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Convex bodies with almost all k-dimensional sections polytopes

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1. It is a well-known result of V. L. Klee (2) that if a convex body K in E^n has all its k-dimensional sections as polytopes ($k \geqslant 2$) then K is a polytope.

In (6, 7) E. Schneider has asked if any properties of a polytope are retained by K if the above assumptions are weakened to hold only for almost all k-dimensional sections of K. In particular, for 2-dimensional sections in E^3 he asked whether $\operatorname{ext} K$ is countable, or, if not, is the 1-dimensional Hausdorff measure of $\operatorname{ext} K$ zero?

Here we shall give an example (Theorem 1) of a convex body K in E^3, almost all of whose 2-sections are polygons but $\operatorname{ext} K$ has Hausdorff dimension 1. Our example does not, however, yield a convex body K with $\operatorname{ext} K$ of positive 1-dimensional Hausdorff measure although we believe that such a body exists.

Using deep measure-theoretic results of J. M. Marstrand (3), as extended by P. Mattila (5), we shall prove a very general result (Theorem 2) showing that a convex body K in E^n has $\operatorname{ext}(K \cap L)$ of dimension at most s for almost all of its k-dimensional sections L, if, and only if, the dimension of the $(n-k)$-skeleton of K is at most $n-k+s$.

THEOREM 1. *There exists, in E^3, an example of a convex body K almost all of whose 2-sections are polygons but* $\operatorname{ext} K$ *has Hausdorff dimension* 1.

THEOREM 2. *K is a convex body in E^n with* $\operatorname{ext}(K \cap L)$ *of dimension at most s for almost all of its k-dimensional sections L if, and only if, the dimension of the $(n-k)$-skeleton of K is at most $n-k+s$.*

Definitions and notations. Let L_{n-k} be the set of $(n-k)$-dimensional subspaces of E^n and for each $L \in L_{n-k}$ let $L^{\perp}$ denote the k-dimensional subspace perpendicular to L. Let A be a measurable subset of E^n and, for each $L \in L_{n-k}$, let $A(L)$ be the orthogonal projection of A into L. Let G be a family of k-flats which meet A and, for each $L \in L_{n-k}$ let $A(L, G)$ denote the set of points $x \in A(L)$ with $x + L^{\perp} \in G$. Then we may ascribe a measure ν_k to G by

$$\nu_k(G) = \int H^{n-k}(A(L,G))\, d\mu_{n-k}(L),$$

where μ_{n-k} is the ordinary Haar measure on the Grassmanian L_{n-k} and H^s is the s-dimensional Hausdorff measure. To say that a property P holds for almost all k-sections of A is to mean, if G denotes those k-sections for which P does not hold,

$$\nu_k(G) = 0.$$

2. *The construction of a convex body K in E^3 which satisfies Theorem* 1.

Let S be the intersection of the unit sphere with the positive octant of E^3, i.e.

$$S = \{(x,y,z) \in E^3,\ x^2+y^2+z^2 = 1,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$$

and let S^* be the intersection of S with the plane $z = 0$.

If $\mathbf{r} \in S$ let $\theta(\mathbf{r})$ denote the angle made by the orthogonal projection of $\mathbf{r}$ into the plane $z = 0$ with the positive direction of the x-axis. If $\mathbf{r}, \mathbf{t} \in S$ we shall say that $\mathbf{r}$ lies to the left of $\mathbf{t}$, and $\mathbf{t}$ lies to the right of $\mathbf{r}$, if $\theta(\mathbf{r}) < \theta(\mathbf{t})$.

Divide S^* into four equal arcs $S^{(4)}(i_4), i_4 = 1, 2, 3, 4$, say. In the plane $z = \frac{1}{2} = z_4$, say, choose points

$$\alpha^{(4)}(i_4) = (\sqrt{3}\cos\theta/2, \sqrt{3}\sin\theta/2, \tfrac{1}{2}), \quad \theta = 3\pi(2i_4 - 1)/2(4!)$$

of S, $i_4 = 1, 2, 3, 4$. Let $T^{(4)}(i_4)$ be the spherical isosceles triangle with base $S^{(4)}(i_4)$ and their vertex $\alpha^{(4)}(i_4)$, $i_4 = 1, 2, 3, 4$.

