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SETS OF CONSTANT WIDTH AND DIAMETRICALLY COMPLETE SETS IN NORMED SPACES

BY

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1. Introduction It is known that in the n-dimensional Euclidean space E^n and in an n-dimensional Minkowski space M^n, a convex compact set of constant width has the property that every two parallel support hyperplanes are at the same distance apart (see [1], [9]).

Closely related to the sets of constant width is a class of sets, called diametrically complete sets which have the property that the mere addition of a single point increases their diameter. In the case of the n-dimensional Euclidean space, it is known that these two classes of sets coincide (see [2], [9]).

In this paper, we generalize the above notions to an arbitrary normed space of infinite dimension and we examine whether several known properties in the case of E^n, hold also true for the above spaces.

In their work, Eggleston [3] and Soltan [8] characterize among the Minkowski spaces those which have the property that the diametrically complete sets are homothetic to the unit ball. Here, it is proved that infinite dimensional spaces with the above property do not exist among the reflexive spaces.

Throughout this paper, E denotes a (real) normed space and E' its topological dual with the supremum norm.

2. The width of a set in a direction

2.1. Definition. *Let C be a bounded convex set in E and $f \in E'$, with $f \neq 0$. We call the non-negative real number*

$$B(f) = \frac{1}{\|f\|}\left[\sup_{x \in C} f(x) - \inf_{x \in C} f(x)\right],$$

*the **width of C in the direction f**.*

By the Riesz Representation Theorem, we can easily conclude that the above definition of the width of a set coincides with the usual definition of the width, given in the n-dimensional Minkowski space M^n, (see [1], [9]).

2.2. Definition. *Let C de a bounded convex set in E. We define the* ***minimal width*** *of C to be the non-negative real number*

$$w = \inf \{B(f) : f \in E' \setminus \{0\} \}$$

2.3. Definition. *Let C be a bounded convex set in E. We say that C is a set of* **constant width** *iff $B(f)$ is constant for every $f \in E'$, with $f \neq 0$.*

Since $B(f) = B\left(\frac{f}{\|f\|}\right)$, we may say that C is a set of constant width iff $B(f)$ is constant for every $f \in E'$ with $\|f\| = 1$.

Remark. The open and closed balls in E are sets of constant width. Obviously, their width $B(f)$ in every direction f, is equal to their diameter.

In an n-dimensional Minkowski space M^n, the width of a compact convex C in a direction u, is the distance between the two supporting hyperplanes of C which are orthogonal to u (see [9]). An analogous result holds in the case of a normed space, as it is shown in the next Proposition.

2.4. Proposition. *Let C be a bounded convex set in E and $B(f)$ be the width of C in the direction f where $f \in E' \setminus \{0\}$. Let also H_f^1 and H_f^2 denote the hyperplanes*

$$\left\{ y \in E : f(y) = \sup_{x \in C} f(x) \right\}, \left\{ z \in E : f(z) = \inf_{x \in C} f(x) \right\}$$

respectively. Then

$$B(f) = d(H_f^1, H_f^2)$$

where $d(H_f^1, H_f^2)$ is the distance between the hyperplanes H_f^1 and H_f^2.

Proof. We consider the points y, z of H_f^1 and H_f^2 respectively, then

$$B(f) = \frac{1}{\|f\|}\left[\sup_{x \in C} f(x) - \inf_{x \in C} f(x) \right] =$$

$$= \frac{1}{\|f\|}[f(y) - f(z)] \leqslant \|y - z\| \quad ,$$

that is,

$$B(f) \leqslant \|y - z\| \quad ,$$

and this for every $y \in H_f^1$ and $z \in H_f^2$. Hence

$$B(f) \leqslant d(H_f^1, H_f^2) \qquad (1).$$

Conversely now, let y be a point of H_f^1 and α be a real number, with $\alpha > 1$. Since

$$\|f\| = \sup_{\|x\|=1} f(x)$$

there exists $e \in E$, with $\|e\| = 1$, such that

$$|f(e)| > \frac{\|f\|}{\alpha} \qquad (2).$$

Clearly, $f(e) \neq 0$. We put

$$z = y + \left[\inf_{x \in C} f(x) - \sup_{x \in C} f(x)\right] \frac{e}{f(e)} \quad ,$$

then

$$f(z) = f(y) + \left[\inf_{x \in C} f(x) - \sup_{x \in C} f(x)\right] = \inf_{x \in C} f(x) \quad ,$$

since $y \in H_f^1$. Thus, z is a point of H_f^2.

