In this section we study the existence of a k-area of a convex body K in the Euclidean space E^d and the "measure" of the k-dimensional subspaces of E^d for which there exist E^k-areas of K.

THEOREM 2.1. *Let K be a convex body in E^d and let E^k be a k-dimensional subspace of E^d, $2 \leqq k \leqq d-1$. Then for every $\varepsilon > 0$ there exists a projection $\omega : E^d \to E^k$ and a k-area $D = g(\omega(K))$ on K such that $\omega(g(t)) = t$, $t \in \omega(K)$ and for every $t, t' \in \omega(K)$, $\| g(t) - g(t') \| \geqq (1-\varepsilon) \| t - t' \|$.*

PROOF. We consider first the case $k = d - 1$. From Theorem 1 in [3] we have that for every $\varepsilon > 0$ there exists a unit vector $p_d \in E^d$, $p_d \notin E^k = E^{d-1}$ such that $\cos \measuredangle (p_d, e_d) \geqq 1 - \varepsilon$ and there are no line segments on the boundary of K in the direction p_d.

Let proj_{p_d} be the projection map on E^{d-1} in the direction p_d. Now for $t \in \mathrm{proj}_{p_d}(K)$ we define $\Lambda_t := \mathrm{proj}_{p_d}^{-1}(t) \cap K$. Then

$$\Lambda_t = \{(t, \lambda) \in E^d : t \in E^{d-1}, \alpha_t \leqq \lambda \leqq \beta_t\}$$

with $\alpha_t < \beta_t$ if $t \in \mathrm{relint}(\mathrm{proj}_{p_d}(K))$ and $\alpha_t = \beta_t$ if $t \in \mathrm{relbd}(\mathrm{proj}_{p_d}(K))$. We define $g_1(t) := (t, \alpha_t)$, $g_2(t) := (t, \beta_t)$, $t \in \mathrm{proj}_{p_d}(K)$.

Because of the convexity of K, g_1 and g_2 are continuous on $\mathrm{relint}(\mathrm{proj}_{p_d}(K))$. From the choice of p_d one can easily see that g_1 and g_2 are continuous on $\mathrm{relbd}(\mathrm{proj}_{p_d}(K))$. Therefore g_1 and g_2 are continuous on $\mathrm{proj}_{p_d}(K)$. Also, if $D_1 := g_1(\mathrm{proj}_{p_d}(K))$ and $D_2 := g_2(\mathrm{proj}_{p_d}(K))$, then $D_1, D_2 \subseteq \mathrm{skel}_k K$ with

$$g_1(\mathrm{relint}(\mathrm{proj}_{p_d}(K))) \cap g_2(\mathrm{relint}(\mathrm{proj}_{p_d}(K)) = \varnothing .$$

Now taking $\omega := \mathrm{proj}_{p_d}$ and $g = g_1$ (or g_2) the result follows.

Consider now the case $2 \leqq k < d-1$. Let $(E^k)^\perp = E^{d-k} := [e_{k+1}, e_{k+2}, \ldots, e_d]$ where e_i, $k+1 \leqq i \leqq d$ is a set of orthonormal vectors, $M^{d-i} := E^k \oplus [e_{k+1}, e_{k+2}, \ldots, e_{d-i}]$, $i = 0, 1, 2, \ldots, d-k-1$ and $M^k = E^k$. As before we may choose unit vectors $p_d, p_{d-1}, \ldots, p_{k+2}$ in $M^d = E^d$, $M^{d-1}, \ldots, M^{k+1}$ respectively with $p_i \notin M^{i-1}$, $i = k+2, k+3, \ldots, d$, arbitrarily close to $e_d, e_{d-1}, \ldots, e_{k+2}$ and there is no line segment on $\mathrm{bd}(K)$ in the direction p_d and also there is no line segment on $\mathrm{relbd}(\mathrm{proj}_{p_{i+1}} \circ \mathrm{proj}_{p_{i+2}} \circ \cdots \circ \mathrm{proj}_{p_d}(K))$ in the direction p_i, $i = k+2, k+3, \ldots, d-1$. Let

$$\omega_1 := \mathrm{proj}_{p_{k+2}} \circ \mathrm{proj}_{p_{k+3}} \circ \cdots \circ \mathrm{proj}_{p_d} \text{ where } \mathrm{proj}_{p_i} : M^i \to M^{i-1},$$

$i = k+2, \ldots, d$ is the projection map in the direction p_i and $\mu > 0$ such that $\| \omega_1(x) \| \leqq \mu \| x \|$, $x \in E^d$. Then $\omega_1(K)$ is a convex body in M^{k+1} and so we may find $p_{k+1} \in M^{k+1}$ and a k-area $B = \{h(t) : t \in \mathrm{proj}_{p_{k+1}}(\omega_1(K))\}$ with $\mathrm{proj}_{p_{k+1}}(h(t)) = t$, $t \in \mathrm{proj}_{p_{k+1}}(\omega_1(K))$, by case $k = d-1$. Let $\omega = \mathrm{proj}_{p_{k+1}} \circ \omega_1$ and $g(t) = \omega_1^{-1}(h(t))$, $t \in \omega(K)$. By the selection of p_i, $i = k+2, k+3, \ldots, d$ the map g is well defined, one to one, $g(t) \in \mathrm{skel}_k(K)$ and $\omega(g(t)) = t$, $t \in \omega(K)$.

