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INCREASING PATHS LEADING TO A FACE OF A CONVEX COMPACT SET IN A HILBERT SPACE

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1. Introduction

Let C be a convex compact set in a normed space E. A point c of C is an *extreme point* of C if it is not contained in the relative interior of a line segment lying in C. The *exposed points* are extreme points that can be expressed as the sole intersection of C with one of its support hyperplanes. The ***r-skeleton of C*** for r a non-negative integer will be defined to be the set of all points of C that do not belong to the relative interior of an $(r+1)$-dimensional convex subset of C. The set of extreme points of C coincides with the 0-skeleton of C. Let l be a continuous functional on E non-constant on C. In [1], D. G. Larman proved the existence of an *l-strictly increasing path* on the *one-skeleton* of C. In a recent paper [2] it is proved that if the face $F=\{x\in C\colon l(x)=\max_{y\in C} l(y)\}$ is of infinite dimension then for every $n=1,2,\ldots$ there are n l-strictly increasing paths on the one-skeleton of C mutually disjoint that lead in F (Corollary 2.1 in [2]). In this paper it is proved that if the dimension of F is k then there are $k+1$ such paths with the above mentioned property for every $k=1,2,\ldots$. Furthermore, we give an example showing that this result is the best possible in a Hilbert space.

2. The results

We quote the following propositions:

PROPOSITION 2.1. *Let C be a compact convex set in a normed space E and let l be a continuous linear functional on E, non-constant on C whose maximum on C is taken on a face F of C with* $\dim F=k$ $(k\geqq 1)$. *Then there are $k+1$ mutually disjoint paths on the one-skeleton of C, leading to F, along each of which the functional l strictly increases.*

PROPOSITION 2.2. *Let $\mathcal{H}$ be a Hilbert space of infinite dimension, l a non-constant continuous linear functional and k an arbitrary positive integer. Then there exists a convex compact set Σ, which is of infinite dimension and on which l is non-constant, such that the face $F=\{x\in\Sigma\colon l(x)=\max_{y\in\Sigma} l(y)\}$ is of dimension k and Σ has the property that on its one-skeleton, it is impossible to find $k+2$ l-strictly increasing paths, mutually disjoint that lead to F.*

PROOF OF PROPOSITION 2.1. We assume that $\dim F=k$. Then there exist $k+1$ $e_1,e_2,\ldots,e_{k+1}$ linearly independent vectors in E and $k+1$ linear functionals $l_1=l,l_2,\ldots,l_{k+1}$ for which the following hold:

(i) $l_i(e_j)=\delta_{ij}$, $i,j=1,2,\dots,k+1$ where δ_{ij} is the *Kronecker delta* and
(ii) $\dim \pi_0(F)=k$, where π_0 is the projection

$$\pi_0(x) = l_1(x)e_1+l_2(x)e_2+\dots+l_{k+1}(x)e_{k+1}, \quad x\in E.$$

From now on, the steps required to find the appropriate $k+1$ paths on the one-skeleton of C are similar to those in the proof of Theorem 2.1 in [2], so we omit them. This concludes the proof of the proposition.

Before we proceed to the proof of Proposition (2.2) we introduce some appropriate notation: Let $\mathscr{H}_0$ be a separable Hilbert space of infinite dimension and let l be a continuous linear functional on $\mathscr{H}_0$ of norm 1. Then there exists a unit vector u_1 such that $l(x)=\langle x, u_1\rangle$ for every $x\in\mathscr{H}_0$. Let $\{u_n\}_{n=1}^{\infty}$ be a complete orthonormal system in $\mathscr{H}_0$. We denote by Q the convex compact set

$$\left\{x\in\mathscr{H}_0:\ x=\sum_{n=1}^{\infty} c_n u_n,\ 0\leqq c_n\leqq\frac{1}{n},\ n=1,2,\dots\right\}.$$

Also, let $H^-(a)=\{x\in\mathscr{H}_0:\ l(x)\leqq a\}$ be the closed half space, where a is a real number and let $H(a)$ be the boundary of $H^-(a)$. Next we quote and prove the following lemma which is essential in the proof of Proposition 2.2.

