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Skeletons of the Unit Ball of a C^*-algebra

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1. Introduction

The purpose of this paper is to investigate the n-skeleton of the closed unit ball B of a C^*-algebra A.

The O-skeleton has been studied by R. Kadison [2], S. Sakai [6] and P. Miles [3]. Let K be a closed convex set in a normed space. We define n-skeleton of K ($n=0, 1, 2, \dots$) the set $\text{skel}_n K$, of all points of K not belonging in the relative interior of $(n+1)$-dimensional subsets of K (where dimensionality is considered in the underlying real space). In particular $\text{skel}_0 K$ coincides with the set ext K of all extreme points of K.

It is well known that a commutative C^*-algebra A with unit is isometrically isomorphic to the space $C(X)$, of all continuous functions on a compact, Hausdorff space X. R. Phelps in [5] has proved that in the commutative case B is the closed convex hull of its extreme points.

In section 2 below we give a complete characterization of $\text{skel}_1 B$ and $\text{skel}_2 B$ for commutative C^*-algebras, as well as some properties of $\text{skel}_n B$, $n>2$. In the case of non-commutative C^*-algebra we give characterization of $\text{skel}_1 B$ and some properties of $\text{skel}_2 B$. The main tool used is the concept of single elements, introduced by J. Erdos in [1]. An element s of an algebra A is called single, iff whenever $asb=0$ (a, b$\in A$), then at least one of as, sb is zero.

Notation and terminology used is the same as in M. Takesaki [7, p. 47-49] and J. Tölke - J. Wills [8, p. 13-20].

2. Commutative case

Theorem 2.1. *Let S be the closed unit ball of $C(X)$. A function $f\in S$ belongs to the $\text{skel}_2 S$ if and only if, $|f|=1-|s|$, where $s\in S$ is a single element of $C(X)$.*

Proof. Suppose $f\in \text{skel}_2 S$. We may assume that $f\notin \text{ext} S$, for if $f\in S$, then we may take $s=0$. Hence, there exists at least one point t in X, such that $|f(t)|<1$. We claim that there exists exactly one t_1 in X, with $|f(t)|<1$. Suppose that there exist at least two distinct points t_1, t_2 in X, with $|f(t_j)|<1, j=1, 2$. Since X is a compact Hausdorff space we can find compact disjoint neighbourhoods V_1, V_2 of t_1, t_2

respectively, with $|f(t)|<1$, $t\in V_1\cup V_2$. By Urysohn's Lemma there exist continuous functions g_j, $j=1,2$ such that $g_j(t_j)=1$, $g_j(V_j^c)=0$ and $g_j(X)=[0,1]$, $j=1,2$. Denote $a_j=\sup\{|f(t)|:t\in V_j\}$, $j=1,2$. Then the functions $h_j^{(1)}=f+(1-a_j)g_j$, $h_j^{(2)}=f-(1-a_j)g_j$, $j=1,2$, $h_3^{(1)}=f+i(1-a_1)g_1$, $h_3^{(2)}=f-i(1-a_1)g_1$ are points in S and $f=\frac{1}{2}(h_j^{(1)}+h_j^{(2)})$, $j=1,2,3$. Since $a_j<1$, $j=1,2$ we can easily see that f, $h_j^{(1)}$, $j=1,2,3$ are affinely independent, contradicting the fact that $f\in \mathrm{skel}_2 S$. Hence for every $f\in \mathrm{skel}_2 S\setminus \mathrm{ext}\, S$ there exists exactly one point t_1 in X with $|f(t_1)|<1$. As f is continuous t_1 is an isolated point of X. We define a function $s:X\to \mathbf{C}$, by $s(t_1)=1-|f(t_1)|$, $s(t)=0$, $t\neq t_1$. Since t_1 is an isolated point of X, s is continuous and clearly a single element in $C(X)$, with $\|s\|\leqq 1$. We now have $|f|=1-|s|$.

Conversely, let $f\in S$ such that $|f|=1-|s|$ where $s\in S$ is a single element in $C(X)$. We may assume that $s\neq 0$, for if $s=0$ then $f\in \mathrm{ext}\, S\subseteq \mathrm{skel}_2 S$. If $t_1, t_2\in X$, $t_1\neq t_2$ and $s(t_1).s(t_2)\neq 0$, then there exist disjoint neighbourhoods U_1, U_2 of t_1, t_2, respectively, and functions $h, g\in C(X)$ with $h(t_1).g(t_2)\neq 0$, $h(V_1^c)=0$, and $g(V_2^c)=0$. Then $h\cdot s\cdot g=0$ and $hs\neq 0$, $sg\neq 0$. This is a contradiction since s is a single element in $C(X)$. Therefore $s(t_1)\neq 0$, for exactly one point $t_1\in X$. This entails $|f(t)|=1$, $t\neq t_1$ and $|f(t_1)|<1$. Suppose now that $f\in \mathrm{relint}\, B$, where B is a convex subset of S and relint B is the relative interior of B. Then the expression of $|f|$ and the strict convexity of the norm of $\mathbf{C}$, implies that $\dim B\leqq 2$ and so $f\in \mathrm{skel}_2 S$. The proof is complete.