In order to prove that $D = g(\omega(K))$ is a k-area of K it remains to prove that g is continuous on $\omega(K)$. Let $t \in \omega(K)$ and let $\{t_n\}_{n=1}^{\infty}$ be a sequence in $\omega(K)$ with $\lim_{n\to\infty} t_n = t$. As K is compact we suppose $\lim_{n\to\infty} g(t_n) = x_0 \in K$. Then from the definition of ω_1 and g we have

$$\begin{aligned} \| \omega_1(g(t)) - \omega_1(x_0) \| &\leqq \| \omega_1(g(t)) - \omega_1(g(t_n) \| + \| \omega_1(g(t_n)) - \omega_1(x_0) \| \\ &\leqq \| h_1(t) - h(t_n) \| + \mu \| g(t_n) - x_0 \| . \end{aligned}$$

The continuity of h on $\omega(K)$ implies that $\omega_1(g(t)) = \omega_1(x_0)$ and from the definition of g, $g(t) = x_0 = \lim_{n\to\infty} g(t_n)$. Therefore g is continuous on $\omega(K)$. Also as $p_d, p_{d-1}, \ldots, p_{k+1}$ can be chosen as close as we please to e_d, $e_{d-1}, \ldots, e_{k+1}$ respectively, we can construct g so that $\| g(t) - g(t') \| \geqq (1-\varepsilon) \| t - t' \|$ with $t, t' \in \omega(K)$. This concludes the proof of the theorem.

From the above theorem we have the following corollaries, where the proof of the first one is obvious.

COROLLARY 2.1. *Let K be a convex body in E^d and E^{d-1} a $(d-1)$-dimensional subspace of E^d. If there are no line segments on* $\mathrm{bd}(K)$ *perpendicular to E^{d-1} then there exist two E^{d-1}-areas on K, $D_i = \{g_i(t) : t \in \pi(K)\}$, $i = 1, 2$,* such that $g_1(\mathrm{relint}\, \pi(K)) \cap g_2(\mathrm{relint}\, \pi(K)) = \emptyset$.

COROLLARY 2.2. *Let K be a convex body in E^d and E^k is a k-dimensional* *subspace* $2 \leqq k < d-1$. *If the directions of line segments on* bd(K) *perpendicular to* E^k *form a subset of* $(d-k-1)$*-dimensional Hausdorff measure zero on the boundary of the unit ball of* $(E^k)^\perp = E^{d-k}$, *then there exists an* E^k*-area on* K.

PROOF. We may select the vectors $p_d, p_{d-1}, \ldots, p_{k+1}$ of Theorem 2.1 to be orthonormal and lying in E^{d-k}. This selection entails ω to be the orthogonal projection on E^k.

COROLLARY 2.3. *Let* K *be a convex body in* E^d *and* E^k *is a* k*-dimensional subspace*, $2 \leqq k \leqq d-1$. *Let* π *be the orthogonal projection on* E^k *and* $\pi(K) = f(I^k)$ *where* $I = [0, 1]$ *and* f *is a continuous one to one map. Then there exists a sequence* $D_r = \{h_r(t) : t \in I^k\}$, $r = 1, 2, \ldots$ *of* k*-areas on* K *with* $\{\pi \circ h_r\}_{r=1}^\infty$ *converging uniformly to* f *on* I^k.

PROOF. Let $(E^k)^\perp = E^{d-k} = [e_{k+1}, e_{k+2}, \ldots, e_d]$. We may select vectors $p_d^{(r)}, p_{d-1}^{(r)}, \ldots, p_{k+1}^{(r)}$ with $\lim_{r\to\infty} p_i^{(r)} = e_i$, $i = k+1, k+2, \ldots, d$ and using $\omega_r = \mathrm{proj}_{p_{k+1}^{(r)}} \circ \cdots \circ \mathrm{proj}_{p_d^{(r)}}$, $r = 1, 2, \ldots$ we construct, as in the proof of Theorem 2.1, k-areas, $D_r = \{g_r(t) : t \in \omega_r(K)\}$ on K with $\omega_r(g_r(t)) = t$, $t \in \omega_r(K)$, for $r = 1, 2, \ldots$. The sequence of projections $\{\omega_r\}_{r=1}^\infty$ converges uniformly to π on the compact body K. Therefore, we may take $\omega_r(K) = \{f_r(t), t \in I^k\}$ where f_r is a continuous injection map on I^k for $r = 1, 2, \ldots$ and such that $\{f_r\}_{r=1}^\infty$ converges uniformly to f on I^k. Then, taking $h_r = g_r \circ f_r$ we have that $D_r = \{h_r(t), t \in I^k\}$ and $\omega_r(h_r(t)) = f_r(t)$, $t \in I^k$ for $r = 1, 2, \ldots$.