Next divide each of the arcs $S^{(4)}(i_4)$ into five equal arcs $S^{(5)}(i_4, i_5)$, $i_5 = 1, 2, 3, 4, 5$. Take $S^{(5)}(i_4, 1)$ as base for an isosceles spherical triangle $T^{(5)}(i_4, 1)$ with the third vertex $\alpha^5(i_4, 1)$ of $T^{(5)}(i_4, 1)$ lying on the left hand edge of $T^{(4)}(i_4)$, $i_4 = 1, 2, 3$. On each of the arcs $S^{(5)}(i_4, i_5)$ $(i_4 = 1, 2, 3;\ i_5 = 1, 2, 3, 4, 5)$ we construct an isosceles spherical triangle $T^{(5)}(i_4, i_5)$ congruent to $T^{(5)}(i_4, 1)$. Each of these triangles has a third vertex $\alpha^{(5)}(i_4, i_5)$ lying in the same plane $z = z_5$ say; $z_5 > 0$. We also choose the points $\alpha^{(5)}(4, 1)$, $\alpha^{(5)}(4, 5)$ on the edges of $T^{(4)}(4)$ and on the plane $z = z_5$, with $\alpha^{(5)}(4, 1)$ lying to the left of $\alpha^{(5)}(4, 5)$.

Proceeding inductively, suppose now that we have constructed, for $n \geq 5$, equal arcs $S^{(n)}(i_4, \ldots, i_n)$, $1 \leq i_k \leq k-1$, $1 \leq k \leq n-1$ and $1 \leq i_n \leq n$, on S^*.

Divide each arc $S^{(n)}(i_4, \ldots, i_n)$, $1 \leq i_k \leq k-1$, $1 \leq k \leq n$ into $n+1$ equal arcs $S^{(n+1)}(i_4, \ldots, i_{n+1})$. Take $S^{(n+1)}(1, \ldots, 1)$ as base for an isosceles spherical triangle $T^{(n+1)}(1, \ldots, 1)$. The third vertex $\alpha^{(n+1)}(1, \ldots, 1)$ of $T^{(n+1)}(1, \ldots, 1)$ lies on the left hand edge of $T^{(k)}(1, \ldots, 1)$, $k = 4, \ldots, n$. On each of the arcs $S^{(n+1)}(i_4, \ldots, i_{n+1})$, $1 \leq i_k \leq k-1$, $1 \leq k \leq n$, $1 \leq i_{n+1} \leq n+1$ we construct a copy $T^{(n+1)}(i_4, \ldots, i_{n+1})$ of $T(^{n+1})(1, \ldots, 1)$ with third vertex $\alpha^{(n+1)}(i_4, \ldots, i_{n+1})$. The points $\alpha^{(n+1)}(i_4, \ldots, i_{n+1})$ lie in the plane $z = z_{n+1} > 0$, $1 \leq i_k \leq k-1$, $1 \leq k \leq n$, $1 \leq i_{n+1} \leq n+1$.

Also we define points $\alpha^{(n+1)}(i_4, i_5, \ldots, i_{\nu-1}, \nu, 1)$, $\alpha^{(n+1)}(i_4, i_5, \ldots, i_{\nu-1}, \nu, n+1)$, $4 \leq \nu \leq n$, $1 \leq i_k \leq k-1$, $1 \leq k \leq \nu - 1$ on the edges of $T^{(\nu)}(i_4, i_5, \ldots, i_{\nu-1}, \nu)$, $4 \leq \nu \leq n$ and lying in the plane $z = z_{n+1}$; we suppose that $\alpha^{(n+1)}(i_4, \ldots, i_{\nu-1}, \nu, 1)$ lies to the left of

$$\alpha^{(n+1)}(i_4, \ldots, i_{\nu-1}, \nu, n+1)$$

We may write down the above points explicitly:

