Group Theory and Representation Theory

(Lecture Notes, Perpetually in progress)

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Overview

This course introduces the theory of group representations as the systematic way of classifying objects on which a group can act. Furthermore, it reveals how this leads to a deeper understanding of symmetry aspects of physical systems and how one can use it to simplify mathematical computations.

Contents

1 Motivation

The common theme of many branches of Mathematical and Physical sciences is symmetry. In this module we will learn the principles and mathematical techniques to exploit symmetry properties of a system. There are two major reasons for doing so: (i) from a practical view point, reduce significantly or organize better the computational tasks involved and (ii) at a more conceptual level, provide a fundamental starting point or classification scheme for a deeper understanding of the system that may lead to further generalizations and advancements.

What we need is a formalism which will tell us how to extract the maximum information from symmetries systematically and avoiding as much as possible the complexities of the problem at hand. Such a formalism has been developed and referred to as *Group Theory*. This module introduces the theory of *Group Representations* as the systematic way of classifying objects on which a group can act. Furthermore, it reveals how this leads to a deeper understanding of symmetry aspects of physical systems that how one can use it to simplify mathematical computations.

At the end of the module a student should:

• Have a firm understanding of the concepts, theorems and techniques of group representation theory.

• Have a clear understanding how to construct irreducible representations of groups.

• Be able to explicitly apply the above to small order groups.

• Have a true appreciation of the importance of the knowledge that they have acquired and its wide range of applications in mathematical and physical sciences.

Abbreviations

- l.h.s. (r.h.s.) : left (right) hand side
- w.r.t. : with respect to
- i.e. : "in a sense" or "in other words".
- n-dim : n-dimensional.
- rep(s) : representation(s)
- irrep(s) : irreducible representation(s)

2 Extended review of Group Theory

To make these notes self-contained we will provide in this section a rather detailed and extended review of group theory with emphasis on discrete groups.

2.1 Basic definitions

By a group G we mean a set of objects or operators (called the *Group Elements*) $(a, b, \dots)$ which possess the following properties:

1. A composition law or "multiplication"

$$A \cdot B = AB \text{ (for simplicity)} , \qquad A, B \in G , \tag{2.1}$$

which is associative

$$(AB)C = A(BC) , \qquad \forall\ A, B, C \in G \tag{2.2}$$

and closed

$$AB = C , \qquad C \in G , \quad \forall\ A, B \in G . \tag{2.3}$$

2. There is an element e called the *identity* or *unit element*, such that

$$EA = AE = A , \qquad \forall A \in G . \tag{2.4}$$

3. Every element $A \in G$ possesses an *inverse* A^{-1} such that

$$A^{-1}A = AA^{-1} = E , \qquad \forall A \in G . \tag{2.5}$$

Remarks:

• This compositions law need not be *commutative*, that is, generically, $AB \neq BA$. A group with a commutative composition law is called *Abelian*. In all other cases is called *non-Abelian*.

• A group contains only one unit element (**Exercise**).

• The inverse of the inverse of $A \in G$ is A itself, that is $(A^{-1})^{-1} = A$ (**Exercise**).

• For each $A \in G$ there is a unique inverse (**Exercise**).

• A group is called *finite* (*infinite*) if the number of its elements is finite (infinite). For finite groups the number of elements is called *order* of the group.

2.2 Some examples of groups

2.2.1 Abelian groups

1. The set of all integers constitutes an Abelian infinite group, if the composition law is the usual addition. The unit element is zero and the inverse of an integer A is $-A$.

2. Similar to the above example, the set of vectors of a linear n-dim space form an infinite Abelian group if the composition law is the usual addition of vectors.

3. The set of all rational numbers of the form $\frac{p}{q}$ excluding zero constitutes an Abelian infinite group, if the composition law is the usual multiplication. The unit element is one and the inverse of $\frac{p}{q}$ is just $\frac{q}{p}$. Unlike, the negative rational numbers, the positive ones form a group by themselves.

2.2.2 Non-Abelian groups

1. The set of all non-singular $n \times n$ matrices over $\mathbb{C}$ comprises the General Linear Group $GL(n)$ under the usual multiplication of matrices, with the unit element the identity matrix. The group is obviously non-Abelian and infinite dimensional. The elements of $GL(n)$ depend on $2n^2$ continuous real parameters and therefore the group is called *continuous*.

2. The covering operations of a symmetric object form a group. A covering operation is a rotation, a reflection or an inversion that sends the object into itself, that is to a form that is indistinguishable optically from the original one. An important example of a finite covering group is the covering group of the equilateral triangle (D_3), which will be used extensively in these notes to illustrate various points.

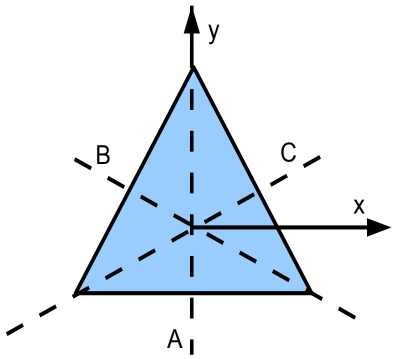

Figure 1: Equilateral triangle and its symmetry axes.

$E:$ Identity operation

$A, B, C:$ Rotations by π about the axes shown

$D:$ Rotation by $-\dfrac{2\pi}{3}$ in z axis(clockwise)

$F:$ Rotation by $+\dfrac{2\pi}{3}$ in z axis(anti − clockwise)

(z points outwards)

By inspection one constructs the following *Group Multiplication Table* (column operation performed first)

	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D .

(2.6)

For this group we will use the notation D_3.

Remarks:

• The group is non-Abelian, since, for instance, $AB = D \neq BA = F$. Its order $h = 6$.

• The covering operations may be represented as 2-dim matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$
$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad B = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} , \qquad C = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} , \tag{2.7}$$
$$D = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} , \qquad F = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} .$$

These matrices act on the 2-dim vector $\begin{pmatrix} x \\ y \end{pmatrix}$. One can readily verify that indeed they reproduce the table (2.6) (**Exercise**).

• One may view this group on its own and simply say that it *isomorphic* to the covering group of the equilateral triangle.

2.3 More theorems and definitions

Rearrangement Theorem: In a group multiplication table each column or row contains each group element once and only once. This implies that if

$$G = \{X_1, X_2, \dots, X_h\}\ , \tag{2.8}$$

then the set

$$GX_k = \{X_1X_k, X_2X_k, \dots, X_hX_k\} = G\ , \qquad \forall\ X_k \in G\ . \tag{2.9}$$

2.3.1 Subgroups

Any subset of group elements which itself forms a group G with the same composition law is called a subgroup H of G. The remaining elements of G cannot form a group since they do not contain the identity.

Any group has at least two subgroups, the group itself and the group consisting only of the identity element. Such subgroups are called *improper subgroups* or *trivial subgroups*. All other subgroups are called *proper subgroups*.

The covering group of the equilateral triangle has four proper subgroups: $\{E, A\}$, $\{E, B\}$, $\{E, C\}$ and $\{E, D, F\}$. The first three are covering groups of three isosceles triangles.

<u>Exercise</u>: Compute the multiplication table of the covering group of a square (D_4) using the following table and notation:

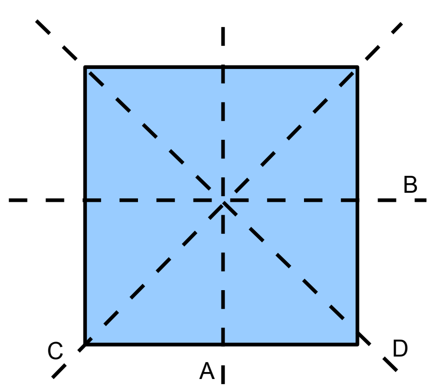

Figure 2: Square and its symmetry axes.

E : Identity operation

A, B, C, D : Rotations about axis A, B, C, D

F, G, H : Clockwise rotations by $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Find its proper subgroups and indicate the ones that correspond to covering groups of rectangles.

2.3.2 Cyclic groups

Consider a finite group G. For any $X \in G$ consider the sequence

$$E\ , \quad X^k\ , \qquad k = 1, 2, \dots\ . \tag{2.10}$$

By closure all element in this sequence belong in G and therefore there is a certain power, say X^n for which $X^n = X^m$, with $m < n$. Indeed, unless $X^n = E$, we have that the group element $X^{n-1} = X^{m-1}$ would have appeared earlier in the sequence, in contrast with our hypothesis. Hence, the elements that appear in the sequence are distinct up to order $n-1$. The integer n is called the order of the element X since it is the smallest integer for which $X^n = E$. The sequence

$$E\ , \quad X\ , \quad X^2\ , \quad \dots\ , \quad X^{n-1}\ . \tag{2.11}$$

is called the period of the element X and forms a group (note that $X^{-1} = X^{n-1}$).

This procedure might not exhaust all elements of the group G we started with.

Definition: A group formed entirely of powers of a single element is called *cyclic*.

Remarks:

• Since $X^n X^m = X^m X^n$, all cyclic groups are Abelian.

• The covering group of the equilateral triangle has three cyclic subgroups of order 2 and one of order 3 (**Exercise**).

2.3.3 Generators of a group

Definition: The elements X_i, $i = 1, 2, \dots, k$ of a group G are called *generators* or *generating elements* if every $X \in G$ can be expressed as a finite product of positive and negative powers of the X_i's. The set of relations

$$g_m(X_1, X_2, \dots, X_k) = E\ , \qquad m = 1, 2, \dots\ , \tag{2.12}$$

which are sufficient to determine the entire multiplication table of G are called *defining relations* or *generating relations* of G.

Remarks:

• The sets of generators and defining relations are by no means unique.

• There is always a minimum set of generators without which one cannot generate the group. It is often practical to use an overdetermined set instead of the minimal one.

Exercise: For the covering group of the equilateral triangle a set of generators is $\{A, D\}$ and the corresponding defining relations are

$$A^2 = E\ , \qquad D^3 = E\ , \qquad ADAD^{-2} = E\ . \tag{2.13}$$

2.4 Cosets

Let $H = \{S_1 = E, S_2, \dots, S_g\}$ be a subgroup of order g of a larger group $G = \{X_1 = E, X_2, \dots, X_h\}$ of order h.

Definition: We call the set of g elements

$$HX_k = \{S_1X_k, S_2X_k, \dots, S_gX_k\}\ , \qquad X_k \notin H\ , \tag{2.14}$$

the *right coset* of H with respect to X_k. There is similar definition for the *left coset*.

Remarks:

• The coset is not a group itself since it does not contain the identity (**Exercise**).

• The coset HX_k has no common elements with H (**Exercise**).

• The element X_k is called a *coset representative* and is by no means special. Any element of the coset may serve as such a representative.

Proof: If we choose an element $A \in HX_k$, then this is of the form $A = S_mX_k$. Then the coset $HA = HS_mX_k = HX_k$ is the same as the original one by the rearrangement theorem.

• The right (or the left) cosets of the subgroup H are either identical or have no elements in common.

Proof: Consider the two right cosets HX_m and HX_n. Assuming that they have one element in common we have a relation of the form $S_rX_m = S_sX_n$, which implies that

$X_m X_n^{-1} \in H$. Then by the rearrangement theorem $HX_m X_n^{-1} = H$ or $HX_m = HX_n$ that is the two cosets are completely identical. A similar proof holds for left cosets as well.

• The order g of the subgroup H is an integral divisor of the order h of the group G. The integer

$$\ell = \frac{h}{g} \,, \tag{2.15}$$

is called the *index* of the subgroup H.

Proof: There are $h - g$ group elements that do not belong to H which we denote by $X_{g+1}, X_{g+2}, \ldots, X_h$. Hence we may construct the following sets of g elements each

$$H\,, \quad HX_{g+1}\,, \quad HX_{g+2}\,, \quad \ldots, \quad HX_h\,. \tag{2.16}$$

Due to the above, no coset has any element in common with H and also the cosets themselves are either identical or have no common elements as well. Hence, after keeping only the distinct cosets we have the sets

$$HX_1\,, \quad HX_2\,, \quad \ldots, \quad HX_\ell\,, \qquad X_1 \equiv E\,, \tag{2.17}$$

where ℓ is a positive integer. Each of these sets has g elements none of them in common. Hence an element of G should appear in the above sequence only once. Therefore $h = \ell g$.

• In this way we see that we can expand the group G in terms of the right cosets of any subgroup H as

$$G = \bigcup_{k=1}^{\ell} HX_k\,, \quad X_1 \equiv E\,, \tag{2.18}$$

with a similar expression for the expansion in terms of the left coset.

• Finite groups whose order h is a prime integer are cyclic Abelian groups.

Proof: If this was not the case the period of some element would have been a subgroup whose order is a divisor of a prime number. Therefore for each prime there is only one distinct group, i.e. only one distinct multiplication table.

Exercise: For the covering group of the equilateral triangle choose the subgroup $H = \{E, A\}$, construct all the right and left cosets and verify the above statements. Repeat the exercise for the square and choose the subgroup H appropriately.

2.5 Conjugate elements and class structure

Definition: Let $A, B \in G$. They are said to be *conjugate elements* if there is $X \in G$ such that $B = XAX^{-1}$.

Remarks:

• One easily shows that if two elements are conjugate to a third element, then they are conjugate to each other as well (**Exercise**).

• All mutually conjugate elements can be collected into a *class*. The class that includes an element A_i is formed by considering the sequence $X_k A_i X_k^{-1}$, $k = 1, 2, \ldots, h$. Of course some elements may occur more than once in this sequence.

• The unit element forms a class on its own. This class is the only one that is also a subgroup of G (**Exercise**).

• All elements of a given class have the same period. Indeed, if $A^n = E$ we have that $(XAX^{-1})^n = \cdots = XA^nX^{-1} = E$.

• In Abelian groups every element is a class by itself.

2.5.1 Class decomposition

The various distinct classes of conjugate elements are mutually exclusive.
Proof: Assuming two classes A and B have one element in common then for some $X, Y \in G$ we have that

$$XA_rX^{-1} = YB_sY^{-1} \quad \Longrightarrow \quad CA_rC^{-1} = ZB_sZ^{-1} , \quad Z = CX^{-1}Y , \quad \forall\, C \in G . \tag{2.19}$$

By ranging over all elements $C \in G$, the l.h.s. generates the entire class A. Similarly by the rearrangement theorem Z runs over all elements of G and generates the entire class B. Hence, the two classes must be identical.

Hence, we have the decomposition of G into classes as

$$G = \bigcup_{a=1}^{k} C_a . \tag{2.20}$$

Consequently, if h_a is the number of elements in the class C_a, we have that

$$h = \sum_{a=1}^{k} h_a \ . \tag{2.21}$$

Exercise: For the equilateral triangle show that the class decomposition leads to

$$D_3 = C_1 \cup C_2 \cup C_3 \ , \qquad C_1 = \{E\} \ , \quad C_2 = \{A, B, C\} \ , \quad C_3 = \{D, F\} \ . \tag{2.22}$$

Exercise: Repeat the previous exercise for D_4, the symmetry group of the square and show that $D_4 = \cup_{a=1}^{5} C_a$, where

$$C_1 = \{E\} \ , \quad C_2 = \{A, B\} \ , \quad C_3 = \{C, D\} \ , \quad C_4 = \{G\} \ , \quad C_5 = \{F, H\} \ . \tag{2.23}$$

2.5.2 Distinct elements in a class

How many distinct elements are there in the class generated by $A \in G$? To answer this question we define the *normalizer*:

Definition: The set of elements of G that commute with a fixed element $A \in G$ is called the *normalizer* of A.[1] **Remarks:**

• The elements of the normalizer form a subgroup N_A of G.

Proof: Let $N_i \in N_A$, $i = 1, 2$ arbitrary elements obeying $[A, N_i] = 0$.[2] We easily establish that

$$[A, N_1 N_2] = N_1 [A, N_2] + [A, N_1] N_2 = 0 \ , \qquad [A, N_i^{-1}] = 0 \ , \quad i = 1, 2 \ . \tag{2.28}$$

[1]More generally let $H \subseteq G$. Then the normalizer of H in G is

$$N_G(H) = \{g \in G \mid Hg = gH\} \ , \tag{2.24}$$

Also the centralizer is

$$C_G(H) = \{g \in G \mid hg = gh \ , \forall \, h \in H\} \ , \tag{2.25}$$

Obviously, $C_G(H) \subseteq N_G(H)$. In our case we use the term normalizer and centralizer indistinguishably since the group element A is fixed, i.e. the subgroup H consists of only one element, that is $H = \{A\}$.

[2]We will use the notation

$$[A, B] = AB - BA \ , \tag{2.26}$$

for any two operators A and B. This is commonly known as the commutator of A and B. Two basic properties are

$$[A, B] = -[B, A] \ , \qquad [A, BC] = B[A, C] + [A, B]C \ . \tag{2.27}$$

Since, also $[A, E] = 0$ and by definition associativity is obeyed, the assertion is proved.

• We may expand G into left cosets with respect to N_A as

$$G = \bigcup_{k=1}^{\ell_A} X_k N_A \ , \qquad X_1 \equiv E \ , \tag{2.29}$$

where $\ell_A = h/n_A$ is the index of N_A and n_A its order. A typical element of $X_k N_A$ is $X_k N$, where $N \in N_A$ and obeys $(X_k N)A(X_k N)^{-1} = X_k A X_k^{-1}$, independent of which element N is chosen. Hence, all elements belonging to the same coset transform A in the same manner. Moreover, if we pick up two elements of different cosets, i.e. $X_i \neq X_j$, we have that $(X_i N_A)A(X_i N_A)^{-1} \neq (X_j N_A)A(X_j N_A)^{-1}$.

Proof: Assume instead equality. Then easily $[A, X_i^{-1} X_j] = 0$, which means that $X_i^{-1} X_j \in N_A$. Then by the rearrangement theorem $X_i^{-1} X_j N_A = N_A$ or $X_i N_A = X_j N_A$, which contradicts our hypothesis.

Therefore elements of two different cosets of the normalizer N_A lead to different transformations. Hence, the number of distinct conjugates of A is the index $\ell_A = h/n_A$ of the normalizer of A. These elements are obtained by applying the transformation XAX^{-1}, $\forall\ X \in G$, but we have just demonstrated that only a number of them, equal to the index of the normalized ℓ_A, are the distinct ones. They can be written as

$$X_i A X_i^{-1} \ , \qquad i = 1, 2, \ldots, \ell_A \ . \tag{2.30}$$

Exercise: For the covering group of the equilateral triangle show that the normalizers corresponding to the various elements are

$$\begin{aligned}
&N_E = \{E, A, B, C, D, F\} \ , && n_E = 6 \ , && \ell_E = 1 \ , \\
&N_A = \{E, A\} \ , && n_A = 2 \ , && \ell_A = 3 \ , \\
&N_B = \{E, B\} \ , && n_B = 2 \ , && \ell_B = 3 \ , \\
&N_C = \{E, C\} \ , && n_C = 2 \ , && \ell_C = 3 \ , \\
&N_D = N_F = \{E, D, F\} \ , && n_D = n_F = 3 \ , && \ell_D = \ell_F = 2 \ .
\end{aligned} \tag{2.31}$$

and verify the above, in particular (2.22).

Exercise: Repeat the previous exercise for the symmetry group of the square.

2.6 Invariant (or normal) subgroups and factor groups

Starting with a subgroup H and its elements S_i, $i = 1, 2, \dots, g$, one may construct the elements XS_iX^{-1} with X any element of G. It is easily seen that they form a subgroup called the *conjugate subgroup* of H in G (**Exercise**).
By choosing different elements $X \in G$ we obtain different conjugate subgroups, symbolically XHX^{-1}.[3]

Definition(s): If $XHX^{-1} = H$, $\forall\ X \in G$, we call H an *invariant subgroup* or *normal subgroup* of G. By writing the above relation as $XH = HX$, we have an equivalent definition of an invariant subgroup as that for which the right and left cosets formed with any element $X \in G$, coincide.

Remarks:

• From the above if $S_1 \in H$ and H is an invariant subgroup of G then all elements $XS_1X^{-1} \in H$. Hence, a subgroup H of G is invariant if and only if it contains complete classes of G, that is it contains either all or none of the members of the class.

• The identity element and the whole group G are trivial invariant subgroups of G.

Definition(s): A group that contains no proper invariant subgroups is said to be *simple* and *semisimple* if none of its invariant subgroups are Abelian.

• All subgroups H of index $\ell = 2$ are invariant subgroups.
Proof: By definition we have the decomposition in terms of right and left cosets

$$G = H \cup HX = H \cup XH \quad \Longrightarrow \quad H = XHX^{-1}\ , \tag{2.32}$$

where the last step follows from the fact that a subgroup and a coset have no elements in common.

The above can be discussed more compactly by introducing the concept of a *complex*, which is a collection of group elements disregarding order, $\mathcal{K} = \{K_1, K_2, \dots\}$. A complex maybe multiplied by a group element or by another complex, i.e.

$$\mathcal{K}X = \{K_1X, K_2X, \dots\}\ , \qquad \mathcal{K}\mathcal{R} = \{K_1R_1, K_1R_2, \dots, K_2R_1, K_2R_2, \dots\}\ . \tag{2.33}$$

Elements are considered to be included once, regardless on how often they are gener-

[3]For instance if for the group D_3 we take $H = \{E, A\}$ and $X = B$ we have that $XHX^{-1} = \{E, C\}$.

ated. In this way, a subgroup H is defined by the property of closure, i.e. $HH = H$. Also, if H is an invariant subgroup then $XHX^{-1} = H, \forall\, X \in G$.

We have seen that there is a finite number $\ell - 1$ of distinct right cosets $\mathcal{K}_i$ for any subgroup H of index ℓ. Each may be denoted as a complex and clearly we have that $\mathcal{K}_i = H\mathcal{K}_i$. If in addition H is an invariant subgroup then we have $\mathcal{K}_i = HK_i = K_iH$. Also $HK_i = HK_j$, if K_i and K_j are members of the same right coset since the order in which the elements appear in the complex are immaterial. Therefore we have:

Definition: Consider a group G and an invariant subgroup H. This and the set of all $\ell - 1$ distinct cosets maybe regarded as members of a smaller group of order $\ell = h/g$. This is called the *factor* group of G with respect to the invariant subgroup H and is denoted by G/H. In this group the identity is played by the invariant subgroup H itself. To see that note

$$H\mathcal{K}_i = H(HK_i) = (HH)K_i = HK_i = \mathcal{K}_i \ . \tag{2.34}$$

Group multiplication also works as in

$$\mathcal{K}_i\mathcal{K}_j = (HK_i)(HK_j) = K_i(HH)K_j = K_iHK_j = H(K_iK_j) = (\mathcal{K}_i\mathcal{K}_j)\ , \tag{2.35}$$

where the last expression refers to the complex associated with the product K_iK_j as a coset representative.

Exercise: For the covering group of the equilateral triangle D_3 show that the only invariant proper subgroup is $H = \{E, D, F\}$ and that $\mathcal{K} = \{A, B, C\}$ is the only distinct coset. Using the multiplication table (2.6) verify the multiplication table of the factor group $\{H, \mathcal{K}\}$

D_3/H	H	$\mathcal{K}$
H	H	$\mathcal{K}$
$\mathcal{K}$	$\mathcal{K}$	H .

