Conformal Field Theory Lecture Notes

Perpetually in progress

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Abstract

Lecture notes prepared for the master level students at the department of Physics of the National and Kapodistrian University of Athens. In particular an introduction is given to those areas of CFT that are most relevant for those interested to take a fist course in the subject with a direction for string theory applications. These lecture notes are perpetually in progress and they will be updated with new material as we may see fit.

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Introduction

Conformal symmetry is a potent tool in the construction of two-dimensional conformal quantum field theories which are very special in the following sense. The symmetry group of transformations which leaves angles invariant in Minkowski space is the conformal group. While in d dimensions the conformal group is isomorphic to the Poincaré group in $d + 2$ dimensions and thus it is finite dimensional, in two dimensions there is an infinite variety of conformal transformations and thus the symmetry algebra corresponding to these conformal transformations is infinite dimensional. This is a very powerful tool since this high degree of symmetry imposes many natural constraints so that any QFT in two dimensions with conformal symmetry has a structure that makes it clearly arranged. There are many examples of such theories which are completely solvable in the sense that one can compute accurately in principle all the correlation functions, from which observable quantities are obtained in field theories. Making some times such exact statements in nontrivial situation without relying on the mysteries of perturbation theory is at least a very satisfying and interesting result.

However, one might say that although all this is true, two dimensions are not quite enough to describe what seems to be the real world in four space-time dimensions and this is a fair argument. This raises the question whether or not two-dimensional conformal field theories are significant, if not at all important, as a language in physics and if their structure can capture, describe and substantiate measurable processes. The answer to this question is threefold, at least to our knowledge.

Firstly, in statistical and condensed matter physics there are many models and theories which take place in two dimensions and thus two-dimensional conformal field theories play and essential role. For example, one might be interesting in phenomena which are confined on the two-dimensional boundary of a three-dimensional object, or a system with one spatial dimension which evolves in time whose history is thus a two-dimensional surface. If these situations are accompanied by ceratin symmetries, the most important of which is scale invariance, then two-dimensional conformal field theory can be utilized. The critical Ising model whose continuum limit is described by a two-dimensional conformal field theory of central charge $c = 1/2$, is probably a famous example of such a situation.

Furthermore, two-dimensional conformal field theory is intimately connected, in a sense even identical with perturbative string theory. String theory is the most famous candidate for a grand unified theory that describes all known physical interactions. That is, electromagnetism, gravity as well as the weak and strong interactions, all in a unified manner. The underlying formulation of string theory is described by an action principle, where the action is an integral over the two-dimensional surface swept out by the superstring as it propagates in space and time. This action is invariant under conformal transformations of the worldsheet coordinates and Weyl transformations of the worldsheet metric, which implements the conformal symmetry.

Last but not least, two-dimensional conformal field theory serves as an interface between physics and mathematics. This might seem tautological in the sense that most of physics is formulated in some kind of mathematical language, but the above statement is meant in the stronger sense that both mathematicians and physicists pursue common research with open mind for the views and ideas of the other side. While in most other QFTs the mathematicians some times cannot make sense of the concepts employed by physicists, in a two-dimensional conformal field theory this goal is much closer to be achieved. For example, in d-dimensional QFTs one usually assumes a path integral description of the theory, however the path integral is not a well defined mathematical object. In other words, the path integral approach has the disadvantage of not being defined rigorously, because it is unclear what measure one may put on the infinite dimensional space the path integral is over. In a two-dimensional conformal field theory, a path integral description is not explicitly needed, although one can always implicitly assume one. String theory was also the motivation for Segal to give his abstract definition of conformal field theory. His work has been highly influential for many mathematicians working on conformal field theory but we will not go any further into this matter. The point here is that mathematicians on one hand can be inspired by the intuition and insight of physicists and use this as a motivation in order to develop new structures or gain better understanding of known ones. Physicists on the other hand can appreciate and use these deeper mathematical structures in order to uncover the fundamental structure of a physical system, otherwise it should not be spoken of true understanding.