$$\alpha^{(n)}(i_4, \ldots, i_n) = ((1-z_n^2)^{\frac{1}{2}}\cos\phi, (1-z^2)^{\frac{1}{2}}\sin\phi, z_n)$$

where $$\phi = (i_4 - 1)\frac{3\pi}{4!} + (i_5 - 1)\frac{3\pi}{5!} + \ldots + (i_{n-1} - 1)\frac{3\pi}{(n-1)!} + (2i_n - 1)\frac{3\pi}{2(n!)}$$

and $$z_n = \sin\frac{3\pi}{2(n!)}\bigg/\left(\sin^2\frac{3\pi}{2(n!)} + 3\sin^2\frac{\pi}{16}\right)^{\frac{1}{2}}$$

for $$1 \leq i_k \leq k-1, \quad 1 \leq k \leq n-1, \quad 1 \leq i_n \leq n.$$

Also, for $$4 \leq \nu \leq n-1, \quad 1 \leq i_k \leq k-1, \quad 1 \leq k \leq \nu - 1,$$

we have $$\alpha^{(n)}(i_4, \ldots, i_{\nu-1}, \nu, 1) = ((1-z^2)^{\frac{1}{2}}\cos\theta_1, (1-z^2)^{\frac{1}{2}}\sin\theta_1, z_n),$$

where $$\theta_1 = (i_4 - 1)\frac{3\pi}{4!} + \ldots + (\nu - 1)\frac{3\pi}{\nu!} + \frac{3\pi}{2(n!)},$$

and $$\alpha^{(n)}(i_4, \ldots, i_{\nu-1}, \nu, n) = ((1-z_n^2)^{\frac{1}{2}} \cos\theta_n, (1-z^2)^{\frac{1}{2}} \sin\theta_n, z_n),$$

where $$\theta_n = (i_4 - 1)\frac{3\pi}{4!} + \ldots + \nu\frac{3\pi}{\nu!} - \frac{3\pi}{2(n!)}.$$

We now suppose that the points $\alpha^{(n)}(i_4, \ldots, i_n)$ have been constructed inductively for $n = 4, 5, \ldots$; and, for fixed n, E_n denotes the set of all points $\alpha^{(n)}(i_4, \ldots, ,i_n)$ constructed above.

For a set F in E^3 let $\operatorname{con} F$ denote its convex hull and $\overline{\operatorname{con}}\, F$ its closed convex hull. Let

$$K_m = \overline{\operatorname{con}}\{(0,0,0) \cup (0,0,z_m) \cup \bigcup_{n=m}^{\infty} E_n\} \cap \{z \leqslant z_m\} \cap A^+ \cap B^+,$$

where $A = \operatorname{con}\{(0,0,0),\ (1,0,0), \alpha_1^{(4)}\}$, $B = \operatorname{con}\{(0,0,0),\ (0,1,0), \alpha_4^{(4)}\}$ and A^+, B^+ are the closed half spaces determined by A and B respectively which contain the point $(1,1,0)$.

The set of extreme points of K_m in the plane $z = 0$ is

$$E_\infty = C \cup D$$

where $$C = \left\{\cos\left(\sum_{r=4}^{\infty} \frac{3\pi u_r}{r!}\right), \sin\left(\sum_{r=4}^{\infty} \frac{3\pi u_r}{r!}\right), 0\right), 0 \leqslant u_r \leqslant r-2\right\}$$

and D is the set of all points of the form $(\cos\theta, \sin\theta, 0)$

where $$\theta = \sum_{r=4}^{k-1} \frac{3\pi u_r}{r!} + (k-1)\frac{3\pi}{k!}, \quad 0 \leqslant u_r \leqslant r-2, \quad k = 4, 5, \ldots$$

or $$\theta = \sum_{r=4}^{k-1} \frac{3\pi u_r}{r!} + k\frac{3\pi}{k!}, \quad 0 \leqslant u_r \leqslant r-2, \quad k = 4, 5, \ldots.$$

The set C is a Cantor-like set with Hausdorff dimension 1 but of zero 1-dimensional Hausdorff measure.