(2.36)

Remark:
It is natural to ask what would have happened if H was not an invariant subgroup. Would this and the corresponding cosets form a group? The answer is emphatically no. As a counter example consider the case of the group D_3 and its non-invariant subgroup $H = \{E, A\}$. Then one easily find that there are two distinct right cosets given by $\mathcal{K}_1 = HB = HD = \{B, D\}$ and $\mathcal{K}_2 = HC = HF = \{C, F\}$. Then one easily

finds that $\mathcal{K}_1\mathcal{K}_1 = H \bigcup \mathcal{K}_2$. Hence, it is clear that the $\{H, \mathcal{K}_1, \mathcal{K}_2\}$ do not form a group.

Exercise: For D_4, the symmetry group of the square show that the invariant subgroups are (for the action of the various elements see fig. 2)

$$H_1 = \{E, G\}\ , \quad H_2 = \{E, F, G, H\}\ , \quad H_3 = \{E, G, A, B\}\ , \quad H_4 = \{E, G, C, D\} \quad (2.37)$$

and that the corresponding cosets are

$$\begin{aligned}
&\text{Cosets of } H_1: \qquad \mathcal{K}_1 = \{A, B\}\ , \quad \mathcal{K}_2 = \{C, D\}\ , \quad \mathcal{K}_3 = \{F, H\}\ ,\\
&\text{Cosets of } H_2: \qquad \mathcal{K} = \{A, B, C, D\}\ ,\\
&\text{Cosets of } H_3: \qquad \mathcal{K} = \{C, D, F, H\}\ ,\\
&\text{Cosets of } H_4: \qquad \mathcal{K} = \{A, B, F, H\}\ .
\end{aligned} \quad (2.38)$$

(What are the other proper subgroups?) Then show that the multiplication table of the factor group D_4/H_i for each $i = 2, 3, 4$ is that in (2.36) and that for D_4/H_1 is the following

D_4/H_1	H_1	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_3$
H_1	H_1	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_3$
$\mathcal{K}_1$	$\mathcal{K}_1$	H_1	$\mathcal{K}_3$	$\mathcal{K}_2$
$\mathcal{K}_2$	$\mathcal{K}_2$	$\mathcal{K}_3$	H_1	$\mathcal{K}_1$
$\mathcal{K}_3$	$\mathcal{K}_3$	$\mathcal{K}_2$	$\mathcal{K}_1$	H_1 .

(2.39)

The multiplication can be represented as $\mathcal{K}_1\mathcal{K}_2 = \mathcal{K}_2\mathcal{K}_1 = \mathcal{K}_3$ and cyclic in $1, 2, 3$.

2.7 Isomorphism and Homomorphism

Definition: Two groups having the same multiplication table are said to be *isomorphic*.

Definition: Two groups G and G' are said to be *homomorphic* if to each element of G' corresponds one and only one element of G and to each element of G correspond several elements of G' and these correspondences are preserved under multiplication.

Remarks:

- Let $G = \{A, B, C, \dots\}$. $G' = \{A'_1, A'_2, \dots, B'_1, B'_2, \dots, C'_1, C'_2, \dots\}$ and the one-to-many correspondence between elements

$$\begin{aligned} A &\leftrightarrow A'_1, A'_2, \ldots\ , \\ B &\leftrightarrow B'_1, B'_2, \ldots\ , \\ C &\leftrightarrow C'_1, C'_2, \ldots\ . \end{aligned} \tag{2.40}$$

If $AB = C$, then the product of any of the A'_i's with any of the B'_j's will belong to the set of the C_k's.

• A homomorphism becomes an isomorphism if the correspondence is one to one.

• The collection of elements in G' that correspond to the identity in G is an invariant subgroup of G'.

Proof: Let the set $H' = \{E'_i\} \in G'$ correspond to E. By definition $EE = E$ implies that this set is closed under multiplication. Let the identity $E' \in G'$ correspond to an element $X \in G$. Noting that $E'X'_k = X'_k = X'_k E'$ corresponds by definition to $XX' = X' = X'X$, which by definition means that $X = E$, the identity element in G, one proves that the identity $E' \in H'$. Let the inverse E'^{-1}_i correspond to an element $Y \in G$. Then $E'^{-1}_i E'_i = E'$ corresponds to $YE = E$, which implies that $Y = E$. Hence, $E'^{-1}_i \in H'$, $\forall\, i$ as well. By definition the multiplication in H' is associative and therefore H' is a subgroup of G. For a $X \in G$, the relation $XEX^{-1} = E$ implies that $X'_m E'_i (X'_k)^{-1} \in H'$. Taking $m = k$ and since X runs over all elements in G this implies that in $X'_k E'_i (X'_k)^{-1} \in H'$ for an arbitrary element $X'_k \in G'$. Hence we prove that H' is an invariant subgroup of G'.

Let G and G' homomorphic groups and H' the set of elements in G' corresponding to the identity in G. Then, the following hold:

• The factor group G'/H' is isomorphic with the group G (the notation as above).

Proof:

a) First note that all the elements $A'_1, A'_2, \ldots$ of G' corresponding to the same element $A \in G$ belong to the same coset of G' w.r.t. H'. To see that note that $A^{-1}A = E$ corresponds to a relation of the form $A'^{-1}_i A'_j = E'_k$, where $E'_k \in H'$. This implies that $A'_j = A'_i E'_k$ which for the coset means that $A'_j H' = A'_i E_k H' = A'_k H'$ (by the rearrangement theorem). Since as we have seen above the identity element E' of G' belongs to H' we have that A'_i and A'_j belong to the same coset.

b) Next we prove that, if two elements of G' lie in the same coset w.r.t. H', then they correspond to the same group element in G. Indeed, let A'_i and A'_j corresponding to A and B in the group elements of G. If they lie in the same coset w.r.t. H' it means that $A'_j = A'_i E'_k$ for some $E'_k \in H'$. From the homomorphism this implies that $A = BE$, or $A = B$, i.e. they correspond to the same element in G. Hence, combined with a) there is a one-to-one correspondence between the elements of G and the elements of the factor group G'/H'.

c) Now we show that the product of two elements of the factor group G'/H' corresponds to the product of the corresponding elements in G. Indeed, if $A'H'B'H' = C'H'$ then $A'E'_iB'E'_j = C'E'_k$ corresponds to $AEBE = CE$ or $AB = C$. Hence G'/H' is isomorphic to G. The order of the latter smaller group is an integral divisor of the order of the larger group G'.

• A group is homomorphic to any one of its factor groups.

Proof: The invariant subgroup H corresponds to all elements of H. The coset $\mathcal{K}_i = HK_i$ corresponds to all elements of the coset $\mathcal{K}_i$. Thus if H is of order g there is a g-to-one correspondence between the original group elements and the elements of the factor group.

Exercise: Consider as the group $G = \{E, P\}$ with $P^2 = E$ and as a group G' the covering group of the equilateral triangle whose multiplication table is in (2.6). Make the correspondences

$$E \leftrightarrow E, D, F\,, \qquad P \leftrightarrow A, B, C \tag{2.41}$$

and verify the above.

Exercise: Formulate an analog of the previous exercise for the case of the group D_4.

2.8 Class multiplication

Class multiplication is nothing but the standard multiplication of complexes with the added feature that we keep track of the number of times each element appears in the product. Let the class C_a have elements $\{X_k^{(a)}\}$, with $k = 1, 2, \ldots, h_a$.

Remarks:

• If C_a is a conjugacy class of a group G and X any element of G, then

$$XC_aX^{-1} = C_a \ . \tag{2.42}$$

Proof: To show that the two sets are equal first note that each element produced in the l.h.s. is contained, by definition, in C_a. Also, all the elements produced on the left are different since $XX_k^{(a)}X^{-1} = XX_l^{(a)}X^{-1}$, only if $X_k^{(a)} = X_l^{(a)}$. Hence, the two sets are identical.

• Any collection C for which

$$XCX^{-1} = C \ , \qquad \forall \ X \in G \ , \tag{2.43}$$

is comprised wholly of complete classes.
Proof: First note that the number of elements on each side is the same. Next, assume that

$$C = C_1 \cup C_2 \cup \cdots \cup C_m \cup \mathcal{R} \ , \tag{2.44}$$

where $\mathcal{R}$ is a non-empty set of elements that do not constitute a complete class. Since, $XC_aX^{-1} = C_a$, for $a = 1, 2, \ldots, m$, then we just have to show that $X\mathcal{R}X^{-1} = \mathcal{R}$, $\forall \ X \in G$. Again, note that the number of elements on each side is the same. If $\mathcal{R}_k \in \mathcal{R}$, then applying $X\mathcal{R}_kX^{-1}$, $\forall \ X \in G$ by definition generates the whole class that contains $\mathcal{R}_k$, which contradicts the assumption that $\mathcal{R}$ is not an empty set.

2.8.1 Product of classes

Using (2.42) we have that

$$C_aC_b = X^{-1}C_aXX^{-1}C_bX = X^{-1}C_aC_bX \ , \qquad \forall \ X \in G \ . \tag{2.45}$$

According to the above theorem C_aC_b consists of complete classes. Therefore

$$C_aC_b = \bigcup_{c=1}^{k} c_{abc}C_c \ , \tag{2.46}$$

where c_{abc} is an integer which tells how many times the complete class C_c appears in the product of C_a with C_b.

Remarks:

• If C_a and C_b are two conjugacy classes then

$$C_a C_b = C_b C_a \qquad \Longleftrightarrow \qquad c_{abc} = c_{bac} \ . \tag{2.47}$$

Proof: In $C_a X = X C_a$, we choose for $X = X_k^{(b)} \in C_b$ and then we sum over k, that is over all members of the class C_b

$$\sum_{k=1}^{h_b} X_k^{(b)} C_a = C_a \sum_{k=1}^{h_b} X_k^{(b)} \qquad \Longrightarrow \qquad C_b C_a = C_a C_b \ . \tag{2.48}$$

Exercise: For the covering group of the equilateral triangle show that the class product leads to the table

D_3	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3C_1 + 3C_3$	$2C_2$
C_3	C_3	$2C_2$	$2C_1 + C_3$,

(2.49)

where the classes C_a, $a = 1, 2, 3$ are given by (2.22) and are labeled in the indicated order.

• Let's denote by $C_{a'}$ the collection of group elements which are the inverses of those appearing in the class C_a. One easily sees that it is itself a class. Indeed, C_a is generated by $X^{-1}AX$, $\forall\ X \in G$. Then the inverse elements $(X^{-1}AX)^{-1} = X^{-1}A^{-1}X$ generate $C_{a'}$ which by a definition is then a class. Note then that

$$c_{ab'1} = h_a \delta_{ab} \ , \tag{2.50}$$

since the identity can only appear as many times as the numbers of elements in the class. Note also that $C_{a'}$ may coincide with C_a as is the case for the group D_3.

Exercise: Construct the analogous table for the symmetry group of the square D_4.

3 Foundations of Group Theory Representations

Now that we have reviewed the general structure of group theory we turn our attention to the objects that the group elements may act on as well as on how this happens. This is the subject of the *theory of group representations*. This theory provides a framework for the specific techniques which are used to exploit symmetries of the various objects and systems of interest.

A *rep*[4] of an abstract group is any group composed of concrete mathematical entities which is homomorphic to the original group. For our purposes we restrict attention mostly to square non-singular matrices with matrix multiplication as the group composition law.

We associate to each group element $X \in G$ a matrix $\Gamma(X)$. The associated set of matrices obeys the multiplication table of G. To the identity E we associate the unit matrix, i.e. $\Gamma(E) = \mathbb{I}$, where $\mathbb{I}_{ij} = \delta_{ij}$.[5]

Remarks:

• The dimensionality of the matrices in the representation is called the *dimensionality of the representation*.

• If each of the matrices in $\{\Gamma(X)\}$ is different, the two groups, i.e. G and its rep, are isomorphic and the rep is said to be *true* or *faithful*.

• According to the results of subsection 2.7, if several group elements correspond to a single matrix, all of the elements corresponding to the identity matrix form an invariant subgroup of the full group. Also. the elements corresponding to each of the other matrices form the distinct cosets of the invariant subgroup and the matrices form a faithful rep of the factor group of the invariant subgroup.

• An example of a faithful rep of dimension 2 for the covering group of the equilateral triangle was given in (2.7). Another rep is obtained by taking the determinant of these matrices since $\det(\Gamma(A)\Gamma(B)) = \det(\Gamma(A))\det(\Gamma(B))$. However, this rep is not faithful. From (2.7) we see that all the determinants equal ± 1, whereas the group has

[4]It is customary for convenience to use the abbreviation rep for representation.

[5]Recall the definition

$$\delta_{ij} = \begin{cases} 1\,, & \text{if } i = j \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.1}$$

six elements.

• Once we have a matrix rep of a group we may generate an infinite number of other reps by just letting all matrices undergo the same similarity transformation, i.e. the set of matrices $\{\Gamma'(A)\}$ with $\Gamma'(A) = S^{-1}\Gamma(A)S$, satisfies the same group multiplication table as the set $\{\Gamma(A)\}$. Such reps are deemed to be *equivalent*.

Exercise: The matrices X_k, $k = 1, \dots 6$ written explicitly in (2.7) provide a rep of the group D_3 since they reproduce the table (2.6). However, they do not produce the rotations as depicted in Fig. 1. Show that the set of matrices

$$X'_k = S^{-1}X_kS\ , \qquad S = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ . \tag{3.2}$$

are needed producing the rotations of Fig. 1.

3.1 Reducible and Irreducible Representations

We can take two or more reps and construct from them a new rep by combining them into larger matrices. A typical example is

$$\Gamma(A) = \begin{pmatrix} \Gamma^{(1)}(A) & \mathbf{0} \\ \mathbf{0} & \Gamma^{(2)}(A) \end{pmatrix}\ , \tag{3.3}$$

where $\Gamma^{(1)}(A)$ and $\Gamma^{(2)}(A)$ are reps, not necessarily of equal dimensionality.
Definition: Such an artificially enlarged rep is said to be *reducible*.

Remarks:

• The reducibility may be concealed by carrying out a similarity transformation on the larger matrix which yields an equivalent rep but not in a diagonal form.

• A criterion for reducibility is that to be possible to reduce all rep matrices in a block diagonal form with the same block structure for all matrices and with the same similarity transformation. If this cannot be done then the rep is said to be *irreducible*.[6] Hence, studying and classifying all irreps of a group is important since these are the building blocks of any other rep.

• A more general definition of reducible rep is that a rep is reducible if all the matrices

[6] It is customary for convenience to use the abbreviation irrep for an irreducible representation.

can be put in the form

$$\Gamma(A) = \begin{pmatrix} \Gamma^{(1)}(A) & Q(A) \\ \mathbf{0} & \Gamma^{(2)}(A) \end{pmatrix} , \tag{3.4}$$

where $\Gamma^{(i)}(A)$, $i = 1, 2$ are $n_i \times n_i$ matrices, while $Q(A)$ is a $n_1 \times n_2$ matrix. It is easy to verify that the $\Gamma^{(i)}(A)$'s also provide reps for the group (**Exercise**).

• For unitary reps the matrix $Q(A) = 0$.

Proof: A unitary matrix M obeys $M^{-1} = M^\dagger$.[7] We have that

$$\Gamma(A^{-1}) = \Gamma(A)^{-1} = \Gamma(A)^\dagger = \begin{pmatrix} \Gamma^{(1)}(A)^\dagger & \mathbf{0} \\ Q(A)^\dagger & \Gamma^{(2)}(A)^\dagger \end{pmatrix} , \tag{3.5}$$

Since all matrices are required to have the form (3.4) we conclude that $Q(A) = 0$.

• The reducibility of the matrices $\{\Gamma(A)\}$ as in (3.4) can be expressed in the form[8]

$$\Gamma(A) = \Gamma^{(1)}(A) \oplus \Gamma^{(2)}(A) . \tag{3.6}$$

In general, the matrices of a reducible rep will be written in the form

$$\Gamma(A) = a_1\Gamma^{(1)}(A) \oplus a_2\Gamma^{(2)}(A) + \cdots = \sum_{k=1}^{r} \oplus a_k \Gamma^{(k)}(A) , \tag{3.7}$$

where the positive integers a_k indicate the times that $\Gamma^{(k)}(A)$ appears in $\Gamma(A)$.

In the following we will consider, unless stated otherwise, reps provided by unitary matrices, the reason being:

• Any rep by square non-singular matrices is equivalent through a similarity transformation to a rep of unitary matrices.

Proof: Construct the Hermitian matrix[9]

$$H = \sum_{i=1}^{h} A_i A_i^\dagger \qquad \Longrightarrow \qquad H_{mn} = \sum_{i=1}^{h} \sum_k (A_i)_{mk} (A_i^*)_{nk} , \tag{3.8}$$

where the range of the indices m and n depends on the dimensionality of the rep. Any

[7]For a matrix M, its Hermitian conjugate is $M^\dagger = (M^T)^*$ and for the matrix elements $(M^\dagger)_{ij} = M^*_{ji}$.

[8]Note that (3.4) is not precisely of the form of direct sum for matrices, in which case $Q(A) = 0$.

[9] For notational convenience in this proof we will denote the rep matrices corresponding to an abstract group element A_i by the same symbol.

Hermitian matrix can be diagonalized by means of a unitary transformation with a matrix U. Denoting this by d we have that

$$d = U^{-1}HU = \sum_{i=1}^{h} A'_i A'^{\dagger}_i \ , \qquad A'_i = U^{-1}A_iU = U^{\dagger}A_iU \ . \tag{3.9}$$

The diagonal matrix d has real positive elements. To show that let the eigenvectors $\{e^{(p)}\}$ and eigenvalues λ_p of H

$$He^{(p)} = \lambda_p e^{(p)} \qquad \Longrightarrow \qquad \sum_n H_{mn}e^{(p)}_n = \lambda_p e^{(p)}_m \ . \tag{3.10}$$

Since $H = H^{\dagger}$ the set $\{e^{(p)}\}$ is orthonormal and complete, these obey the orthogonality and completeness

$$\sum_m e^{(p)*}_m e^{(q)}_m = \delta_{pq} \ , \qquad \sum_p e^{(p)*}_m e^{(p)}_n = \delta_{mn} \ . \tag{3.11}$$

Using that we obtain from (3.10) that

$$\lambda_p = \sum_{m,n} e^{(p)*}_m H_{mn} e^{(p)}_n = \cdots = \sum_{i=1}^{h} \sum_m |f^{(p)}_{im}|^2 > 0 \ , \tag{3.12}$$

where

$$f^{(p)}_{im} = \sum_n e^{(p)*}_n (A_i)_{nm} \ . \tag{3.13}$$

We define the diagonal matrix $d_{1/2}$ whose entries are the square roots of the entries of d and then the matrices

$$\tilde{A}_i = d^{-1}_{1/2} A'_i d_{1/2} \ . \tag{3.14}$$

One may show with the help of the rearrangement theorem that the matrices $\tilde{A}_i$ are unitary (**Exercise**). Recalling all steps

$$\tilde{A}_i = S^{-1}A_iS \ , \qquad S = Ud_{1/2} \ . \tag{3.15}$$

through the indicated similarity transformation.

• If two unitary reps are equivalent through a similarity transformation then a unitary matrix can be found which proves the same equivalence when used in the similarity transformation.

Proof: Proceeds along similar lines as above.

• If a rep is in its reduced form (3.4) then a similarity transformation can bring it into a unitary reduced form with $Q(A) = 0$ that is

$$\Gamma(A) = \begin{pmatrix} \Gamma^{(1)}(A) & \mathbf{0} \\ \mathbf{0} & \Gamma^{(2)}(A) \end{pmatrix} , \tag{3.16}$$

with $\Gamma^{(i)}(A)$, $i = 1, 2$ unitary reps.

Proof: Proceeds along similar lines as above.

3.2 Criteria for (ir)reducibility of a representation; Schur's lemmas

Next we state and prove several theorems which will lead to criteria for deciding whether a given rep is reducible or not.

• **Schur's first lemma:** Any matrix which commutes with all matrices in an irrep is proportional to the unit matrix.

Proof: It is clearly sufficient to restrict to unitary reps. By assumption if C is such a matrix it obeys

$$[C, \Gamma(A)] = 0 , \qquad \forall\ A \in G . \tag{3.17}$$

It is sufficient to consider a Hermitian matrix C. The reason is that by taking the Hermitian adjoint of (3.17) we see that $[C^\dagger, \Gamma(A)] = 0$, $\forall\ A \in G$, as well. The matrices $C + C^\dagger$ and $i(C - C^\dagger)$ are Hermitian and commute with $\Gamma(A)$. Hence, if we can show that these matrices are proportional to the identity then so is C. Then C can be brought into a diagonal form by a unitary transformation. This transformation acts also on the rep matrices $\Gamma(A)$ transforming then into another equivalent rep.

Assuming the above we write (3.17) in components to obtain

$$(C_{mm} - C_{nn})\, \Gamma(A)_{mn} = 0 , \tag{3.18}$$

where we used the same symbols for the matrices after the similarity transformation. With no loss of generality assume that the first k of the diagonal entries of C are equal and different than the rest. Then, the indices m, n split as $(\hat{m}, m')$ and $(\hat{n}, n')$, with $\hat{m} = 1, 2, \dots k$ and $m' = k + 1, \dots$ and similarly for $\hat{n}$ and n'. Specializing (3.18) to $(m, n) = (\hat{m}, n')$ and $(m, n) = (m', \hat{n})$ we learn that, since by hypothesis $C_{\hat{m}\hat{m}} \neq C_{m'm'}$, the rep matrix elements $\Gamma(A)_{\hat{m}n'} = \Gamma(A)_{m'\hat{n}} = 0$. Hence the rep matrices $\Gamma(A)$ take a reducible form for all $A \in G$. Hence, C should be proportional to the identity which will also hold for the original matrix C before the similarity transformation.

It follows that if C commutes with all rep matrices and is not proportional to the identity the rep is reducible.

• **The converse:** If the only matrices that commute with all matrices of a rep is proportional to the unit matrix, then the rep is irreducible.
Proof: Let's assume that the rep is reducible, that is with a similarity transformation it can be put into the form (3.16). Assume that S is the matrix that does this, i.e.

$$\Gamma'(A) = S^{-1}\Gamma(A)S = \begin{pmatrix} \Gamma^{(1)}(A) & \mathbf{0} \\ \mathbf{0} & \Gamma^{(2)}(A) \end{pmatrix} , \tag{3.19}$$

where the $\Gamma^{(i)}(A)$'s are unitary irrespectively of the $\Gamma(A)$, with dimensions n_1 and n_2. We would like a matrix C such that $[C, \Gamma'(A)] = 0, \forall\, A \in G$. An obvious choice is the block diagonal one

$$C = \begin{pmatrix} c_1 I_{n_1} & \mathbf{0} \\ \mathbf{0} & c_2 I_{n_2} \end{pmatrix} , \tag{3.20}$$

where the constants $c_1 \neq c_2$. This implies that

$$[\Gamma(A), SCS^{-1}] = 0\,, \qquad \forall\, A \in G\,. \tag{3.21}$$

By assumption SCS^{-1} and consequently C, should be proportional to the identity. This cannot be the case since $c_1 \neq c_2$. Hence, there is a contradiction with the original assumption that $\Gamma(A)$ is a reducible rep. Therefore the rep $\Gamma(A)$ is irreducible.