As I^k is compact, in order to prove the uniform convergence of $\{\pi \circ h_r\}_{r=1}^\infty$ to f on I^k, it suffices to prove $\lim_{r\to\infty} \pi(h_r(t_r)) = f(t_0)$ for any sequence $\{t_r\}_{r=1}^\infty$ of points of I^k whose limit is $t_0 \in I^k$. As K is compact we may suppose that $\lim_{r\to\infty} h_r(t_r) = x_0 \in K$. Then the uniform convergence of the sequences $\{f_r\}_{r=1}^\infty$ and $\{\omega_r\}_{r=1}^\infty$ implies that

$$f(t_0) = \lim_{r\to\infty} f_r(t_r) = \lim_{r\to\infty} \omega_r(h_r(t_r)) = \pi(x_0) = \lim_{r\to\infty} \pi(h_r(t_r)).$$

Therefore the proof is complete.

We may remark that from the proof of Theorem 2.1 for any convex body K in E^d, there exists always a k-area D that is not necessarily an E^k-area for a fixed subspace E^k, $2 \leqq k \leqq d-1$. For a further support of this assertion we give a simple example of a convex body in E^3 that has not an E^2-area for a fixed subspace E^2. Define the following set:

$$K = \mathrm{conv}(\{(x, y, 0) : (x-1)^2 + y^2 \leqq 1\} \cup \{(0, 0, 1), (0, 0, -1)\})$$

and let E^2 be the plane $z = 0$. Then it is easy to see that there does not exist an E^2-area on K.

From the above remark the following question arises: For a convex body in E^d "how many" k-dimensional subspaces E^k, $2 \leqq k \leqq d-1$ are there, such that no E^k-area exists on K? Of course the expression "how many" must be defined properly. To this end, for each E^k in E^d there is associated a point pair $\pm G(E^k)$ in $E^{\binom{d}{k}}$ (see [8]). The Grassmanian I_k^d will be taken to be the collection of all these pairs corresponding to the different k-dimensional subspaces E^k of E^d. The set I_k^d is an algebraic manifold in $E^{\binom{d}{k}}$ of real dimension $k(d-k)$, of positive $k(d-k)$-dimensional Hausdorff measure in $E^{\binom{d}{k}}$ and certainly of non-σ-finite $(k(d-k)-1)$-dimensional Hausdorff measure. With the above notation we have the following theorem.

THEOREM 2.2. *Let K be a convex body in E^d and let*

$$A = \{\pm G(E^k) \text{ such that there is no } E^k\text{-area on } K\}.$$

Then A forms a set in I_k^d of σ-finite $(k(d-k)-1)$-dimensional Hausdorff measure.

PROOF. Let E^k be a subspace with its orthogonal complement E^{d-k} of non-singular direction, i.e., there are no line segments on the bd(K) parallel to E^{d-k}. Then from Corollaries 2.1 and 2.2 there exists an E^k-area on K. Hence for any E^k with E^{d-k} non-singular, $\pm G(E^k) \notin A$ so $A \subseteq \{\pm G(E^k) : E^{d-k}$ singular$\}$. The set $\{\pm G(E^{d-k}) : E^{d-k}$ singular) in $E^{\binom{d}{d-k}} = E^{\binom{d}{k}}$ is a set of σ-finite $(k(d-k)-1)$-dimensional Hausdorff measure (see [9] Theorem A). As the map $G(E^k) \to G(E^{d-k})$ is an isometry (see [8]) the set $\{\pm G(E^k) : E^{d-k}$ singular$\}$ is of σ-finite $(k(d-k)-1)$-dimensional Hausdorff measure. Therefore, by the above inclusion, A has the same property. This ends the proof.

3. Existence of k-areas in Hilbert space

In this section we investigate the existence of k-areas on a convex compact set of infinite dimension in a Hilbert space.

The main result is included in the following theorem.