The set of extreme points A_m of K_m is $A_m = E_\infty \cup \bigcup_{n=m}^{\infty} E_n \cup (0,0,0) \cup (x_m, y_m, z_m)$ where the point (x_m, y_m, z_m) is the intersection of the planes A, B and $z = z_m$.

Since $\cup_{n=m}^{\infty} E_n$ is countable, $\dim A_m = \dim C = 1$ and $H^1(A_m) = 0$, where $\dim A$ denotes the Hausdorff dimension of A.

We shall now describe the 2-faces of K_m.

There will of course be the four faces $F_1 = A \cap K_m$, $F_2 = B \cap K_m$, $F_3 = \{z = 0\} \cap K_m$, $F_4 = \{z = z_m\} \cap K_m$.

For a set A in E^3 let $\operatorname{aff} A$ denote the affine hull of A. For $n = 4, 5, \ldots$, let

$$\Pi_1^n = \operatorname{aff}\{\alpha^{(n)}_{(i_4, \ldots i_n)}, \alpha^{(n+1)}_{(i_4, \ldots, i_n, \sigma)}), \alpha^{(n+1)}_{(i_4, \ldots, i_n, \sigma+1)}\}, \quad 1 \leqslant \sigma \leqslant n,$$

$$\Pi_2^n = \operatorname{aff}\{\alpha^{(n)}_{(i_4, \ldots, i_n)}, \alpha^{(n)}_{(i_4, \ldots, i_n+1)}, \alpha^{(n+1)}_{(i_4, \ldots, i_n, n+1)}, \alpha^{(n+1)}_{(i_4, \ldots, i_n+1, 1)}\},$$

$$\Pi_3^n = \operatorname{aff}\{\alpha^{(n)}_{(i_4, \ldots, i_{n-1}, n)}, \alpha^{(n+1)}_{(i_4, \ldots, i_{n-1}, n, 1)}, \alpha^{(n+1)}_{(i_4, \ldots, i_{n-1}, n, n+1)}\},$$

$$\Pi_4^n = \operatorname{aff}\{\alpha^{(n)}_{i_4, \ldots, i_{n-1}, n)}, \alpha^{(n)}_{(i_4, \ldots, i_{n-1}+1, 1)}, \alpha^{(n+1)}_{(i_4, \ldots, i_{n-1}, n, n+1)}, \alpha^{(n+1)}_{i_4, \ldots, i_{n-1}+1, 1, 1)}\},$$

$$\Pi_5^n = \operatorname{aff}\{\alpha^{(n)}_{(i_4, \ldots, i_{\nu-1}, \nu, 1)}, \alpha^{(n)}_{(i_4, \ldots, i_{\nu-1}, \nu, n)}, \alpha^{(n+1)}_{(i_4, \ldots, i_{\nu-1}, \nu, 1)}, \alpha^{(n+1)}_{(i_4, \ldots, i_{\nu-1}, n+1)}\}.$$

We claim that the 2-faces of K_m other than F_1, F_2, F_3, F_4 are of the form $K_m \cap \Pi_i^n$, $i = 1, 2, 3, 4, 5$, m sufficiently large. To show this, we need only show that there exists m_0 such that, for $m \geq m_0$ and $n \geq m$, Π_i^n does not lie below any member of E_j, $j \geq n+2$.

This is certainly true for Π_i^n, $i = 2, 3, 4, 5$, since the spherical triangle or spherical rectangle determined by the corresponding vertices $\alpha^{(n)}(i_4, \ldots, i_n)$ of $\Pi_i^n \cap K_m$ can be extended to S^* without containing any further vertex of K_m in its interior.

