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A note on the Fermat-Torricelli point of a d-simplex

Dedicated to Professor Theophilos Cacoullos

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Abstract. The isogonal property of the Fermat-Torricelli point for the vertex set of a d-simplex is examined. For $d = 2$ and 3, it is known that the property holds true if the point is in the interior of the simplex, but for $d \geq 4$ an example is given, proving that the Fermat-Torricelli point is not necessarily isogonal.

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1. Introduction

Motivated by a letter of R. Descartes, Pierre de Fermat, in 1643 raised the following question: *"Given three points in the plane, find the point having the minimal sum of distances to these three points"*. The initial solution to the problem was given by E. Torricelli. Several other solutions were also given and many contributions to the problem and its generalizations followed since then. This problem is usually named as "Fermat problem" or "Fermat-Torricelli problem" and the corresponding point, let it be x_o, as "Fermat point" or "Torricelli point". Some authors, like R. Courant, named it "Steiner point", but look [3] for comments about the name of this point. Following [2] we call x_o the Fermat-Torricelli point.

To find this point, we consider the following cases.

1. All the angles of the triangle formed by the given three points are less then 120 degrees.
2. One of the two angles is greater or equal to 120 degrees.

In Case 1, the required point is characterized by the property that its angle to every side of the triangle is equal to 120 degrees and it is in the interior of the triangle.

In Case 2, it turns out to be the vertex of the obtuse angle. This was first explicitly stated by B. Cavallieri.

Due to the large number of authors and its applications, the relevant literature is voluminous. The interested reader may consult the recent article [3] where many other aspects of the problem are discussed. It also contains historical notes and a very large number of references.

The Fermat-Torricelli problem in a more general setting is the following: Let α_1, $\alpha_2, \ldots, \alpha_n$ be n points in $\mathbb{R}^d$ and

$$\varphi(x) = \sum_{i=1}^{n} \|x - \alpha_i\|,$$

where $\|\cdot\|$ is the usual Euclidean norm. Then we look for $x_o \in \mathbb{R}^d$ which minimizes $\varphi(x)$.

In the case $d = 1$, the points $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers. Let

$$\alpha_{(1)} \leq \alpha_{(2)} \leq \cdots \leq \alpha_{(n)}$$

be the previous numbers in increasing order. Then if $n = 2k + 1$, the Fermat-Torricelli point is $x_o = \alpha_{(k)}$ and if $n = 2k$ it is any point in the closed interval $[\alpha_{(k)}, \alpha_{(k+1)}]$. In statistical terms, x_o is the median of the sample $\alpha_1, \alpha_2, \ldots, \alpha_n$.

When $d \geq 2$ we can not arrange the sample in increasing order so the above definition of the median is not valid any more. In this case one may define as the median of the sample $\alpha_1, \alpha_2, \ldots, \alpha_n$ the point x_o which minimizes $\varphi(x)$. For $d = 2, n = 3$ and in the Case 1, where x_o is in the interior of the triangle, x_o is the isogonal point of the triangle, that is $(x_o; \alpha_i, \alpha_j),\ i \neq j$ are all equal to 120 degrees. The same also holds true when $d = 3, n = 4$ for the solid angles $(x_o; \alpha_i, \alpha_j, \alpha_k),\ i \neq j \neq k$ and for the corresponding Case 1, see [2]. Prof. Th. Cacoullos, raised the question whether this isogonality property of the Fermat-Torricelli point (the median) holds for any d and $n = d + 1$.

In this note an analytic proof is given for the cases $d = 2, 3$ and a counterexample for $d \geq 4$ is given which proves that the isogonal property of the Fermat-Torricelli point does not hold for the simplex, of dimension higher than 3. (For $d = 2, 3$ see [3] for further discussion about the solution of the problem).

2. The results

Let $\alpha_1, \alpha_2, \ldots, \alpha_{d+1}$ points in $\mathbb{R}^d$, $d \geq 2$, affinely independent and let S be the d-simplex with vertices these points. We define

$$\varphi_S(x) = \sum_{i=1}^{d+1} \|x - \alpha_i\|, \quad x \in \mathbb{R}^d$$

with $\|\cdot\|$ the usual Euclidean norm. As $\varphi_S(\cdot)$ is strictly convex and $\lim_{\|x\| \to +\infty} \varphi_S(x) = \infty$ there exists a unique minimum point which we denote by x_o.

For x_o we have the following dichotomy: Either $x_o \in \{\alpha_1, \alpha_2, \ldots, \alpha_{d+1}\}$ or x_o belongs to the interior of the simplex. In this paper we study some aspects of the latter case. In particular we study the solid angles formed by x_o and the $(d-1)$-dimensional faces (facets) of the simplex S. We quote the following definition.