THEOREM 3.1. *Let C be a convex compact set in a Hilbert space $\mathscr{H}$ and let E^k, $k \geqq 2$ be a k-dimensional subspace of $\mathscr{H}$. Suppose that $\pi(C) = g(I^k)$ where g is a continuous injection map and* dim $\pi(C) = k$. *Then there exists a sequence $D_r = \{h_r(t) : t \in I^k\}$, $r = 1, 2, \ldots$ of k-areas on C where $\{\pi \circ h_r\}_{r=1}^{\infty}$ converges uniformly to g on I^k.*

Before proceeding in the proof of Theorem 3.1 we quote and prove some auxiliary lemmas.

LEMMA 3.1. *Let C be a convex compact set in a Hilbert space $\mathscr{H}$ and let π be the orthogonal projection on the k-dimensional subspace E^k. Then given $\eta > 0$, there exists a d-dimensional subspace E^d containing E^k and a projection σ_1 such that the orthogonal projection π_1 on E^d satisfies the conditions* (i) $\pi = \pi_1 \circ \sigma_1$, (ii) $\operatorname{diam} C \cap \pi_1^{-1}(\pi_1(x)) < \eta$ *for each x in C and* (iii) *there exists* $\delta = \delta(\eta) > 0$ *such that* $\operatorname{diam}(C \cap \pi_1^{-1}(D)) < \eta$ *where D is a subset of $\pi_1(C)$ with* $\operatorname{diam} D < \delta$.

PROOF. See Lemma 1 in [7]

LEMMA 3.2. *Let E^{d-1} be a hyperplane of E^d and let $\{K_n\}_{n=0}^{\infty}$ be a sequence of convex bodies of E^d that converges in the Hausdorff metric to K_0. Suppose that $\{p_n\}_{n=0}^{\infty}$ is a sequence of unit vectors of E^d, not lying in E^{d-1}, with* $\lim_{n\to\infty} p_n = p_0$ *and $D^{(n)} = \{f_n(t) : t \in I^k\}$ is a k-area of the convex body,* $\operatorname{proj}_{p_n}(K_n)$ $n = 0, 1, \ldots$ *with $\{f_n\}_{n=0}^{\infty}$ converging uniformly to f_0 on I^k. If we can construct (as in Theorem* 2.1) *a k-area on K_n, $\{h_n(t) : t \in I^k\}$ with* $\operatorname{proj}_{p_n} h_n(t) = f_n(t)$, $t \in I^k$, $n = 0, 1, \ldots$ *then the sequence $\{h_n\}_{n=0}^{\infty}$ converges uniformly to h_0 on I^k.*

PROOF. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence in I^k with $\lim_{n\to\infty} t_n = t_0 \in I^k$ and let S be a closed ball of E^d with $K_n \subseteq S$, $n = 0, 1, \ldots$.

As $\lim_{n\to\infty} K_n = K_0$ we may suppose that $\lim_{n\to\infty} h_n(t_n) = x_0$ with $x_0 \in K_0$. We also have

$$\begin{aligned} &\| \operatorname{proj}_{p_0} x_0 - \operatorname{proj}_{p_0} h_0(t_0) \| \\ &\leqq \| \operatorname{proj}_{p_n} h_n(t_n) - \operatorname{proj}_{p_0} h_0(t_0) \| + \| \operatorname{proj}_{p_n} h_0(t_n) - \operatorname{proj}_{p_0} x_0 \| \\ &= \| f_n(t_n) - f_0(t_0) \| + \| \operatorname{proj}_{p_n} h_n(t_n) - \operatorname{proj}_{p_0} x_0 \| . \end{aligned}$$

As $\lim_{n\to\infty} p_n = p_0$ the sequence $\{\operatorname{proj}_{p_n}\}_{n=0}^{\infty}$ converges uniformly to $\operatorname{proj}_{p_0}$ on S. Using this and the assumption that $\{f_n\}_{n=0}^{\infty}$ converges uniformly in the above relations we find that $\operatorname{proj}_{p_0} x_0 = \operatorname{proj}_{p_0} h_0(t_0)$. Therefore the construction of h_0 entails $h_0(t_0) = x_0 = \lim_{n\to\infty} h_n(t_n)$.

LEMMA 3.3. *Let K be a convex body in E^d, E^{d-1} be a hyperplane and let τ be the orthogonal projection on E^{d-1}. Suppose that $B = g(I^k)$ is a k-area on $\tau(K)$ (constructed as in Theorem 2.1). Then there exists a sequence* $D_r = \{h^{(r)}(t) : t \in I^k\}$, $r = 1, 2, \ldots$ *of k-areas on K with $\{\tau \circ h_r\}_{r=1}^{\infty}$ converging uniformly to g on I^k. Also, for any $\varepsilon > 0$ there exists an integer r_0 such that*

$$\| h_r(t) - h_r(t') \| \geqq (1-\varepsilon) \| g(t) - g(t') \|$$

for any $r \geqq r_0$ *and every* $t, t' \in I^k$.