Now, if $d(y, z)$ is the distance between the points y, z, we have

$$d(H_f^1, H_f^2) \leqslant d(y, z) = \left| \inf_{x \in C} f(x) - \sup_{x \in C} f(x) \right| \frac{\|e\|}{|f(e)|} =$$

$$= B(f) \cdot \|f\| \frac{1}{|f(e)|} \quad ,$$

and from (2) we get

$$d(H_f^1, H_f^2) \leqslant B(f) \qquad (3).$$

Hence, the result follows from (1) and (3).

3. Properties of sets of constant width

In this section we shall be concerned with the relation between sets of constant width and diametrically complete sets. We also study the boundary of sets of constant width.

3.1. Definition. *Let K be a non-empty, bounded set in E. The set K is said to be* ***diametrically complete*** *iff it is not contained strictly in any set of the same diameter.*

Remark. As far as we know, the diametrically complete sets are refered also as **complete** sets (see for example [1], [2], [9]), or as **diametrically maximal** sets (see [3]), to avoid confusion with the topological use of the word "complete".

It is known that in the n-dimensional Euclidean space E^n, a diametrically complete set is a point or it has non-empty interior (see [2], p. 125). As it follows from the Proposition we quote below, the diametrically complete sets of an arbitrary normed space E have the same property.

3.2. Proposition. *If K is a diametrically complete set in E, then, it is either a point or it has non-empty interior.*

Proof. Since the convex hull operation and the clocure operation do not increase the diameter, we conclude that K is convex and closed.

Let d be the diameter of K. If $d = 0$ then, since K is non-empty, it is a point.

Now, if $d > 0$, we suppose that K has empty interior. So for every $x \in K$ and $\varepsilon > 0$, there exists $y \notin K$ such that

$$\|x - y\| \leqslant \varepsilon$$

Furthermore, since K is convex, we may suppose that for every $x \in K$ and $\varepsilon > 0$, there exists $y \notin K$ such that

$$\|x - y\| = \varepsilon \tag{1}.$$

Finally, we may assume without loss of generality that the origin o belongs to the relative interior of K. Then, there exists $\varepsilon_0 > 0$, with $\varepsilon_0 < 2d$, such that

$$\forall x \in A(K), \ \|x\| \leqslant \varepsilon_0 \Rightarrow x \in K \tag{2},$$

where $A(K)$ is the affine hull of K.

Since $o \in K$, from (1) follows that there exist $y_n \notin K$, with

$$\|y_n\| = \frac{\varepsilon_0}{n}$$

for every $n \in \mathbb{N}$. But K is a diametrically complete set, so for every $n \in \mathbb{N}$, there exists a point $z_n \in K$, such that

$$\|y_n - z_n\| > d \quad .$$

Hence,

$$d \geqslant \|o - z_n\| = \|z_n\| \geqslant \|z_n - y_n\| - \|y_n\| > d - \frac{\varepsilon_0}{n} ,$$

and taking $n \to +\infty$, we have

$$\lim_{n \to \infty} \|z_n\| = d ,$$

so there exists $k \in \mathbb{N}$, such that

$$\|z_k\| > d - \frac{\varepsilon_0}{2} > 0 \tag{3}$$

Put now

$$w = -\varepsilon_0 \cdot \frac{z_k}{\|z_k\|} ,$$

then, $w \in A(K)$ and $\|w\| = \varepsilon_0$. Thus, by (2) w is a point of K.

Finally, from (3) we have

$$d \geqslant \|w - z_k\| = \left\| \varepsilon_0 \cdot \frac{z_k}{\|z_k\|} + z_k \right\| = \varepsilon_0 + \|z_k\| >$$

$$> \varepsilon_0 + d - \frac{\varepsilon_0}{2} = d + \frac{\varepsilon_0}{2}$$

which is a contradiction. Therefore, the set K has non-empty interior.

Next, we give a Corollary the proof of which is obvious.