In these lecture notes a short introduction to Conformal Field Theory (CFT) is presented. It should be noted however, that it is beyond the scope of these notes to present a full summary of CFT. Conformal field theory is a highly developed subject with many connections to different areas of physics and mathematics as well as with many excellent reviews and textbooks available. A selection recommended by the authors, in alphabetical order is

Recommended literature on 2-dimensional CFT

- [ASG89] An introduction by Alvarez-Gaume, Sierra and Gomez, written with an emphasis on the connection to knots and quantum groups.
- [BPZ] The original paper by Belavin, Polyakov and Zamolodchikov.
- [BYB] The book by Di Francesco, Mathieu and Sénéchal, which develops CFT from first principles. The treatment is self-contained, pedagogical, exhaustive and includes background material on QFT, statistical mechanics, Lie and affine Lie algebras, WZW models, the coset construction e.t.c.
- [Ca08] Lectures given at Les Houches (2008) by John Cardy with emphasis to statistical mechanics.
- [Gab99] An overview of CFT centered on the role of the symmetry generating chiral algebra by Matthias Gaberdiel.
- [Gin] Lectures given at Les Houches (1988) by Paul Ginsparg.
- [Se02] The axiomatic formulation of CFT by Segal in the language of category theory and modular functors.

In the following, an introduction is given to those areas of CFT that are most relevant for those interested to take a fist course in the subject with a direction for string theory applications. In a few cases the results might just be stated since they are considered as standard in the literature and the readers may refer themselves to the recommendations mentioned above or to citations within the main text, for further details.

Exercise 0.1. We certainly did not manage to remove all the errors from these notes. The first exercise is to find all the errors and send them to us.

1 Conformal Invariance

1.1 Symmetry in Physics

This subsection is somewhat of general interest since we will explain in some detail what does one mean by a symmetry in physics. The ideas developed here will be used later on when we will discuss the consequence of an infinitesimal continuous symmetry transformation on the correlation functions of the theory and the Ward identities.

Symmetries are an important concept in physics. Recent theories are almost entirely constructed from symmetry considerations alone. Some notable examples are gauge theories, supergravity theories and two-dimensional conformal field theories. In this approach one demands the existence of a certain symmetry and wonders what theories with this property one can construct.

Recall for example, in quantum mechanics the states of a quantum system are elements of the Hilbert space $\mathcal{H}$. Given a state $\psi(0) \in \mathcal{H}$ at time zero, its time evolution is described by a self-adjoint operator H, the Hamiltonian on $\mathcal{H}$. Thus, at time t the system will be in the state

$$\psi(t) = e^{\frac{t}{i\hbar}H}\psi(0). \tag{1.1}$$

In general, given a self adjoint operator $A \in \mathcal{H}$, such that $[A, H] = 0$, one can consider a one parameter family of operators $U_A(s) = e^{isA}$, for $s \in \mathbb{R}$. The operators $U_A(s)$ are unitary so they preserve probabilities and commute with time-evolution:

$$\begin{tikzcd} \psi \arrow[rr, "\text{evolve}"] \arrow[d, "U_A(s)"'] \arrow[drr, phantom, "\circlearrowleft"] & & e^{\frac{t}{i\hbar}H}\psi \arrow[d, "U_A(s)"] \\ U_A(s)\psi \arrow[rr, "\text{evolve}"'] & & U_A(s)e^{\frac{t}{i\hbar}H}\psi = e^{\frac{t}{i\hbar}H}U_A(s)\psi \end{tikzcd} \tag{1.2}$$

In a QFT on the other hand, the symmetries will act on the fields of the theory. These fields are scalar fields, vector fields and spinor fields. In these notes we will mostly be concerned with scalar fields ϕ. A *local field* $\phi(x) \equiv \phi(x, t)$ arises from giving the time-zero field $\phi(\vec{x})$, time dependence generated by a local Hamiltonian H,

$$\phi(x) = U_H^\dagger(t)\phi(\vec{x})U_H(t), \tag{1.3}$$

where $U_H(t) = e^{-itH}$, (in units where $\hbar = 1$) is the time evolution operator. Then, a *symmetry* is an invertible map f on the space of fields (or space of states) which commutes with the time evolution map ρ, where ρ is known as a *projective representation* and it acts on the time-zero fields as in (1.3) i.e. $\phi \mapsto U^\dagger\phi U$:

$$\begin{tikzcd} \phi \arrow[r, "\rho"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & \phi(x) \arrow[d, "f"] \\ \phi' \arrow[r, "\rho"'] & \phi'(x) \end{tikzcd} \tag{1.4}$$

In words, what the above commutative diagram says is the following. If one started from ϕ and evolved in time with ρ to arrive to $\phi(x)$ and then performed a symmetry transformation

f, to finally arrive to $\phi'(x) = f(\phi(x))$, it would have been the same as if one started from ϕ and first performed the symmetry transformation f, followed by ρ. That is $f \circ \rho = \rho \circ f$.