• **Schur's second lemma:** If $\Gamma^{(1)}(A)$ and $\Gamma^{(2)}(A)$ are two irreps with dimensions $n_1 \leqslant n_2$ and a $n_1 \times n_2$ matrix S is found obeying

$$\Gamma^{(1)}(A)S = S\Gamma^{(2)}(A)\,, \qquad \forall A \in G\,, \tag{3.22}$$

then either S is the null matrix, or $n_1 = n_2$ in which case the irreps are equivalent.
Proof: We assume from the outset the $\Gamma^{(i)}(A)$'s are irreps. By taking the Hermitian adjoint of (3.22) and using the fact that ${\Gamma^{(i)}}^\dagger(A) = \Gamma^{(i)}(A^{-1})$ we have that

$$S^\dagger\Gamma^{(1)}(A^{-1}) = \Gamma^{(2)}(A^{-1})S^\dagger \quad \Longrightarrow \quad S^\dagger\Gamma^{(1)}(A^{-1})S = \Gamma^{(2)}(A^{-1})S^\dagger S\,, \tag{3.23}$$

or using (3.22) we obtain

$$S^\dagger S\Gamma^{(2)}(A) = \Gamma^{(2)}(A)S^\dagger S\ , \qquad \forall A \in G\ . \tag{3.24}$$

where at the end we' ve replaced A^{-1} by A since it holds for all $A \in G$. From Schur's 1st lemma the $n_2 \times n_2$ matrix $S^\dagger S = c\mathbb{I}$.

First consider the case with $n_1 = n_2$. If $c \neq 0$, then $\det S \neq 0$ and S^{-1} exists. Hence from (3.22) $\Gamma^{(1)}(A) = S^{-1}\Gamma^{(2)}(A)S$ so that the reps are equivalent. If $c = 0$, then

$$S^\dagger S = 0 \qquad \Longrightarrow \qquad \sum_k S^*_{ki}S_{kj} = 0\ , \qquad \forall\, i, j\ . \tag{3.25}$$

Taking $j = i$ this becomes $\sum_k |S_{ki}|^2 = 0$, $\forall\, i$, implying that $S_{ki} = 0$, $\forall\, k, i$ and therefore S is the null square matrix.

Finally consider the case with $n_1 < n_2$. We enlarge the matrix S by adding $n_2 - n_1$ rows with all elements zero so that it becomes a square $n_2 \times n_2$ matrix. Denoting this matrix by S' we easily establish that $S'^\dagger S' = S^\dagger S$. Hence $\det(S'^\dagger S') = \det(S^\dagger S) = c^{n_2}$. But since $\det S' = 0$ we have that $\det S^\dagger S = 0$ and therefore $c = 0$. Taking the element $(S^\dagger S)_{ii} = \sum_k |S_{ki}|^2 = 0$, $\forall\, i$, we establish that S is the null matrix.

3.3 The great orthogonality theorem

If we consider all the inequivalent unitary irreps of a group G, then

$$\sum_{A\in G} \Gamma^{(i)}(A)^*_{\mu\nu}\Gamma^{(j)}(A)_{\alpha\beta} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}\ , \tag{3.26}$$

where h is the order of G and l_i is the dimensionality of the rep $\Gamma^{(i)}(A)$. We may have n irreps so that $i = 1, 2, \ldots, n$. This is called the *great orthogonality theorem*.

Proof: First we consider two inequivalent reps $\Gamma^{(i)}(A)$, with dimensions l_i, $i = 1, 2$ and we construct the $l_2 \times l_1$ matrix

$$M = \sum_{A\in G} \Gamma^{(2)}(A)X[\Gamma^{(1)}(A)]^{-1}\ , \tag{3.27}$$

where X is an arbitrary $l_2 \times l_1$ matrix. This matrix has the property

$$\Gamma^{(2)}(B)M = M\Gamma^{(1)}(B)\ , \qquad \forall\, B \in G\ . \tag{3.28}$$

To see that start with the l.h.s

$$\begin{aligned}\Gamma^{(2)}(B)M &= \sum_{A\in G}\Gamma^{(2)}(B)\Gamma^{(2)}(A)X[\Gamma^{(1)}(A)]^{-1} \\ &= \sum_{A\in G}\Gamma^{(2)}(BA)X[\Gamma^{(1)}(A)]^{-1} \\ &= \sum_{A\in G}\Gamma^{(2)}(BA)X[\Gamma^{(1)}(BA)]^{-1}\Gamma^{(1)}(B) \\ &= M\Gamma^{(1)}(B)\ ,\end{aligned} \tag{3.29}$$

where in the last step we have used the rearrangement theorem. Due to (3.28) and Schur's 2nd lemma if the reps are inequivalent then $M = 0$ or in terms of its elements

$$M_{\alpha\mu} = 0 = \sum_{A\in G}\sum_{\kappa,\lambda}\Gamma^{(2)}(A)_{\alpha\kappa}X_{\kappa\lambda}\Gamma^{(1)}(A^{-1})_{\lambda\mu}\ . \tag{3.30}$$

Since X is arbitrary we may set all of its element to zero except for $X_{\beta\nu}$, i.e. $X_{\kappa\lambda} = \delta_{\kappa\beta}\delta_{\lambda\nu}$. That implies

$$\sum_{A\in G}\Gamma^{(2)}(A)_{\alpha\beta}\Gamma^{(1)}(A^{-1})_{\nu\mu} = 0\ . \tag{3.31}$$

Consider next the case of the same reps $\Gamma^{(1)}(A)$. Then the matrix M is defined as

$$M = \sum_{A\in G}\Gamma^{(1)}(A)X[\Gamma^{(1)}(A)]^{-1} \tag{3.32}$$

and following similar steps as above we prove that

$$\Gamma^{(1)}(B)M = M\Gamma^{(1)}(B)\ , \qquad \forall\ B\in G\ . \tag{3.33}$$

Then from Schur's first lemma this is a multiple of the unit matrix, i.e.

$$M = \sum_{A\in G}\Gamma^{(1)}(A)X[\Gamma^{(1)}(A)]^{-1} = c(X)\mathbb{I}\ , \tag{3.34}$$

where the argument in $c(X)$ emphasizes that this constant depends on the choice of the matrix X. Choosing as before $X_{\kappa\lambda} = \delta_{\kappa\beta}\delta_{\lambda\nu}$, means that $c(X)$ can be labeled in this case by $c_{\beta\nu}$. Then we have in components

$$M_{\alpha\mu} = \sum_{A\in G}\Gamma^{(1)}(A)_{\alpha\beta}\Gamma^{(1)}(A^{-1})_{\nu\mu} = c_{\beta\nu}\delta_{\alpha\mu}\ . \tag{3.35}$$

To determine $c_{\beta\nu}$ we set $\alpha = \mu$ and sum over $\alpha = 1, 2, \ldots, l_1$. We obtain

$$\begin{aligned} l_1 c_{\beta\nu} &= \sum_{\alpha=1}^{l_1} \sum_{A\in G} \Gamma^{(1)}(A)_{\alpha\beta}\Gamma^{(1)}(A^{-1})_{\nu\alpha} \\ &= \sum_{A\in G} \Gamma^{(1)}(A^{-1}A)_{\nu\beta} = \sum_{A\in G} \Gamma^{(1)}(E)_{\nu\beta} = \sum_{A\in G} \delta_{\nu\beta} = h\delta_{\nu\beta}\ . \end{aligned} \tag{3.36}$$

Therefore

$$c_{\beta\nu} = \frac{h}{l_1}\delta_{\nu\beta}\ . \tag{3.37}$$

Thus for general reps

$$\sum_{A\in G} \Gamma^{(i)}(A)_{\alpha\beta}\Gamma^{(j)}(A^{-1})_{\nu\mu} = \frac{h}{l_i}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}\ , \tag{3.38}$$

For unitary reps

$$\Gamma^{(j)}(A^{-1})_{\nu\mu} = [\Gamma^{(j)}(A)]^{-1}_{\nu\mu} = [\Gamma^{(j)}(A)]^{\dagger}_{\nu\mu} = \Gamma^{(j)}(A)^{*}_{\mu\nu} \tag{3.39}$$

and we obtain (3.26).

• One may establish the following inequality (later in (3.62) this will be proven to be a strict equality)

$$\sum_{i=1}^{n} l_i^2 \leqslant h \tag{3.40}$$

and hence that the dimensionality of an irrep is bounded as

$$l_i \leqslant \sqrt{h}\ , \qquad i = 1, 2, \ldots, n\ . \tag{3.41}$$

<u>Proof</u>: The great orthogonality theorem (3.26) can be viewed as an orthonormality relation of the vectors

$$V^{(i)}_{\alpha\beta}(A) = \sqrt{\frac{l_i}{h}}\Gamma^{(i)}_{\alpha\beta}(A)\ . \tag{3.42}$$

in the h-dimensional vector space of group elements in G. The number of these vectors is found by summing over the indices α, β giving, for each i, l_i and then summing over the inequivalent irreps which is precisely the l.h.s. of (3.40). This number has of course to be less than the dimensionality of the vector space, thus proving (3.40). Then (3.41) follows trivially.

3.4 The character of a representation

Because all matrix reps related by a similarity transformation are equivalent there is a certain degree of arbitrariness in the actual form of the rep matrices. It is therefore worthwhile to develop a method of characterizing a given rep in a manner which is invariant under the similarity transformation.

Clearly this is achieved by taking the traces of the reps matrices since these are independent of any similarity transformation. We will denote them by

$$\chi(A) = \mathrm{Tr}(\Gamma(A)) = \sum_{i=1}^{l} \Gamma(A)_{ii} \,, \qquad A \in G \tag{3.43}$$

and we will call the *character system* of the rep the set of these h numbers.

Remarks:

• The characters of group elements in the same class are equal.

Proof: Since in a class all group elements are related by a similarity transformation the proof is immediate. Thus, for a given rep there are as many distinct characters as classes. Hence, we better use the symbol $\chi^{(i)}(C_a) \equiv \chi_a^{(i)}$ to indicate the character in a given class of conjugate elements.

• The *character orthogonality theorem* can be written as

$$\sum_{A\in G} \chi^{(i)}(A)\chi^{(j)}(A)^* = \sum_{a=1}^{k} h_a \chi_a^{(i)} \chi_a^{(j)*} = h\delta_{ij} \,, \tag{3.44}$$

where the second summation is over the distinct k classes of the group, the ath one having h_a elements.

Proof: We set in (3.26) $\nu = \mu$ and $\beta = \alpha$ and then sum over μ and α. Then we obtain

$$\sum_{A\in G} \chi^{(i)}(A)\chi^{(j)}(A)^* = \frac{h}{l_i}\delta_{ij} \sum_{\alpha,\beta,\mu,\nu} \delta_{\mu\alpha}\delta_{\nu\beta}\delta_{\mu\nu}\delta_{\alpha\beta} = \frac{h}{l_i}\delta_{ij} \sum_{\alpha} \delta_{\alpha\alpha} = h\delta_{ij} \,. \tag{3.45}$$

To finish the proof we use the fact that the character of a group element in a rep depends only on the class in which it belongs.

• The number n of irreps of G is at most equal to the number of k conjugacy classes in

G (will be proven later in (3.63) that this is a strict equality), i.e.

$$n \leqslant k\ . \tag{3.46}$$

Proof: By defining

$$e_a^{(i)} = \sqrt{\frac{h_a}{h}}\chi_a^{(i)}\ , \tag{3.47}$$

one may view (3.44) as the orthonormality relation of the above unit vectors in the k-dimensional space of the classes of G. Hence, the number of linearly independent such vectors, that is the number of irreps of G, cannot exceed the dimensionality of this space k.

• In an irrep, the sum of the matrices of elements of a conjugacy class C_a is proportional to the identity, that is

$$M_a^i = \sum_{X\in C_a} \Gamma^{(i)}(X) = \frac{h_a}{l_i}\chi_a^{(i)}\mathbb{I}\ . \tag{3.48}$$

Proof: We first compute

$$\begin{aligned}\Gamma^{(i)}(A)M_a^i[\Gamma^{(i)}(A)]^{-1} &= \sum_{X\in C_a}\Gamma^{(i)}(A)\Gamma^{(i)}(X)[\Gamma^{(i)}(A)]^{-1} = \sum_{X\in C_a}\Gamma^{(i)}(AXA^{-1})\\ &= M_a^i\ , \qquad \forall\ A\in G\ .\end{aligned} \tag{3.49}$$

Hence from Schur's 1st lemma $M_a^i = \lambda_a^i\mathbb{I}$. The proportionality constant is computed by taking the trace of both sides

$$\text{Tr}(M_a^i) = \sum_{X\in C_a}\chi^{(i)}(X) = h_a\chi_a^{(i)} = \lambda_a^i l_i \quad \Longrightarrow \quad \lambda_a^i = \frac{h_a}{l_i}\chi_a^{(i)}\ . \tag{3.50}$$

• A necessary and sufficient condition for the equivalence of two irreps is the equality of their character systems.

Proof: From the invariance of the characters under a similarity transformation it is clear that we have a necessary condition. To show sufficiency we must show that if the characters of two irreps are identical then the reps are equivalent. To prove that, assume that the irreps $\Gamma(R)$ and $\Gamma'(R)$ are inequivalent with equal characters $\chi(R) = \chi'(R)$. Then from (3.44) we have that $\sum_{A\in G}\chi(R)\chi'(R)^* = 0$, which implies that $\sum_{A\in G}|\chi(R)|^2 = 0$ which contradicts the fact that from (3.44) the r.h.s. should equal h. Hence, for equal characters the corresponding irreps are also equivalent.

3.5 Decomposition of reducible representations

We have seen that every reducible rep can be brought into the form (3.7). Then we immediately have that

$$\chi(A) = \sum_{i=1}^{n} a_i \chi^{(i)}(A) \ . \tag{3.51}$$

The number of times that an irrep $\Gamma^{(i)}(A)$ appears in the reducible rep $\Gamma(A)$ is

$$a_i = \frac{1}{h}\sum_{A\in G}\chi(A)\chi^{(i)}(A)^* = \frac{1}{h}\sum_{a=1}^{k} h_a \chi_a \chi_a^{(i)*} \ , \tag{3.52}$$

which follows immediately by using the character orthogonality theorem (3.44).

Remarks:

• The sum for a reducible, in general, rep

$$\sum_{A\in G}|\chi(A)|^2 = \sum_{a=1}^{k} h_a|\chi_a|^2 = h\sum_{i=1}^{n} a_i^2 \geqslant h \ . \tag{3.53}$$

<u>Proof</u>: We compute

$$\begin{aligned}\sum_{A\in G}|\chi(A)|^2 &= \sum_{A\in G}\sum_{i,j=1}^{n} a_i a_j \chi^{(i)}(A)\chi^{(j)}(A)^* \\ &= \sum_{i,j=1}^{n} a_i a_j \underbrace{\sum_{A\in G}\chi^{(i)}(A)\chi^{(j)}(A)^*}_{h\delta_{ij}} = h\sum_{i=1}^{n} a_i^2 \ . \end{aligned}\tag{3.54}$$

The inequality follows from the fact that at least one of the a_i's equals one.

• A necessary and sufficient condition for rep $\Gamma(A)$ to be irreducible is that

$$\sum_{A\in G}|\chi(A)|^2 = \sum_{a=1}^{k} h_a|\chi_a|^2 = h \ . \tag{3.55}$$

<u>Proof</u>: The necessity follows from the fact that if the rep is irreducible say $\Gamma^{(j)}(A)$ then from (3.52) we have that $a_i = \delta_{ij}$. Then the r.h.s. of (3.53) equals h. The sufficiency is also clear, since if the rep were reducible then the r.h.s. of (3.53) would have been strictly larger than h.

Hence we have a criterion for deciding whether a rep is reducible that depends only on the characters of the rep and the order of the group.

• From the relation between conjugacy classes and class products we have that

$$M^i_a M^i_b = \sum_{c=1}^{k} c_{abc} M^i_c \ , \qquad a,b = 1,2,\ldots,k \ , \qquad i = 1,2,\ldots,n \ . \tag{3.56}$$

Substituting (3.48) we obtain

$$h_a h_b \chi^{(i)}_a \chi^{(i)}_b = l_i \sum_{c=1}^{k} c_{abc} h_c \chi^{(i)}_c \ . \tag{3.57}$$

3.6 The regular representation

From the multiplication table of a group G we can always form a reducible rep called the *regular representation*. Let the elements of G be $\{X_1, X_2, \ldots, X_h\}$.
Definition: The regular rep is defined as[10]

$$\Gamma^{\rm reg}(X_m) : \qquad [\Gamma^{\rm reg}(X_m)]_{kl} = \delta\left(X_m - X_k X_l^{-1}\right) \equiv \begin{cases} 1 & \text{if } X_m = X_k X_l^{-1} \\ 0 & \text{otherwise} \ . \end{cases} \tag{3.58}$$

To prove that this is a rep we must show that $\Gamma^{\rm reg}(A)\Gamma^{\rm reg}(B) = \Gamma^{\rm reg}(AB)$. For the matrix elements we have

$$\begin{aligned} [\Gamma^{\rm reg}(A)\Gamma^{\rm reg}(B)]_{ml} &= \sum_{k=1}^{h} [\Gamma^{\rm reg}(A)]_{mk} [\Gamma^{\rm reg}(B)]_{kl} = \sum_{k=1}^{h} \delta\left(A - X_m X_k^{-1}\right) \delta\left(B - X_k X_l^{-1}\right) \\ &= \delta\left(A - X_m X_l^{-1} B^{-1}\right) = \delta\left(AB - X_m X_l^{-1}\right) \\ &= [\Gamma^{\rm reg}(AB)]_{ml} \ . \end{aligned} \tag{3.59}$$

The characters of the rep are

$$\begin{aligned} \chi^{\rm reg}(X) &= \sum_{i=1}^{h} [\Gamma^{\rm reg}(X)]_{ii} = \sum_{i=1}^{h} \delta\left(X - X_i X_i^{-1}\right) = \sum_{i=1}^{h} \delta\left(X - E\right) \\ &= h\delta\left(X - E\right) \ . \end{aligned} \tag{3.60}$$

Hence, it is nonzero only for the identity for which equals the order of the group h.

Exercise: Construct the regular rep for the covering group of the equilateral triangle.

[10] The definition of $\delta(A - B)$ for matrices A and B is a direct generalization of (3.1).

3.7 The celebrated theorem

Using (3.52) we may compute the number of times the ith irrep appears in the regular rep as

$$a_i = \frac{1}{h}\sum_{A\in G}\chi^{\rm reg}(A)\chi^{(i)}(A)^* = \frac{1}{h}\chi^{\rm reg}(E)\chi^{(i)}(E)^* = \frac{1}{h}hl_i = l_i\ , \tag{3.61}$$

where we used (3.60). Thus, the ith irrep appears as often as its dimension. Since the dimension of the regular rep is h we have that

$$h = \sum_{i=1}^{n} a_i l_i \qquad \Longrightarrow \qquad h = \sum_{i=1}^{n} l_i^2\ . \tag{3.62}$$

Remarks:

• The number n of irreps in a group equals the number of classes k, i.e.

$$n = k\ . \tag{3.63}$$

Proof: We start from (3.57) and sum over all irreps obtaining

$$h_a h_b \sum_{i=1}^{n}\chi_a^{(i)}\chi_b^{(i)} = \sum_{c=1}^{k} c_{abc}h_c\sum_{i=1}^{n} l_i\chi_c^{(i)}\ . \tag{3.64}$$

However

$$\sum_{i=1}^{n} l_i\chi_c^{(i)} = \chi_c^{\rm reg} = h\delta_{c,1}\ , \tag{3.65}$$

where the index $c = 1$ corresponds to the identity element which is a class on its own. Therefore

$$\sum_{i=1}^{n}\chi_a^{(i)}\chi_b^{(i)} = \frac{h}{h_a h_b}c_{ab1} = \frac{h}{h_a}\delta_{ba'}\ , \tag{3.66}$$

where $C_{a'}$ denotes the class whose elements are the inverses of the elements in C_a and where we used (2.50). For unitary reps $\chi_a^{(i)} = \chi_{a'}^{(i)*}$ and therefore

$$\sum_{i=1}^{n}\chi_a^{(i)}\chi_b^{(i)*} = \frac{h}{h_a}\delta_{ab}\ . \tag{3.67}$$

This is a new orthogonality relation. Summing up (3.44) over $i = 1, 2, \ldots, n$ we get that

$$\sum_{i=1}^{n}\sum_{a=1}^{k} h_a\chi_a^{(i)}\chi_a^{(j)*} = hn\ . \tag{3.68}$$

Similarly, summing up (3.67) (after multiplying both sides by h_a) over $a = 1, 2, \ldots, k$ we obtain

$$\sum_{a=1}^{k} \sum_{i=1}^{n} h_a \chi_a^{(i)} \chi_a^{(j)*} = hk \, . \tag{3.69}$$

Comparing the above two expressions we immediately see that (3.63) holds.

An alternative proof arises by following the argument that led to (3.46) but now with the role of the vector space played by the space of conjugate classes and the number of linear independent vectors is k. Hence, $k \leqslant n$ and compatibility with (3.46) requires (3.63).

3.8 Character tables

It is convenient to display the characters of the various irreps of a given group in a *character table*. The rows are labeled by the irreps and the columns are labeled by the classes preceded by the number of the class elements. The entries are the characters. The form of a character table is

$$\begin{array}{c|cccc} G & h_1 C_1 & h_2 C_2 & h_3 C_3 & \cdots \\ \hline \Gamma^{(1)} & \chi_1^{(1)} & \chi_2^{(1)} & \chi_3^{(1)} & \cdots \\ \Gamma^{(2)} & \chi_1^{(2)} & \chi_2^{(2)} & \chi_3^{(2)} & \cdots \\ \Gamma^{(3)} & \chi_1^{(3)} & \chi_2^{(3)} & \chi_3^{(3)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array} \quad , \qquad C_1 \equiv \{E\} \, , \quad h_1 = 1 \, . \tag{3.70}$$

Although a character table contains much less information about a given group than the rep matrices themselves, it has enough information for many purposes. In addition, it is much easier to construct without prior knowledge of the matrices of the irreps.

3.8.1 The rules for constructing character tables

The construction of the character table is based on the following ingredients which we have already encountered, but we repeat here for convenience:

Rule 1: The number of irreps equal the number of classes, i.e. (3.63). Hence the character tables have equal numbers of rows and columns.

Rule 2: The dimensionalities of l_i of the irreps are constrained by the fact that $\sum_{i=1}^{n} l_i^2 = h$, that is (3.62). In many cases this equation has a unique solution subject to the constraint of rule 1. Then the first column of the character table is determined by the fact that $\chi^{(i)}(E) = l_i$. Also since there is always the one-dimensional irrep in which all elements are represented by $+1$, the first row can be written as $\chi_k^{(1)} = 1, \forall\, k$.