3.3. Corollary. *A diametrically complete set in E has dimension either equal to the dimension of E, or equal to zero.*

In the n-dimensional Euclidean space E^n, a closed convex set of constant width is a diametrically complete set and vice versa (see [2], p. 123). In an n-dimensional Minkowki space M^n the direct implication holds (see [1]), but the converse is not true in general (see [3]). In a normed space E, the direct implication is also true, as we can see from the following Theorem.

3.4. Theorem. *Let K be a bounded, closed and convex set in E. If K is a set of contant width then it is diametrically complete.*

In order to prove the Theorem 3.4. we shall need the following Lemma.

3.5. Lemma. *Let C be a bounded, convex set in E. If d is the diameter of C, then*

$$d = \sup \{B(f) : f \in E', \|f\| = 1\}$$

Proof: Let $f \in E'$, with $\|f\| = 1$. For an arbitrary $\varepsilon > 0$, there exist $x_1, x_2 \in C$, such that

$$f(x_1) + \frac{\varepsilon}{2} > \sup_{x \in C} f(x), \qquad f(x_2) - \frac{\varepsilon}{2} < \inf_{x \in C} f(x) \ .$$

Then

$$B(f) < f(x_1) + \frac{\varepsilon}{2} - f(x_2) + \frac{\varepsilon}{2} = f(x_1 - x_2) + \varepsilon \leqslant$$

$$\leqslant \|x_1 - x_2\| + \varepsilon \leqslant d + \varepsilon \ .$$

So

$$B(f) \leqslant d \qquad (1).$$

Conversely now, let $\varepsilon > 0$, then there exist $x_0, y_0 \in C$, such that

$$\|x_0 - y_0\| > d - \varepsilon \ .$$

If ℓ is the 1-dimensional linear subspace of E generated by $x_0 - y_0$, we define the linear functional

$$f_0 : \ell \longrightarrow \mathbb{R}^n ,$$

with

$$f_0 (\lambda(x_0 - y_0)) = \lambda \cdot \|x_0 - y_0\| \ , \ \lambda \in \mathbb{R} .$$

Clearly,

$$\|f_0\| = 1 .$$

and by the Hahn-Banach Theorem, f_0 extends to a linear functional f_1 on E that satisfies

$$\|f_1\| = 1 .$$

Then,

$$B(f_1) = \sup_{x \in C} f_1(x) - \inf_{x \in C} f_1(x) \geqslant f_1(x_0) - f_1(y_0) = f_1(x_0 - y_0) =$$
$$= f_0(x_0 - y_0) = \|x_0 - y_0\| > d - \varepsilon ,$$

and hence,

$$\sup \{B(f) : f \in E', \|f\| = 1\} \geqslant B(f_1) > d - \varepsilon ,$$

for arbitrary $\varepsilon > 0$. So

$$\sup \{B(f) : f \in E', \|f\| = 1\} \geqslant d \qquad (2).$$

From (1) and (2), we get the desired result.

3.6. Corollary. *A bounded, convex set in E is a set of constant width iff its diameter is equal to its minimal width.*

Proof. This is an immediate consequence of the Lemma 3.5 and the definition of the minimal width.

The proof of Theorem 3.4. Let d be the diameter of K, then, since K is a set of constant width, we have from Lemma 3.5 that

$$d = B(f) ,$$

for every $f \in E'$, with $\|f\| = 1$. Let also z be a point of E, such that $z \notin K$. By the hypothesis that K is a convex and closed set, there exist a linear functional $f \in E'$, with $\|f\| = 1$ and a real number α such that

$$K \subseteq \{x \in E : f(x) < \alpha\} \qquad (1)$$

and

$$f(z) > \alpha \qquad (2).$$

We put

$$\varepsilon = f(z) - \alpha > 0 \; .$$

Then there exist $x_0, y_0 \in K$, such that

$$f(x_0) > \sup_{x \in C} f(x) - \frac{\varepsilon}{2} \quad , \quad f(y_0) < \inf_{x \in C} f(x) + \frac{\varepsilon}{2}$$

Now, we have

$$\|z - y_0\| \geqslant |f(z - y_0)| = f(z - y_0) \qquad (3),$$

because $y_0 \in K$ and so $f(z) > \alpha > f(y_0)$, as it follows from the relations (1) and (2).