LEMMA. *Let l be a non-constant continuous linear functional on a separable Hilbert space $\mathscr{H}_0$, $k\geqq 1$ be an integer and $\mu<(k+1)^{-1}$ a positive real number. Then there is a sequence of convex compact sets $(\Sigma_i)_{i=1}^{\infty}$ with $\Sigma_i\subseteqq Q$ such that*

1) *If* $\alpha^{(i)}=\max\limits_{x\in\Sigma_i} l(x)$, *then the sequence* $\{\alpha^{(i)}\}_{i=1}^{\infty}$ *is strictly increasing.*

2) $\Sigma_{i+1}\cap H^-(\alpha^{(i)})=\Sigma_i,\quad i=1,2,\dots.$

3) *The sets* $\Delta^{(0)}=\Sigma_1\cap H(0)=\overline{\mathrm{con}}\{\bigcup\limits_{n=1}^{\infty} x_n^{(0)}\}$ *and* $\Delta^{(i)}=\Sigma_i\cap H(\alpha^{(i)})=\overline{\mathrm{con}}\{\bigcup\limits_{n=1}^{\infty} x_n^{(i)}\}$,

$i=1,2,\dots$ *are convex compact sets of co-dimension* 1.

4) $F^{(i)}=\mathrm{con}\,\{x_1^{(i)},\dots,x_{k+1}^{(i)}\}$ *is a k-dimensional face of* $\Delta^{(i)}$ *with*

$$\min_{1\leqq j<n\leqq k+1}\|x_j^{(i)}-x_n^{(i)}\|>\mu \quad i=0,1,2,\dots.$$

5) $\max\limits_{1\leqq j\leqq k+1}\|x_j^{(i)}-x_j^{(i+1)}\|<\mu^{k+i},\quad i=0,1,2,\dots.$

6) *The point* $x_j^{(i)}$ *is joined to the point* $x_j^{(i+1)}$ *by a single edge of the convex compact set* Σ_{i+1}, $1\leqq j\leqq k+1$, $i=0,1,2,\dots$.

7) $$\lim_{j\to\infty}\|x_j^{(i)}-x_{\lambda(i)}^{(i)}\|=0,\ \lambda(i)=\begin{cases}1 & \text{for}\quad i=2n\quad \text{where}\quad n=0,1,2,\dots\\ 2 & \text{for}\quad i=2n+1\end{cases}$$

and

$$0<\|x_j^{(i)}-x_{\lambda(i)}^{(i)}\|<\mu^{k+i},\quad j=k+2,\dots,\quad i=0,1,\dots.$$

8) *If $k+2$ disjoint l-strictly increasing paths in the one-skeleton of Σ^{i+1} lead from $\Delta^{(i)}$ to $\Delta^{(i+1)}$ then one must contain a line segment of length exceeding $\mu-3\mu^{k+i}$,* $i=0,1,2,\dots$.

PROOF. Let $x_1^{(0)}=0, x_2^{(0)}=u_2/2, \ldots, x_{k+1}^{(0)}=u_{k+1}/(k+1)$ and $x_{k+j}^{(0)}=\mu^{k+j}u_{k+j}$, $j\geqq 2$. Then $\lim_{n\to\infty} x_n^{(0)}=x_1^{(0)}$, hence $\Delta^0=\overline{\text{con}}\,(\bigcup_{n=1}^{\infty} x_n^{(0)})$ is a compact convex subset of Q with $\text{ext}\,\Delta^{(0)}=\exp\Delta^{(0)}=\bigcup_{n=1}^{\infty} x_n^{(0)}$, where $\text{ext}\,\Delta^{(0)}$ and $\exp\Delta^{(0)}$ are the set of extreme and exposed points of $\Delta^{(0)}$, respectively.

Conditions 4) and 7) are satisfied for the points $\{x_n^{(0)}\}_{n=1}^{\infty}$. Let $u_1\in Q$ and $P'=\overline{\text{con}}\,\{\{\bigcup_{n=1}^{\infty} x_n^{(0)}\}\cup\{u_1\}\}=\text{con}\,(\Delta^{(0)}\cup\{u_1\})$. We can choose $0<\alpha^{(1)}<1$ sufficiently small, so that with $x_j^{(1)}=\text{con}\,(x_j^{(0)}, u_1)\cap H(\alpha^{(1)})$, the inequalities of conditions 4) and 5) are satisfied for $1\leqq j\leqq k+1$. Let $y_j^{(1)}=\text{con}\,(x_j^{(0)}, u_1)\cap H(\alpha^{(1)}), j\geqq k+2$. Then

$$T = P'\cap H(\alpha^{(1)}) = \overline{\text{con}}\,(\bigcup_{j=1}^{k+1}\{x_j^{(1)}\}\cup\bigcup_{j=k+2}^{\infty}\{y_j^{(1)}\})$$

is a compact convex subset of Q.