PROOF. For $k = d-1$ the result is contained in Corollary 2.3. Assume now $k \leqq d-2$. Let $E^k = [e_1, e_2, \ldots, e_k]$, $\mathrm{proj}_{e_d} = \tau$ and let $p_{k+1}, p_{k+2}, \ldots, p_{d-1}$ be the vectors used in the construction of the k-area B on $\tau(K)$. We may choose a unit vector p'_d as close as we please to e_d in such a way that there are no line segments on bd K in the direction p'_d where $p'_d \notin [e_1, e_2, \ldots, e_k, p_{k+1}, p_{k+2}, \ldots, p_{d-1}]$.

In a similar way we may choose unit vectors $p'_{d-1}, p'_{d-2}, \ldots, p'_{k+1}$ as close as we please to $p_{d-1}, p_{d-2}, \ldots, p_{k+1}$ respectively and in such a way that

$$[e_1, e_2, \ldots, e_k, p'_{k+1}, p'_{k+2}, \ldots, p'_{d-1}] = E^{d-1},$$

$$p_i \in [e_1, e_2, \ldots, e_k, p_{k+1}, p_{k+2}, \ldots, p_i],$$

$$p_i \notin [e_1, e_2, \ldots, e_k, p_{k+1}, p_{k+2}, \ldots, p_{i-1}], \quad i = k+1, k+2, \ldots, d-1$$

with $p_k = e_k$ and such that there are no line segments on the boundary of $\mathrm{proj}_{p'_{i+1}} \circ \mathrm{proj}_{p'_{i+2}} \circ \cdots \circ \mathrm{proj}_{p'_d}(K)$ in the direction p'_i. Then using p'_{k+1}, $p'_{k+2}, \ldots, p'_d$ we construct a k-area $D = \{h(I^k)\}$ on K.

For each $r \in \mathrm{N}$ we may choose a system of unit vectors $\{p^{(r)}_{k+1}, p^{(r)}_{k+2}, \ldots, p^{(r)}_d\}$ satisfying the additional conditions $\lim_{r\to\infty} p^{(r)}_d = e^d$ and $\lim_{r\to\infty} p^{(r)}_i = p_i$ for $i = k+1, k+2, \ldots, d-1$. Then the sequence of projections $\omega^{(r)} = \mathrm{proj}_{p^{(r)}_{k+1}} \circ \cdots \circ \mathrm{proj}_{p^{(r)}_d}$, $r = 1, 2, \ldots$ converges uniformly to $\mathrm{proj}_{p_{k+1}} \circ \cdots \circ \mathrm{proj}_{p_{d-1}} \circ \mathrm{proj}_{e_d} = \omega \circ \tau$ on K and so we take the k-dimensional sets on E^k $\omega^{(r)}(K) = f_r(I^k)$, $(\omega \circ \tau)(K) = f(I^k)$ with $\{f_r\}_{r=1}^{\infty}$ converging uniformly to f on I^k. Because of the condition for the sequences $\{p^{(r)}_d\}_{r=1}^{\infty}$ and $\{p^{(r)}_i\}_{r=1}^{\infty}$, the sequence of sets $\{\mathrm{proj}_{p^{(r)}_i} \circ \cdots \circ \mathrm{proj}_{p^{(r)}_d}(K)\}_{r=1}^{\infty}$ converges in the Hausdorff metric to the sets $\mathrm{proj}_{p_i} \circ \cdots \circ \mathrm{proj}_{p_d}(K)$, $i = k+1$, $k+2, \ldots, d-1$. Now from Lemma 3.3 we deduce that the sequence of functions $\{g_r\}_{r=1}^{\infty}$, related respectively to the k-areas $A^{(r)} = g_r(I^k)$ on $\mathrm{proj}_{p^{(r)}_d}(K)$, $r = 1, 2, \ldots$, converges uniformly to g on I^k. Let $D^r = h_r(I^k)$ be k-areas on K. For these k-areas we have that $\mathrm{proj}_{p_d} h_r(t) = g_r(t)$, $t \in I^k$. Then this property and the above-mentioned convergence imply that $\{\tau \circ h_r\}_{r=1}^{\infty}$ converges uniformly to g on I^k. Also as τ is the orthogonal projection and $\lim_{r\to\infty} \tau \circ h_r(t) = g(t)$, $t \in I^k$, then there exists an r_0 such that

$$\| h_r(t) - h_r(t') \| \geqq \| \tau(h_r(t)) - \tau(h_r(t')) \|$$

$$\geqq (1-\varepsilon) \| g(t) - g(t') \|$$

for $r \geqq r_0$ and any $t, t' \in I^k$.