From Theorem 1 we have the following corollaries.

Corollary 2.2 *Let $f\in S$. Then $f\in skel_2 S$ if and only if $f^n\in skel_2 S$, $n=1,2,3,\ldots$.*

Corollary 2.3. *If $f\in skel_2 S$, then there exist an exreme point g of S and a single element $s\in S$ such that $f=g-s$. Also for any extreme point g of S, there exists a single element $s\in S$ such that the point $g-s=f$ belongs to $skel_2 S$.*

It is known that if $g\in S$, then g is not in ext S if and only if there exists $h\in C(X)$, $h\neq 0$, such that $|g(t)|+|h(t)|\leqq 1$ for each t in X (see [4]). From the proof of Theorem 1, we may obtain a corresponding characterization for points not belonging to $\mathrm{skel}_2 S$. This is stated in the following corollary.

Corollary 2.4. *If $g\in S$, then g is not in* $\mathrm{skel}_2 S$ *if and only if there exist $h_j\in C(X)$,* $j=1,2,3$ *linearly indepentend, such that $|g(t)|+|\mathrm{h}_j(t)|\leqq 1$ for each t in X.*

Proposition 2.5. *Let S be the closed unit ball in $C(X)$. Then* $\mathrm{skel}_1 S=\mathrm{ext}\, S$.

Proof. Clearly ext $S\subseteq \mathrm{skel}_1 S$. Let $f\in \mathrm{skel}_1 S$. Since $\mathrm{skel}_1 S\subseteq \mathrm{skel}_2 S$ from Theorem 1 we have $|f|=1-|s|$, for some single element $s\in S$ of $C(X)$. If $s\neq 0$, then $|f(t_1)|<1$ and $|f(t)|=1$, $t\neq t_1$. The functions $\varphi_k^{(1)}$, $\varphi_k^{(2)}$, $k=1,2$ defined by

$$\varphi_k^{(1)}(t)=\varphi_k^{(2)}(t)=f(t) \quad \text{for} \quad t\neq t_1,\ t_2,\quad k=1,2,$$

$$\varphi_1^{(1)}(t_1)=f(t_1)+\varepsilon,\quad \varphi_1^{(2)}(t_1)=f(t_1)-\varepsilon,$$

$$\varphi_2^{(1)}(t_1)=f(t_1)+i\varepsilon,\quad \varphi_2^{(2)}(t_1)=f(t_1)-i\varepsilon$$

with $\varepsilon=1-|f(t_1)|>0$, are continuous, belong to S and $f=\frac{1}{2}(\varphi_k^{(1)}+\varphi_k^{(2)})$, $k=1,2$. It is clear that f, $\varphi_k^{(1)}$, $k=1, 2$ are affinely independent, contradicting the fact that $f\in \mathrm{skel}_1 S$. Hence $s=0$, so $|f|=1$. This implies $\mathrm{skel}_1 S \subseteq \mathrm{ext}\ S$.

Proposition 2.6. 1) *If* $\mathrm{skel}_n S = \mathrm{ext}\ S$ *for some* $n \geqq 2$, *then* $C(X)$ *does not contain non-zero single elements.*

2) *If* $C(X)$ *does not contain non-zero single elements, then* $\mathrm{skel}_n S = \mathrm{ext}\, S$, *for any integer* $n \geqq 2$.

Proof. 1) As $\mathrm{skel}_2 S \subseteq \mathrm{skel}_n S$, then $\mathrm{skel}_2 S = \mathrm{ext}\, S$ (1). From Theorem 1 it follows that if $s \neq 0$ is a single element in $C(X)$, then the functions $f\in S$, $|f| = 1 - \frac{|s|}{\|s\|}$ are in $\mathrm{skel}_2 S \setminus \mathrm{ext}\, S$, contradicting (1).