Example 1.1. Lets take f to be the map $U_\theta(q)\colon \theta \mapsto e^{iq\theta} \in U(1)$ for $q \in \mathbb{R}$. Then (1.4) simply says that

$$U_\theta(q)\left(U_H(t)^\dagger \phi(\vec{x}) U_H(t)\right) = \phi'(x) = U_H(t)^\dagger \left(U_\theta(q)\phi(\vec{x})\right) U_H(t). \tag{1.5}$$

This equality holds because θ and H commute. This example is a $U(1)$ symmetry, one can gauge such a symmetry if one supposes that the parameter θ is allowed to be a function of space-time: $\theta(x)$. We see that the map $U_\theta(q)$ provides a representation of $U(1)$. One can now try to generalize this by replacing $U(1)$ by any Lie group G and taking the fields to take values in a space that carries a representation $R(g)$ of G, for some group element $g \in G$. The group G is usually called the *gauge group* and this generalization is known as a Yang-Mills theory.

Consider now the family of operators $U_T(\epsilon) = e^{i\epsilon_a T^a}$. From the commutative diagram (1.4) we see that in order for this to be a symmetry, (i.e. so that one must be able to write an equality of the form (1.5)) the operators T^a have to commute with H. If the parameters ϵ_a are very small then we take the *infinitesimal symmetry transformation*

$$U_T(\epsilon) = 1 + i\epsilon_a T^a + \mathcal{O}\left(\epsilon^2\right). \tag{1.6}$$

The operators T^a are elements of the Lie algebra $\mathfrak{g}$ of G and they can be thought of as the generators of the infinitesimal symmetry transformation (1.6) since they generate $\mathfrak{g}$. We recall that a Lie algebra can be thought of as the tangent space at the identity of a continuous group G, see figure 1 for example. Then $\{T^a \mid a = 1 \ldots \dim \mathfrak{g}\}$ form a basis, note however, that if $\mathfrak{g}$ is infinite-dimensional, we cannot be sure to find a basis.

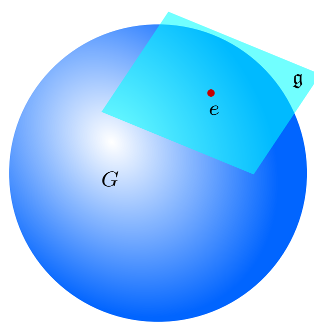


Figure 1: S^3 as a Lie group is isomorphic to the group $SU(2)$ whose Lie algebra is $\mathfrak{su}(2)$.

The infinitesimal symmetry transformations are easier to deal with than the whole family. Therefore, one usually describes continuous symmetries in terms of their generators. The relation between a continuous family of symmetries and their generators, is in essence the relation between Lie groups and Lie algebras, the latter being an infinitesimal version of the former. It turns out that Lie algebras are much easier to work with and still capture most of the structure.

1.2 The Conformal Group in d Dimensions

In this and the following subsection we will talk about some aspects of CFT in d dimensions. The rest of the notes will concentrate in two dimensional CFT which we will develop in more detail. Consider a flat metric $g_{\mu\nu}$ on a space-time manifold $\mathscr{M}$. We say that the transformation $x^\mu \mapsto x'^\mu$ is a *conformal transformation* of the coordinates if it leaves the metric tensor invariant

$$g_{\mu\nu}(x) \mapsto g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x), \tag{1.7}$$

up to a scale factor $\Omega^2(x)$, called the *conformal factor*. This means that the physics of the theory under consideration looks the same at all length scales. In other words, conformal field theories preserve angles but not necessarily lengths.

Example 1.2. Consider the Minkowski metric in two dimensions in light cone coordinates $\sigma_\pm = x \pm t$

$$ds^2 = d\sigma_+ d\sigma_-.$$

Using the conformal transformation $\sigma_\pm = \tan\sigma'_\pm$, with $\sigma'_\pm \in (-\pi/2, \pi/2)$ one obtains

$$ds'^2 = \cos^2\sigma_+ \cos^2\sigma_- ds^2 = d\sigma'_+ d\sigma'_-,$$

from which we immediately see that $\Omega^2 = \cos^2\sigma_+\cos^2\sigma_-$.