DEFINITION 2.1. Let K be a convex set of dimension $(d-1)$ and $\alpha \in \mathbb{R}^d, \alpha \notin \alpha ff\ K$. The $cone_\alpha K$, generated by K, with apex α is the set

$$\bigcup_{\beta \in K} \{\alpha + \lambda(\beta - \alpha), \quad \lambda \geq 0\}.$$

The solid angle $s(K, \alpha)$ of the $cone_\alpha K$ at α, is the ratio of the $(d-1)$-content of $S \cap cone_\alpha$ K, to the $(d-1)$-content of S, where S is the surface of a ball centered at α. Analytically,

$$s(K, \alpha) = c(d) \int_{\mathrm{cone}_\alpha \mathrm{K}} e^{-\|x\|^2} dv$$

where $c(d)$ is a constant, dependent only from d.

PROPOSITION 2.1. *Let* $S = conv\{\alpha_1, \ldots, \alpha_{d+1}\}$ *be a d-simplex in* $\mathbb{R}^d$ *and let* $x_o \in int S$ *be the minimum point of* $\varphi_S(\cdot)$. *If* $K_i = \mathrm{conv}\{\alpha_\mathrm{j},\ \mathrm{j} \neq \mathrm{i}\}, \mathrm{i} = 1, 2, \ldots, d+1$ *are the* $(d-1)$*-faces of S then:*

1. *For* $d = 2, 3$ *the angles* $s(K_i, x_o), i = 1, 2, \ldots, d+1$ *are all equal.*
2. *For* $d \geq 4$ *the angles* $s(K_i, x_o), i = 1, 2, \ldots, d+1$ *are not necessarily equal.*

To prove the above proposition we need the following lemmata.

LEMMA 2.1. *Let* $x_1, \ldots, x_{d+1} \in \mathbb{R}^d$, *with* $\|x_i\| = 1,\ i = 1, 2, \ldots, d+1$. *The following are equivalent:*

(i) $\sum_{i=1}^{d+1} x_i = 0$.

(ii) $\sum_{i=1}^{d+1} \sum_{j=1}^{d+1} \|x_i - x_j\|^2 = 2(d+1)^2$

(iii) $\sum_{\substack{i=1 \\ i\neq\nu}}^{d+1} \sum_{\substack{j=1 \\ j\neq\nu}}^{d+1} \|x_i - x_j\|^2 = 2(d^2-1) \quad$ *for* $\quad \nu = 1, 2, \ldots, d+1$.

Proof. If $c_1, c_2, \ldots, c_n \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$ then the following identity holds:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \|c_i - c_j\|^2 = 2n \cdot \sum_{i=1}^{n} \|c_i - x\|^2 - 2\|(c_1 + \cdots + c_n) - nx\|^2.$$

Assume that (ii) holds. Then setting in the above identity $x = \frac{1}{d+1} \sum_{i=1}^{d+1} x_i$, we arrive at $\sum_{i=1}^{d+1} \|x_i - x\|^2 = d+1$. As $\|x_i\| = 1, i = 1, 2, \ldots, d+1$ expanding the left hand side using inner products, we have $x = 0$. Hence (i) follows.

Assume that (iii) holds true. Then summing the resulting equations, (ii) follows.

Suppose now that (i) is true. Without loss of generality, let $\nu = d + 1$. If y is the center of the circumscribed sphere of $\mathrm{conv}\{\mathrm{x_i}, \mathrm{i} = 1, 2, \ldots, d\}$, by Pythagorean theorem and the fact that

$$-\frac{x_{d+1}}{d} = \frac{x_1 + \cdots + x_d}{d} \in \mathrm{conv}\{\mathrm{x_i}, \mathrm{i} = 1, 2, \ldots, d\}$$

we have that

$$\left\| y + \frac{x_{d+1}}{d} \right\|^2 = \frac{1}{d^2} - \|y\|^2.$$

Setting inthe initial identity $x = y$ and $n = d$ we have

$$\begin{aligned}
\sum_{i=1}^{d} \sum_{j=1}^{d} \|x_i - x_j\|^2 &= 2d \sum_{i=1}^{d} \|x_i - y\|^2 - 2\| - x_{d+1} - dy\|^2 \\
&= 2d \sum_{i=1}^{d} (1 - \|y\|^2) - 2(1 - d^2\|y\|^2) \\
&= 2(d^2 - 1).
\end{aligned}$$

Hence (iii) follows. This completes the proof of the lemma. □

LEMMA 2.2. *Let K be a $(d-1)$-simplex lying on $H \cap B$ where B is the unit ball of $\mathbb{R}^d$ and H is a hyperplane not passing through the origin. Then the angle $s(K, 0)$ takes its maximum value if and only if K is the regular simplex inscribed in $H \cap B$.*