2) Let $n \geqq 2$ and f $\mathrm{skel}_n S \setminus \mathrm{ext}\ S$. A similar argument as in the proof of Theorem 1 gives m points $t_1, t_2, \ldots, t_m$ in X, $1 \leqq m \leqq n-1$, with $|f(t_j)| < 1, j=1, 2, \ldots, m$. The continuity of f implies that $t_1, t_2, \ldots, t_m$ are isolated points of X. Hence the functions $s_j \in C(X)$ defined by $s_j(t_j) \neq 0$, $s_j(t) = 0$, $t \neq t_j$, $j = 1, 2, \ldots, m$ are single elements, contradicting our hypothesis. Hence $\mathrm{skel}_n S = \mathrm{ext}\, S$ for any $n \geqq 2$.

Corollary 2.7. *The following propositions are equivalent.*
i) $\mathrm{skel}_n S = \mathrm{ext}\ S$, for any $n \geqq 2$;
ii) $C(X)$ *does not contain non-zero single elements;*
iii) X *is a perfect set, i. e. does not contain isolated points;*
iv) $C(X)$ *does not contain minimal ideals.*

Corollary 2.8. *If* X *is a perfect set and* H *is a support hyperplane of the ball* S *of* $C(X)$, *then either* $\dim(S \cap H) = 0$ *or* $\dim(S \cap H) = +\infty$.

Proposition 2.9. *If* $\mathrm{skel}_2 S$ *contains a single element, then either* $\dim C(X) = 2$ *or* $\dim\ C(X) = 4$.

Proof. Let s be a single element in $C(X)$ and $s \in \mathrm{skel}_2 S$. If $s \in \mathrm{ext}\ S$, then $s(t_1) \neq 0$, $s(t) = 0$ for $t \in X \setminus \{t_1\}$ and also $|s| = 1$, which proves that X is a singleton, $X = \{t_1\}$ and so $\dim\ C(\{t_1\}) = 2$. If $s \in \mathrm{skel}_2 S \setminus \mathrm{ext}\ S$, then by Theorem 1 $|s(t_1)| < 1$ and $|s(t)| = 1$ for $t \in X \setminus \{t_1\}$. On the other hand, there exists $t_2 \in X_2$ such that $|s(t_2)| \neq 0$ and $|s(t)| = 0$ for $t \in X \setminus \{t_2\}$. Hence X contains exactly two points t_1, t_2, and $\dim\ C(\{t_1, t_2\}) = 4$.

Corollary 2.10. *If* $\mathrm{skel}_2 S$ *contains a single element then* $C(X) = \mathrm{soc}(C(X))$, *where* $\mathrm{soc}(C(X))$ *is the socle of the algebra* $C(X)$.

3. Non-commutative case

Throughout in this section A denotes a non-commutative C^*-algebra with unit. We also denote by A_+ the convex cone of all positive elements of A. In order to find properties of the 1 and 2 skeleton of the closed unit ball B of A, we quote the following Lemma.

Lemma 3.1. *Let* S *be the closed unit ball in* $C(X)$. *Then we have*
1) *A function* $f \in S \cap C_+(X)$ *belongs to* $\mathrm{skel}_1(S \cap C_+(X))$ *if and only if, there exist* $s \in S \cap C_+(X)$, *a single element of* $C(X)$ *and* P, *a projection of* $C(X)$, *such that* $f = P - s$.

2) *If* $f \notin \mathrm{skel}_1(S \cap C_+(X))$, *then there exist* $h_1, h_2 \in C_+(X)$, *with* $h_1 \neq h_2$ *and* fh_1, fh_2 *linearly independent, such that*

$$\|f(1 \pm h_j)^2\| \leq 1, \quad j=1, \ 2.$$

3) *If* $f \in \mathrm{skel}_1(S \cap C_+(X))$, *then there exist projections* $g_1, g_2 \in C(X)$ *and* $0 \leq \lambda \leq$ ...*iquely determined such that,* $f = \lambda g_1 + (1-\lambda) g_2$.

Proof. 1) Let $f \in \mathrm{skel}_1(S \cap C_+(X))$. Since $\mathrm{ext}(S \cap C_+(X))$ is the set of projections of $C(X)$, (see [7, p. 47]). We may assume that $f \notin \mathrm{ext}(S \cap C_+(X))$ (otherwise take $s=0$) and therefore, there exists at least one point t in X, with $0 < f(t) < 1$. We now claim that there exists exactly one point $t_1 \in X$, such that $0 < f(t_1) < 1$. If $t_1, t_2 \in X$, $t_1 \neq t_2$ and $0 < f(t_j) < 1$ $j=1, 2$, we can find V_1, V_2 disjoint compact neigbourhoods of t_1, t_2, respectively and $g_j \in C(X)$, with the property $g_j(t_j) = 1$ $g_j(V_j^c) = \{0\}$ and $g(X) = [0, 1]$. Then we can find $0 < \varepsilon < 1$ such that

$$f(1 \pm \varepsilon g_j)^2, \quad f(1 \pm \varepsilon g_j) \in S \cap C_+(X), \quad j=1, \ 2. \tag{1}$$