LEMMA 3.4. *Let K be a convex body in E^d and let $D = \{h(t) : t \in I^k\}$ be a k-area on K constructed using the projection ω on E^k. Then for any $\delta > 0$ there exists an $\varepsilon = \varepsilon(\delta) > 0$ such that whenever $\| h(t) - \frac{1}{2}(\mu + \nu) \| < \varepsilon$ for some $t \in I^k$ and $\mu, \nu \in K$ with $\omega(\mu) = \omega(\nu)$ then $\| \mu - \nu \| < \delta$.*

PROOF. Suppose on the contrary that for each $n \in \mathbf{N}$ there exists $t_n \in I^k$ and $\mu_n, \nu_n \in K$ with the property that $\| h(t_n) - \frac{1}{2}(\mu_n + \nu_n) \| < 1/n$, $\omega(\mu_n) = \omega(\nu_n)$ and $\| \mu_n - \nu_n \| \geqq \delta$. Let

$$\omega = \mathrm{proj}_{p_{k+1}} \circ \cdots \circ \mathrm{proj}_{p_d} = \mathrm{proj}_{p_{k+1}} \circ \omega_1 \quad \text{and} \quad \omega(K) = f(I^k)$$

where f is a continuous one to one map on I^k and $\omega \circ h = f$. As I^k and K are compact we may suppose that

$$\lim_{n \to \infty} t_n = t_0 \in I^k, \quad \lim_{n \to \infty} \mu_n = \mu \quad \text{and} \quad \lim_{n \to \infty} \nu_n = \nu \quad \text{with} \quad \mu, \nu \in K.$$

These imply $\| \mu - \nu \| \geqq \delta$, $\omega(\mu) = \omega(\nu)$ and $h(t_0) = \frac{1}{2}(\mu + \nu)$. As $\omega(\mu) = \omega(\nu)$ we have $\omega_1(\mu) = \mu_1 e_1 + \cdots + \mu_k e_k + \mu_{k+1} p_{k+1}$ and $\omega_1(\nu) = \mu e_1 + \cdots + \mu_k e_k + \nu_{k+1} p_{k+1}$. Therefore

$$\omega_1(h(t_0)) = \mu_1 e_1 + \cdots + \mu_k e_k + \tfrac{1}{2}(\mu_{k+1} + \nu_{k+1}) p_{k+1}.$$

By the construction of the k-area the point $\omega_1(h(t_0))$ has the minimum value of the p_{k+1}-coordinate on the line segment $\mathrm{proj}_{p_{k+1}}^{-1}(\omega(h(t_0)) \cap \omega_1(K))$. Hence $\mu_{k+1} = \nu_{k+1}$ and this entails $\omega_1(\mu) = \omega_1(\nu) = \omega_1(h(t_0))$. As ω_1 is one to one from $\omega_1(D)$ to K we have $\mu = \nu$ and this contradicts $\| \mu - \nu \| \geqq \delta$.

Now we give the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. Let $\varepsilon > 0$, $E_0 = E^k$, $\pi = \pi_0$ and $g = g_0$. By Lemma 3.2 for $\eta_1 = \frac{1}{2}$ we choose $\delta_1 > 0$ and π_1 an orthogonal projection on $\mathscr{H}$ with finite dimensional range E_1 and a continuous projection σ_1 with $\sigma_1 \circ \pi_1 = \pi_0$ and $\mathrm{diam}(C \cap \pi_1^{-1}(D)) < \eta_1$ whenever D is a subset of $K_1 = \pi_1(C)$ with $\mathrm{diam}(D) < \delta_1$. Next, we choose a coordinate system $(x_1, x_2, \ldots, x_{n(1)})$ for E_1 which extends the coordinate system $(x_1, x_2, \ldots, x_k)$ in $E_0 = E^k$ so that

$$\sigma_1(x_1, x_2, \ldots, x_{n(1)}) = (x_1, x_2, \ldots, x_k, 0, \ldots, 0).$$

Because of Lemma 3.3 we may find, with the aid of a projection ω_1, a k-area $A_1 = g_1(I^k)$ on K_1 such that $\| \sigma_1(g_1(t)) - g_0(t) \| < \varepsilon/6$ and

$$\| g_1(t) - g_1(t') \| \geqq (1 - \tfrac{1}{4}) \| g_0(t) - g_0(t) \| \qquad \text{for any } t, t' \in I^k.$$

Now applying Lemma 3.4 for the convex body $K_1 = \pi_1(C)$, the projection ω_1 and the number δ_1, we may find a positive number $e_1 < \min\{\varepsilon, \delta_1\}$ with the property that there do not exist points μ, ν in K_1 and $t \in I^k$ such that $\omega_1(\mu) = \omega_1(\nu)$, $\| \mu - \nu \| \geqq \delta_1$ and $\| g_1(t) - \frac{1}{2}(\mu + \nu) \| < \varepsilon_1$.