Exercise 1.3. Start with the flat space metric in $\mathbb{R}^{1,d-1}$ in polar coordinates

$$ds^2_{\mathbb{R}^{1,d-1}} = -dt^2 + dr^2 + r^2 dS^2_{d-2},$$

where dS^2_{d-2} is the metric on S^{d-2}. Consider now the coordinate transformation

$$t = R\frac{\sinh\left(\frac{\tau}{R}\right)}{\cosh u + \cosh\left(\frac{\tau}{R}\right)}, \qquad r = R\frac{\sinh u}{\cosh u + \cosh\left(\frac{\tau}{R}\right)},$$

where R is the constant radius of S^{d-2}. Find the conformally transformed metric and show that the conformal factor is given by

$$\Omega^2 = \frac{1}{\left(\cosh u + \cosh\left(\frac{\tau}{R}\right)\right)^2}.$$

Identify the topology of the conformally transformed metric. What does R now represents in this new metric?

The set of all conformal transformations in d dimensions forms a group, the *conformal group*, which is isomorphic to the group of Poincaré transformations $SO(d+1,1)$ in $d+2$ dimensions (we will see this isomorphism in a moment), with $\frac{1}{2}(d+1)(d+2)$ parameters and thus it is finite dimensional. For an infinitesimal transformation $x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu(x)$ to be conformal the metric tensor, at first order in ϵ changes as follows

$$\delta g_{\mu\nu} = (\Omega^2 - 1) g_{\mu\nu} = 2\partial_{(\mu}\epsilon_{\nu)}. \tag{1.8}$$

Exercise 1.4. Convince yourself that under the infinitesimal transformation the metric tensor indeed changes as stated in (1.8). Hint: use the transformation rule for the metric up to first order in ϵ.

The conformal factor is determined by taking traces

$$\Omega^2 = 1 + \frac{2}{d}\partial_\mu \epsilon^\mu. \tag{1.9}$$

Combining equations (1.8) and (1.9) we get

$$\partial_{(\mu}\epsilon_{\nu)} = \frac{1}{d}\partial_\rho \epsilon^\rho g_{\mu\nu}. \tag{1.10}$$

The last equation implies that

$$\left(g_{\mu\nu}\partial^2 + (d-2)\partial_\mu\partial_\nu\right)\Omega^2 = 0, \tag{1.11}$$

which after contracting with $g^{\mu\nu}$ reduces to

$$(d-1)\partial^2\Omega^2 = 0. \tag{1.12}$$

Exercise 1.5. Derive equation (1.11) from (1.10). Hint: Apply an extra derivative ∂_ρ on (1.10) and permute indices. Then take a linear combination to arrive at $2\partial_\mu\partial_\nu\epsilon_\rho = (g_{\nu\rho}\partial_\mu + g_{\mu\rho}\partial_\nu - g_{\mu\nu}\partial_\rho)\Omega^2$. Finally, contract with $g^{\mu\nu}$ and apply ∂_ν to arrive to (1.11).

Clearly, the case $d = 1$ is trivial and it simply means that everything is conformal in one dimension since there are no angles. The case $d = 2$ will be treated in more detail later on. Now, for $d > 2$ we see that ϵ is at most quadratic in x, so we have the following possibilities:

- For ϵ zeroth order in x: *translations* $\epsilon^\mu = a^\mu$.
- For ϵ linear in x we have two possibilities:
  1. *scale transformations* $\epsilon^\mu = \lambda x^\mu$
  2. *rotations* $\epsilon^\mu = \omega^\mu{}_\nu x^\nu$, $(\omega_{(\mu\nu)} = 0)$
- For ϵ quadratic in x: *special conformal transformations* or briefly SCTs

$$\epsilon^\mu = b^\mu x^2 - 2x^\mu b\cdot x.$$

Manifestly, the SCTs are nothing but an inversion plus a translation, $\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu$.

More abstractly, with think of the infinitesimal transformations as being generated by the linear operators

$$\begin{aligned} P_\mu &= -i\partial_\mu \\ M_{\mu\nu} &= 2ix_{[\mu}\partial_{\nu]} \\ D &= -ix^\mu\partial_\mu \\ K_\mu &= -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) \end{aligned} . \tag{1.13}$$

The factors of i are chosen to ensure that the generators are Hermitian. These generators can then also be though of as applying on different objects, e.g. space-time fields rather than space-time points. In other words we have an abstract algebra and its action on x^μ is merely one representation.

