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# The socle and finite-dimensionality of a semiprime Banach algebra

by

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**Abstract.** All finite-dimensional semiprime Banach algebras are semisimple.

The purpose of this paper is to give a characterization of the elements of the socle of a semiprime Banach algebra. If $A$ is a semiprime Banach algebra we prove that $\operatorname{soc} A \cap \operatorname{rad} A = \{0\}$, and $t \in \operatorname{soc} A$ if and only if $\dim(tAt) < +\infty$ (i.e. $tAt$ has finite dimension). This extends a result of Alexander in [1] concerning semisimple Banach algebras, and is used to prove that the elements of $\operatorname{soc} A$ are algebraic and that $A$ is finite-dimensional if and only if $A = \operatorname{soc} A$ (and in this case $A$ is forced to be semisimple). This completes Tullo's assertion in Theorem 5 of [8]. We also give a different proof of Tullo's result.

An element $s$ of $A$ is called *single* if whenever $asb = 0$ for some $a, b$ in $A$, at least one of $as$ or $sb$ is zero. We say that an element $t$ of $A$ *acts compactly* if the map $a \to tat$ $(A \to A)$ is compact. If the algebra $A$ has no minimal ideals we define $\operatorname{soc} A = \{0\}$.

In general, notation and terminology used are as in [3]. All the algebras and subspaces considered will be over the complex field.

Single elements that act compactly have proved to have a close connection with the elements of the minimal ideals of the algebra.

More precisely, with a slight modification (see e.g. [5] or [6]) in the proofs of Theorem 4 and Corollary 5 in [4] one can easily deduce Theorem 1 and Corollary 2 below (see also [7] for an alternative approach).

THEOREM 1. *Let $s$ and $t$ be nonzero compactly acting single elements of a semiprime Banach algebra $A$, and $s,t \notin \operatorname{rad} A$. Then:*

(i) *There exist minimal idempotents $e$ and $f$ such that $s = se$ and $t = ft$.*

(ii) *The dimension of $tAs$ is at most* 1.

From Theorem 1 we find that if $s \notin \operatorname{rad} A$ and $s$ is a compactly acting

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single element of $A$ then $As$ ($sA$) is a minimal left (right) ideal of $A$. The converse is also valid and so we have

COROLLARY 2. *The socle of $A$ consists of all finite sums of compactly acting single elements of $A$ that are not in* $\mathrm{rad}\, A$.

PROPOSITION 3. *Let $A$ be a semiprime Banach algebra. Then* $\mathrm{soc}\, A \cap \mathrm{rad}\, A = \{0\}$.

Proof. Suppose on the contrary that there exists a nonzero element $t$ in $\mathrm{soc}\, A \cap \mathrm{rad}\, A$. Then $t \notin \mathrm{lan}(\mathrm{soc}\, A)$ ([3], p. 162, Lemma 4). This implies that $ty \neq 0$ for some $y \in \mathrm{soc}\, A$ and so from Corollary 2, $ts \neq 0$ for some compactly acting single element $s$ of $A$ which is not in $\mathrm{rad}\, A$. $As$ is then a left minimal ideal of $A$ and since $\{0\} \neq Ats \subseteq As$ we have $As = Ats \subset \mathrm{soc}\, A \cap \mathrm{rad}\, A$, which is a contradiction (since $\mathrm{rad}\, A$ does not contain nonzero idempotents).

COROLLARY 4. *Let $A$ be a semiprime Banach algebra. Then:*
(i) $\mathrm{soc}\, A \subset \mathrm{lan}(\mathrm{rad}\, A) = \mathrm{ran}(\mathrm{rad}\, A)$.
(ii) $\mathrm{rad}\, A \subset \mathrm{lan}(\mathrm{soc}\, A) = \mathrm{ran}(\mathrm{soc}\, A)$.

Proof. This follows immediately from Proposition 3 above and Lemma 4, p. 162 of [3].

LEMMA 5. *Let $A$ be an algebra that has proper left ideals, let $X$ be a finite-dimensional subspace of $A$, and let $t$ be a nonzero element of $A$. If $X \cap L \neq \{0\}$ for every left ideal $L$ of $A$ contained in $At$, then $At$ contains a left minimal ideal of $A$.*

Proof. We distinguish two cases:

*Case* 1: $X \cap L = X \cap L'$ for any two left ideals $L, L' \subset At$. Since $X \cap L \neq \{0\}$ for every left ideal $L \subset At$, it is clear that the intersection of all such $L$'s is nonzero and therefore it is a minimal left ideal of $A$.

*Case* 2: There exist two left ideals $L_1$, $L_2$ of $A$ contained in $At$ such that $\{0\} \neq X \cap L_2 \subsetneqq X \cap L_1$. Then from the assumptions we have $\{0\} \neq X \cap L$ for every left ideal $L \subset L_2 \subset At$.