It is easy to see that $\varepsilon f g_1$, $\varepsilon f g_2$ are linearly independent. Hence $f = \frac{1}{2}((f + \varepsilon f g_j) + (f - \varepsilon f g_j))$, $j=1, 2$, with $(f \pm \varepsilon f g_j) \in S \cap C_+(X)$ $j=1, 2$, which contradicts the fact that $f \in \mathrm{skel}_1(S \cap C_+(X))$. Let now $f \in \mathrm{skel}_1(S \cap C_+(X))$. Then if we take the projection P with $P(t) = f(t)$, $t \neq t_1$, $P(t_1) = 1$ and consider the single element s such that $s(t) = 0$ $t \neq t_1$, $s(t) = 1 - f(t_1)$, then we have $f = P - s$.

Conversely, let $f \in S \cap C_+(X)$ and $f = P - s$ for some projection P of $C(X)$ and $s \in S \cap C_+(X)$ a single element of $C(X)$. Then, if $f \in \mathrm{relint}\, G$, where G is a convex subset of $S \cap C_+(X)$, we can easily check that $\dim G \leq 1$, and so $f \in \mathrm{skel}_1(S \cap C_+(X))$. 2) If $f \notin \mathrm{skel}_1(S \cap C_+(X))$ then there exist at least two points $t_1, t_2 \in X$ with $0 < f(t_j) < 1, j=1, 2$. We take $h_j = \varepsilon g_i$ $j=1, 2$, as in relation (1) of part 1). By construction, h_j, $j=1, 2$ belong in $S \cap C_+(X)$ are linearly independent and $\|f(1 \pm h_j)^2\| \leq 1$, $j=1, 2$.

3) If $f \in \mathrm{ext}(S \cap C_+(X))$, then take $\lambda = 1$ and $f = g_1$. If $f \in \mathrm{skel}_1(S \cap C_+(X)) \setminus \mathrm{ext}(S \cap C_+(X))$, by part 1) we have $f^2(t) = f(t)$, $t \neq t_1$ and $0 < f(t_1) < 1$. Consider the projections defined by $g_1(t) = g_2(t) = f(t)$, $t \neq t_1$ $g_1(t_1) = 1$ and $g_2(t_1) = 0$ and for $\lambda = f(t_1)$ we have $f = \lambda g_1 + (1-\lambda) g_2$. The uniqueness of λ, g_1, g_2 is obvious.

Theorem 3.2. *If* B *is the closed unit ball of the* C^**-algebra* A *with unit, then* $\mathrm{skel}_1 B = \mathrm{ext}\, B$.

Proof. The set $\mathrm{ext}\, B$ is the collection of points $x \in B$ such that xx^*, x^*x are projections and $(1 - x^*x) A (1 - xx^*) = \{0\}$ (see [7, p. 48]). Let $x \in \mathrm{skel}_1 B$. Suppose that x^*x is not a projection. We take E be the C^*-subalgebra of A generated by x^*x and the unit 1. Then E is a commutative C^*-algebra with unit and is isometrically isomorphic to $C(X)$, for some X compact, Hausdorff space. Then we can find (in the same way as in Lemma 1, 1)) a point $a \in E_+$, such that $x^*xa \neq 0$, $\|x^*x(1 \pm a)^2\| \leq 1$, $\|x^*x(1 + a^2)\| \leq 1$. Then $x = \frac{1}{2}\{(x + xa) + (x - xa)\} = \frac{1}{2}\{(x + ixa) + (x - ixa)\}$ where $\|x \pm xa\| = \|x^*x(1 + a^2)\|^{1/2} \leq 1$, $\|x \pm ixa\| = \|x^*x(1 + a^2)\|^{1/2} \leq 1$, and x, $x + xa$, $x + ixa$ are affinely independent. This is a contradiction, as

$x \in \operatorname{skel}_1 B$. Hence x^*x and therefore xx^* are projections. We claim that $(1-x^*x)\, A\, (1-xx^*)=\{0\}$. For if not, then there exists a point $a=(1-x^*x)a\,(1-xx^*) \in B$ with $a \neq 0$. Then as x^*x, xx^* are projections we have that $\|x \pm a\| = \|x \pm ia\| = 1$ (see [7, p. 48]) and $x=\frac{1}{2}((x+a)+(x-a))= \frac{1}{2}((x+ia)+(x-ia))$. This is a contradiction, as $x \in \operatorname{skel}_1 B$. Hence $(1-x^*x)B\,(1-xx^*)=\{0\}$ and this implies $(1-x^*x)A\,(1-xx^*)=\{0\}$. The proof is now complete.