Proof. See Lemmata 4, 5, 6 in [1]. □

Proof of Proposition 2.1. We may assume without loss of generality that $x_o = 0$. As the minimum point $x_o = 0$ of $\varphi_S(\cdot)$ is in the interior of S, the gradient

$$\nabla\varphi_S(0) = \sum_{i=1}^{d+1} \frac{\alpha_i}{\|\alpha_i\|} = 0. \text{ (see [3])}$$

If S' is the simplex with vertices $\lambda_1\alpha_1, \lambda_2\alpha_2, \ldots, \lambda_{d+1}\alpha_{d+1}$ with $\lambda_i > 0, i = 1, 2, \ldots, d + 1$ we can easily prove $\nabla\varphi_{S'}(0) = 0$. As $\varphi_{S'}(\cdot)$ is strictly convex then 0 is the minimum point of $\varphi_{S'}(\cdot)$. As the solid angles of interest are the same for S, S' we may assume that $\|\alpha_i\| = 1, i = 1, 2, \ldots, d + 1$ which entails $\sum_{i=1}^{d+1} \alpha_i = 0$.

1. For $d = 2$, from Lemma 2.1 (iii) we have $\|\alpha_i - \alpha_j\|^2 = 3$, $i \neq j$. Hence the triangle formed by α_1, α_2 and α_3 is equilateral and the angles $s(K_i, 0)$, $i = 1, 2, 3$ are all equal.

   For $d = 3$ from Lemma 2.1 (iii), we have

$$\begin{aligned} \|\alpha_1 - \alpha_2\| &= \|\alpha_3 - \alpha_4\|, \\ \|\alpha_1 - \alpha_3\| &= \|\alpha_2 - \alpha_4\| \text{ and} \\ \|\alpha_1 - \alpha_4\| &= \|\alpha_2 - \alpha_3\|. \end{aligned}$$

   Therefore the 2-faces of S are congruent and so are the corresponding solid angles $s(K_i, 0)$, $i = 1, 2, 3, 4$. Such a 3-simplex, with all the 2-faces congruent is called isosceles (see [2]).

2. For $d = 4$ we shall construct a d-simplex such that $s(K_i, x_o)$ are not equal.

   Consider the unit ball B of $\mathbb{R}^d$ and a $(d-1)$-simplex S_{d-1} in $B \cap \{x_d = 0\}$ which is not regular, with vertices α_i, such that $\|\alpha_i\| = 1$, $i = 1, 2, \ldots, d$ and $\sum_{i=1}^{d} \alpha_i = 0$. (Relating to 1, there always exists such a simplex being non-regular but isosceles). Set now

$$\begin{aligned} \beta_i &= (1 - d^{-2})^{\frac{1}{2}} \alpha_i + d^{-1} e_d, i = 1, 2, \ldots, d, \quad \text{and} \\ \beta_{d+1} &= -e_d = -(0, \ldots, 0, 1). \end{aligned}$$

   Then $\|\beta_i\| = 1$, $i = 1, 2, \ldots, d+1$ and $\sum_{i=1}^{d+1} \beta_i = 0$.

   Hence 0 is the minimum point of $\varphi_S(\cdot)$ with $S = \mathrm{conv}\{\beta_\mathrm{i}, \mathrm{i} = 1, 2, \ldots, \mathrm{d}+1\}$. Let $H = \alpha ff\{\beta_i, i = 1, 2, \ldots, d\}$ and $K_{d+1} = \mathrm{conv}\{\beta_\mathrm{i}, \mathrm{i} = 1, 2, \ldots, \mathrm{d}\}$. Then $dist(H, 0) = d^{-1}$ and for the regular $(d-1)$-simplex K inscribed in $H \cap B$ we have $s(K, 0) = \frac{1}{d+1}$ and since K_{d+1} is not regular we see that

$$s(K_{d+1}, 0) < \frac{1}{d+1} \qquad \text{(due to Lemma 2.2).}$$

   Hence the angles $s(K_i, 0)$, $i = 1, 2, \ldots, d+1$ cannot be equal. For $d \geq 5$ we repeat the above process with the simplex S_{d-1} already constructed in $\mathbb{R}^{d-1}$.

COROLLARY 2.1. *Let S be a d-simplex, $S = \mathrm{conv}\{\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{d}+1}\}$ with $B(x_o, r)$ its circumsphere and x_o lying in the interior of S. Then the following statements are equivalent:*

(i) *x_o is the Fermat-Torricelli point of S.*

(ii) $\sum_{i=1}^{d+1} \sum_{j=1}^{d+1} \|x_i - x_j\|^2 = 2(d+1)^2 r^2$.

(iii) $\sum_{\substack{i=1 \\ i \neq \nu}}^{d+1} \sum_{\substack{j=1 \\ j \neq \nu}}^{d+1} \|x_i - x_j\|^2 = 2(d^2 - 1) r^2$, *for* $\nu = 1, 2, \ldots, d+1$.

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