Rule 3: The rows of the table must be orthogonal and normalized to h with weight factors h_a as dictated by (3.44)

$$\sum_{a=1}^{k} h_a \chi_a^{(i)} \chi_a^{(j)*} = h\delta_{ij}\ . \tag{3.71}$$

Rule 4: The columns of the table must be orthogonal and normalized to h/h_a as dictated by (3.67)

$$\sum_{i=1}^{n} \chi_a^{(i)} \chi_b^{(i)*} = \frac{h}{h_a}\delta_{ab}\ . \tag{3.72}$$

Rule 5: The elements of the ith row are related by (3.57)

$$h_a h_b \chi_a^{(i)} \chi_b^{(i)} = l_i \sum_{c=1}^{k} c_{abc} h_c \chi_c^{(i)}\ . \tag{3.73}$$

3.8.2 Example of character table construction

We consider the covering group of the equilateral triangle D_3 for which we have three classes and three irreps. Therefore the character table has the form

D_3	C_1	$3C_2$	$2C_3$
$\Gamma^{(1)}$	1	1	1
$\Gamma^{(2)}$	1	α	β
$\Gamma^{(3)}$	2	γ	δ ,

(3.74)

where the complex constants α, β, γ and δ are to be determined. Below we will assume them to be real, which will be justified by the end result.

• We apply rule 3 to rows and obtain

$$\begin{aligned}
&\text{rows 1 \& 2}: \quad 1+3\alpha+2\beta=0\,,\\
&\text{rows 1 \& 3}: \quad 2+3\gamma+2\delta=0\,,\\
&\text{rows 2 \& 3}: \quad 2+3\alpha\gamma+2\beta\delta=0\,,\\
&\text{row 2}: \quad 1+3\alpha^2+2\beta^2=6\,,\\
&\text{row 3}: \quad 4+3\gamma^2+2\delta^2=6\,.
\end{aligned} \tag{3.75}$$

From the 1st and the 4th eqs. we obtain that $(\alpha,\beta)=(-1,1)$ or $(\frac{3}{5},-\frac{7}{5})$. Similarly from the 2nd and the 5th eqs. we obtain that $(\gamma,\delta)=(0,-1)$ or $(-\frac{4}{5},\frac{1}{5})$. Out of the four in total possible solutions, the 3rd eq. selects $(\alpha,\beta,\gamma,\delta)=(-1,1,0,-1)$ or $(\frac{3}{5},-\frac{7}{5},-\frac{4}{5},\frac{1}{5})$.

• We apply rule 4 to columns and obtain

$$\begin{aligned}
&\text{columns 1 \& 2}: \quad 1+\alpha+2\gamma=0\,,\\
&\text{columns 1 \& 3}: \quad 1+\beta+2\delta=0\,,\\
&\text{columns 2 \& 3}: \quad 1+\alpha\beta+\gamma\delta=0\,,\\
&\text{column 2}: \quad 1+\alpha^2+\gamma^2=2\,,\\
&\text{column 3}: \quad 1+\beta^2+\delta^2=3\,,
\end{aligned} \tag{3.76}$$

a system which is satisfied by both of the above two solutions.

• Finally we apply rule 5 for $a=b=2$, $i=2$ and using (2.49) we obtain $\beta=1$ which singles out the first of our solutions. The rest of the eqs. arising for different values of a,b and i are then trivially satisfied.

Hence the resulting character table is

D_3	C_1	$3C_2$	$2C_3$
$\Gamma^{(1)}$	1	1	1
$\Gamma^{(2)}$	1	-1	1
$\Gamma^{(3)}$	2	0	-1 .

(3.77)

Note: In the above example all the entries of the character table turned out to be not only real but also integer numbers. I emphasize that generically this is not the case.

Exercise: Construct the character table for the group D_4.

4 Direct product groups

It often happens that the complete symmetry group of a system can be broken into two or more types such that all the operators of one type commute with all the operators of any other type. When this occurs then we can simplify our work by introducing the concept of the *direct product groups*.

If we have two groups $A = \{A_1, A_2, \ldots, A_h\}$ and $B = \{B_1, B_2, \ldots, B_{h'}\}$, such that all elements of A commute with all elements of B and that the only element in common is the identity, then we associate to them the direct product group

$$A \times B = \{A_m B_n\ , \quad m = 1, 2, \ldots, h\ , \quad n = 1, 2, \ldots, h'\}\ , \tag{4.1}$$

that is a group whose elements are all possible products of the elements of A with those of B and has order hh'. Clearly the product of two elements of the direct product group preserves closeness.

In order to proceed further we define below the direct product of matrices and present its essential properties.

4.1 The direct (or Kronecker) product of matrices

Let A and B be $m \times n$ and $p \times q$ matrices, respectively. Then, their direct product is an $(mp) \times (nq)$ matrix given by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1q} & \cdots & \cdots & a_{1n}b_{11} & a_{1n}b_{12} & \cdots & a_{1n}b_{1q} \\ a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{11}b_{2q} & \cdots & \cdots & a_{1n}b_{21} & a_{1n}b_{22} & \cdots & a_{1n}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{p1} & a_{11}b_{p2} & \cdots & a_{11}b_{pq} & \cdots & \cdots & a_{1n}b_{p1} & a_{1n}b_{p2} & \cdots & a_{1n}b_{pq} \\ \vdots & \vdots & & \vdots & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1q} & \cdots & \cdots & a_{mn}b_{11} & a_{mn}b_{12} & \cdots & a_{mn}b_{1q} \\ a_{m1}b_{21} & a_{m1}b_{22} & \cdots & a_{m1}b_{2q} & \cdots & \cdots & a_{mn}b_{21} & a_{mn}b_{22} & \cdots & a_{mn}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{p1} & a_{m1}b_{p2} & \cdots & a_{m1}b_{pq} & \cdots & \cdots & a_{mn}b_{p1} & a_{mn}b_{p2} & \cdots & a_{mn}b_{pq} \end{pmatrix} \quad (4.2)$$

A useful way to represent the elements of the matrix $A \otimes B$ is with a double set of indices

$$(A \otimes B)_{ia,jb} = A_{ij}B_{ab} \ . \quad (4.3)$$

Note that

$$\text{number of rows :} \quad \sum_{i=1}^{m}\sum_{a=1}^{p} \cdot 1 = mp \ ,$$

$$\text{number of columns :} \quad \sum_{j=1}^{n}\sum_{b=1}^{q} \cdot 1 = nq \ . \quad (4.4)$$

Remarks:

• The direct product is bilinear and associative

$$\begin{aligned} A\otimes(B+C) &= A\otimes B + A\otimes C\,,\\ (A+B)\otimes C &= A\otimes C + B\otimes C\,,\\ (kA)\otimes B &= A\otimes(kB) = k(A\otimes B)\,,\\ (A\otimes B)\otimes C &= A\otimes(B\otimes C)\,, \end{aligned} \tag{4.5}$$

where A, B and C are matrices and k is a number.

Proof: The proof of the first three is trivial. To prove the last one we write

$$\begin{aligned} ((A\otimes B)\otimes C)_{ijk,mnl} &= (A\otimes B)_{ij,mn}C_{kl} = A_{im}B_{jn}C_{kl}\,,\\ (A\otimes(B\otimes C))_{ijk,mnl} &= A_{im}(B\otimes C)_{jk,nl} = A_{im}B_{jn}C_{kl}\,. \end{aligned} \tag{4.6}$$

• The direct product is not commutative as in general, $A\otimes B \neq B\otimes A$.

• If A, B, C and D are matrices of such size that one can form the matrix products AC and BD, then

$$(A\otimes B)(C\otimes D) = AC\otimes BD\,, \tag{4.7}$$

assuming that the elements B_{jn} and C_{mk} commute, i.e. $[B_{jn}, C_{mk}] = 0$. This is called the *mixed-product property*, because it mixes the ordinary matrix product and the direct product.

Proof: Note that

$$\begin{aligned} ((A\otimes B)(C\otimes D))_{ij,kl} &= \sum_{m,n}(A\otimes B)_{ij,mn}(C\otimes D)_{mn,kl}\\ &= \sum_{m,n} A_{im}B_{jn}C_{mk}D_{nl} = (AC)_{ik}(BD)_{jl} = ((AC)\otimes(BD))_{ij,kl}\,. \end{aligned} \tag{4.8}$$

If $[B_{jn}, C_{mk}] \neq 0$ then there is an additional contribution (**Exercise**).

• Note the property

$$(A\oplus B)\otimes C = A\otimes C\oplus B\otimes C\,,\qquad A\otimes(B\oplus C) = A\otimes B\oplus A\otimes C\,. \tag{4.9}$$

• The matrix $A\otimes B$ is invertible if and only if A and B are invertible, in which case the inverse is given by

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} . \tag{4.10}$$

• The operation of transposition acts as

$$(A \otimes B)^T = A^T \otimes B^T . \tag{4.11}$$

• Suppose that A and B are square matrices of size n and q, respectively. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A and $\mu_1, \dots, \mu_q$ be those of B. Then the eigenvalues of $A \otimes B$ are

$$\lambda_i \mu_j , \qquad i = 1, \dots, n , \quad j = 1, \dots, q . \tag{4.12}$$

Proof: Let ψ_1 and ψ_2 eigenvectors of A and B respectively with corresponding eigenvalues λ_i and μ_j. Defining $\Psi = \Psi_1 \otimes \Psi_2$ we have that

$$(A \otimes B)\Psi = (A \otimes B)(\Psi_1 \otimes \Psi_2) = (A\Psi_1) \otimes (B\Psi_2) = \lambda_i \mu_j \Psi . \tag{4.13}$$

• The trace and determinant of a direct product are given by (**Exercise**)

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}A \, \mathrm{Tr}B , \qquad \det(A \otimes B) = (\det A)^q (\det B)^n . \tag{4.14}$$

4.2 Direct product representations of different groups

The direct product of the rep matrices of the irreps of the groups A and B form irreps of the direct product group.

Proof: By definition we have that

$$\Gamma^{(A\times B)}(A_k B_l) = \Gamma^{(A)}(A_k) \otimes \Gamma^{(B)}(B_l) , \tag{4.15}$$

from which

$$\begin{aligned} \Gamma^{(A\times B)}(A_k B_l)\Gamma^{(A\times B)}(A_{k'} B_{l'}) &= \left(\Gamma^{(A)}(A_k) \otimes \Gamma^{(B)}(B_l)\right) \left(\Gamma^{(A)}(A_{k'}) \otimes \Gamma^{(B)}(B_{l'})\right) \\ = \Gamma^{(A)}(A_k A_{k'}) \otimes \Gamma^{(B)}(A_l A_{l'}) &= \Gamma^{(A\times B)}(A_k A_{k'} B_l B_{l'}) , \end{aligned} \tag{4.16}$$

where in the second line we 've used (4.7).

All of the irreps of the direct product groups can be found by taking direct products of the irreps of the individual groups.

Proof: Let l_i and l'_i be the dimensionalities of the irreps of the groups A and B, satisfying $\sum\limits_{i=1} l_i^2 = h$ and $\sum\limits_{i=1} l_i'^2 = h'$. The dimensionalities of the direct product matrix will be $l_{ij} = l_i l'_j$. Therefore we have that $\sum_{i,j} l_{ij}^2 = \cdots = hh'$, which is the order of $A \times B$. Hence, there cannot be any other irreps in addition to those expressed as a direct product.

The number of conjugate classes of the direct product group equals the product of classes of the component groups.

The character of any direct product rep is the product of the characters of the component reps.

Proof: We have that

$$\begin{aligned}\chi^{(A\times B)}(A_k B_l) &= \sum_{i,j} \Gamma^{(A\times B)}(A_k B_l)_{ij,ij} = \sum_{i,j} \Gamma^{(A)}(A_k)_{ii} \Gamma^{(B)}(B_l)_{jj} \\ &= \chi^{(A)}(A_k)\chi^{(B)}(B_l) \,. \end{aligned} \tag{4.17}$$

4.2.1 Example of character table for direct product representations

Consider the group D_{3h} which is the direct product of the covering group of the equilateral triangle D_3 with multiplication table in (2.6) with the group $\mathcal{H} = \{E, \sigma_h\}$ of reflections in the plane of the triangle whose multiplication table is

$\mathcal{H}$	E	σ_h
E	E	σ_h
σ_h	σ_h	E .

(4.18)

This group is Abelian, has two classes $\{E\}$ and $\{\sigma_h\}$ and two 1-dim irreps.

$\mathcal{H}$	E	σ_h
Γ^+	1	1
Γ^-	1	-1 .

(4.19)

The character table of the direct product $D_{3h} = \mathcal{H} \times D_3$ is

D_{3h}	E	$\{A,B,C\}$	$\{D,F\}$	σ_h	$\sigma_h\{A,B,C\}$	$\sigma_h\{D,F\}$
$\Gamma^{(1+)}$	1	1	1	1	1	1
$\Gamma^{(2+)}$	1	−1	1	1	−1	1
$\Gamma^{(3+)}$	2	0	−1	2	0	−1
$\Gamma^{(1-)}$	1	1	1	−1	−1	−1
$\Gamma^{(2-)}$	1	−1	1	−1	1	−1
$\Gamma^{(3-)}$	2	0	−1	−2	0	1 .

(4.20)

It can be readily verified that the above character table for D_{3h} satisfies all rules governing character tables.

4.3 Direct product representations within the same group

In many applications we take the direct product of irreps within the same group. Let $\Gamma^{(\alpha)}(R)$ and $\Gamma^{(\beta)}(R)$ be two such irreps. Then, as in (4.15), the direct product of the irrep matrices

$$\Gamma^{(\alpha)}(R) \otimes \Gamma^{(\beta)}(R) \equiv \Gamma^{(\alpha\times\beta)}(R) , \tag{4.21}$$

is still a rep with character given by the product of the characters of the two irreps.

An importance difference with the case we encountered before where the irreps were for different groups, is that now the new rep is in general reducible. We present this by writing the decomposition

$$\Gamma^{(\alpha)}(R) \otimes \Gamma^{(\beta)}(R) = \sum_{\gamma}{}^{\oplus} a_{\alpha\beta\gamma}\Gamma^{(\gamma)}(R) , \tag{4.22}$$

where the notation means that if with a similarity transformation we bring the l.h.s. into a block diagonal form, then the rep $\Gamma^{(\gamma)}(R)$ appears $a_{\alpha\beta\gamma}$ times in the r.h.s. The coefficients $a_{\alpha\beta\gamma}$ are called *Clebsch–Gordan coefficients* and can be computed by first tracing (4.22). We find that

$$\chi^{(\alpha\times\beta)}(R) = \chi^{(\alpha)}(R)\chi^{(\beta)}(R) = \sum_{\gamma} a_{\alpha\beta\gamma}\chi^{(\gamma)}(R) , \tag{4.23}$$

which clearly shows that they are symmetric in the first two indices. Then using the character orthogonality theorem (3.44) we obtain that

$$a_{\alpha\beta\gamma} = \frac{1}{h}\sum_{R\in G}\chi^{(\alpha)}(R)\chi^{(\beta)}(R)\chi^{(\gamma)}(R)^* = \frac{1}{h}\sum_{a=1}^{k} h_a \chi_a^{(\alpha)}\chi_a^{(\beta)}\chi_a^{(\gamma)*} \ . \tag{4.24}$$

If the characters are real, then $a_{\alpha\beta\gamma}$ is independent of the order of the subscripts. Note that the decomposition of an irrep and its conjugate rep contains the identity rep only once. Indeed,

$$\alpha_{\alpha\alpha^* E} = \frac{1}{h}\sum_{a=1}^{k} h_a \chi_a^{(\alpha)} \underbrace{\chi_a^{(\alpha^*)}}_{\chi_a^{(\alpha)*}} \underbrace{\chi_a^{(E)*}}_{1} = \frac{1}{h}\sum_{a=1}^{k} h_a \chi_a^{(\alpha)}\chi_a^{(\alpha)*} = 1 \ , \tag{4.25}$$

where in the last step I used (3.72).

<u>Exercise</u>: Compute the decomposition of the direct product of the irreps for the covering group of the equilateral triangle D_3 and show that

$$\begin{aligned} &\Gamma^{(1)}\otimes\Gamma^{(i)} = \Gamma^{(i)} \ , \qquad i = 1,2,3 \ , \\ &\Gamma^{(2)}\otimes\Gamma^{(2)} = \Gamma^{(1)} \ , \qquad \Gamma^{(2)}\otimes\Gamma^{(3)} = \Gamma^{(3)} \ , \\ &\Gamma^{(3)}\otimes\Gamma^{(3)} = \Gamma^{(1)}\oplus\Gamma^{(2)}\oplus\Gamma^{(3)} \ . \end{aligned} \tag{4.26}$$

5 Applications of representation theory

In most applications of rep theory the group of interest is that of the symmetry operations that leave invariant a set of operators $\{T_i\}$, $i = 1, 2, \ldots$. A notable example of such an operator is the Hamiltonian of a quantum system. In what follows, we will restrict our discussion to the case that these operations are rotations, reflections or inversions. Nevertheless, the techniques and the results obtained will be applicable to other kinds of symmetry operations.

5.1 Transformation operators

Denoting by $\mathbf{x}$ the position of a point in an d-dimensional space w.r.t. a fixed coordinate system and by $\mathbf{x}'$ the position of the same point in the transformed system w.r.t. the same coordinate system we have a relation of the form

$$\mathbf{x}' = R\mathbf{x} , \tag{5.1}$$

where R is a real, orthogonal $d \times d$ matrix. The inverse transformation is $\mathbf{x} = R^{-1}\mathbf{x}' = R^T\mathbf{x}'$. Obviously, the set of matrices R form a group, also to be denoted for convenience by R, with the conventional matrix multiplication as the composition law.

We would like to act on functions of $\mathbf{x}$. For that we introduce a new group with elements P_R defined operationally on functions as

$$P_R f(\mathbf{x}) = f(R^{-1}\mathbf{x}) . \tag{5.2}$$

This group is isomorphic to R since

$$P_R P_S f(\mathbf{x}) = P_R f(S^{-1}\mathbf{x}) = f(S^{-1}R^{-1}\mathbf{x}) = f((RS)^{-1}\mathbf{x}) = P_{RS} f(\mathbf{x}) . \tag{5.3}$$

Note that $P_R^{-1} = P_{R^{-1}}$ and $P_I = I$.

Assume that a system contains an operator H with eigenfunctions and eigenvalues $\Psi^{(n)}$ and E_n, respectively. If a transformation leaves it invariant, i.e,

$$[H, P_R] = 0 \qquad \Longleftrightarrow \qquad P_R H P_R^{-1} = H , \tag{5.4}$$

then it is said that R is a symmetry group of H. In addition, any function $\tilde{\Psi}^{(n)}(\mathbf{x}) = P_R\Psi^{(n)}(\mathbf{x}) = \Psi^{(n)}(R^{-1}\mathbf{x})$ will be an eigenfunction of H with the same eigenvalue, i.e.

$$\begin{aligned} H\tilde{\Psi}^{(n)} &= HP_R\Psi^{(n)}(\mathbf{x}) = P_R H\Psi^{(n)}(\mathbf{x}) = E_n P_R\Psi^{(n)}(\mathbf{x}) \\ &= E_n\tilde{\Psi}^{(n)} \,. \end{aligned} \tag{5.5}$$

Remarks:

• There is a *degeneracy* in the spectrum of H associated with the symmetry group.

• By acting on a representative function $\Psi_n(\mathbf{x})$ with all operators P_R we generate all degenerate eigenfunctions. This degeneracy is called *normal*.

• If there are more degenerate functions that cannot be obtained this way, then we have a case of *hidden degeneracy*. In many instances the explanation of this is that there is more symmetry called *hidden symmetry*.

5.2 Constructing representations

Let a set of l_n degenerate eigenfunctions $\{\Psi_\nu^{(n)}\}$, $\nu = 1, 2, \dots, l_n$ of an operator H with eigenvalue E_n. Excluding accidental degeneracy we have that the action of P_R on a member of this set should be a linear combination of members of the same degenerate set of eigenfunctions. Mathematically this means that

$$P_R\Psi_\nu^{(n)}(\mathbf{x}) = \Psi_\nu^{(n)}(R^{-1}\mathbf{x}) = \sum_{\kappa=1}^{l_n} \Gamma^{(n)}(R)_{\kappa\nu}\Psi_\kappa^{(n)} \,. \tag{5.6}$$

The l_n-dim matrices obtained in this way provide the l_n-dim rep of R.
<u>**Proof**</u>**:** We have that

$$\begin{aligned} P_R P_S\Psi_\nu^{(n)} &= \sum_{\kappa=1}^{l_n} \Gamma^{(n)}(S)_{\kappa\nu} P_R\Psi_\kappa^{(n)} \\ &= \sum_{\kappa,\lambda=1}^{l_n} \Gamma^{(n)}(S)_{\kappa\nu}\Gamma^{(n)}(R)_{\lambda\kappa}\Psi_\lambda^{(n)} \\ &= \sum_{\lambda=1}^{l_n} \left[\Gamma^{(n)}(R)\Gamma^{(n)}(S)\right]_{\lambda\nu} \Psi_\lambda^{(n)} \,. \end{aligned} \tag{5.7}$$

However the l.h.s also equals

$$P_R P_S \Psi_\nu^{(n)} = P_{RS}\Psi_\nu^{(n)} = \sum_{\lambda=1}^{l_n} \Gamma^{(n)}(RS)_{\lambda\nu}\Psi_\lambda^{(n)} . \tag{5.8}$$

Therefore since the $\Psi_\nu^{(n)}$'s form a basis

$$\Gamma^{(n)}(R)\Gamma^{(n)}(S) = \Gamma^{(n)}(RS) , \tag{5.9}$$

thus proving the assertion.

Definition: We will denote the inner product of two functions Ψ and Φ by

$$\langle\Psi|\Phi\rangle \equiv \int d[\mathbf{x}]\Psi^*(\mathbf{x})\Phi(\mathbf{x}) , \tag{5.10}$$

where $d[\mathbf{x}]$ denotes the measure of integration. Also if T is an operator

$$\langle\Psi|T|\Phi\rangle \equiv \int d[\mathbf{x}]\Psi^*(\mathbf{x})T\Phi(\mathbf{x}) . \tag{5.11}$$

If R is a symmetry operation, the following is an important identity

$$\langle\Psi|\Phi\rangle = \langle P_R\Psi|P_R\Phi\rangle . \tag{5.12}$$

Proof: By changing coordinates as $\mathbf{x} = R^{-1}\mathbf{x}'$ we have that

$$\begin{aligned} \langle\Psi|\Phi\rangle &= \int d[\mathbf{x}']J[\frac{\partial x}{\partial x'}]\Psi^*(R^{-1}\mathbf{x}')\Phi(R^{-1}\mathbf{x}') , \quad [\text{but } J[\frac{\partial x}{\partial x'}] = 1, \text{ since } |\det(R)| = 1] \\ &= \int d[\mathbf{x}'] \left[P_R\Psi(\mathbf{x}')\right]^* P_R\Phi_(\mathbf{x}') \\ &= \langle P_R\Psi|P_R\Phi\rangle . \end{aligned} \tag{5.13}$$

If the basis eigenfunctions are orthonormal, i.e.

$$\langle\Psi_\kappa|\Psi_\nu\rangle = \int d[\mathbf{x}]\Psi_\kappa^*(\mathbf{x})\Psi_\nu(\mathbf{x}) = \delta_{\kappa\nu} , \tag{5.14}$$

then the rep $\Gamma^{(n)}(R)$ is unitary.