Corollary 3.3. *Let E be a normed space. If the closed unit ball of E contains an edge in its boundary, then there is no way to define multiplication and an involution on E making it a C*-algebra.*

Theorem 3.4. *Let $x \in \operatorname{skel}_2 B$, where B is the closed unit ball of the C*-algebra A. Then*

$$x^*x=\lambda x_1+(1-\lambda)x_2,$$

for some projections x_1, x_2 and $0 \leqq \lambda \leqq 1$. If in addition x^*x is a projection then

$$\dim (1-x^*x)\, A\, (1-xx^*) \leqq 2.$$

Proof. Let $x \in \operatorname{skel}_2 B \backslash \operatorname{ext} B$, for if $x \in \operatorname{ext} B$ then we have nothing to prove. We take the C^*-subalgebra E of A generated by x^*x and the unit. If $x^*x \notin \operatorname{skel}_1 (B \cap E_+)$, then by Lemma 1, 2) there exist a_1, $a_2 \in E_+$ with x^*xa_1, x^*xa_2 independent and $\| x^*x(1+a_j)^2 \| \leqq 1$, $j=1, 2$. Then $x=\frac{1}{2}((x+xa_1)+(x-xa_1))= \frac{1}{2}((x+xa_2)+(x-xa_2))= \frac{1}{2}((x+ixa_1)+(x-ixa_1))$ where $\|x \pm xa_j\| = \|x^*x(1+a_j)^2\|^{1/2} \leqq 1$, $j=1, 2$, $\|x \pm ixa_1\| = \|x^*x(1+a_1^2)\|^{1/2} \leqq 1$ and x, $x+xa_1$, $x+xa_2$, $x+ixa_1$ are affinely independent, contradicting the assumtion $x \in \operatorname{skel}_2 B$. Hence $x^*x \in \operatorname{skel}_1 (B \cap E_+)$, therefore by Lemma 1, 3) there exist projections x_1, $x_2 \in A$, $0<\lambda<1$ with $x^*x=\lambda x_1+(1-\lambda)x_2$. Suppose, now, that x^*x is itself a projection. Then xx^* is a projection, too. If b_1, b_2, $b_3 \in (1-x^*x)B\,(1-xx^*)$ are linearly independent then $\|x \pm b_j\|=1$, $j=1, 2, 3$ (see [7, p. 48]) and $x=\frac{1}{2}((x+b_j)+(x-b_j))$, $j=1, 2, 3$, which is impossible as $x \in \operatorname{skel}_2 B$. This entails $\dim (1-x^*x)B\,(1-xx^*) \leqq 2$ and $\dim (1-x^*x)A\,(1-xx^*) \leqq 2$.

4. Questions

The following problems might be of interest:

1. Characterize the n-skeleton of the closed unit ball B, of a C^*-algebra for $n \geqq 3$.
2. Can one answer in the affirmative that $x \in \operatorname{skel}_2 B$ iff $x^*x=\lambda x_1+(1-\lambda)x_2$ where x_1, x_2 are projections and $\dim (1-x^*x)\, A\, (1-xx^*) \leqq 2$?

3. For the commutative case it was proved that the boundary of the closed unit ball B of a C^*-algebra A contains a 2-face iff A contains a non-zero single element (see prop. 1.6). Can we say the same the for non-commutative case?

References

1. J. A. Erdos. On certain elements of C^*-algebras. *Illinois J. Math.*, **15**, 1971, 682-697.
2. R. V. Kadison. Isometries of operator algebras. *Annals of Math.*, (2) **54**, 1951, 325-338.
3. P. Miles. B^*-algebra unit ball extremal points. *Pacific J. Math.*, **14**, 1964, 627-637.
4. R. R. Phelps. Extreme positive operators and homomorphisms. *Trans. Amer. Math. Soc.*, **108**, 1963, 265-274.
5. R. R. Phelps. Extreme points in function algebras. *Duke Math. J.*, **32**, 1965, 267-277.
6. S. Sakai. Theory of W^*-algebras. Yale, 1962.
7. M. Takesaki. Theory of Operator algebras I. Berlin, 1979.
8. J. Tölke, J. M. Wills. Contributions to Geometry. Basel, 1979.

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