These linear operators generate the *conformal algebra*, which is locally isomorphic to $SO(p+1,q+1)$. To see this a little counting will help. If we set $p+q=d$ then we see that there are $p+q$ generators for P_μ (translations), $\frac{1}{2}(p+q)(p+q-1)$ for $M_{\mu\nu}$ (rotations), 1 for D (dilations) and finally $p+q$ for K_μ (SCTs). In total, the conformal algebra has $\frac{1}{2}(p+q+1)(p+q+2)$ generators. The conformal algebra is defined by the commutators

$$\begin{aligned}
[D,P_\mu] &= -iP_\mu \\
[D,K_\mu] &= -iK_\mu \\
[K_\mu,P_\nu] &= 2i(\eta_{\mu\nu}D - M_{\mu\nu}) \\
[K_\rho,M_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\
[P_\rho,M_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
[M_{\mu\nu},M_{\rho\sigma}] &= i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho})
\end{aligned} \quad . \tag{1.14}$$

Exercise 1.6. Show this using (1.13).

Redefining now

$$\begin{aligned}
J_{\mu\nu} &= M_{\mu\nu} \\
J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu) \\
J_{-1,0} &= D \\
J_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu)
\end{aligned} \quad , \tag{1.15}$$

with $J_{ab} = -J_{ba}$, and $a,b \in \{-1,0,1,\ldots,d\}$, we see that the new generators satisfy the $SO(d+1,1)$ commutation relations

$$[J_{ab},J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}), \tag{1.16}$$

which shows the isomorphism between the conformal group in d dimensions and $SO(d+1,1)$ in $d+2$ dimensions as mentioned above.

One can integrate to finite conformal transformations. Translations and rotations form the Poincaré group

$$\begin{aligned}
x'^\mu &= x^\mu + a^\mu \\
x'^\mu &= \Lambda^\mu{}_\nu x^\nu, \quad (\Lambda^\mu{}_\nu \in SO(p,q))
\end{aligned} \qquad (\Omega^2 = 1). \tag{1.17}$$

Next for the dilations we have

$$x'^\mu = \lambda x^\mu, \qquad (\Omega^2 = \lambda^{-2}), \tag{1.18}$$

while for the SCT's

$$x'^\mu = \frac{bx^2 + x}{b^2x^2 + 2bx + 1}, \qquad (\Omega^2 = (b^2x^2 + 2bx + 1)^2). \tag{1.19}$$

1.2.1 Representations of the Conformal Group

The linear operators in (1.13) are not the full generators of the conformal symmetry since by the discussion in subsection 1.1 they must include the continuous symmetry generators T^a, which form a representation of the conformal group. Under an infinitesimal symmetry transformation the fields transform as

$$\phi(x) \to \phi'(x) = \phi(x) + \delta_\epsilon \phi(x) = (1 - i\epsilon_a T^a)\phi(x). \tag{1.20}$$

where $\delta_\epsilon \equiv \frac{\delta}{\delta \epsilon_a}$, expresses the variation of the field with respect to the infinitesimal parameter ϵ_a. Therefore, one must add T^a to the space-time part of (1.13) in order to obtain the full symmetry. To proceed, it is customary to rewrite (1.20) as

$$\delta_\epsilon \phi(x) \equiv \phi'(x) - \phi(x) = -i\epsilon_a T^a \phi(x), \tag{1.21}$$

Furthermore, the coordinates under a general infinitesimal transformation change as

$$x^\mu \to x'^\mu = x^\mu + \epsilon^a(x)\delta_\epsilon x^\mu. \tag{1.22}$$

Under this change of coordinates the various fields change also as

$$\phi_a(x) \to \phi'_a(x') = X_a[\phi(x)]. \tag{1.23}$$

This means that, the field considered as a mapping $\phi \colon \mathbb{R}^d \to \mathcal{M}$, from space-time to some target space $\mathcal{M}$ is affected in two ways, first by the functional change $\phi' = X[\phi]$ and second by the change of argument $x \to x'$. Expanding to first order in ϵ^a we have