Proof: Using (5.12), the l.h.s. of the above orthonormality condition can be written as

$$\begin{aligned}
&\sum_{\kappa',\nu'} \int d[\mathbf{x}] \, (\Gamma_{\kappa'\kappa}(R)\Psi_{\kappa'}(\mathbf{x}))^* \, (\Gamma_{\nu'\nu}(R)\Psi_{\nu'}(\mathbf{x})) \\
&\quad = \sum_{\kappa',\nu'} \Gamma^*_{\kappa'\kappa}(R)\Gamma_{\nu'\nu}(R)\delta_{\kappa'\nu'} \\
&\quad = \sum_{k'} \Gamma^*_{\kappa'\kappa}(R)\Gamma_{\kappa'\nu}(R) = \sum_{k'} \Gamma^\dagger_{\kappa\kappa'}(R)\Gamma_{\kappa'\nu}(R) \qquad (5.15) \\
&\quad = (\Gamma^\dagger(R)\Gamma(R))_{\kappa\nu} \, .
\end{aligned}$$

Hence $\Gamma^\dagger(R)\Gamma(R) = I, \forall \, R \in G$.

Remarks:

• The reps arising in this way are irrreducible. Only in special circumstances where an accidental symmetry occurs there are more degenerate eigenfunctions and one gets reducible reps. These however could be irreducible w.r.t. a larger symmetry group.

• What would we have if a different basis set instead of Ψ_μ is used? How this would affect the rep? Using a matrix-vector notation to hide indices, if the two basis are related as

$$\Psi' = S^T\Psi \qquad \Longleftrightarrow \qquad \Psi = (S^{-1})^T\Psi' \, , \qquad (5.16)$$

we have that

$$P_R\Psi' = S^T P_R\Psi = S^T\Gamma^T(R)\Psi = S^T\Gamma^T(R)(S^{-1})^T\Psi' = \Gamma^{'T}(R)\Psi' \, , \qquad (5.17)$$

where $\Gamma^{'}(R) = S^{-1}\Gamma(R)S$. Hence the two reps are equivalent by a similarity transformation.

• The dimensionalities of the irreps of the symmetry group that commutes with H give the *degree of degeneracy* for each eigenvalue (barring accidental degeneracies).

• If one modifies the operator H by adding a new term, then this degeneracy can be lifted only if the symmetry group is reduced.

5.3 Basis functions for irreducible representations

Using (5.6) one may construct the irrep matrices if the set of basis functions is provided. Here we address the reverse issue, namely, if we know the irrep matrices how do we construct the basis functions? Starting from (5.6), then multiplying it by $\Gamma^{(m)}_{\kappa'\nu'}(R)^*$ and summing over all $R \in G$ we obtain that

$$\sum_{R\in G} \Gamma^{(m)}_{\kappa'\nu'}(R)^* P_R \Psi^{(m)}_\nu(\mathbf{x}) = \sum_{R\in G}\sum_{k=1}^{l_n} \Gamma^{(n)}_{k\nu}(R)\Gamma^{(m)}_{\kappa'\nu'}(R)^*\Psi^{(n)}_k(\mathbf{x}) = \frac{h}{l_m}\delta_{mn}\delta_{\nu\nu'}\Psi^{(n)}_{\kappa'}(\mathbf{x}) \ , \tag{5.18}$$

where we have made use on the great orthogonality theorem for reps (3.26). Defining

$$\mathcal{P}^{(i)}_{\mu\nu} = \frac{l_i}{h}\sum_{R\in G}\Gamma^{(i)}_{\mu\nu}(R)^* P_R \ , \tag{5.19}$$

we may write the above as

$$\mathcal{P}^{(i)}_{\mu\nu}\Psi^{(j)}_\lambda(\mathbf{x}) = \delta_{ij}\delta_{\nu\lambda}\Psi^{(j)}_\mu(\mathbf{x}) \ . \tag{5.20}$$

Hence, when we act on a given basis function $\Psi^{(j)}_\lambda(\mathbf{x})$ with (5.19) the result is non-zero if they belong to the same rep and moreover the basis function belongs to the νth row of the rep.

Remarks:

• A practical way to construct all the partners of the given basis function $\Psi^{(i)}_\nu(\mathbf{x})$ is to compute

$$\mathcal{P}^{(i)}_{\mu\nu}\Psi^{(i)}_\nu(\mathbf{x}) = \Psi^{(i)}_\mu(\mathbf{x}) \ , \qquad \mu = 1, 2, \ldots, l_i \ . \tag{5.21}$$

• Clearly if $\Gamma^{(i)}$ are all the distinct irreps of the group G with dimensionalities l_i, then any function in the space operated on by P_R can be decomposed as

$$\Psi(\mathbf{x}) = \sum_{i=1}^{n}\sum_{\mu=1}^{l_i}\Psi^{(i)}_\mu(\mathbf{x}) \ . \tag{5.22}$$

Any member of this decomposition can be obtain from

$$\Psi^{(i)}_\mu(\mathbf{x}) = \mathcal{P}^{(i)}_{\mu\mu}\Psi(\mathbf{x}) \ . \tag{5.23}$$

• If we do not have all the matrices of the ith rep then we may still obtain some less detailed information from the corresponding characters. We define

$$P^{(i)} \equiv \sum_{\mu=1}^{l_i} \mathcal{P}^{(i)}_{\mu\mu} = \frac{l_i}{h} \sum_{R\in G} \chi^{(i)}(R)^* P_R \ . \tag{5.24}$$

Then from (5.23) we obtain

$$P^{(i)}\Psi(\mathbf{x}) = \sum_{\mu=1}^{l_i} \Psi^{(i)}_\mu(\mathbf{x}) \equiv \Psi^{(i)}(\mathbf{x}) \ , \tag{5.25}$$

that is $P^{(i)}$ projects out of any function the part that belongs to the ith irrep.

• The inner product of two functions $\Psi^{(i)}_\mu$ and $\Psi^{(j)}_\nu$ that belong to the μ and ν rows of the irreps $\Gamma^{(i)}$ and $\Gamma^{(j)}$, respectively, satisfy

$$\langle \Psi^{(i)}_\mu | \Psi^{(j)}_\nu \rangle = \frac{1}{l_i} \delta_{ij}\delta_{\mu\nu} \sum_{\lambda=1}^{l_i} \langle \Psi^{(i)}_\lambda | \Psi^{(j)}_\lambda \rangle \ . \tag{5.26}$$

Proof: Using (5.12) we immediately have that

$$\langle \Psi^{(i)}_\mu | \Psi^{(j)}_\nu \rangle = \sum_{\mu',\nu'} \Gamma^{(i)}_{\mu'\mu}(R)^* \Gamma^{(j)}_{\nu'\nu}(R) \langle \Psi^{(i)}_{\mu'} | \Psi^{(j)}_{\nu'} \rangle \ . \tag{5.27}$$

The l.h.s. should be independent of the group element R. Hence, by summing over all $R \in G$ we just get a factor of h multiplying the l.h.s., whereas for the r.h.s. we may use the great orthogonality theorem (3.26). Then the result follows immediately.

Note that $\langle \Psi^{(i)}_\mu | \Psi^{(i)}_\mu \rangle$ is independent of the row the two functions belong to.

5.3.1 An example

We will use the previous analysis to assign the function xz and into the appropriate irrep of D_3. Recall that for that group there are two 1-dim irreps and one 2-dim irrep. Hence, if this function belongs to the latter irrep we have to find its partner as well. The best way to proceed is to see how the components of the 3-dim vector $\mathbf{x} = (x, y, z)$

transform. We easily identify the set of 3-dim transformation matrices R as

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
$$B = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{5.28}$$
$$D = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad F = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which form a reducible rep as it is clearly an extension of the irrep (2.7).[11] Using (5.2) we find that

$$P_E\mathbf{x} = E\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad P_A\mathbf{x} = A\mathbf{x} = \begin{pmatrix} x \\ -y \\ -z \end{pmatrix},$$
$$P_B\mathbf{x} = B\mathbf{x} = \begin{pmatrix} -\frac{1}{2}x + \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x + \frac{1}{2}y \\ -z \end{pmatrix}, \qquad P_C\mathbf{x} = C\mathbf{x} = \begin{pmatrix} -\frac{1}{2}x - \frac{\sqrt{3}}{2}y \\ -\frac{\sqrt{3}}{2}x + \frac{1}{2}y \\ -z \end{pmatrix}, \tag{5.29}$$
$$P_D\mathbf{x} = F\mathbf{x} = \begin{pmatrix} -\frac{1}{2}x - \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x - \frac{1}{2}y \\ z \end{pmatrix}, \qquad P_F\mathbf{x} = D\mathbf{x} = \begin{pmatrix} -\frac{1}{2}x + \frac{\sqrt{3}}{2}y \\ -\frac{\sqrt{3}}{2}x - \frac{y}{2} \\ z \end{pmatrix}.$$

From the above one easily finds the action of the P_R's on x, y and z. For example, $P_D x = -\frac{1}{2}(x + \sqrt{3}y)$ and $P_F y = -\frac{1}{2}(\sqrt{3}x + y)$. Hence, we know the action of the P_R's on any function of them. For the two 1-dim irreps of D_3 we simply have $\Gamma^{(1)}(R) = 1$, $\forall\, R$ and $\Gamma^{(2)}(E, D, F) = 1$ and $\Gamma^{(2)}(A, B, C) = -1$. For the 2-dim irrep the matrices are given by (2.7). Using those and (5.19) we obtain that

$$\mathcal{P}^{(1)} = \frac{1}{6}(P_E + P_D + P_F + P_A + P_B + P_C)\,, \tag{5.30}$$

[11]The matrices E, D, F are extended by just adding unity to the 3rd column and row, the reason being that under these operations z remains inert. However, for A, B, C we have added minus unity precisely because under a rotation by π about the corresponding axes, as depicted in Fig. 1, $z \to -z$.

that

$$\mathcal{P}^{(2)} = \frac{1}{6}(P_E + P_D + P_F - P_A - P_B - P_C) \tag{5.31}$$

and that

$$\mathcal{P}^{(3)} = \frac{1}{3}\begin{pmatrix} P_E + P_A - \frac{1}{2}(P_B + P_C + P_D + P_F) & \frac{\sqrt{3}}{2}(P_B - P_C + P_D - P_F) \\ \frac{\sqrt{3}}{2}(P_B - P_C - P_D + P_F) & P_E - P_A + \frac{1}{2}(P_B + P_C - P_D - P_F) \end{pmatrix} . \tag{5.32}$$

Letting $\Psi(\mathbf{x}) = xz$ and using (5.23) we may project to the various irreps and to the corresponding rows. We easily compute that $\mathcal{P}^{(1)}\Psi(\mathbf{x}) = \mathcal{P}^{(2)}\Psi(\mathbf{x}) = 0$, so that xz has no component in anyone of the 1-dim irreps. Similarly we compute that $\mathcal{P}^{(3)}_{11}\Psi(\mathbf{x}) = 0$, so that xz does not belong to the first row of the 2-dim irrep. Hence, it has to belong to the second row of the 2-dim irrep. Indeed, we easily compute that $\mathcal{P}^{(3)}_{22}\Psi(\mathbf{x}) = \Psi(\mathbf{x})$. Hence we may write that

$$\Psi^{(3)}_2 = xz \, . \tag{5.33}$$

To compute its partner in this irrep we use the off diagonal element of $\mathcal{P}^{(3)}$ in (5.21) as

$$\Psi^{(3)}_1 = \mathcal{P}^{(3)}_{12}\Psi^{(3)}_2 = \cdots = -yz \, . \tag{5.34}$$

Using the spherical coordinates

$$\begin{aligned} &(x, y, z) = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \, , \\ &r \geqslant 0 \, , \quad 0 \leqslant \theta \leqslant \pi \, , \quad 0 \leqslant \phi \leqslant 2\pi \, , \end{aligned} \tag{5.35}$$

we may easily verify the orthonormality condition (5.14) with integration measure the solid angle $d\Omega = d\theta d\phi \sin\theta$, provided that we multiply the two functions by $\sqrt{15/(4\pi)}$ (**Exercise**).[12]

Exercise: Show that the functions $\frac{1}{2}(y^2 - x^2)$ and xy can be identified as basis functions for the 2-dim irrep of D_3.

Exercise: Show that the functions $x^2 + y^2$ and z^2 belong to the $\Gamma^{(1)}$ irrep of D_3 and that the function z belong to the $\Gamma^{(2)}$ irrep of the same group.

Exercise: Decompose the function $x^2 z$ into irreps of the group D_3.

[12]In principle one should include the integral over r, since in spherical coordinates the measure of integration is $d[\mathbf{x}] = drd\theta d\phi r^2 \sin\theta$. However, since we are dealing with monomials of the same degree the integral over r always gives the same factor and thus it can be consistently omitted.

Remark: The character table can be slightly enlarged to include information on simple functions forming a basis for the various irreps. For instance, given the above findings for D_3 the character table in (3.77) can be enlarged to

	D_3	C_1	$3C_2$	$2C_3$
(x^2+y^2, z^2)	$\Gamma^{(1)}$	1	1	1
z	$\Gamma^{(2)}$	1	-1	1
$(x,y),\ (yz,xz),\ (y^2-x^2,xy)$	$\Gamma^{(3)}$	2	0	-1 .

(5.36)

Note that any normalization factors have been omitted in order to simplify the writing.

5.4 Representations of Abelian groups

Since in an Abelian group every element is a class by itself and since the number of distinct irreps equals the number of classes, the number of distinct irreps equals the order of the group h. Then (3.62) immediately gives that $l_i = 1,\ \forall\ i = 1,2,\dots,$ hence all the irreps of an Abelian group are 1-dim.

An important consequence is that no degeneracies can occur if the symmetry group of an operator H is Abelian.

5.4.1 Cyclic groups

In cyclic groups all elements are of the form $X_k = X^k$, $k = 1,2,\dots,h$, with $X^h = E$. Then $\Gamma(X_k) = (\Gamma(X))^k$. Since $\Gamma(X)^h = I$, we have that $\Gamma(X) = e^{2\pi i p/h}$, $p = 1,2,\dots,h$. Thus we have that the irreps of a cyclic group are of the form

$$\Gamma^{(p)}(X_k) = (\Gamma^{(p)}(X))^k = e^{2\pi i pk/h}\ , \qquad p = 1,2,\dots,h\ , \quad k = 1,2,\dots,h\ . \tag{5.37}$$

5.4.2 Bloch's theorem

Assume shifts in a 1-dim space with the coordinate $x \in [0, ha]$, where $h = 1,2,\dots$ and $a \in \mathbb{R}$, generated by A as $Ax = x - a$, with inverse $A^{-1}x = x + a$. One may think of it an a 1-dim crystal with equidistantly spaced atoms.

Figure 3: 1-dim crystal of length $L = ha$.

Symmetry generated by A : $Ax = x - a$,

$$P_A \Psi(x) = \Psi(A^{-1}x) = \Psi(x + a) \ . \qquad (5.38)$$

Assume that we have an operator H whose symmetry group is generated by A and that its eigenfunctions are periodic with period $L = ha$. Hence, they should belong to irreps of the cyclic group of order h and satisfy the property

$$P_A \Psi^{(p)}(x) = \Psi^{(p)}(x + a) = e^{2\pi i p/h} \Psi^{(p)}(x) = e^{ika} \Psi^{(p)}(x) \ , \qquad (5.39)$$

where $k = 2\pi p/L$. Since there is a one to one correspondence between p and k we may relabel $\Psi_p \to \Psi_k$ and have

$$\Psi_k(x + a) = e^{ika} \Psi_k(x) \ . \qquad (5.40)$$

Obviously, any function satisfying this condition is of the form

$$\Psi_k(x) = e^{ikx} u_k(x) \ , \qquad u(x + a) = u(x) \ . \qquad (5.41)$$

These results for the eigenfunctions do not depend on the details of the operator H. Of course play their role in the construction of the periodic function $u(x)$.

The extension to an d-dim space is trivial. In that case we have the vectors $\mathbf{a}$ and $\mathbf{k}$. Then a function

$$\Psi_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} u_{\mathbf{k}}(\mathbf{x}) \ , \qquad u(\mathbf{x} + \mathbf{a}) = u(\mathbf{x}) \ . \qquad (5.42)$$

In quantum mechanics the above is referred to as the Bloch's theorem for Hamiltonians corresponding to periodic potentials one encounters in examining 3-dim solids.

5.4.3 The 2-dim rotation group $SO(2)$

Consider an d-dimensional space and the group of rotations about an axis perpendicular to a plane, e.g. the z axis perpendicular to the $x - y$ plane, in a Cartesian coordinate system. The group is of infinite order since it corresponds to additions of successive rotations of arbitrary angles. The irreps are just numbers and satisfy

$\Gamma(\phi_1)\Gamma(\phi_2) = \Gamma(\phi_1 + \phi_2)$ with solution $\Gamma^{(m)}(\phi) = e^{-im\phi}$, for some number m. In many applications one requires periodicity over a full rotation, i.e. $\Gamma^{(m)}(\phi + 2\pi) = \Gamma^{(m)}(\phi)$. Then we necessarily have that $m \in \mathbb{Z}$. Instead, demanding anti-periodicity i.e. $\Gamma^{(m)}(\phi + 2\pi) = -\Gamma^{(m)}(\phi)$, requires that $m \in \mathbb{Z} + \frac{1}{2}$.

We define the operator P_{ϕ_0} by

$$P_{\phi_0}\Psi(\phi) = \Psi(\phi - \phi_0) \tag{5.43}$$

and assume that it belongs to the symmetry group of an operator H. The eigenfunctions of the latter must satisfy that

$$P_{\phi_0}\Psi^{(m)}(\phi) = \Gamma^{(m)}(\phi_0)\Psi^{(m)}(\phi) \qquad \Longrightarrow \qquad \Psi^{(m)}(\phi - \phi_0) = e^{-im\phi_0}\Psi^{(m)}(\phi) \ . \tag{5.44}$$

Any function satisfying the above requirements should be of the form

$$\Psi^{(m)}(\phi, \dots) = e^{im\phi} f_m(\dots) \ , \tag{5.45}$$

where the dots represent all coordinates but ϕ. The detailed form of the function f_m depends on the operator H.

Remarks:

• Note that the identities

$$\int_0^{2\pi} d\phi e^{i(m-m')\phi} = 2\pi\delta_{m\,m'} \ , \qquad \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = 2\pi\delta(\phi - \phi') \ , \tag{5.46}$$

can be interpreted as the character orthogonality and completeness relations (3.44) and (3.67), respectively, with "order" of the group $h = \int_0^{2\pi} d\phi = 2\pi$.

• The continuous group $SO(2)$ consists of matrices of the form

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \ , \qquad 0 < \phi \leqslant 2\pi \ . \tag{5.47}$$

A point with Cartesian coordinates (x, y) is rotated so that after the rotation has coordinates (x', y') related to the original ones as depicted below.

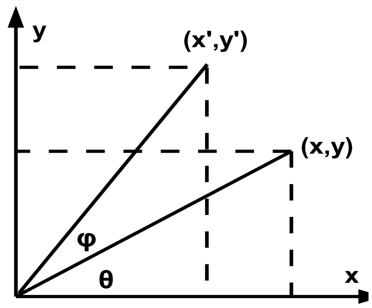

Figure 4: $SO(2)$ rotation by an angle $\phi \in (0, 2\pi]$.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R(\phi) \begin{pmatrix} x \\ y \end{pmatrix} . \qquad (5.48)$$

The rep (5.47) is reducible since its character

$$\chi(R) = 2\cos\phi = e^{i\phi} + \epsilon^{-i\phi} = \chi^{(1)} + \chi^{(-1)} . \qquad (5.49)$$

5.4.4 The Orthogonal group $O(2)$

The group $O(2)$ of 2×2 orthogonal matrices as $R(\phi)$ above with $\det(R) = 1$, but also of the form

$$S(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} , \qquad 0 < \phi \leqslant 2\pi , \qquad (5.50)$$

with $\det(S) = -1$. The group identity is $R(0)$ and the element $S(\phi)$ is disconnected from the identity.

Exercise:

a) Show that $S(\phi)$ represents the reflection of the point (x, y) w.r.t. the straight line $y = \tan(\phi/2)x$.

b) Verify that the the inverses $R^{-1}(\phi)$ and $S^{-1}(\phi)$ are given by

$$R^{-1}(\phi) = R(-\phi) , \qquad S^{-1}(\phi) = S(\phi) \qquad (5.51)$$

and that the multiplication table for $O(2)$ is

$$\begin{array}{c|cc} O(2) & R(\phi') & S(\phi') \\ \hline R(\phi) & R(\phi+\phi') & S(\phi+\phi') \\ S(\phi) & S(\phi-\phi') & R(\phi-\phi') \end{array} , \qquad (5.52)$$

hence $SO(2)$ is a proper subgroup of $O(2)$.

c) The group $O(2)$ is non-Abelian so it should have 2-dim irreps. Show that indeed the 2×2 matrices

$$\Gamma^{(m)}(R(\phi)) = \begin{pmatrix} e^{im\phi} & 0 \\ 0 & e^{-im\phi} \end{pmatrix} , \qquad \Gamma^{(m)}(S(\phi)) = \begin{pmatrix} 0 & e^{im\phi} \\ e^{-im\phi} & 0 \end{pmatrix} , \tag{5.53}$$

form such an irrep. Also show that the great orthogonality theorem (3.26) is indeed obeyed.

5.5 Selection rules

We may use rep theory and the fact that any function can be decomposed as in (5.22) to obtain information on integrals without actually computing them or by computing just a minimum set of them in specific applications. An elementary example is the fact that if a function is odd, i.e. $f(-x) = f(x)$ then $\int_{-a}^{a} dx f(x) = 0$. We have already seen a generalization of this in (5.26). In particular, take j to be the identity rep labeled by $j = 1$. Since it is invariant under all group operations we may take this to be a constant. It is normalized to unity so that

$$\Psi_1^{(1)} = \frac{1}{\sqrt{h_G}} , \qquad h_G = \int d[\mathbf{x}] , \tag{5.54}$$

where h_G is the order of the group. Using the decomposition (5.22), for any function

$$\int d[\mathbf{x}] \Psi(\mathbf{x}) = \int d[\mathbf{x}] \Psi_1^{(1)}(\mathbf{x}) , \tag{5.55}$$

hence if a function has no part belonging to the identity rep its integral will be zero. More generally

$$F_{\mu\nu}^{ij} = \langle \Psi_\mu^{(i)} | F | \Phi_\nu^{(j)} \rangle = \int d[\mathbf{x}] \Psi_\mu^{(i)*}(\mathbf{x}) F(\mathbf{x}) \Phi_\nu^{(j)}(\mathbf{x}) , \tag{5.56}$$

will be non-zero only if the decomposition of the function $F(\mathbf{x})\Phi_\nu^{(j)}(\mathbf{x})$ into irreps has a component in the ith irrep and its μth row. Clearly using (5.22) we may consider a more specialized form than (5.56), namely

$$F_{\mu\rho\nu}^{ikj} = \int d[\mathbf{x}] \Psi_\mu^{(i)*}(\mathbf{x}) F_\rho^{(k)}(\mathbf{x}) \Phi_\nu^{(j)}(\mathbf{x}) . \tag{5.57}$$

Accordingly, the integral will be zero if the direct product of the irreps $\Gamma^{(k)}$ and $\Gamma^{(j)}$ does not contain the irrep $\Gamma^{(i)}$, no matter what the detailed structure of the various functions is. This can be done by using the corresponding character decomposition and (4.17) or (4.24).

