AN ISOPERIMETRIC INEQUALITY IN THE CLASS OF SIMPLICIAL POLYTOPES

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ABSTRACT. The classical isoperimetric inequality relates the $(d-1)$-th root of the area of the surface of a convex body K in R^d with the d-th root of its volume. A similar problem has been considered in the case of polyhedra for the quantities $\zeta_s^{1/s}$, $\zeta_r^{1/r}$, $1 \le r \le s \le d$ where ζ_s is the s-dimensional Hausdorff measure of the s-skeleton of K. Several inequalities are proved for specific values of r, s, d. In the case where the faces of the polyedra are simplices, an optimal inequality is proved for arbitrary values of r, s, d $(1 \le r \le s \le d)$.

1. Introduction. Let P be a convex polytope in the d-dimensional Euclidean space E^d. The k-skeleton of P, with $1 \le k \le d$, is defined to be the union of all its k-dimensional faces. The k-dimensional Hausdorff measure of the k-skeleton of the polytope P will be denoted by $\zeta_k(P)$.

The classical isoperimetric problem for convex polytopes is the following:

For any integers r, s, with $1 \le r \le s \le d$, determine the least number $\gamma(d,s,r)$ such that the inequality

$$\frac{\zeta_s^{1/s}(P)}{\zeta_r^{1/r}(P)} \le \gamma(d,s,r)$$

holds for all convex polytopes P in E^d.

Eggleston, Grunbaum and Klee have proved in [4] that $\gamma(d,d,r) < \infty$, $\gamma(d,d,-1,r) < \infty$ and if r is a divisor of s then $\gamma(d,s,r) \le 1$. Aberth has given the following bounds in [1] and [2],

$$\gamma(3,2,1) \le (6\pi)^{-1/2}, \quad \gamma(3,3,1) \le (432\pi)^{-1/3}.$$

Also Schneider proved in [6] a result which implies that

$$\gamma(d,d,-1,r) \le (d\alpha(d))^{1/r(d-1)} \cdot \left[\frac{\alpha(d-r)}{(d-r+1)\alpha(d)\binom{d}{r}}\right]^{1/r},$$

where $\alpha(k)$ denotes the volume of the k-dimensional unit ball. Finally from a paper of Burton and Larman [3] we have that

$$\gamma(4,2,1) \le \frac{1}{2}$$

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and for $d \geq 5$

$$\begin{aligned}&\gamma(d, d-2, d-3)\\ &\leq \left[\frac{5(d-3)\sqrt{d-4}((d-3)!/4)^{1/(d-3)}}{\alpha(d-2)}\left(\frac{2\alpha(d-3)}{d-2}+\frac{9(d-1)(d-4)2^d\pi}{64}\right)\right]^{1/(d-2)}\\ &\approx \left[\frac{45\pi^{5/2}d^4}{32e^2}\left(\frac{2d}{\pi e}\right)^{d/2}\right]^{1/(d-3)}\end{aligned}$$

In the general case it is not known whether $\gamma(d, s, r)$ exists.

Here, we restrict the problem in the class of the simplicial polytopes (see for example [5]), i.e. convex polytopes the facets of which are $(d-1)$-simplices, and we prove that if r and s are integers, such that $1 \leq r \leq s \leq d$, the inequality

$$\frac{\zeta_s^{1/s}(P)}{\zeta_r^{1/r}(P)} \leq \binom{d-r}{d-s}^{1/r} \cdot \Theta,$$

holds for any simplicial polytope P in E^d, with equality if, and only if, P is a regular s-simplex, where $\Theta = \dfrac{\left(\frac{1}{s!}\sqrt{\frac{s+1}{2^s}}\right)^{1/s}}{\left[\binom{s+1}{r+1}\frac{1}{r!}\sqrt{\frac{r+1}{2^r}}\right]^{1/r}}$. This notation is used in the rest of this paper.

2. An inequality for the r-skeleton of an s-simplex.

In this section we prove that if F is an s-simplex of a given s-dimensional volume, then for any integer r, with $1 \leq r < s$, $\zeta_r(F)$ gets its minimum value exactly when F is regular. So conclude that

$$\frac{\zeta_s^{1/s}(F)}{\zeta_r^{1/r}(F)} \leq \Theta$$

with equality if, and only if, the simplex F is regular.

Theorem 2.1. *Let F be an s-simplex with given s-dimensional volume, and let r be an integer such that $1 \leq r < s$. Then, $\zeta_r(F)$ gets its minimum value exactly when the s-simplex F is regular.*

Proof. Let $x_0, x_1, \ldots, x_s$ be the vertices of F and R be the $(s-1)$-simplex formed by the vertices $x_1, x_2, \ldots, x_s$. We assume that R is fixed and since F is of a given s-dimensional volume the distance h of x_0 from the flat L formed by $x_1, x_2, \ldots, x_s$ is fixed. We consider the following two cases:

CASE I: $r = 1$

For $i = 1, 2, \ldots, s$, let α_i denote the length of the line segment ℓ_i with endpoints the vertices x_0, x_i and φ_i be the acute angle between ℓ_i and the perpendicular from x_0 to L. Then, since the function $(\cos x)^{-1}$ is strictly convex in $\left[0, \frac{\pi}{2}\right)$, we have

$$\sum_{i=1}^{s} \alpha_i = h \sum_{i=1}^{s} (\cos\varphi_i)^{-1} \geq hs \left[\cos\left(\frac{1}{s}\sum_{i=1}^{s}\varphi_i\right)\right]^{-1}$$

with equality only if $\varphi_1 = \varphi_2 = \ldots \varphi_s$ namely $\alpha_1 = \alpha_2 = \ldots = \alpha_s$.