So far we have been through the initial step of an inductive process of choosing the following: a sequence of positive numbers $\eta_1 = \frac{1}{2}, \eta_2 = 1/2^2, \ldots$, a sequence of finite dimensional linear spaces $E_0, E_1, \ldots$, three sequences of projection maps $\pi_0, \pi_1, \ldots, \sigma_1, \sigma_2, \ldots$ and $\omega_1, \omega_2, \ldots$, a sequence of k-area $A_0, A_1, \ldots$ with $A_i = g_i(I^k)$, $i = 0, 1, \ldots$ and finally two sequences of real numbers $\delta_1, \delta_2, \ldots$ and $\varepsilon > \varepsilon_1 > \varepsilon_2 \cdots$.

First, let $\eta_{r+1} = 1/2^{r+1}$. By Lemma 3.1 there exists an orthogonal projection π_{r+1} defined on $\mathscr{H}$ with a finite dimensional range E_{r+1} containing E_r and a second projection σ_{r+1} with $\sigma_{r+1} \circ \pi_{r+1} = \pi_r$ and there will be a $\delta_{r+1} > 0$ such that $\operatorname{diam}(C \cap \pi_{r+1}^{-1}(D)) < \eta_{r+1}$ whenever $D \subseteq \pi_{r+1}(C)$ with $\operatorname{diam} D > \delta_{r+1}$. Then by Lemma 3.3 for the projection σ_{r+1} and the k-area $A_r = g_r(I^k)$ on $\pi_r(C)$ we may find using a projection ω_{r+1} a k-area $A_{r+1} = g_{r+1}(I^k)$ on $\pi_{r+1}(C)$ with the properties

$$\| \sigma_{r+1} \circ g_{r+1}(t) - g_r(t) \| < \frac{\varepsilon_r}{6(r+1)^2}, \qquad t \in I^k$$

and

$$\| g_{r+1}(t) - g_{r+1}(t') \| \geqq \left(1 - \frac{1}{4^{r+1}}\right) \| g_r(t) - g_r(t') \| \qquad \text{for all } t, t' \in I^k.$$

Applying Lemma 3.4 to the projection ω_{r+1} and the number δ_{r+1} we may find a positive number $\varepsilon_{r+1} < \min\{\varepsilon_r, \delta_{r+1}\}$ and with the property that there are no $\mu, \nu \in \pi_{r+1}(C)$ and $t \in I^k$ such that $\omega_{r+1}(\mu) = \omega_{r+1}(\nu)$, $\| \mu - \nu \| \geqq \delta_{r+1}$ and $\| g_{r+1}(t) - \frac{1}{2}(\mu + \nu) \| \leqq \varepsilon_{r+1}$. This completes the inductive step of the construction.

For each $r = 0, 1, \ldots$ we select $z_r(t) \in C$ with the property $\pi_r z_r(t) = g_r(t)$, $t \in I^k$. We shall prove that for any $t \in I^k$, $\{z_r(t)\}_{r=1}^{\infty}$ is a Cauchy sequence. Indeed, for $r \geqq s$ we have

$$\| \pi_s z_s(t) - \pi_s z_r(t) \| = \| g_s(t) - \sigma_{s+1} \circ \cdots \circ \sigma_r g_r(t) \| < \varepsilon_s/3, \qquad s = 0, 1, \ldots .$$

As $\varepsilon_s < \delta_s$, the choice of δ_s implies $\| z_s(t) - z_r(t) \| < \eta_s = 1/2^s$ for $t \in I^k$.

The compactness of C and the fact that $\{z_r(t)\}_{r=0}^{\infty}$ is Cauchy allow us to define for each $t \in I^k$ the point $h(t) = \lim_{r \to \infty} z_r(t)$ belonging to C. We shall prove that $D = \{h(t) : t \in I^k\}$ is a k-area of C. For this purpose we prove the following:

(i) *$h(I^k)$ is a subset of the k-skeleton of C.*

Suppose, on the contrary, that there exists $t_0 \in I^k$ such that $h(t_0) \notin \mathrm{skel}_k C$. Then there exists a $(k+1)$-dimensional ball B with centre the point $h(t_0)$ and radius $\gamma > 0$ such that $B \subseteq C$. Let $s \in \mathbf{N}$ such that $1/2^s < \gamma$ and let $\pi_s : \mathscr{H} \to E_s$ be the corresponding projection and $B_0 = \pi_s(B)$. We have that

$$\operatorname{diam}(B \cap \pi_s^{-1}(\pi_s(h(t_0))) < \eta_s$$

and as $\eta_s < \gamma$ we get $B \cap \pi_s^{-1}(\pi_s(h(t_0))) = \{h(t_0)\}$. If $\dim B_0 = n$, the point $\pi_s(h(t_0))$ has co-dimension n relative to B_0 therefore the set $B \cap \pi_s^{-1}(\pi_s(h(t_0)))$ has also co-dimension n relative to B. This implies