As an application let's compute the non-vanishing integrals (5.56) for the function

$$F(\mathbf{x}) = \mathbf{V} \cdot \mathbf{x} = V_1 x + V_2 y + V_3 z \,, \tag{5.58}$$

where the symmetry group is D_3. According to the character table (5.36) the functions (x, y) are basis partners in the 2-dim irrep $\Gamma^{(3)}$ for D_3. Using also (4.26) we obtain

$$\begin{array}{c|l} \Gamma^{(j)} & \Gamma^{(3)} \otimes \Gamma^{(j)} \\ \hline \Gamma^{(1)} & \Gamma^{(3)} \\ \Gamma^{(2)} & \Gamma^{(3)} \\ \Gamma^{(3)} & \Gamma^{(1)} \oplus \Gamma^{(2)} \oplus \Gamma^{(3)} \;. \end{array} \tag{5.59}$$

Hence, the integrals in (5.56) of the form F^{13}, F^{23} and F^{33}, where we have suppressed row indices, can be non-zero. In addition, since z belongs to the 1-dim irrep $\Gamma^{(2)}$ we obtain

$$\begin{array}{c|c} \Gamma^{(j)} & \Gamma^{(2)} \otimes \Gamma^{(j)} \\ \hline \Gamma^{(1)} & \Gamma^{(2)} \\ \Gamma^{(2)} & \Gamma^{(1)} \\ \Gamma^{(3)} & \Gamma^{(3)} \;. \end{array} \tag{5.60}$$

Hence, the integrals in (5.56) of the form F^{12} and F^{33} can be non-zero. Overall, we conclude that the integrals of the form F^{11} and F^{22} vanish.

The preceding results in this particular example do not guarantee the non-vanishing of the remaining integrals. This is reasonable since no-where in the analysis the row indices were used. To appreciate that take $\Phi_\nu^{(j)} \sim y$. Then $F\Phi_\nu^{(j)}$ has components proportional to xy, y^2 and yz, which means that it belongs to the rep $\Gamma^{(1)} \oplus \Gamma^{(3)}$. Hence, only integrals for which $\Psi^{(i)}$ belongs to the $\Gamma^{(1)}$ and $\Gamma^{(3)}$ irreps can be non-zero. Moreover, if we further assume that the vector components $V_2 = V_3 = 0$, then we only have the xy factor which is proportional to the basis function $\Psi_2^{(3)}$. In that case only if $\Psi_\mu^{(i)}$ is proportional to xy, i.e. the 2nd row of the irrep $\Gamma^{(3)}$ the integral is non-zero.

We can give a more precise criterion by applying to (5.57) the identity (5.12). We easily find that (**Exercise**)

$$F^{ikj}_{\mu\rho\nu} = \sum_{\mu'=1}^{l_i} \sum_{\rho'=1}^{l_k} \sum_{\nu'=1}^{l_j} \Gamma^{(i)*}_{\mu'\mu}(R)\Gamma^{(k)}_{\rho'\rho}(R)\Gamma^{(j)}_{\nu'\nu}(R)F^{ikj}_{\mu'\rho'\nu'} , \qquad (5.61)$$

where R is any element of the group leaving H invariant. The relations (5.61) provide constraints between the $F^{ijk}_{\mu\rho\nu}$'s which are found by evaluating them for different group elements

We may apply the above for the group D_3. Let's consider the case in which the indices i, j and k all correspond to the irrep $\Gamma^{(3)}$ given in (2.7). Then the Greek indices μ, μ' etc, necessarily take the values 1 and 2. There are 8 independent integrals

$$F_{111}, \quad F_{112}, \quad F_{121}, \quad F_{211}, \quad F_{122}, \quad F_{212}, \quad F_{221}, \quad F_{222} , \qquad (5.62)$$

where we have omitted for convenience the superscripts as they all correspond to the irrep $\Gamma^{(3)}$. Applying (5.61) for $R = A$ immediately give that all the $F_{\mu\rho\nu}$'s with odd number of 2's vanish. Hence we are left with $F_{111}, F_{122}, F_{212}$ and F_{221}. Next we apply (5.61) for $(\mu, \rho, \nu) = (1, 1, 1), (1, 2, 2)$ and $(2, 2, 1)$ and in all cases for $R = D$, obtaining the relations (**Exercise**)

$$\begin{aligned} F_{111} &= -\frac{1}{8}F_{111} - \frac{3}{8}(F_{122} + F_{221} + F_{212}) , \\ F_{122} &= \frac{3}{8}(-F_{111} + F_{221} + F_{212}) - \frac{1}{8}F_{122} , \\ F_{221} &= \frac{3}{8}(-F_{111} + F_{122} + F_{212}) - \frac{1}{8}F_{221} . \end{aligned} \qquad (5.63)$$

Consequently, we may express three of the F's in terms of the fourth one.[13] Hence, we have obtained just from group rep theory that

$$F_{111} \equiv F, \qquad F_{112} = F_{121} = F_{211} = F_{222} = 0, \qquad F_{122} = F_{212} = F_{221} = -F . \qquad (5.64)$$

In other words only one integral out of eight is really independent. This result is independent of the particular details of the functions in the integrand!

[13] Using other group elements for R should just give a rearrangement of (5.63) (**Exercise**).

6 The 3-dim rotation group

The set of all rotations in 3-dim space forms an infinite group, the covering group of a sphere. All rotations though the same angle form a class irrespectively of the axis they are performed about, the reason being that any two axis can be connected via another rotation. Hence there are infinite many irreps.

6.1 General rotations

We have seen that a rotation in the $x-y$ plane can be represented by the matrix (5.47). From a 3-dim view point the transformation matrix is $(0 \leqslant \phi < 2\pi)$

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.1}$$

and similarly

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} , \qquad R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} , \tag{6.2}$$

for rotations about the x and y axes. A full rotation can be made by taking three successive such elementary rotations. The way this is done is to a certain extent arbitrary, the only limitation being that no two successive rotations are made about the same axis. Among the 12 such allowed rotations we choose

$$R(\alpha,\beta,\gamma) = R_z(\gamma)R_y(\beta)R_z(\alpha) , \tag{6.3}$$

where the angles α,β and γ are known as Euler angles. Similarly to the case of rotations about a single axis, these rotations are proper in the sense that they have unit determinant. For that reason the group they form is called $SO(3)$.

Unlike $SO(2)$ rotations, these rotations generally do not commute. In fact two infinitesimal $SO(3)$ rotations around the axes x and y performed in opposite order differ by a single rotation in the z-axis. To see that first we expand the commutator of $R_x(\alpha)$ and $R_y(\beta)$ around the identity to second order in the rotation angles α and β. We find that (**Exercise**)

$$
\begin{aligned}
[R_x(\alpha), R_y(\beta)] &= \mathrm{diag}(1,1,1) - \begin{pmatrix} 1 & -\alpha\beta & 0 \\ \alpha\beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \cdots \\
&= \mathbb{I} - R_z(\alpha\beta) + \mathcal{O}(\alpha,\beta)^3 \, . \qquad (6.4)
\end{aligned}
$$

Within the same quadratic order approximation this can be rewritten as

$$
R_x(\alpha)R_y(\beta)R_x^{-1}(\alpha)R_y^{-1}(\beta)R_z(\alpha\beta) = I + \mathcal{O}(\alpha,\beta)^3 \, , \qquad (6.5)
$$

which clearly shows that even after performing a full circle of transformations, an additional infinitesimal rotation around the z axis is still needed in order to return to the original point.

6.2 Group generators; reps in terms of differential operators

We would like to represent R in the transformation (5.1) in terms of differential operators. Consider the special case where $R = R_z(\phi_z)$, that is a rotation around the z-axis. Then, infinitesimally expanding around $\phi_z = 0$ we obtain that

$$
R_z(\phi_z) = \mathrm{diag}(1,1,1) + \begin{pmatrix} 0 & -\delta\phi_z & 0 \\ \delta\phi_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\delta\phi_z^2) \, . \qquad (6.6)
$$

Denoting the infinitesimal difference $\delta\mathbf{x} = \mathbf{x}' - \mathbf{x}$ we have that

$$
\delta x = -\delta\phi_z y \, , \qquad \delta y = \delta\phi_z x \, , \qquad \delta z = 0 \, . \qquad (6.7)
$$

One observes that if we define the vector $\delta\boldsymbol{\phi} = (0,0,\delta\phi_z)$, then the above is the z-component of the cross product

$$
\delta\mathbf{x} = \delta\boldsymbol{\phi} \times \mathbf{x} \, . \qquad (6.8)
$$

The above is a definition of a general rotation with $\delta\boldsymbol{\phi} = (\delta\phi_x, \delta\phi_y, \delta\phi_z)$. In general we have for the components of (6.8)[14]

$$\delta x_i = -\epsilon_{ijk} x_j \delta\phi_k \ , \qquad i = 1,2,3 \ . \tag{6.10}$$

The infinitesimal action on a function is

$$P_{\delta\mathbf{x}} f(\mathbf{x}) = f(\mathbf{x} - \delta\mathbf{x}) = f(\mathbf{x} - \delta\boldsymbol{\phi} \times \mathbf{x}) \simeq f(\mathbf{x}) - (\delta\boldsymbol{\phi} \times \mathbf{x}) \cdot \nabla f(\mathbf{x}) \ . \tag{6.11}$$

We identify this with

$$P_{\delta\mathbf{x}} f(\mathbf{x}) \simeq f(\mathbf{x}) - i\delta\phi_i J_i f(\mathbf{x}) \ , \tag{6.12}$$

where

$$J_i = -i\epsilon_{ijk} x_j \partial_k \ , \tag{6.13}$$

is a first order differential operator. From the infinitesimal transformation it is easy to infer the finite one. A finite rotation around the z axis at an angle ϕ_z can be obtained in n steps such that $\delta\phi_z = \phi_z/n$ with $n \to \infty$. We have that

$$P_{\phi_z} f(\mathbf{x}) = \lim_{n\to\infty} \left(1 - i\frac{\phi_z}{n} J_z\right)^n f(\mathbf{x}) = e^{-i\phi_z J_z} f(\mathbf{x}) \ . \tag{6.14}$$

We note that the corresponding rotation matrix can be represented as

$$R_z(\phi_z) = e^{i\phi_z J_z} \ . \tag{6.15}$$

Consequently from (6.3) we have that the general rotation can be represented as

$$R(\alpha, \beta, \gamma) = e^{i\gamma J_z} e^{i\beta J_y} e^{i\alpha J_z} \ . \tag{6.16}$$

In addition, from (6.5) we find that

$$[J_x, J_y] = iJ_z \ , \quad \text{more generally} \ \ [J_i, J_j] = i\epsilon_{ijk} J_k \ . \tag{6.17}$$

[14] We use the notation $1,2,3$ for the components x, y, z, respectively. Also recall that ϵ_{ijk} is the totally antisymmetric tensor, Levi–Civita symbol, in 3-dims. Note the useful properties

$$\begin{aligned} \epsilon_{ijk}\epsilon_{kmn} &= \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \ , \\ \epsilon_{imn}\epsilon_{mnj} &= 2\delta_{ij} \ , \\ \epsilon_{ijk}\epsilon_{ijk} &= 6 \ . \end{aligned} \tag{6.9}$$

We will also use the convention that repeated indices are summed over.

To prove that we simply substitute the expressions for $R_x(\alpha,\ R_y(\beta)$ and $R_z(\alpha\beta)$ and expand for small α and β, keeping up to quadratic terms as this is consistent with the fact that in the r.h.s. of (6.5) we neglect cubic terms. The commutation relations (6.17) follow by comparing the terms proportional to $\alpha\beta$ **(Exercise)**.

Exercise: Using (6.13) verify that the above commutation relations are obeyed.

This set of commutation relations for the group can be viewed in the abstract sense as providing the definition of a *Lie algebra*. In this case the $SU(2)$ Lie algebra.

6.3 Elements of Lie-algebras

A Lie algebra is defined in terms of quantities called *structure constants* which are obtained from the commutation relations for a set of operators $\{J_a\}$, i.e.[15]

$$[J_a, J_b] = C_{ab}{}^c J_c\ , \tag{6.18}$$

where the set of $C_{ab}{}^c$ are the above mentioned structure constants (not to be confused with the coefficients c_{abc} that appear in the class multiplication (2.46)).

Remarks:

• By definition

$$C_{ab}{}^c = -C_{ba}{}^c\ . \tag{6.19}$$

• Also by making use of the Jacobi identity for commutators, i.e.

$$[[J_a, J_b], J_c] + [[J_c, J_a], J_b] + [[J_b, J_c], J_a] = 0\ , \tag{6.20}$$

one finds that

$$C^d_{ab}C^e_{dc} + C^d_{ca}C^e_{db} + C^d_{bc}C^e_{da} = 0\ , \tag{6.21}$$

which should be obeyed if the structure constants are to define a Lie algebra.

Definition: Define a symmetric matrix g with elements

$$g_{ab} = C_{ac}{}^d C_{bd}{}^c = g_{ba} \tag{6.22}$$

and, assuming it exists, denote by g^{ab} the components of the inverse matrix g^{-1}, i.e.

[15]We employ as before the convention that repeated indices are summed over.

$(g^{-1})_{ab} \equiv g^{ab}$. The matrix g_{ab} is called the Killing metric and is used to lower indices, e.g. $A_a = g_{ab}A^b$. Similarly its inverse g^{ab} is used to raise indices, i.e. $A^a = g^{ab}A_b$.

Definition: Define structure constants with all three lower indices as

$$C_{abc} = C_{ab}{}^d g_{dc} \ . \tag{6.23}$$

• The above quantities are completely antisymmetric in all three indices.

Proof: Indeed, for a and b this follows from the definition of the structure constants. For the index c we have that

$$C_{abc} + C_{acb} = C_{ab}{}^d g_{dc} + (b \leftrightarrow c) = C_{ab}{}^d C_{df}{}^e C_{ce}{}^f + (b \leftrightarrow c) \ . \tag{6.24}$$

Next we use for the first two factors the Jacobi identity in the indices a, b and f. Then

$$C_{abc} + C_{acb} = -C_{fa}{}^d C_{db}{}^e C_{ce}{}^f - C_{bf}{}^d C_{da}{}^e C_{ce}{}^f - C_{fa}{}^d C_{dc}{}^e C_{be}{}^f - C_{cf}{}^d C_{da}{}^e C_{be}{}^f \ . \tag{6.25}$$

We can easily see that the 1st and 4th term cancel each other and similarly for the 2nd and 3rd. Hence $C_{abc} = -C_{acb}$. Using that $C_{bac} = -C_{abc}$ we also find that $C_{cba} = -C_{abc}$ as well.

• Note also that

$$C_{ab}{}^c = C_{abd} g^{dc} \ . \tag{6.26}$$

Definition: The *quadratic Casimir operator* is defined as

$$C_2 = g^{ab} J_a J_b \ . \tag{6.27}$$

• Its usefulness lies on the fact that

$$[C_2, J_a] = 0 \ , \qquad \forall \ a \ , \tag{6.28}$$

that is, it commutes with all generators.

Exercise: Prove (6.28).

Exercise: Show that for the $SU(2)$ algebra (6.17) structure constants obey (6.21). Also show that in this case $g_{ab} \sim \delta_{ab}$.

Exercise: Prove that

$$C_{ab}{}^{c}C_{cd}{}^{d} = 0\ , \qquad \forall a, b\ . \tag{6.29}$$

7 $SU(2)$: The group of 2×2 unitary matrices

The set of 2×2 unitary matrices with unit determinant form a group (**Exersise**), called $SU(2)$, which is a three parameter group. The most general such matrix is of the form

$$g = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} \alpha_0 + i\alpha_3 & \alpha_2 + i\alpha_1 \\ -\alpha_2 + i\alpha_1 & \alpha_0 - i\alpha_3 \end{pmatrix} , \tag{7.1}$$

where $a, b \in \mathbb{C}$ whereas $(\alpha_0, \alpha_i) \in \mathbb{R}, i = 1,2,3$. The constraint imposed by the requirement that $\det(g) = 1$, implies the conditions

$$|a|^2 + |b|^2 = 1 \quad \Longleftrightarrow \quad \alpha_0^2 + \boldsymbol{\alpha}^2 = 1 \ . \tag{7.2}$$

Let's also introduce the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{7.3}$$

which are obviously traceless.

Exercise: Show that the Pauli matrices obey

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{I} + i\epsilon_{ijk}\sigma_k \ , \qquad \mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij} \ . \tag{7.4}$$

Remarks:

• The group element (7.1) and its inverse can be written as

$$g = \alpha_0 \mathbb{I} + i\alpha_i\sigma_i \ , \qquad g^{-1} = g^\dagger = \alpha_0 \mathbb{I} - i\alpha_i\sigma_i \ . \tag{7.5}$$

• Any 2×2 traceless matrix h is a linear combination of the Pauli matrices

$$h = x_i \sigma_i \ , \tag{7.6}$$

where we have, for reasons that will become apparent shortly, denoted the coefficients of the expansions with x_i, $i = 1,2,3$. Using (7.4) these are found as

$$x_i = \frac{1}{2}\mathrm{Tr}(h\sigma_i) \ . \tag{7.7}$$

A transformation of h by a unitary matrix g as in (7.1) induces a rotation on the vector $\mathbf{x}$ with components x_i of the form (5.1) with the rotation matrix having components

$$R(g)_{ij} = (\alpha_0^2 - \boldsymbol{\alpha}^2)\delta_{ij} + 2\alpha_i\alpha_j + 2\alpha_0\epsilon_{ijk}\alpha_k\ . \tag{7.8}$$

__**Proof:**__ We have that for a unitary transformation

$$h' = ghg^\dagger = g\mathbf{x}\cdot\boldsymbol{\sigma}g^{-1} = x_i'\sigma_i \quad \Longrightarrow \quad x_i' = \frac{1}{2}\mathrm{Tr}(\sigma_i g\mathbf{x}\cdot\boldsymbol{\sigma}g^{-1}) \equiv R(g)_{ij}x_j\ , \tag{7.9}$$

where

$$R(g)_{ij} = \frac{1}{2}\mathrm{Tr}(\sigma_i g\sigma_j g^{-1})\ . \tag{7.10}$$

Using (7.5) and (7.4) one proves (7.8) (**Exercise**).

Remarks: Some properties of the matrix $R(R)$ are:

• It is manifestly real and obeys

$$R^T(g) = R(g^{-1})\ . \tag{7.11}$$

• It is orthogonal, i.e.

$$R(g)R^T(g) = R^T(g)R(g) = \mathbb{I}\ . \tag{7.12}$$

• It has

$$\det(R(g)) = 1\ . \tag{7.13}$$

Hence $R(g) \in SO(3)$, the group of proper rotations.

• We may solve the constraint (7.2) as

$$\alpha_0 = -\cos\frac{\theta}{2}\ , \qquad \boldsymbol{\alpha} = \sin\frac{\theta}{2}\hat{\mathbf{n}}\ . \tag{7.14}$$

where $\hat{\mathbf{n}}$ is the unit vector. The resulting vector after the rotation is

$$\mathbf{x}' = R\mathbf{x} = \cos\theta\mathbf{x} + (1-\cos\theta)(\hat{\mathbf{n}}\cdot\mathbf{x})\hat{\mathbf{n}} + \sin\theta(\hat{\mathbf{n}}\times\mathbf{x}) \tag{7.15}$$

and represents a rotation in a plane with normal $\hat{\mathbf{n}}$ through an angle θ. In particular, if we take $\hat{\mathbf{n}} = \hat{z}$, i.e. the unit vector along the z-axis, we find that $\mathbf{x}' = R_z(\theta)\mathbf{x}$, where $R_z(\theta)$ is the rotation matrix (6.1).

7.1 Homomorphism between $SO(3)$ with $SU(2)$

Thus to every 2-dim unitary matrix g of unit determinant corresponds a 3-dim rotational matrix R with elements given by (7.10). Moreover, the above correspondence is such that if

$$g_1 g_2 = g \quad \Longrightarrow \quad R(g_1)R(g_2) = R(g) \ . \tag{7.16}$$

Proof: We have that

$$\begin{aligned} [R(g_1)R(g_2)]_{ij} &= (R(g_1))_{ik}(R(g_2))_{kj} = \frac{1}{4}\mathrm{Tr}(g_1^{-1}\sigma_i g_1 \sigma_k)\mathrm{Tr}(\sigma_k g_2 \sigma_j g_2^{-1}) \\ &= \frac{1}{2}\mathrm{Tr}(g_1^{-1}\sigma_i g_1 g_2 \sigma_j g_2^{-1}) = R(g_1 g_2)_{ij} = R(g)_{ij} \ , \end{aligned} \tag{7.17}$$

where we used that $\mathrm{Tr}(A\sigma_i)\mathrm{Tr}(B\sigma_i) = 2\mathrm{Tr}(AB)$ (valid for traceless matrices).

Hence there is a homomorphism between the group of 2-dim unitary unimodular matrices ($SU(2)$) and the 3-dim rotations. To show that the homomorphism exists between the group $SU(2)$ and the whole group of proper rotations $SO(3)$ one should show that $R(g)$ ranges over all proper rotations as g covers the entire unitary group. This will be proven shortly.

We would like to know the $SU(2)$ matrices giving rise to the elementary rotations $R_{y,z}$ that enter in the general rotation (6.3). We easily see that

$$\exp\left(\frac{i}{2}\alpha\sigma_3\right) = \begin{pmatrix} e^{\frac{i}{2}\alpha} & 0 \\ 0 & e^{-\frac{i}{2}\alpha} \end{pmatrix} \quad \text{corresponds to} \quad R_z(\alpha) \tag{7.18}$$

and that[16]

$$\exp\left(\frac{i}{2}\beta\sigma_2\right) = \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \quad \text{corresponds to} \quad R_y(\beta) \ . \tag{7.19}$$

Therefore the matrix

$$g(\alpha,\beta\,\gamma) = e^{\frac{i}{2}\gamma\sigma_3}e^{\frac{i}{2}\beta\sigma_2}e^{\frac{i}{2}\alpha\sigma_3} = \begin{pmatrix} \cos\frac{\beta}{2}e^{\frac{i}{2}(\alpha+\gamma)} & \sin\frac{\beta}{2}e^{\frac{i}{2}(\gamma-\alpha)} \\ -\sin\frac{\beta}{2}e^{-\frac{i}{2}(\gamma-\alpha)} & \cos\frac{\beta}{2}e^{-\frac{i}{2}(\alpha+\gamma)} \end{pmatrix} \ , \tag{7.20}$$

[16]One way to prove these expressions is to use the Cayley–Hamilton theorem which allows for an $N \times N$ matrix A to write that $e^A = \sum_{n=0}^{N-1} c_n A^n$. The coefficients c_n are obtained by solving the algebraic system $e^{\lambda_i} = \sum_{n=0}^{N-1} c_n \lambda_i^n$, where the λ_i's are the eigenvalues of the matrix A.

corresponds to the rotation (6.3). The relation to the parametrization with a and b or the α_i's follows by simply comparing with (7.1).