$$\begin{aligned} \phi'(x') &\overset{(1)}{=} \phi(x) + \epsilon^a(x) X_a[\phi(x)] \\ &\overset{(2)}{=} \phi(x'^\mu - \epsilon^a \delta_\epsilon x'^\mu) + \epsilon^a X_a[\phi(x')] \\ &\overset{(3)}{=} \phi(x') - \epsilon^a \delta_\epsilon x'^\mu \partial'_\mu \phi(x') + \epsilon^a X_a[\phi(x')] \end{aligned} \quad , \tag{1.24}$$

where in step (1) we wrote (1.23) in infinitesimal form, in step (2) we did an inverse transformation of the coordinates $x = x' - \epsilon^a(x)\delta_\epsilon x'^\mu$ and finally in step (3) we did an expansion to first order in ϵ^a. From (1.24) it can also be seen that

$$\epsilon^a X_a[\phi(x)] = \phi'(x') - \phi(x). \tag{1.25}$$

We can treat x' as a dummy variable in (1.24) to finally take

$$\delta_\epsilon \phi(x) = -\epsilon^a \delta_\epsilon x^\mu \partial_\mu \phi(x) + \epsilon^a X_a[\phi(x)] = -i\epsilon^a T_a \phi(x). \tag{1.26}$$

Therefore, the explicit expression for the generator is

$$\boxed{T_a \phi(x) = i\left(X_a[\phi(x)] - \delta_\epsilon x^\mu \partial_\mu \phi(x)\right).} \tag{1.27}$$

For an infinitesimal translation generated by $\epsilon^\mu = a^\mu$ we see that $x'^\mu = x^\mu + a^\mu$ and therefore, $\phi'(x') = \phi'(x + a) = \phi(x)$, thus $X_a[\phi] = 0$. So we conclude that

$$\boxed{P_\mu = -i\partial_\mu.} \tag{1.28}$$

For an infinitesimal Lorentz transformation $\epsilon^\mu = \omega^\mu{}_\nu x^\nu$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad \Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \qquad (\omega^\mu{}_\nu \ll 1). \tag{1.29}$$

Plugging this into the condition

$$\eta^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} \quad \Longrightarrow \quad \omega_{(\mu\nu)} = 0. \tag{1.30}$$

Exercise 1.7. Show this by keeping ω to first order.

One can use this to write (1.29) as

$$x'^\mu = x^\mu + \omega_{[\nu\sigma]}\eta^{\mu[\nu}x^{\sigma]}, \tag{1.31}$$

which implies

$$\delta_\omega x^\mu = \eta^{\mu[\nu}x^{\sigma]}. \tag{1.32}$$

Under the infinitesimal Lorentz transformations (1.29) the field $\phi(x)$ will transform as

$$\phi'(x') = \phi(\Lambda^\mu{}_\nu x^\nu) = U(\Lambda)\phi(x), \tag{1.33}$$

where $U(\Lambda)$, is a matrix representation of the Lorentz group depending on Λ and to first order in ω, is given by

$$U(\Lambda) = 1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}, \tag{1.34}$$

where the factor of 1/2 compensates for the double counting of transformation parameters caused by the full contraction of indices and $S^{\mu\nu}$ is some Hermitian matrix obeying the Lorentz algebra. Thus, we see that $X^a[\phi] = -\frac{i}{2}S^{\mu\nu}\phi(x)$. Finally, using (1.27) the full generator $M^{\mu\nu}$, of infinitesimal Lorentz transformations can be written as

$$\frac{i}{2}M^{\mu\nu}\phi(x) = \left(x^{[\mu}\partial^{\nu]} - \frac{i}{2}S^{\mu\nu}\right)\phi(x), \tag{1.35}$$

from which we immediately see that

$$\boxed{M_{\mu\nu} = 2ix_{[\mu}\partial_{\nu]} + S_{\mu\nu}.} \tag{1.36}$$

Exercise 1.8. Show that for an infinitesimal dilation $x'^\mu = (1+\lambda)x^\mu$, for which

$$\phi'(x') = (1 - \lambda\Delta)\phi(x), \qquad (\lambda \ll 1), \tag{1.37}$$

with Δ the generator of dilations, the corresponding symmetry generator is given by

$$\boxed{D = -i\left(x^\mu\partial_\mu + \Delta\right).} \tag{1.38}$$

We will use these later on to derive the Ward identities that correspond to each one of these generators.