We shall show that this is also true of Π_i^n, $n \geq m \geq m_0$, by showing that Π_i^n does not meet the quarter circle,

$$S_{n+2} = S \cap \{z = z_{n+2}\}, \quad n \text{ sufficiently large.}$$

Consider the plane $F_{\sigma,n}$ defined by $\alpha^{(n+1)}(1, \ldots, 1, \sigma)$, $\alpha^{(n+1)}(1, \ldots, 1, \sigma+1)$ and α', where α' is the point on the plane $z = z_n$ which makes the three points into an isosceles spherical triangle with $\alpha^{(n+1)}(1, \ldots, 1, \sigma)$, $\alpha^{(n+1)}(1, \ldots, 1, \sigma+1)$ as base. Let $F_{\sigma,m}^+$ denote that closed half-space determined by $F_{\sigma,m}$ which does not contain $(0, 0, 0)$. Then it is enough to show that $F_{\sigma,m}^+$ does not meet S_{n+2}, n sufficiently large; since Π_1^n meets S_{n+2} in a subset of $F_{\sigma,m}^+$.

Let (x_0, y_0, z_{n+2}) be the point lying in $z = z_{n+2}$ and on the line through α' and $(\alpha^{(n+1)}(1, \ldots, 1, \sigma) + \alpha^{(n+1)}(1, \ldots, 1, \sigma+1))/2$. Then $F_{\sigma,n}^+$ does not meet S_{n+2} if

$$(x_0^2 + y_0^2)^{\frac{1}{2}} > (1 - z_{n+2}^2)^{\frac{1}{2}}. \tag{1}$$

Now

$$(\alpha^{(n+1)}(1, \ldots, 1, \sigma) + \alpha^{(n+1)}(1, \ldots, 1, \sigma+1))/2$$

$$= \left((1 - z_{n+1}^2)^{\frac{1}{2}} \cos\frac{3\sigma\pi}{(n+1)!} \cos\frac{3\pi}{2(n+1)!}, (1 - z_{n+1}^2)^{\frac{1}{2}} \sin\frac{3\sigma\pi}{(n+1)!} \cos\frac{3\pi}{2(n+1)!}, z_{n+1}\right)$$

and

$$\alpha' = \left((1 - z_n^2)^{\frac{1}{2}} \cos\frac{3\sigma\pi}{(n+1)!}, (1 - z_n^2)^{\frac{1}{2}} \sin\frac{3\sigma\pi}{(n+1)!}, z_n\right).$$

Consequently

$$x_0 = \left[\left(\frac{z_n - z_{n+2}}{z_n - z_{n+1}}\right)(1 - z_{n+1}^2)^{\frac{1}{2}} \cos\frac{3\pi}{2(n+1)!} - \left(\frac{z_{n+2} - z_{n+1}}{z_{n+1} - z_n}\right)(1 - z_n^2)^{\frac{1}{2}}\right] \cos\frac{3\sigma\pi}{(n+1)!},$$

$$y_0 = \left[\left(\frac{z_n - z_{n+2}}{z_n - z_{n+1}}\right)(1 - z_{n+1}^2)^{\frac{1}{2}} \cos\frac{3\pi}{2(n+1)!} - \left(\frac{z_{n+2} - z_{n+1}}{z_{n+1} - z_n}\right)(1 - z_n^2)^{\frac{1}{2}}\right] \sin\frac{3\sigma\pi}{(n+1)!}.$$

For (1) to be satisfied we must have

$$(z_n - z_{n+2})(1 - z_{n+1}^2)^{\frac{1}{2}} \cos\frac{3\pi}{2(n+1)!} > (z_n - z_{n+1})(1 - z_{n+2}^2)^{\frac{1}{2}} + (z_{n+1} - z_{n+2})(1 - z_n^2)^{\frac{1}{2}}. \tag{2}$$

For n large,

$$z_n \sim \beta \frac{3\pi}{2(n!)} \quad \text{where} \quad \beta = \left(3 \sin^2 \frac{\pi}{16}\right)^{-\frac{1}{2}}$$

So, using the approximation

$$(1 - z_n^2)^{\frac{1}{2}} \sim 1 - \tfrac{1}{2} z_n^2, \quad n \text{ large,}$$

we establish (2) and hence (1) for $n \geq m_0$ by seeing that the first-order terms cancel but that the dominating second-order term is $-z_{n+1} z_n^2/2$ occurring in the right-hand side of (2).