Note that in the rep (6.16) the operator $J_i = \frac{\sigma_i}{2}$. Thus an arbitrary rotation can be induced by an arbitrary $SU(2)$ matrix as in (7.20). However, as it is clear from (7.10) the matrix $(-g)$ leads to the same rotation as well, so that the representation in not unique but rather *doubled valued*. Hence, we have that

$$R(g_1)R(g_2) = R(g) \quad \Longrightarrow \quad g_1 g_2 = \pm g \,. \tag{7.21}$$

Which of the two signs appears cannot be deduced from general arguments and one has to explicitly check by performing the multiplication $g_1 g_2$.

Hence we have a 2-fold homomorphism between the group $SU(2)$ and the group $SO(3)$. Two matrices $\pm g$ are associated to a proper rotation $R(g)$.

7.2 Representations of $SU(2)$

The matrices g in (7.1) form a 2-dim rep of $SU(2)$. Let the 2-dim vector

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} , \qquad \xi_i \in \mathbb{C} \,, \quad i = 1,2 \,, \tag{7.22}$$

transforming as

$$\xi' = g\xi \,, \qquad \xi'_1 = a\xi_1 + b\xi_2 \,, \quad \xi'_2 = -b^*\xi_1 + a^*\xi_2 \,, \tag{7.23}$$

so that it is a basis for this 2-dim rep with the partners being ξ_1 and ξ_2. The inverse transformation is

$$\xi = g^{-1}\xi' \,, \qquad \xi_1 = a^*\xi'_1 - b\xi'_2 \,, \quad \xi_2 = b^*\xi'_1 + a\xi'_2 \,, \tag{7.24}$$

We may construct higher dimensional reps by taking as a basis the $n+1$ monomials of degree n

$$g_p^{(n)} = \xi_1^p \xi_2^{n-p} \,, \qquad p = 0, 1, \ldots, n \,. \tag{7.25}$$

Defining as in (5.2) the transformation operator

$$P_g f(\xi) = f(g^{-1}\xi) \,, \quad \text{or} \quad P_g f(\xi_1, \xi_2) = f(a^*\xi_1 - b\xi_2, b^*\xi_1 + a\xi_2). \tag{7.26}$$

Clearly, applying this transformation to $g_p^{(n)}$ we obtain a linear combination of the same monomials with the same n but different p's.

Definition: Reparametrizing as $p = j + m$ and $n = 2j$, where $j \in \mathbb{Z}/2$ we define

$$f_m^{(j)} = \frac{g^{(2j)}}{\sqrt{(j+m)!(j-m)!}} = \frac{\xi_1^{j+m}\xi_2^{j-m}}{\sqrt{(j+m)!(j-m)!}} \,, \qquad m = -j, -j+1, \dots, j \,. \tag{7.27}$$

Remarks:

• The action of P_g on $f_m^{(j)}$ induces the transformation

$$P_g f_m^{(j)} = \sum_{m'} U_{m'm}^{(j)}(g) f_{m'}^{(j)} \,, \tag{7.28}$$

with

$$U_{m'm}^{(j)}(g) = \sum_k B_{m,m',k}^j (a^*)^{j+m-k} a^{j-m'-k} (b^*)^{k-m+m'} b^k \,, \tag{7.29}$$

where the coefficients are given by

$$B_{m,m',k}^j = \frac{(-1)^k \sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{k!(j+m-k)!(j-k-m')!(k+m'-m)!} \,. \tag{7.30}$$

Recalling that $0! \equiv 1$ and the factorial of a negative integer is infinite, the above sum over k ranges over all integer values for which no factorial in the denominator has negative argument. One easily sees that $\max(0, m-m') \leqslant k \leqslant \min(j+m, j-m')$.

<u>Proof</u>: The proof is straightforward (**Exercise**) with the use of (7.26) and the expansion of the binomial

$$(a+b)^n = \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} a^{n-m} b^m \,. \tag{7.31}$$

• The matrix $U^{(j)}$ with elements (7.29) provides, a unitary irrep.

<u>Proof</u>: First consider

$$\sum_{m=-j}^{j} |f_m^{(j)}|^2 = \sum_{m=-j}^{j} \frac{|\xi_1|^{2(j+m)}|\xi_2|^{2(j-m)}}{(j+m)!(j-m)!} = \frac{(|\xi_1|^2 + |\xi_2|^2)^{2j}}{(2j)!} \,. \tag{7.32}$$

and

$$\sum_{m=-j}^{j} \left|P_g f_m^{(j)}\right|^2 = \sum_{m=-j}^{j} \frac{|a^*\xi_1 - b\xi_2|^{2(j+m)}|b^*\xi_1 + a\xi_2|^{2(j-m)}}{(j+m)!(j-m)!}$$
$$= \frac{(|a^*\xi_1 - b\xi_2|^2 + |b^*\xi_1 + a\xi_2|^2)^{2j}}{(2j)!} = \cdots \tag{7.33}$$
$$= \frac{(|\xi_1|^2 + |\xi_2|^2)^{2j}}{(2j)!} .$$

The intermediate step is left as an (**Exercise**). Hence we have established that

$$\sum_{m=-j}^{j} \left|P_g f_m^{(j)}\right|^2 = \sum_{m=-j}^{j} \left|f_m^{(j)}\right|^2 , \tag{7.34}$$

that is the invariance of the r.h.s. Using (7.28) we find that $UU^\dagger = U^\dagger U = \mathbb{I}$ follows (**Exercise**) if the $(2j+1)^2$ functions $f_m^{(j)} f_{m'}^{(j)*}$ are linearly independent. To prove the latter statement one needs to show that demanding

$$\sum_{m,m'} C_{m,m'}^{(j)} f_m^{(j)} f_{m'}^{(j)} = \sum_{m,m'} C_{m,m'}^{(j)} \frac{\xi_1^{j+m}\xi_2^{j-m}(\xi_1^*)^{j+m'}(\xi_2^*)^{j-m'}}{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}} = 0 , \tag{7.35}$$

for all ξ_i's, necessarily implies that the all coefficients $C_{m,m'}^{(j)} = 0$ (**Exercise**).

To prove that the rep provided by $U^{(j)}$ is irreducible we use Schur's (converse) first lemma. Setting $b = 0$ and $a = e^{\frac{i}{2}\alpha}$ we find that

$$U_{mm'}^{(j)}(\alpha) = e^{-im\alpha}\delta_{m,m'} , \qquad m = -j, -j+1, \ldots, j . \tag{7.36}$$

Assuming that a matrix M commutes with $U^{(j)}$ matrices we immediately find that

$$M_{mm'}(e^{im\alpha} - e^{im'\alpha}) = 0 , \qquad \forall \alpha . \tag{7.37}$$

Hence, $M_{mm'} = 0$ for $m \neq m'$. Considering a more general matrix $U^{(j)}$ we have that

$$U_{mm'}^{(j)}(M_{mm} - M_{m'm'}) = 0 , \qquad \forall\, m, m' . \tag{7.38}$$

Since $U_{mm'}^{(j)} \neq 0$ we have that $M_{mm} = M_{m'm'}$, $\forall m, m'$. Hence, M is proportional to the identity matrix and the rep is irreducible.

To show that $U^{(j)}$ exhausts all distinct irreps, i.e. there no addition inequivalent irreps, we construct their characters and check their orthogonality and completeness. This is proven in the next subsection.

Note that

$$U^{(1/2)} = \begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix} \neq g \,, \tag{7.39}$$

where g is given in (7.1). This is achieved by the a unitary transformation, i.e.

$$g = M^\dagger U^{(1/2)} M \,, \qquad M = i\sigma_3 \,. \tag{7.40}$$

In general we define the equivalent rep

$$\widetilde{U}^{(j)} = M^\dagger U^{(j)} M \,, \qquad M_{mm'} = i^{-2m}\delta_{mm'} \,. \tag{7.41}$$

The expression for the matrix elements $\widetilde{U}^{(j)}_{mm'}$ is similar to that in (7.29) but with coefficients given by (7.30) multiplied by $(-1)^{m-m'}$. Then one verifies that $\widetilde{U}^{(1/2)} = g$.

<u>Exercise</u>: Show that

$$\begin{aligned} U^{(j)}_{j,m} &= \begin{pmatrix} 2j \\ j-m \end{pmatrix}^{1/2} (a^*)^{j+m}(b^*)^{j-m} \,, \\ U^{(j)}_{-j,m} &= \begin{pmatrix} 2j \\ j-m \end{pmatrix}^{1/2} a^{j-m}b^{j+m} \,. \end{aligned} \tag{7.42}$$

Also note that these "edge values" of $m' = \pm j$ are the only ones for which the contribution of the sum over k in (7.29) collapses to a single term. The case with $m = \pm j$ follows from unitarity.

7.3 The characters

Let's consider the finite dimensional irreps provided by $U^{(j)}$ with dimensionality $l_j = 2j+1$, with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. Any rotation about an arbitrary axis can be brought into a rotation about the z-axis into the form (7.36). Hence the character is (**Exercise**)

$$\chi^{(j)}(\phi) = \sum_{m=-j}^{j} U^{(j)}_{m,m}(\phi) = \sum_{m=-j}^{j} e^{-im\phi} = \cdots = \frac{\sin\left(j+\frac{1}{2}\right)\phi}{\sin\frac{\phi}{2}} \,. \tag{7.43}$$

Note that $\chi^{(j)}(0) = 2j+1$, i.e. $\chi^{(j)}(0)$ equals the dimensionality of the rep as it should. In addition, for j half-integer we have that

$$\chi^{(j)}(\phi + 2\pi) = (-1)^{2j}\chi^{(j)}(\phi) \ . \tag{7.44}$$

Hence for $j = 0, 1, \ldots$, periodicity of the character is warranted by taking $0 < \phi \leqslant 2\pi$ whereas for $j = 0, \frac{1}{2}, \ldots$ we have to extend the range as $0 < \phi \leqslant 4\pi$.

Remarks:

• With $\phi \in (0, 4\pi]$ one may show that these characters obey the completeness relation

$$\begin{aligned}\sum_{j=0,\frac{1}{2}}^{\infty} \chi^{(j)}(\phi)\chi^{(j)}(\phi')^* &= \frac{\pi}{\sin\frac{\phi}{2}\sin\frac{\phi'}{2}}\left[\delta(\phi-\phi') - \delta(\phi+\phi'-4\pi)\right] \\ &= \frac{\pi}{\sin^2\frac{\phi}{2}}\left[\delta(\phi-\phi') + \delta(\phi+\phi'-4\pi)\right] \ , \qquad 0 < \phi \leqslant 4\pi \ . \end{aligned} \tag{7.45}$$

This is the continuous analog of (3.67).

<u>**Proof**</u>**:** Letting $\phi = 2\theta$, with $0 < \theta \leqslant 2\pi$ and $n = 2j = 0, 1, \ldots$, it is clear that one has to prove that

$$\sum_{n=0}^{\infty} \sin(n+1)\theta \sin(n+1)\theta' = \frac{\pi}{2}\left[\delta(\theta-\theta') - \delta(\theta+\theta'-2\pi)\right] \ , \tag{7.46}$$

where we've used the property $\delta(ax) = \delta(x)/|a|$, $a \in \mathbb{R}$. However the l.h.s. can be rewritten as (**Exercise**)

$$\frac{1}{4}\sum_{n=-\infty}^{\infty} e^{in(\theta-\theta')} - e^{in(\theta+\theta')} = \frac{\pi}{2}\left[\delta(\theta-\theta') - \delta(\theta+\theta'-2\pi)\right] \ , \tag{7.47}$$

where for the final step to be valid it is crucial that $\theta \in (0, 2\pi]$.

• In addition we may prove the orthogonality relations (**Exercise**)

$$\int_0^{4\pi} d\phi \sin^2\frac{\phi}{2}\chi^{(j)}(\phi)\chi^{(j')*}(\phi) = 2\pi\delta_{jj'} \ , \qquad j = 0, \frac{1}{2}, \ldots \ , \tag{7.48}$$

which is the continuous analog of the character great orthogonality theorem (3.44). Note that both δ-function terms in (7.45) are necessary for compatibility with (7.48).

• If we restrict to integer j's then $\phi \in (0, 2\pi]$. Then the completeness and the orthogonality relations are given by (7.45) and (7.48), respectively provided we replace on the r.h.s. π by $\pi/2$ (**Exercise**).

7.3.1 Character decomposition

We consider the tensor product of two irreps labelled by j and j' and decompose it using the character decomposition formula (4.23). Indeed we have that

$$\chi^{(j)}(\phi)\chi^{(j')}(\phi) = \sum_{\tilde{j}=|j-j'|}^{j+j'} \chi^{(\tilde{j})}(\phi)\ , \qquad j, j' = 0, \frac{1}{2}, \dots\ . \tag{7.49}$$

<u>Proof</u>: In general there should be a decomposition of the form

$$\chi^{(j)}(\phi)\chi^{(j')}(\phi) = \sum_{\tilde{j}=0}^{\infty} a_{jj'\tilde{j}}\chi^{(\tilde{j})}(\phi)\ . \tag{7.50}$$

Using (7.48) we have that the coefficients are given by

$$\begin{aligned} a_{jj'\tilde{j}} &= \frac{1}{2\pi}\int_0^{4\pi} d\phi \sin^2\frac{\phi}{2}\chi^{(j)}(\phi)\chi^{(j')}(\phi)\chi^{(\tilde{j})}(\phi)^* \\ &= \frac{1}{\pi}\int_0^{2\pi} d\theta \frac{\sin n\theta \sin n'\theta \sin \tilde{n}\theta}{\sin\theta}\ , \end{aligned} \tag{7.51}$$

where $n = 2j + 1 = 1, 2, \dots$, etc. To evaluate the integral we use the theory of residues in complex analysis. Letting $z = e^{i\theta}$ after some algebra we obtain that

$$a_{jj'\tilde{j}} = -\frac{1}{2}\oint_C \frac{dz}{2\pi i}\frac{(z^{2n}-1)(z^{2n'}-1)(z^{2\tilde{n}}-1)}{(z^2-1)z^n z^{n'} z^{\tilde{n}}}\ , \tag{7.52}$$

where C is the unit circle in the complex plane traversed counterclockwise. Clearly the integrand is everywhere an analytic function inside the contour C except for a potential pole at $z = 0$. Hence, by expanding into the various terms we have that

$$\begin{aligned} a_{jj'\tilde{j}} = -\frac{1}{2}\text{Res}\Big(\frac{1}{z^2-1}\big(&z^{n+n'+\tilde{n}} - z^{n+n'-\tilde{n}} - z^{\tilde{n}+n-n'} - z^{n'+\tilde{n}-n} - z^{-\tilde{n}-n-n'} \\ &+ z^{\tilde{n}-n-n'} + z^{n'-\tilde{n}-n} + z^{n-n'-\tilde{n}}\big)\Big)\ . \end{aligned} \tag{7.53}$$

Without loss of generality we may assume that $n \geqslant n'$. We also note that in order to obtain a non-vanishing result the sum $n + n' + \tilde{n}$ should be odd. Since the contribution of the prefactor $1/(z^2 - 1)$ to the residue will be though monomials of the form z^{2m} it is clear that the 1st and the 3rd terms above do not contribute. Also the contributions of the 5th and 7th terms to the residue cancel each other. Therefore

$$a_{jj'\tilde{j}} = -\frac{1}{2}\text{Res}\Big(\frac{1}{1-z^2}\big(z^{n+n'-\tilde{n}} + z^{n'+\tilde{n}-n} - z^{\tilde{n}-n-n'} - z^{n-n'-\tilde{n}}\big)\Big) . \tag{7.54}$$

Next consider the cases: (i) $\tilde{n} \geqslant n+n'$. Then, taking the above remark into account the exponents of the above four terms are, respectively, $\leqslant -1$, $\geqslant 3$, $\geqslant 1$ and $\leqslant -3$. Therefore the total contribution is zero. (ii) $\tilde{n} \leqslant n-n'$. Similarly the exponents are $\geqslant 3$, $\leqslant -1$, $\leqslant -3$ and $\geqslant 1$. Hence, again the total contribution vanishes. (iii) The remaining case when $n-n'+1 \leqslant \tilde{n} \leqslant n+n'-1$. Clearly only the last two terms contribute, each with -1. Hence, we conclude that, in general

$$a_{jj'\tilde{j}} = \begin{cases} 1\,, & |j-j'| \leqslant \tilde{j} \leqslant j+j' \\ 0\,, & \text{otherwise} \end{cases} \tag{7.55}$$

and therefore (7.49) is proven.

7.3.2 The great orthogonality theorem for $SU(2)$ irreps

We would like to write a continuous version of the great orthogonality theorem (3.26). Clearly the sum over the group elements is replaces by an integral with measure dR. Its precise form is not important as this "detail" is irrelevant for the computation we will need.[17] In addition the order of the group is $h_{SU(2)} = \int dR =$ and the dimensionality of the irrep is $l_j = 2j+1$. Hence we have that

$$\int dR U^{(i)*}_{mn}(R) U^{(j)}_{m'n'} = \frac{h_{SU(2)}}{2j+1}\delta_{ij}\delta_{mm'}\delta_{nn'} . \tag{7.56}$$

<u>Exercise</u>: Specializing (7.56) for $m = m' = j = i$ and $n = n' \to m$ show that

$$\int dR \left(\cos\frac{\beta}{2}\right)^{2(j+m)} \left(\sin\frac{\beta}{2}\right)^{2(j-m)} = \frac{h_{SU(2)}}{2j+1} \begin{pmatrix} 2j \\ j-m \end{pmatrix}^{-1/2} . \tag{7.57}$$

[17]Essentially it involves an integration over the Euler angles.

7.4 The Wigner or Clebsch–Gordan coefficients

The decomposition (7.49) and (4.22) imply that the direct product of irreps $D^{(j)}$ is

$$D^{(j_1)} \otimes D^{(j_2)} = \sum_{j=|j_1-j_2|}^{j_1+j_2} {}^{\oplus} D^{(j)} = D^{(j_1+j_2)} \oplus \cdots \oplus D^{(|j_1-j_2|)} . \tag{7.58}$$

This implies that there is a similarity transformation to write the rep matrices as

$$U^{(j_1)} \otimes U^{(j_2)} = A^{-1} M A \, , \qquad M_{j'm';jm} = \delta_{jj'} U^{(j)}_{m'm} . \tag{7.59}$$

The components of the unitary matrix A are labeled by A_{jm,m_1m_2} and have yet to be determined. Consistency requires that it is a square matrix. The double index $(m_1 m_2)$ takes $(2j_1+1)(2j_2+1)$ values. Similarly, the double index (jm) takes the same number of values since we have that (**Exercise**)

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} \cdot 1 = \sum_{|j_1-j_2|}^{j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1) . \tag{7.60}$$

Hence in components

$$U^{(j_1)}_{m_1' m_1} U^{(j_2)}_{m_2' m_2} = \sum_{j,m,m'} A^{-1}_{m_1' m_2', jm'} U^{(j)}_{m'm} A_{jm,m_1m_2} . \tag{7.61}$$

Remarks:

• Consider a basis of vectors labeled as before by $\psi^{(j)}_m$ that transform as in (7.28). Then the decomposition (7.58) implies that we may write them as a linear combination of the product $\psi^{(j_1)}_{m_1} \psi^{(j_2)}_{m_2}$. Indeed if (7.61) is obeyed we have that

$$\psi^{(j)}_m = \sum_{m_1,m_2} A^{-1}_{m_1m_2,jm} \psi^{(j_1)}_{m_1} \psi^{(j_2)}_{m_2} \, , \qquad |j_1 - j_2| \leqslant j \leqslant j_1 + j_2 . \tag{7.62}$$

<u>Proof</u>: It suffices to show that $\psi^{(j)}_m$ as defined above obeys (7.28). We have that

$$\begin{aligned} P_g \psi^{(j)}_m &= \sum_{m_1,m_2} A^{-1}_{m_1m_2,jm} (P_g \psi^{(j_1)}_{m_1})(P_g \psi^{(j_2)}_{m_2}) \\ &= \sum_{m_1,m_2} \sum_{m_1',m_2'} A^{-1}_{m_1m_2,jm} U^{(j_1)}_{m_1' m_1} U^{(j_2)}_{m_2' m_2} \psi^{(j_1)}_{m_1} \psi^{(j_2)}_{m_2} . \end{aligned} \tag{7.63}$$

Using the inverse of (7.62)

$$\psi^{(j_1)}_{m_1}\psi^{(j_2)}_{m_2} = \sum_{j'm'} A_{j'm',m'_1m'_2}\psi^{(j')}_{m'} \ , \tag{7.64}$$

we find that

$$P_g\psi^{(j)}_m = \sum_{j'm'}\left[A(U^{(j_1)}\otimes U^{(j_2)})A^{-1}\right]_{j'm',jm}\psi^{(j')}_{m'} = \sum_{m'} U^{(j)}_{m'm}\psi^{(j)}_{m'} \ .$$

• To determine A we first consider for $U^{(j)}$ the diagonal matrix in (7.36). Then

$$\begin{aligned} P_g\psi^{(j)}_m &= e^{-im\alpha}\psi^{(j)}_m = e^{-im\alpha}\sum_{m_1,m_2} A^{-1}_{m_1m_2,jm}\psi^{(j_1)}_{m_1}\psi^{(j_2)}_{m_2} \\ &= P_g\sum_{m_1,m_2} A^{-1}_{m_1m_2,jm}\psi^{(j_1)}_{m_1}\psi^{(j_2)}_{m_2} = \sum_{m_1,m_2} A^{-1}_{m_1m_2,jm} e^{-i(m_1+m_2)\alpha}\psi^{(j_1)}_{m_1}\psi^{(j_2)}_{m_2} \ . \end{aligned} \tag{7.65}$$

Since the $\psi^{(j_1)}_{m_1}\psi^{(j_2)}_{m_2}$'s form a basis we should have that $m = m_1 + m_2$. Hence the matrix elements are of the form

$$A_{jm,m_1m_2} = a_{jm_1m_2}\delta_{m,m_1+m_2} \ , \qquad A^{-1}_{m_1m_2,jm} = a^*_{jm_1,m_2}\delta_{m,m_1+m_2} \ , \tag{7.66}$$

due to the unitary of the matrix A. Then (7.61) can be written as

$$U^{(j_1)}_{m'_1m_1}U^{(j_2)}_{m'_2m_2} = \sum_{j=|j_1-j_2|}^{j_1+j_2} a^*_{jm'_1m'_2}a_{jm_1m_2}U^{(j)}_{m'_1+m'_2,m_1+m_2} \ . \tag{7.67}$$

• There is a certain freedom in determining the coefficients $a_{jm_1m_2}$. Indeed, a matrix of the form

$$S_{j'm',jm} = s_j\delta_{jj'}\delta_{mm'} \ , \qquad |s_j| = 1 \tag{7.68}$$

is unitary and commutes with M. Therefore if we replace in (7.59) A by SA the relation still holds. This freedom allows to choose, for fixed j, the phase of one of the coefficients $a_{jm_1m_2}$. We choose $a_{j,-j_1,j_2}$ to be real.