We choose $x_j^{(1)}\in\text{con}\,(y_j^{(1)}, x_2^{(1)}), j\geqq k+2$ such that $0<\|x_j^{(1)}-x_2^{(1)}\|<\mu^{k+1}$ and $\lim_{j\to\infty} x_j^{(1)}=x_2^{(1)}$ and so that $\Delta^{(1)}=\overline{\text{con}}\,(\bigcup_{j=1}^{\infty} x_j^{(1)})$ is a convex compact set with co-dimension 1. Then we define

$$\Sigma_1 = \overline{\text{con}}\,(\Delta^{(0)}\cup\Delta^{(1)}) = \overline{\text{con}}\,(\bigcup_{n=1}^{\infty}\{x_n^{(0)}\}\cup\bigcup_{n=1}^{\infty}\{x_n^{(1)}\}).$$

Then Σ_1 is a compact convex subset of Q, which satisfies conditions 3) to 8).

Assume now that we have constructed a finite sequence of m $(m\geqq 1)$ compact convex subsets of Q, having properties 1)—8). It will be shown that a compact convex set Σ_{m+1} can be constructed so that the enlarged sequence also satisfies the required conditions. Let $d\in Q$ be a point with $\alpha^{(m)}<l(d)<1$ and $\overline{\text{con}}\,(d\cup\Sigma_m)=\Sigma_m\cup\text{con}\,(d\cup\Delta^{(m)})\subseteqq Q$. Let $\alpha^{(m+1)}$ be chosen greater than $\alpha^{(m)}$, so that with $x_j^{(m+1)}$ defined for, $1\leqq j\leqq k+1$ by $x_j^{(m+1)}=\text{con}\,(x_j^{(m)}, d)\cap H(\alpha^{(m+1)})$ the inequalities of Conditions 4) and 5) are satisfied. Let now

$$T = \overline{\text{con}}\,(\Sigma_m\cup d)\cap H(\alpha^{(m+1)}) = \overline{\text{con}}\,(\bigcup_{j=1}^{k+1} x_j^{(m+1)}\cup\bigcup_{j=k+2}^{\infty} y_j^{(m+1)})$$

where $y_j^{(m+1)}$, $j\geqq k+2$ is joined to $x_j^{(m)}$ by a single edge of the compact convex set $\overline{\text{con}}\,(\Sigma_m\cup d)\cap H^-(\alpha^{(m+1)})$. Let $x_j^{(m+1)}$, $j\geqq k+2$ be a point of the edge $\text{con}\,(y_j^{(m+1)}, x_\lambda^{(m+1)})$ ($\lambda=1$ or 2 iff $m+1$ is even or odd) of T chosen so that

$$0<\|x_j^{(m+1)}-x_\lambda^{(m+1)}\|<\mu^{k+(m+1)},\ \lim_{j\to\infty} x_j^{(m+1)} = x_\lambda^{(m+1)} \text{ and } \Delta^{(m+1)} = \overline{\text{con}}\,(\bigcup_{n=1}^{\infty} x_n^{(m+1)})$$

is a compact convex set of co-dimension 1. Next, we define

$$\Sigma_{m+1} = \overline{\text{con}}\,(\Sigma_m\cup\Delta^{(m+1)}) = \overline{\text{con}}\,(\bigcup_{i=0}^{m+1}\bigcup_{n=1}^{\infty} x_n^{(i)}).$$

Then $\text{ext}\,\Sigma_{m+1}=\exp\Sigma_{m+1}=\bigcup_{i=0}^{m+1}\bigcup_{n=1}^{\infty} x_n^{(i)}$. We observe that conditions 1)—7) apply to the sequence of convex compact sets $\Sigma_1, \Sigma_2, \ldots, \Sigma_{m+1}$ by construction. But condition 8) requires proof.