$$0 = \dim(B \cap \pi_s^{-1}(\pi_s(h(t_0))) = (k+1) - n,$$

i.e., $\dim B_0 = n = k+1$. Hence there exist points $\mu, \nu \in B_0, \mu = \nu$, such that $\omega_s(\mu) = \omega_s(\nu)$ and $\pi_s(h(t_0)) = (\mu + \nu)/2$. For the corresponding k-area $\{g_s(t) : t \in I^k\}$ on $\pi_s(C)$, $\| g_s(t_0) - \pi_s(h(t_0)) \| \leqq \varepsilon_s/3$ holds, so $\| g_s(t_0) - \frac{1}{2}(\mu + \nu) \| \leqq \varepsilon_s/3$ and the choice of ε_s implies $\| \mu - \nu \| < \delta_s$. Then

$$2\gamma = \operatorname{diam}(\pi_s^{-1}[\mu, \nu] \cap B) < \eta_s = 1/2^s < \gamma.$$

This contradiction proves the assertion.

(ii) *h is continuous on I^k.*

Let $\varepsilon > 0$, $t_0 \in I^k$ and $s \in \mathbf{N}$ with $1/2^s < \varepsilon$. As g_s is continuous for the corresponding $\varepsilon_s > 0$ we can find a $\delta > 0$ such that $\| g_s(t) - g_s(t_0) \| < \varepsilon_s/3$ for $\| t - t_0 \| < \delta$, $t \in I^k$. On the other hand, for $r > s$ we have $\| g_s(t) - \pi_s z_r(t) \| < \varepsilon_s/3$ so, for any $t \in I^k$, $\| g_s(t) - \pi_s h(t) \| \leqq \varepsilon_s/3$. Therefore, for $\| t - t_0 \| < \delta$ we have $\| \pi_s h(t) - \pi_s h(t_0) \| < \varepsilon_s$ and by the choice of ε_s we find $\| h(t) - h(t_0) \| < n_s = 1/2^s < \varepsilon$. This proves the continuity of h.

(iii) *h is a one to one map.*

As π_r is orthogonal we have

$$\| z_r(t) - z_r(t') \| \geqq \| \pi_r z_r(t) - \pi_r z_r(t') \| = \| g_r(t) - g_r(t') \|$$
$$\geqq \left(1 - \frac{1}{4^r}\right) \| g_{r-1}(t) - g_{r-1}(t') \|$$
$$\geqq \left(1 - \frac{1}{4^r}\right)\left(1 - \frac{1}{4^{r-1}}\right) \cdots \left(1 - \frac{1}{4}\right) \| g_0(t) - g_0(t') \|$$
$$\geqq \left(1 - \sum_{n=1}^{r} \frac{1}{4^n}\right) \| g_0(t) - g_0(t') \| .$$

Taking limits for $r \to \infty$ we find $\| h(t) - h(t') \| \geqq \frac{2}{3} \| g_0(t) - g_0(t') \|$. As g_0 is one to one so is h.

Finally we have $\| \pi_0 z_r(t) - g_0(t) \| < \varepsilon/3$ and taking the limit for $r \to \infty$ we have $\| \pi_0 h(t) - g_0(t) \| \leqq \varepsilon/3$ for any $t \in I^k$. Hence for every $\varepsilon > 0$ we have found a k-area $A = h(I^k)$ on C with $\| \pi \circ h(t) - g(t) \| < \varepsilon$, $t \in I^k$ which proves the result.

4. Conjecture

Let E^k be a k-dimensional subspace of E^d and let π be the orthogonal projection on E^k. Next, let Σ_d be the set of convex bodies K in E^d with the property that the set of directions of line-segments on the boundary of K perpendicular to E^k forms a set of $(d-k-1)$-dimensional Hausdorff measure zero. For any $K \in \Sigma_d$ let $\gamma(K, d, k)$ be the following number: There exist $D_i = \{g_i(t) : t \in \pi(K)\}$, $i = 1, 2, \ldots, \gamma(K, d, k)$, E^k-areas on K such that

$$g_i(\text{relint } \pi(K)) \cap g_j(\text{relint } \pi(K)) = \emptyset, \qquad i \neq j.$$

Set $\gamma(d, k) = \min\{\gamma(K, d, k) : K \in \Sigma_d\}$. Now we observe the following:

If $k = 1$ we have $\gamma(d, 1) = d$ (see [4]).

If $k = d$ we have $\gamma(d, d) = 1$ (obvious).

If $k = d - 1$ we have $\gamma(d, d-1) = 2$ (Corollary 2.1).

If $1 < k < d - 1$, we conjecture that $\gamma(d, k) = d - k + 1$.

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