So, if $K^* = K_{m_0}$, the 2-faces of K^*, other than F_1, F_2, F_3, F_4, are of the form $K^* \cap \Pi_i^n$, $i = 1, 2, 3, 4, 5$, $n \geqslant m_0$. In particular, for any $\epsilon > 0$, there are only finitely many edges of K which meet the half space $z \geqslant \epsilon$.

Consider $K^* \cap \{z = 0\}$. This is a convex 2-dimensional set whose boundary consists of the line segments

$$\Gamma_1 = [(0,0,0), (1,0,0)], \quad \Gamma_2 = [(0,0,0, (0,1,0)];$$

the points of E_∞ and line segments joining points of D. Let

$$F = bd[K^* \cap \{z = 0\}] - [\Gamma_1 \cup \Gamma_2] \quad \text{and} \quad [\mathbf{x}, y(\mathbf{x})]$$

be a line segment in F where $\mathbf{x}$ and $y(\mathbf{x})$ belong to D. Then

$$\mathbf{x} = (\cos\theta, \sin\theta, 0), \quad y(\mathbf{x}) = (\cos\phi, \sin\phi, 0)$$

where, for suitable choice of $k, u_4, \ldots, u_{k-1}$, $0 \leqslant u_j \leqslant j-2$, $j = 4, \ldots, k-1$,

$$\theta = \sum_{r=4}^{k-1} \frac{3\pi u_r}{r!} + (k-1)\frac{3\pi}{k!},$$

$$\phi = \sum_{r=4}^{k-1} \frac{3\pi u_r}{r!} + \frac{3\pi}{(k-1)!}$$

i.e. $\mathbf{x}, y(\mathbf{x})$ are the base vertices of the triangle $T^{(k)}(u_1, \ldots, u_{k-1}, k)$. The line segment $[\mathbf{x}, y(\mathbf{x})]$ is a 1-face of K^* and it is the limit of 1-faces defined by the points

$$\alpha^{(\lambda)}(u_1, \ldots, u_k, 1), \quad \alpha^{(\lambda)}(u_1, \ldots, u_k, \lambda)$$

as $\lambda \to \infty$.

We define the plane $\Pi_k(\mathbf{x})$ which passes through $[\mathbf{x}, y(\mathbf{x})]$ and which contains a translate of the z axis. Let $\Pi_k^+(\mathbf{x})$, $\Pi_k^-(\mathbf{x})$ denote the closed half spaces determined by $\Pi_k(\mathbf{x})$, $(0,0,0) \in \Pi_k^+(\mathbf{x})$. Then the extreme points of K^* lying in $\Pi_k^-(x)$ are of the form $\alpha^{(\lambda)}(u_4, \ldots, u_{k-1}, k, 1)$, $\alpha^{(\lambda)}(u_4, \ldots, u_{k-1}, k, \lambda)$ for λ sufficiently large. Also the diameter of $K^* \cap \Pi_k^-(\mathbf{x})$ tends to zero as $k \to \infty$.

Let

$$K = \bigcap_{\mathbf{x} \in D} (K^* \cap \Pi_k^+(\mathbf{x})).$$

Then

(i) for any $\epsilon > 0$, only finitely many edges of K meet the closed half space $z \geqslant \epsilon$.

(ii) An edge $[\mathbf{x}, y(\mathbf{x})]$ of K, as defined above, is contained in the 2- face $K \cap \Pi_k(\mathbf{x})$.

We show that K satisfies the conditions of Theorem 1.

Let L be a 2-plane meeting K. If L fails to meet F then L is a polygon by (i). If L meets F but L does not meet $C \cup D$ then $L \cap K$ is a polygon by (i) and (ii). So it is only if L meets $C \cup D$ that $L \cap K$ is (perhaps) a non-polygon. As $H^1(C \cup D) = 0$, this happens in a set of measure zero. Hence Theorem 1 is proved.