Also,

$$f(z - y_0) = [f(z) - f(x_0)] + [f(x_0) - f(y_0)] >$$
$$> f(z) - \alpha + B(f) - \varepsilon = \varepsilon + d - \varepsilon = d \qquad (4).$$

From (3) and (4), we get

$$\|z - y_0\| > d \; ,$$

which shows that the diameter of the set $K \cup \{z\}$ is strictly greater than the diameter of K. Hence, since the point z is chosen arbitrarily, it follows that K is a diametrically complete set.

3.7. Corollary. *A bounded convex and closed set of constant width in E is either a point or it has non-empty interior.*

Proof. See Proposition 3.2 and Theorem 3.4.

In the n-dimensional Euclidean space E^n the boundary points of a bounded, closed and convex set of constant width are exposed points (see [2]). In a normed space this is not true in general, since a bounded convex and closed set of constant width (for example a closed ball) may not have extreme points. Nevertheless, we shall show that this property holds for a reflexive Banach space.

3.8. Theorem. *Let E be a reflexive Banach space with a strictly convex norm. If K is a bounded, convex and closed set of constant width in E, then, every boundary point of K is an exposed point.*

Proof. Corollary 3.7 implies that the interior of K is either non-empty or K is exactly a point. In the latter case we have nothing to prove.

We assume that x_0 is a boundary point of K Then, (see for example [4], p. 64) there exist $f \in E'$ and $\alpha \in \mathbb{R}$, such that

$$x_0 \in \{y \in E : f(y) = \alpha\} \quad , \quad K \subseteq \{y \in E : f(y) \leqslant \alpha\} .$$

Since K is weakly compact in E

$$f(x_0) = \sup_{x \in K} f(x) .$$

Let also y_0 be a point of K, such that

$$f(y_0) = \inf_{x \in K} f(x) ,$$

then

$$B(f) = \frac{1}{\|f\|} [f(x_0) - f(y_0)] .$$

If d is the diameter of K, since K is a set of constant width, we have from Lemma 3.5

$$B(f) = d$$

This implies that

$$d = \frac{f(x_0 - y_0)}{\|f\|} \leqslant \|x_0 - y_0\| \leqslant d$$

and so

$$\|x_0 - y_0\| = d \qquad (1).$$

We suppose now that there exist $x_1, x_2 \in K$, with $x_1 \neq x_2$, such that

$$x_0 = \frac{1}{2}(x_1 + x_2)$$

then,

$$d = \|x_0 - y_0\| = \left\| \frac{1}{2}(x_1 + x_2) - y_0 \right\| = \frac{1}{2}\|x_1 - y_0 + x_2 - y_0\| \leqslant$$

$$\leqslant \frac{1}{2}\|x_1 - y_0\| + \frac{1}{2}\|x_2 - y_0\| \leqslant d .$$

Hence, the equality holds, i.e.

$$\|x_1 - y_0 + x_2 - y_0\| = \|x_1 - y_0\| + \|x_2 - y_0\| = 2d .$$

Since, $x_1, x_2, y_0 \in K$ necessarily

$$\|x_1 - y_0\| = \|x_2 - y_0\| = d$$

But, by the hypothesis that the norm is strictly convex, we get

$$\|x_0 - y_0\| = \left\| \frac{x_1 + x_2}{2} - y_0 \right\| < d$$

which contradicts to (1). Thus, x_0 is an extreme point of K.

Finally, we shall show that x_0 is an exposed point. We consider the set

$$H = \left\{ x \in K : f(x) = f(x_0) = \sup_{x \in C} f(x) \right\} ,$$

where f is the support functional of K at x_0.

If x_1 is a point of $H \cap K$, we put

$$z = \frac{x_0 + x_1}{2} .$$

Clearly, z is a point of H and so it must be a boundary point of K, for otherwise, the interior of K and H would have non-empty intersection. But, by the previous, result z is also an extreme point of K, which implies that

$$x_0 = x_1 .$$

Thus, x_0 is an exposed point of K.

3.9. Corollary. *In a uniformly normed Banach space, every boundary point of a closed convex set of constant width is an exposed point.*

Proof. See for example [4], p. 162.

4. Diametrically complete sets in reflexive banach spaces

We know (see [3]) that there exist finite dimensional Banach spaces E (n-Minkowski spaces), in which every diametrically complete set is "homothetic" to the unit ball of E, i.e. if K is a diametrically complete set in E, then, there exist $x \in E$ and $\lambda \geqslant 0$, such that

$$K = x + \lambda S,$$

where S is the unit ball of E.