Let $S_{m+1} = \bigcup_{i=0}^{m+1} \bigcup_{j=1}^{k+1} x_j^{(i)}$ be the vertices of Σ_{m+1} which satisfied conditions 4) and 5). Suppose that $P_1, P_2, \ldots, P_{k+2}$ are $k+2$ disjoint l-strictly-increasing paths in the one-skeleton of Σ_{m+1}, joining $\Delta^{(m)}$ to $\Delta^{(m+1)}$. Suppose that none of these paths contain a line segment of length exceeding $\mu-3\mu^{k+m}$. Let $x_j^{(m)}, x_i^{(m+1)} \in \operatorname{ext}\Sigma_{m+1} - S_{m+1}$. Then using conditions 7), 5) and 4) we can see that $\|x_j^{(m)} - x_i^{(m+1)}\| > \mu - 3\mu^{k+m}$ and

$$\|x_j^{(m)} - x_{\lambda(m)}^{(m+1)}\| < \mu - 3\mu^{k+m}, \quad \|x_i^{(m+1)} - x_{\lambda(m+1)}^{(m)}\| < \mu - 3\mu^{k+m}.$$

Hence $x_j^{(m)}$ can only be joined to $x_{\lambda(m)}^{(m+1)}$ and the vertex $x_i^{(m+1)}$ with $x_{\lambda(m+1)}^{(m)}$. There remain k disjoint paths between $\Delta^{(m)}$ and $\Delta^{(m+1)}$ which do not pass through $x_{\lambda(m+1)}^{(m)}$ and $x_{\lambda(m)}^{(m+1)}$. If one of these paths joins two vertices $x_j^{(m)}$, $x_i^{(m+1)}$, $i \neq j$ with $x_j^{(m)}, x_i^{(m+1)} \in S_{m+1}$ then we can show that

$$\|x_j^{(m)} - x_i^{(m+1)}\| > \mu - 3\mu^{k+m}.$$

If one of these paths join two vertices $x_j^{(m)} \notin S_{m+1}, x_i^{(m+1)} \in S_{m+1}, x_i^{(m+1)} \neq x_{\lambda(m)}^{(m+1)}$ then again $\|x_j^{(m)} - x_i^{(m+1)}\| > \mu - 3\mu^{k+m}$. Similarly, if one of these paths join two vertices then $x_j^{(m+1)} \notin S_{m+1}, x_i^{(m)} \in S_{m+1}, x_i^{(m)} \neq x_{\lambda(m+1)}^{(m)}$. This contradiction establishes Condition 8) for the sequence of convex compact sets $\Sigma_1, \ldots, \Sigma_{m+1}$.

Proof of Proposition 2.2. Let l be a non-constant continuous linear functional on $\mathscr{H}$. We may assume without loss of generality that l is of unit norm. Then $l(x) = \langle x, u_1 \rangle$ for some unit vector u_1 in $\mathscr{H}$. We select a closed separable subspace $\mathscr{H}_0$ of infinite dimension such that $u_1 \in \mathscr{H}_0$. Then l is non-constant on $\mathscr{H}_0$. Define $\Sigma = \overline{\text{con}}\,(\bigcup_{n=1} \Sigma_n) \subseteq Q$ where Σ_n, $n=1, 2, \ldots$ and Q are as in the previous lemma. Then Σ is compact convex and the functional l assumes its maximum value on Σ over the whole of a face F_1 with $\dim F_1 = k$ by the construction of Σ_n, $n=1, 2, \ldots$ and Σ. It is impossible to find $k+2$ paths in the one-skeleton of Σ which lead to F_1, yet which are disjoint outside F_1. If such paths did exist then, by condition 8) of lemma, one of the paths would contain a sequence of disjoint line-segments of length exceeding $\frac{\mu}{2}$ (taking the limit). This is impossible for a path P, since P is the continuous image of $[0, 1]$ on the one-skeleton of Σ. This completes the proof of Proposition 2.2.

References

[1] D. G. Larman, On the one-skeleton of a compact convex set in a Banach space, *Proc. London Math. Soc.*, **34** (1977), 117—144.

[2] L. Dalla, Increasing paths on the one-skeleton of a convex compact set in a normed space, *Pacific J. Math.* **124** (1986), 289—294.

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