Therefore, either $X \cap L = X \cap L_2$ for all such $L$'s, which leads to case 1, or there exists a left ideal $L_3 \subset L_2$ such that

$$\{0\} \neq X \cap L_3 \subsetneqq X \cap L_2 \subsetneqq X \cap L_1.$$

Since $X$ has finite dimension we deduce the existence of a finite sequence $L_1, \ldots, L_k$ of left ideals of $A$ contained in $At$ such that

$$\{0\} \neq X \cap L_k \subsetneqq \ldots \subsetneqq X \cap L_2 \subsetneqq X \cap L_1 \quad \text{and} \quad X \cap L_k = X \cap L \neq \{0\}$$

for every left ideal $L \subsetneqq L_k$. The intersection of such $L$'s is clearly a minimal left ideal of $A$.

LEMMA 6. *Let $t$ be a nonzero element of a semiprime Banach algebra $A$ such that $tAt$ has finite dimension. Then $t \in \operatorname{soc} A$.*

*In particular, there exist a finite set $\{e_1, \ldots, e_n\}$ of minimal idempotents and $t_1, \ldots, t_{n-1} \in A$ such that $t = te_1 + t_1 e_2 + \ldots + t_{n-1} e_n$.*

Proof. We put $X = tAt$, and suppose that $\dim(tAt) = n$. Then from Lemma 5 the left principal ideal $At$ contains a minimal idempotent $e_1 = a_1 t$, for some $a_1 \in A$. We put $t_1 = t - te_1$. Then

$$t_1 At_1 = (t - te_1) A (t - te_1) \subset tA(t - te_1) = tA(t - ta_1 t) \subset tAt.$$

However, the inclusion $tAt(1 - e_1) \subset tAt$ is strict because $te_1 = ta_1 t$ is in $tAt$, is nonzero and cannot be of the form $tat(1 - e_1)$. Hence

$$\dim(t_1 At_1) < \dim(tAt).$$

If $t_1 \neq 0$, by a similar argument we can find a minimal idempotent $e_2$ so that

$$\dim(t_2 At_2) < \dim(t_1 At_1)$$

for $t_2 = t_1 - t_1 e_2$.

A finite number of repetitions of this process gives an element

$$t^* = te_1 + t_1 e_2 + \ldots + t_{k-1} e_k \in \operatorname{soc} A,$$

for some $k \leqslant n$, such that $(t - t^*) A (t - t^*) = \{0\}$. Hence $t = t^*$, since $A$ is semiprime.

THEOREM 7. *Let $A$ be a semiprime Banach algebra. Then $t \in \operatorname{soc} A$ if and only if the dimension of $tAt$ is finite.*

Proof. If $t \in \operatorname{soc} A$, then $t = s_1 + \ldots + s_n$ is a finite sum of compactly acting single elements not in $\operatorname{rad} A$ (Corollary 2) and therefore $tAt \subset \sum_{i,j=1}^{n} s_i As_j$. Since $s_i As_j$ has dimension at most 1 (Theorem 1) it follows that $\dim(tAt) < +\infty$.

Conversely, if $\dim(tAt) < +\infty$ then $t \in \operatorname{soc} A$ from Lemma 6. In particular, $t = 0$ if and only if $tAt = \{0\}$, since $A$ is semiprime.

COROLLARY 8. *Let $A$ be a semiprime Banach algebra. If $A$ has finite dimension then:*

(i) $A = \operatorname{soc} A$.

(ii) *$A$ is semisimple.*

Proof. If $A$ has finite dimension, then $tAt$ has finite dimension for every $t \in A$. Hence $A \subset \operatorname{soc} A$. Since the reverse inclusion is also valid we have $A = \operatorname{soc} A$.

From Proposition 3 above and Corollary 20, p. 126 of [3] we now have $\operatorname{rad} A = \operatorname{rad}(\operatorname{soc} A) = \operatorname{soc} A \cap \operatorname{rad} A = \{0\}$, i.e. $A$ is semisimple.

COROLLARY 9. *Let $A$ be a semiprime Banach algebra. Then every element of* $\operatorname{soc} A$ *is algebraic.*

Proof. If $t \in \operatorname{soc} A$, then $\dim(tAt) = n < +\infty$. Hence the elements $t^3$, $t^4, \ldots, t^{n+4}$ are linearly dependent and therefore there are $\lambda_1, \lambda_2, \ldots, \lambda_{n+1} \in C$ not all zero with $\lambda_1 t^3 + \lambda_2 t^4 + \ldots + \lambda_{n+1} t^{n+4} = 0$.

COROLLARY 10 (Tullo). *Let $A$ be a semiprime Banach algebra and $A = \operatorname{soc} A$. Then $A$ is semisimple and finite-dimensional.*

Proof. $A$ is semisimple as in the proof of Corollary 8. Also, every element of $A$ is algebraic (Corollary 9) and therefore $A$ is finite-dimensional (Corollary 1 in [2]).

We summarize the results concerning the finite-dimensionality of the algebra in the following

THEOREM 11. *Let $A$ be a semiprime Banach algebra. Then the following conditions are equivalent:*

(i) *$A$ has finite dimension.*

(ii) $A = \operatorname{soc} A$.

*Moreover, if* (i) *(and therefore* (ii)*) is valid, then $A$ is semisimple.*

Remark. If $A$ is a semiprime Banach algebra and $A = \overline{\operatorname{soc} A}$, then $A$ is compact (Lemma 12, p. 177 of [3]) and therefore $A$ has a discrete structure space (Theorem 18, p. 180 of [3]).

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