3. The following result is due to P. Mattila (5).

LEMMA. *Let A be a subset of E^n which is measurable with respect to the s-dimensional Hausdorff measure H^s with $0 < H^s(A) < \infty$. Then, for k a positive integer, $k < n$,*

(i) If $n-k < s$ the Hausdorff measure $H^{n-k}(A(L))$ is positive for almost all orthogonal projections $A(L)$ of A into an $(n-k)$-subspace L of L_{n-k}.

(ii) *If $n-k < s$ then at H^s almost all points x of A the following is true: for almost all k-flats L through x, $H^{s-(n-k)}(A \cap L) < \infty$ and the Hausdorff dimension of $A \cap L$ is equal to $s-(n-k)$.*

Proof of Theorem 2. We suppose that the $(n-k)$-skeleton of K has dimension greater than t, where $t > n-k+s$. Then there is an r, $0 \leqslant r \leqslant n-k$, such that the union of the r-faces of K has dimension greater than t.

Let L_{n-k} denote the Grassmanian of $(n-k)$-dimensional subspaces of E^n and μ_{n-k} denote the usual Haar measure on L_{n-k}. Since $r \leqslant n-k$, the orthogonal projection of an r-face of K into a $(n-k)$-subspace L is almost always, with respect to μ_{n-k}, an r-dimensional compact convex set. So we may pick a compact subfamily $\mathscr{K}_r$ of the r-dimensional faces of K and a compact subfamily $\mathscr{F}$ of L_{n-k} such that

$$K_r^* = \bigcup_{F \in \mathscr{K}_r} F$$

has dimension greater than t; the orthogonal projection of any member of $\mathscr{K}_r$ into any member of $\mathscr{F}$ is an r-dimensional compact convex set and $\mu_{n-k}(\mathscr{F}) > 0$. So, if M is a k-flat meeting K_r^* and M is perpendicular to some member of $\mathscr{F}$ then $M \cap K_r^*$ is contained within the extreme points of $K \cap M$.

By using the results of A. S. Besicovitch (1) we may select a compact subset K_r of K_r^* with

$$0 < H^t(K_r) < \infty.$$

By the lemma:

(i) $H^{n-k}(K_r(L))$ is positive for almost all orthogonal projections $K_r(L)$ of K_r into an $(n-k)$-subspace L of L_{n-k}.

(ii) For H^t almost all points x of K_r the following is true: for almost all k-flats L through x, $H^{t-(n-k)}(K_r \cap L) < \infty$ and the Hausdorff dimension of $K_r \cap L$ is equal to $t-(n-k)$. So, in particular, the Hausdorff dimension of $K_r \cap L$ is greater than s.

Now let G denote those k-flats arising in (ii) above and which are orthogonal to an $(n-k)$-subspace in $\mathscr{F}$. Then

$$\nu_k(G) = \int_{\mathscr{F}} H^{n-k}(K_r(L, G)\, d\mu_{n-k}(L).$$

As $\mu_{n-k}(\mathscr{F}) > 0$ and $H^{n-k}(K_r(L, G)) > 0$ for almost all $L \in L_{n-k}$ we conclude that $\nu_k(G) > 0$.

However, if $E \in G$, $E \cap K_r$ has dimension greater than s and is a subset of the extreme points of $E \cap K$. So this contradicts the hypothesis that the extreme points of almost all k-sections of K have dimension of at most s.

So the $(n-k)$-skeleton of K has dimension of at most $n-k+s$.

Now suppose that the dimension of the $(n-k)$-skeleton of K is at most $n-k+s$, but that the dimension of ext $(K \cap L)$ has dimension greater than s for a set of k-flats G of positive ν_k measure. So we may pick $L \in L_{n-k}$, say $L = E^{n-k}$ such that, if K^* is the $(n-k)$-skeleton of K,

$$H^{n-k}(K^*(E^{n-k}, G)) > 0.$$

As each $E \in G$, E perpendicular to E^{n-k}, has $\text{ext}(E \cap K)$ of dimension greater than s, it follows, using results of Marstrand (4),

$$\dim(K^*) > n-k+s;$$

contradiction.

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