• The rest of the computation is by "brute force" (it can be safely skipped and jump directly to the result (7.76)) .

Multiplying (7.67) with $U^{(j')*}_{m_1'+m_2',m_1+m_2}$, integrating over and using (7.56) we have that

$$a^*_{jm_1'm_2'}a_{jm_1m_2}\frac{h_{SU(2)}}{2j+1} = \int dRU^{(j_1)}_{m_1'm_1}(R)U^{(j_2)}_{m_2'm_2}(R)U^{(j)}_{m_1'+m_2',m_1+m_2}(R) \tag{7.69}$$

To proceed further recall (7.42) and set $m_1' = j_1$, $m_2' = -j_2$. Then

$$\begin{aligned} a^*_{j,j_1,-j_2}a_{jm_1m_2}\frac{h_{SU(2)}}{2j+1} &= \begin{pmatrix} 2j_1 \\ j_1-m_1 \end{pmatrix}^{1/2} \begin{pmatrix} 2j_2 \\ j_2-m_2 \end{pmatrix}^{1/2} \\ &\times \sum_k (-1)^{j_2+m_2} B^j_{m_1+m_2,j_1-j_2,k}\int dR\left(\cos\frac{\beta}{2}\right)^{2(j+j_2+m_1-k)}\left(\sin\frac{\beta}{2}\right)^{2(j_1-m_1+k)} \end{aligned} \tag{7.70}$$

To compute the integral we may use (7.57) with the replacements

$$j \to \frac{1}{2}(j+j_1+j_2)\ , \qquad m \to \frac{1}{2}(j+j_2-j_1)+m_1-k\ . \tag{7.71}$$

By doing so the dependence on $h_{SU(2)}$ cancels out. Therefore we have that (**Exercise**)

$$\begin{aligned} \frac{a^*_{j,j_1,-j_2}a_{jm_1m_2}}{2j+1} &= \frac{\sqrt{(2j_1)!(2j_2)!(j+m_1+m_2)!(j-m_1-m_2)!(j+j_1-j_2)!(j-j_1+j_2)!}}{(j+j_1+j_2+1)!\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}} \\ &\times \sum_k (-1)^{k+j_2+m_2}\frac{(j+j_2+m_1-k)!(j_1-m_1+k)!}{(j-j_1+j_2-k)!(j+m_1+m_2-k)!k!(k+j_1-j_2-m_1-m_2)!} \end{aligned} \tag{7.72}$$

Specializing to $m_1 = j_1$ and $m_2 = -j_2$ we have that

$$|a_{j,j_1,-j_2}|^2 = (2j+1)\frac{(j+j_1-j_2)!(j-j_1+j_2)!}{(j+j_1+j_2+1)!}S_{j_1,j_2,j}\ , \tag{7.73}$$

where the sum turns out to be

$$\begin{aligned} S_{j_1,j_2,j} &= \sum_k (-1)^k\frac{(j+j_1+j_2-k)!}{k!(j-j_1+j_2+k)!(j+j_1-j_2-k)!} \\ &= (2j_1)!(2j_2)!/((j+j_1-j_2)!(j_1+j_2-j)!(j-j_1+j_2)!)\ . \end{aligned} \tag{7.74}$$

Hence, remembering our choice of real $a_{j,j,-j_2}$, we arrive at

$$a_{j,j_1,-j_2} = \sqrt{\frac{(2j+1)(2j_1)!(2j_2)!}{(j_1+j_2-j)!(j+j_1+j_2+1)!}}\ . \tag{7.75}$$

Hence, we obtain the final result that $A_{j_1 j_2, m_1 m_2} = a_{j m_1 m_2} \delta_{m, m_1+m_2}$, where

$$a_{jm_1m_2} = \sqrt{\frac{(j+m_1+m_2)!(j-m_1-m_2)!(j_1+j_2-j)!(j+j_1-j_2)!(j-j_1+j_2)!}{(j+j_1+j_2+1)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}}$$
$$\sum_k (-1)^{k+j_2+m_2} \frac{\sqrt{2j+1}(j+j_2+m_1-k)!(j_1-m_1+k)!}{(j-j_1+j_2-k)!(j+m_1+m_2-k)!k!(k+j_1-j_2-m_1-m_2)!} \tag{7.76}$$

where we again recall that k takes all integer values for which all the arguments of the factors are non-negative. Also with our phase choice the matrix A is real and orthogonal.

7.4.1 Wigner $3-j$ symbols

We have seen that the components of the matrix A_{jm,m_1m_2} depend on j_1 and j_2. Hence, it is perhaps better to denote it by $A^{j_1j_2j}_{m_1m_2m} = a_{jm_1m_2}\delta_{m,m_1+m_2}$. Since $m = m_1 + m_2$ the last subscript is sometimes omitted for notational convenience.

We are interested in determining the independent components of the $A^{j_1j_2j}_{m_1m_2m}$'s. One way to find them is to exhibit their symmetries under interchanges of the various indices.

<u>Definition</u>: The Wigner's $3-j$ symbol is defined as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{2j_3+j_2-j_1-m_3}}{2j_3+1} A^{j_1j_2j_3}_{m_1m_2-m_3} \,. \tag{7.77}$$

This symbol has the following properties:

• Equality under cyclic permutations of columns

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} \,. \tag{7.78}$$

• Possible sign change under interchange of two columns

$$\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix} = \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}$$
$$= (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{7.79}$$

• Possible sign change on flipping the sign of all the m's

$$\begin{pmatrix} j_2 & j_1 & j_3 \\ -m_2 & -m_1 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} . \tag{7.80}$$

7.5 Tensors operators

For applications of rep theory it is convenient to classify operators according to the irreps they belong to.

7.5.1 Scalars

A scalar S is any quantity which is invariant under all rotations, that is S belongs to the identity rep $D^{(0)}$. Hence, referring to the notation (5.11) the integral

$$\langle \Psi_{j'm'} | S | \Psi_{jm} \rangle = \delta_{jj'} \delta_{mm'} \Lambda_j \,, \tag{7.81}$$

where the r.h.s. emphasizes the fact that the result is non-zero for the same irreps and for the same row in the rep and also is independent of m. The proof of that is immediate and essentially follows that of (5.26).

7.5.2 Vectors

A vector V is any quantity whose transformation under rotations is the same as that for the coordinates, i.e. $\mathbf{x}' = R\mathbf{x}$. Hence if its components are $(V_0, V_\pm)$ it transforms as

$$V'_m = \sum_{m'=-1}^{1} U^{(1)}_{m'm} V_{m'} \,. \tag{7.82}$$

7.5.3 General tensors

In the case of vectors the index m coincides with the Cartesian coordinates index provided a suitable coordinate transformation is performed (see exercise below). However, consider a Cartesian tensor of rank 2 with 9, in general, components T_{ij}. Then under a rotation it will transform as

$$T'_{ij} = \sum_{k,l=1}^{3} R_{ki} R_{lj} T_{kl} \ . \tag{7.83}$$

By treating the indices (ij) as a double index a we may write

$$T_a = \sum_{b=1}^{9} \Gamma(R)_{ba}(R) T_b \ , \qquad \Gamma(R) = R \otimes R \ . \tag{7.84}$$

Indeed, $\Gamma(R)_{kl,ij} = (R \otimes R)_{kl,ij} = R_{ki} R_{lj}$. The rep $\Gamma(R)$ is not irreducible under the rotation group, since according to (7.58)

$$R \otimes R = D^{(1)} \otimes D^{(1)} = D^{(0)} \oplus D^{(1)} \oplus D^{(2)} \ . \tag{7.85}$$

Motivated by the above we define an irreducible tensor of rank j to be any operator $T_m^{(j)}$ with $2j+1$ components which transforms in the j-th irrep of the rotation group, i.e.

$$T_m'^{(j)} = \sum_{m'=1}^{2j+1} U^{(j)}(R)_{m'm} T_{m'}^{(j)} \ . \tag{7.86}$$

Remarks:

• A systematic way to form tensors with specific transformation properties from products of tensors is by utilizing (7.62) in the form

$$T_\mu^{(j)} = \sum_{m_1} A^{j_1 j_2 j}_{m_1, m-m_1} T_{m_1}^{(j_1)} T_{m-m_1}^{(j_2)} \ . \tag{7.87}$$

• In particular, one may obtain a zero rank tensor (a scalar) by combining two tensors of equal rank j as

$$T_0^{(0)} = \sum_{m=-j}^{j} A^{jj0}_{m,-m} T_m^{(j)} T_{-m}^{(j)} \ . \tag{7.88}$$

However, one finds (**Exercise**) that

$$A^{jj0}_{m,-m} = (-1)^{j-m}(2j+1)^{-1/2} \ , \tag{7.89}$$

so that

$$T_0^{(0)} = (2j+1)^{-1/2} \sum_{m=-j}^{j} (-1)^{j-m} T_m^{(j)} T_{-m}^{(j)} \ . \tag{7.90}$$

• Consider two tensors of rank 1, denoted by V and U with components V_m, U_m, with $m = 0, \pm$. Using these we are able to form tensors of rank $j = 0, 1, 2$.

To form a tensor of rank 0 we use (7.90) and obtain

$$T_0^{(0)} = -\frac{1}{\sqrt{3}}(V_0 U_0 - V_1 U_{-1} - V_{-1} U_1) \ . \tag{7.91}$$

To form a vector (tensor of rank 1) we have again from (7.87)

$$T_m^{(1)} = \sum_{m_1=-1}^{1} A^{111}_{m_1, m-m_1} V_{m_1} U_{m-m_1} \ . \tag{7.92}$$

One finds (**Exercise**)

$$T_{\pm 1}^{(1)} = \mp\frac{1}{\sqrt{2}}(V_0 U_{\pm 1} - V_{\pm 1} U_0) \ , \qquad T_0^{(1)} = \frac{1}{\sqrt{2}}(V_1 U_{-1} - V_{-1} U_1) \ . \tag{7.93}$$

Note that there is no term $V_0 U_0$ as the coefficient $A^{111}_{0,0} = 0$.

To form a tensor of rank 2 we use (7.87) and obtain

$$T_m^{(2)} = \sum_{m_1=-2}^{2} A^{112}_{m_1, m-m_1} V_{m_1} U_{m-m_1} \ . \tag{7.94}$$

One finds (**Exercise**)

$$\begin{aligned} T_{\pm 2}^{(2)} &= V_{\pm 1} U_{\pm 1} \ , \qquad T_{\pm 1}^{(2)} = \frac{1}{\sqrt{2}}(V_{\pm 1} U_0 + V_0 U_{\pm 1}) \ , \\ T_0^{(2)} &= \frac{1}{\sqrt{6}}(2 V_0 U_0 + V_1 U_{-1} + V_{-1} U_1) \ . \end{aligned} \tag{7.95}$$

Exercise: We have seen that vectors are defined as those quantities transforming in the $D^{(1)}$ irrep of the $SO(3)$. Hence, there should be an explicit unitary transformation relating $\widetilde{U}^{(1)}_{mm}$, $m, n = 0, \pm 1$ to the rotation matrix R_{ij}, $i, j = 1, 2, 3$. Let the coordinate transformation be

$$\begin{pmatrix} x_1 \\ x_3 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} V_{-1} \\ V_0 \\ V_{+1} \end{pmatrix} \tag{7.96}$$

and denote by C_{im} the components of the above 3×3 unitary matrix. Show that

$$R_{ij} = \sum_{m,n=-1}^{1} C_{jm} \widetilde{U}^{(1)}_{nm} C^{-1}_{ni} , \tag{7.97}$$

or explicitly

$$R = \begin{pmatrix} \frac{1}{2}(a^2 + a^{*2} - b^2 - b^{*2}) & \frac{i}{2}(a^2 - a^{*2} + b^2 - b^{*2}) & ab + a^* b^* \\ -\frac{i}{2}(a^2 - a^{*2} - b^2 + b^{*2}) & \frac{1}{2}(a^2 + a^{*2} + b^2 + b^{*2}) & i(a^* b^* - ab) \\ -(a^* b + ab^*) & i(a^* b - ab^*) & aa^* - bb^* \end{pmatrix} \tag{7.98}$$

and that this is precisely the matrix (7.8) (after flipping the signs of α_0 and α_1).

7.6 The Wigner–Eckart theorem

We are interested in deriving self-consistency relations between integrals of the form

$$T^{j_1 j j_2}_{m_1 m m_2} = \langle \Psi_{j_1 m_1} | T^{(j)}_m | \Phi_{j_2 m_2} \rangle , \tag{7.99}$$

in a notation introduced in (5.11). With steps identical to those that led to (5.61) we have that

$$T^{j_1 j j_2}_{m_1 m m_2} = \sum_{m_1', m', m_2'} U^{(j_1)*}(R)_{m_1' m_1} U^{(j)}(R)_{m' m} U^{(j_2)}(R)_{m_2' m_2} T^{j_1 j j_2}_{m_1' m' m_2'} , \tag{7.100}$$

Next we use the Wigner coefficients and (7.61) to combine the last two U's as

$$U^{(j_2)}(R)_{m_2' m_2} U^{(j)}(R)_{m' m} = \sum_{\tilde{j}=|j-j_2|}^{j+j_2} A^{j_2 j \tilde{j}}_{m_2' m'} A^{j_2 j \tilde{j}}_{m_2 m} U^{(\tilde{j})}(R)_{m_2'+m', m_2+m} . \tag{7.101}$$

Then we obtain

$$T^{j_1 j j_2}_{m_1 m m_2} = \sum_{m_1', m', m_2', \tilde{j}} A^{j_2 j \tilde{j}}_{m_2' m'} A^{j_2 j \tilde{j}}_{m_2 m} U^{(j_1)*}(R)_{m_1' m_1} U^{(\tilde{j})}(R)_{m_2'+m', m_2+m} T^{j_1 j j_2}_{m_1' m' m_2'} \tag{7.102}$$

and we integrate next over dR. The l.h.s. is independent of R so that we simply pick up a factor of $h_{SU(2)}$. For the r.h.s. we use the orthogonality theorem (7.56). We obtain

$$T^{j_1 j j_2}_{m_1 m m_2} = \sum_{m_1', m', m_2', \tilde{j}} A^{j_2 j \tilde{j}}_{m_2' m'} A^{j_2 j \tilde{j}}_{m_2 m} \delta_{\tilde{j} j_1} \frac{\delta_{m_1', m_2'+m'} \delta_{m_1, m_2+m}}{2j_1 + 1} T^{j_1 j j_2}_{m_1' m' m_2'} . \tag{7.103}$$

The $\delta_{\tilde{j}j_1}$ and the fact that $|j-j_2| \leqslant \tilde{j} \leqslant j+j_2$ implies that

$$|j-j_2| \leqslant j_1 \leqslant j+j_2 \tag{7.104}$$

and that in the sum in (7.103) over $\tilde{j}$ only one term survives. Then

$$T^{j_1 j j_2}_{m_1 m m_2} = A^{j_2 j j_1}_{m_2 m}\delta_{m_1,m_2+m}\sum_{m_1',m',m_2'}\frac{\delta_{m_1',m_2'+m'}}{2j_1+1}A^{j_2 j\tilde{j}}_{m_2' m'}T^{j_1 j j_2}_{m_1' m' m_2'}\ . \tag{7.105}$$

The sum appearing in the r.h.s. depends only on j_1, j_2 and j. Hence, we have that

$$T^{j_1 j j_2}_{m_1 m m_2} = \langle\Psi_{j_1 m_1}|T^{(j)}_m|\Phi_{j_2 m_2}\rangle = A^{j_2 j j_1}_{m_2 m}\delta_{m_1,m_2+m}\Lambda_{j_1 j j_2}\ , \qquad |j-j_2| \leqslant j_1 \leqslant j+j_2\ . \tag{7.106}$$

This is the analog of (7.81) which was obtained for the scalar case. The quantity $\Lambda_{j_1 j j_2}$ denotes the triple sum in (7.105). It can be determined by computing just one of the integrals on the l.h.s., for instance the one corresponding to $m=0$ and then for $m_1 = m_2$.

Exercise: Show that for a vector V

$$\begin{aligned}\langle\Psi_{j,m\pm1}|V_\pm|\Phi_{jm}\rangle &= \mp\sqrt{\frac{(j\mp m)(j\pm m+1)}{2j(j+1)}}M_j^{(\pm)}\ ,\\ \langle\Psi_{j,m}|V_0|\Phi_{jm}\rangle &= \frac{m}{\sqrt{j(j+1)}}N_j\ ,\end{aligned} \tag{7.107}$$

for some $M_j^{(\pm)}$ and N_j that do not depend on m.

7.6.1 Example of tensor decomposition

Any tensor of rank j can be decomposed into irreps of the rotation group. We will explicitly work out all the details for how to assign these reps for rank 2 tensor T_{ij}. Such a tensor can be rewritten as

$$T_{ij} = \Psi^{(0)}\delta_{ij} + \Psi^{(1)}_{ij} + \Psi^{(2)}_{ij}\ , \qquad i = 1,2,3\ , \tag{7.108}$$

where

$$\Psi^{(0)} = \frac{T}{3}\ , \qquad \Psi^{(1)}_{ij} = \frac{1}{2}(T_{ij}-T_{ji})\ , \qquad \Psi^{(2)}_{ij} = \frac{1}{2}(T_{ij}+T_{ji}) - \frac{T}{3}\delta_{ij} \tag{7.109}$$

with the definition

$$T = \sum_{i=1}^{3} T_{ii} \ . \tag{7.110}$$

In accordance with (7.85), the superscripts imply that the $\Psi^{(j)}$'s belong to the irrep j of the rotation group. To show that we note that T_{ij} transforms as $x_i y_j$ under such rotations, where x_i and y_i are the Cartesian components of two vectors. We will transform from x_i to V_m using eqnx123u and similarly from y_i to U_m. This will facilitate to determine the transformation rules of the various pieces in the $\Psi^{(j)}$'s in (7.109). In this way we will associate the $\Psi^{(j)}$'s with tensors of the type $T_m^{(j)}$ which transform in the m-th row of the j irrep of the rotation group.

We start by rewriting (7.91) using (7.96) to obtain that

$$T_0^{(0)} = -\frac{1}{\sqrt{3}} \sum_{i=1}^{3} x_i y_i \ . \tag{7.111}$$

Hence we have

$$T_0^{(0)} = -\frac{1}{\sqrt{3}} T \tag{7.112}$$

and therefore the association

$$\Psi^{(0)} = -\frac{1}{\sqrt{3}} T_0^{(0)} \ , \tag{7.113}$$

which shows that indeed $\Psi^{(0)}$ transforms in the $j = 0$ irrep.

Similarly, rewriting (7.93) we obtain that

$$T_{\pm 1}^{(1)} = \frac{1}{2}(x_3 y_1 - x_1 y_3) \pm \frac{i}{2}(x_3 y_2 - x_2 y_3) \ , \qquad T_0^{(1)} = \frac{i}{\sqrt{2}}(x_1 y_2 - x_2 y_1) \ . \tag{7.114}$$

Therefore

$$T_{\pm 1}^{(1)} = \frac{1}{2}(T_{31} - T_{13}) \pm \frac{i}{2}(T_{32} - T_{23}) \ , \qquad T_0^{(1)} = \frac{i}{\sqrt{2}}(T_{12} - T_{21}) \ . \tag{7.115}$$

Hence we have for the 3 functions that

$$\Psi_{12}^{(1)} = -\frac{i}{\sqrt{2}} T_0^{(1)} \ , \qquad \Psi_{23}^{(1)} = \frac{i}{2}(T_{+1}^{(1)} - T_{-1}^{(1)}) \ , \qquad \Psi_{31}^{(1)} = \frac{1}{2}(T_{+1}^{(1)} + T_{-1}^{(1)}) \ , \tag{7.116}$$

which explicitly demonstrates that they indeed form a basis for the $j = 1$ irrep.

Using the above procedure we may also rewrite (7.95). We find that (**Exercise**)

$$\begin{aligned} T^{(2)}_{\pm 2} &= \frac{1}{2}(T_{11} - T_{22}) \pm \frac{i}{2}(T_{12} + T_{21}) \,, \\ T^{(2)}_{\pm 1} &= \mp\frac{1}{2}(T_{13} + T_{31}) - \frac{i}{2}(T_{23} + T_{32}) \,, \\ T^{(2)}_{0} &= \frac{1}{\sqrt{6}}(2T_{33} - T_{11} - T_{22}) \,. \end{aligned} \tag{7.117}$$

There are 5 states in the $j = 2$ irrep (symmetric and traceless). A convenient basis is

$$\begin{aligned} &\Psi^{(2)}_1 = \Psi^{(2)}_{23} \,, \qquad \Psi^{(2)}_2 = \Psi^{(2)}_{31} \,, \qquad \Psi^{(2)}_3 = \Psi^{(2)}_{12} \,, \\ &\Psi^{(2)}_4 = \frac{1}{\sqrt{2}}(\Psi^{(2)}_{11} - \Psi^{(2)}_{22}) \,, \qquad \Psi^{(2)}_5 = \frac{1}{\sqrt{6}}(2\Psi^{(2)}_{33} - \Psi^{(2)}_{11} - \Psi^{(2)}_{22}) \,. \end{aligned} \tag{7.118}$$

Using (7.117) we find that (**Exercise**)

$$\begin{aligned} &\Psi^{(2)}_1 = \frac{i}{2}(T^{(2)}_{+1} + T^{(2)}_{-1}) \,, \quad \Psi^{(2)}_2 = -\frac{1}{2}(T^{(2)}_{+1} - T^{(2)}_{-1}) \,, \quad \Psi^{(2)}_3 = -\frac{i}{2}(T^{(2)}_{+2} - T^{(2)}_{+2}) \,, \\ &\Psi^{(2)}_4 = \frac{1}{\sqrt{2}}(T^{(2)}_{+2} + T^{(2)}_{-2}) \,, \qquad \Psi^{(2)}_5 = T^{(2)}_0 \,, \end{aligned} \tag{7.119}$$

indeed verifying that they form a basis for the $j = 2$ irrep.

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Suggested bibliography

[1] W. Ledermann, *Introduction to Group Characters*, Cambridge University Press, (1977).

[2] M. Tinkham, *Group Theory and Quantum Mechanics*, Dover, (2003).

[3] M. Hamermesh, *Group Theory and its Applications to Physical Problems*, Addison-Wesley, (1989).