Here, we prove that reflexive Banach spaces of infinite dimension with the above property, do not exist.

4.1. Theorem. *Let E be an infinite dimensional reflexive Banach space. Then, there exists a diametrically complete set which is not homothetic to the unit ball of E.*

For the proof of the above Theorem, we shall need the following:

4.2. Lemma. *Every bounded set in a normed space is contained in a diametrically complete set of the same diameter.*

Proof. Let E be a normed space and K be a non-empty bounded subset of E. Let also d(M) denote the diameter of a bounded set M in E. We consider the set

$$\mathscr{E} = \left\{ B \subseteq E : B \supseteq K,\ d(B) = d(K) \right\}.$$

$\mathscr{E}$ is a non-empty family of sets, partially ordered by the relation of the inclusion

If $(B_i)_{i \in I}$ is a chain in $\mathscr{E}$ then the set

$$B = \bigcup_{i \in I} B_i$$

is an upper bound of the chain in $\mathscr{E}$, since $B \supseteq K$ and $d(B) = d(K)$. Hence, by Zorn's Lemma, there exists a maximal set, A say, in $\mathscr{E}$.

Now, it suffices to show that the set A is diametrically complete. If this is not true, then there would exist a point x in E, such that $x \notin A$ and

$$d(A \cup \{x\}) = d(A) = d(K) .$$

But

$$A \cup \{x\} \supset A \supseteq K ,$$

and so the set $A \cup \{x\}$ belongs to $\mathscr{E}$ contradicting the maximality of the set A.

The proof of Theorem 4.1. For the space E there exists an infinite dimensional separable Banach subspace, say E_0 (see [4], [7]).

If every diametrically complete set in E_0 is homothetic to the unit ball S_0 of E_0, then, by modifying the methods in [8] we prove analogously that the distance of any two extreme points of S_0 is equal to 2. But by [6] this is impossible for separable reflexive Banach spaces of infinite dimension. Hence, there exists a set K_0 in E_0, which is diametrically complete in E_0 but not homothetic to the unit ball S_0 of E_0.

By Lemma 4.2, K_0 is contained in a diametrically complete set K in E with the same diameter, d say. We suppose that K is homothetic to the unit ball S of E, i.e. there exist $x_0 \in E$, $\lambda \geqslant 0$ such that

$$K = x_0 + \lambda S .$$

Then

$$K \cap E_0 = (x_0 + \lambda S) \cap E_0 = x_0 + \lambda S_0 \qquad (1).$$

Now, if d(M) denotes the diameter of a bounded set M, we have

$$d = d(K_0) \leqslant d(K \cap E_0) \leqslant d(K) = d ,$$

which implies that

$$d(K_0) = d(K \cap E_0) .$$

But K_0 is a diametrically complete set in E_0 and so

$$K_0 = K \cap E_0 .$$

Hence, by (1) we have that K_0 is homothetic to the unit ball S_0 of E_0 which is a contradiction.

Thus, the diametrically complete set K in E is not homothetic to the unit ball of E.

Concluding this paper we remark that the following problem remains still open:

"Characterize the infinite dimensional normed spaces in which every diametrically complete set is homothetic to the unit ball".

REFERENCES

1. G. D. Chakerian and H. Groemer, "Convex Bodies of Constant Width", Convexity and its Applications. Birkhäuser, Basel (1983).

2. H.G. Eggleston, "Convexity", Cambridge Univ. Press (1966).

3. H.G. Eggleston, "Sets of Constant Width in Finite Dimensional Banach Spaces", Israel J. Math. 3(1965), 163-172.

4. R. Holmes, "Geometric Functional Analysis and its Applications", Springer -Verlag (1975).

5. G.J. Jameson, "Topology and Normed Spaces", Chapman and Hall, London (1974).

6. J. Linderstrauss and R. Phelps, "Extreme Point Properties of Convex Bodies in Reflexive Banach Spaces" Israel J. Math. 6(1968), 39-48.

7. J. T. Marti, "Introduction to the Theory of Bases", Springer, Berlin (1969).

8. V.P. Soltan, "A Theorem on Complete Sets", Soviet Math. Dokl. Vol. 18(1977), No. 3, 680-682.

9. F.A. Valentine, "Convex Sets", Mc Graw Hill, New York (1964).

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