Now, if the simplex F is not regular, there exists a vertex x_0 say which is not equidistant from the other vertices of F. Considering a point x_0' in E^s which is equidistant from x_i, $i = 1, 2, \ldots, s$ are at the same distance h from the flat L as above, we form the s-simplex F' with vertices $x_0', x_1, \ldots, x_s$. Then, F and F' are of the same s-dimensional volume, but $\zeta_1(F) > \zeta_1(F')$.

Thus among the s-simplices of given s-dimensional volume only the regular s-simplex realises the minimum 1-dimensional Hausdorff measure of the 1-skeleton.

CASE II: $r > 1$

Let $A_1, A_2, \ldots, A_k$ with $k = \binom{s}{r-1}$, be the $(r-1)$-faces of F which do not contain the vertex x_0 and $E_1, E_2, \ldots, E_k$ be the r-faces of F containing the vertex x_0. Finally, let φ_i be the acute angle between the perpendicular lines form x_0 to L and L_i where L_i is the flat containing the face A_i, for $i = 1, 2, \ldots, s$. Then, if V_t denotes the t-dimensional volume function, since the function $(\cos x)^{-1}$ is strictly convex in $\left[0, \frac{\pi}{2}\right)$, we have

$$\sum_{i=1}^{k} V_r(E_i) = \frac{h}{r}\sum_{i=1}^{k} V_{r-1}(A_i)(\cos\varphi_i)^{-1} \geq \frac{h}{r}\zeta_{r-1}(R)\left[\cos\left(\sum_{i=1}^{k}\frac{V_{r-1}(A_i)}{\zeta_{r-1}(R)}\varphi_i\right)\right]^{-1}$$

with equality only if $\varphi_1 = \varphi_2 = \ldots \varphi_k$, that is only when the vertex x_0 is equidistant from the faces A_i, $i = 1, 2, \ldots, k$. So, if F is regular $\zeta_r(F)$ gets its minimum value.

Conversely if an s-simplex F of given s-dimensional volume realises the minimum $\zeta_r(F)$, working as in the case I proceeding inductively we get that every vertex of F is equidistant from the 2-faces of F which do not contain x_0, and this implies that F is regular.

Corollary 2.2. *If F is an s-simplex and r is an integer such that $1 \leq r < s$, then*

$$\frac{\zeta_s^{1/s}(F)}{\zeta_r^{1/r}(F)} \leq \Theta$$

Proof. If R is a regular k-simplex of edge 1, then

$$V_k(R) = \frac{1}{k!}\sqrt{\frac{k+1}{2^k}},$$

where V_k denotes the k-dimensional volume function. Hence, from the Theorem 2.1 we get the result.

3. The isoperimetric problem for simplicial polytopes.

A polytope P in the d-dimensional Euclidean space E^d, is said to be a simplicial polytope, if every $(d-1)$-face of P is a $(d-1)$-simplex. Here we prove the following:

Theorem 3.1. *Let P be a simplicial polytope in E^d and r, s be integers such that $1 \leq r < s \leq d$. Then,*

$$\frac{\zeta_s^{1/s}(P)}{\zeta_r^{1/r}(P)} \leq \Theta\binom{d-r}{d-s}^{1/r}$$

with equality if, and only if, P is a regular s-simplex.

Proof. Let $F_1, F_2, \ldots, F_k$ be the s-faces of P. Since P is a simplicial polytope, the s-faces F_i, for $i = 1, 2, \ldots, k$, are s-simplices. Therefore, by the Corollary 2.2, for $i = 1, 2, \ldots, k$, we have relations

$$\zeta_r(F_i) \geq \Theta^{-r} \cdot \zeta_s^{r/s}(F_i), \tag{1}$$

with equality if, and only if, the s-simplex F_i is regular.

Also we have

$$\zeta_r(P) = \binom{d-r}{d-s}^{-1} \cdot \sum_{i=1}^{k} \zeta_r(F_i)$$

and from (1) we get

$$\zeta_r(P) \geq \binom{d-r}{d-s}^{-1} \cdot \Theta^{-r} \sum_{i=1}^{k} \zeta_s^{r/s}(F_i). \tag{2}$$

Now, since $1 \leq r < s$, it holds

$$\sum_{i=1}^{k} \zeta_s^{r/s}(F_i) \geq \left[\sum_{i=1}^{k} \zeta_s(F_i)\right]^{r/s}$$

with equality if, and only if, $k = 1$. Hence (2) becomes

$$\zeta_r(P) \geq \binom{d-r}{d-s}^{-1} \cdot \Theta^{-r} \left[\sum_{i=1}^{k} \zeta_s(F_1)\right]^{r/s} \tag{3}$$

with equality if, and only if P is a regular s-simplex.

Finally, since

$$\zeta_r(P) = \sum_{i=1}^{k} \zeta_r(F_i),$$

we have from (3)

$$\zeta_r^{1/r}(P) \geq \binom{d-r}{d-s}^{-1/r} \cdot \Theta^{-1} \zeta_s^{1/s}(P),$$

with equality exactly when P is a regular s-simplex, which proves the Theorem.

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