1.3 Conformal Invariance in 2 Dimensions

In two dimensions there exists an infinite variety of coordinate transformations that, although not everywhere well defined, are locally conformal and they are holomorphic mappings from the complex plane to itself. The local conformal symmetry is of special importance in two dimensions since the corresponding symmetry algebra is infinite-dimensional. These statements will become more clear in a moment.

Note that for $d = 2$ and $g_{\mu\nu} = \delta_{\mu\nu}$, equations (1.10) are just the Cauchy-Riemann equations

$$\partial_1\epsilon_1 = \partial_2\epsilon_2 \quad \text{and} \quad \partial_1\epsilon_2 = -\partial_2\epsilon_1. \tag{1.39}$$

These equations are the conditions for a function to be conformal. If we identify the two dimensional Euclidean space with the complex plane we may write

$$\epsilon(z) = \epsilon_1 + i\epsilon_2, \quad \bar{\epsilon}(\bar{z}) = \bar{\epsilon}_1 + i\bar{\epsilon}_2, \tag{1.40}$$

in the complex coordinates $z = x + iy$ and $\bar{z} = x - iy$. If we denote the metric tensor in complex coordinates as $g_{\alpha\beta}$, where the indices α, β take the values z and $\bar{z}$ in that order and we set

$$\partial \equiv \partial_z \quad \text{and} \quad \bar{\partial} \equiv \partial_{\bar{z}}, \tag{1.41}$$

then the following table summarizes the relation between some quantities in Cartesian and complex coordinates.

Quantity	**Cartesian coordinates** (x, y)	**Complex coordinates** $(z, \bar{z})$
x^μ	$x = \frac{1}{2}(z + \bar{z}),\ y = -\frac{i}{2}(z - \bar{z})$	$z = x + iy,\ \bar{z} = x - iy$
∂_μ	$\partial_x = \partial + \bar{\partial},\ \partial_y = i(\partial - \bar{\partial})$	$\partial = \frac{1}{2}(\partial_x - i\partial_y),\ \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$
$g_{\mu\nu}$	$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$g_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},\ g^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

Table 1: Relation between some quantities in Cartesian and complex coordinates

Exercise 1.9. Show the relations in table 1.

In this language, the holomorphic Cauchy-Riemann equations become

$$\bar{\partial}w(z, \bar{z}) = 0, \tag{1.42}$$

whose solution is any holomorphic mapping $z \mapsto w(z) = z + \epsilon(z)$. Analytic functions automatically preserve angles and we see that there are infinitely many independent such transformations.

Exercise 1.10. Show the holomorphic Cauchy-Riemann equations (1.42) from (1.39) using table 1.

Everything we have said up to now is purely local, we have not yet imposed any conditions for the conformal transformations to be everywhere well defined and invertible. Strictly speaking, in order to form a group, the mappings must be invertible and must map the whole plane to itself (more precisely the Riemann sphere $\mathbb{C} \cup \infty$). One, therefore, must distinguish *global conformal transformations*, which satisfy these requirements, from the local ones, which are not everywhere well defined. In order to proceed and find these global conformal transformations we need to find first the commutator relations for the infinite dimensional local conformal algebra and then mod out the non invertible transformations. We start by taking the basis

$$z \mapsto w(z) = z + \epsilon_n(z), \quad \bar{z} \mapsto \bar{w}(\bar{z}) = \bar{z} + \bar{\epsilon}_n(\bar{z}), \qquad n \in \mathbb{Z}, \tag{1.43}$$

where, $\epsilon_n(z)$ is a polynomial in z of degree $n+1$

$$\epsilon_n(z) = -z^{n+1}, \quad \bar{\epsilon}_n(\bar{z}) = -\bar{z}^{n+1}. \tag{1.44}$$

The corresponding infinitesimal generators are

$$\ell_n = -z^{n+1}\partial, \quad \bar{\ell}_n = -\bar{z}^{n+1}\bar{\partial}. \tag{1.45}$$

These satisfy the algebra

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \quad [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}, \quad [\ell_m, \bar{\ell}_n] = 0. \tag{1.46}$$

Exercise 1.11. Show that the ℓ's satisfy the above algebra.

The holomorphic and antiholomorphic infinitesimal generators, generate the two isomorphic subalgebras $\mathcal{W}$ and $\overline{\mathcal{W}}$ respectively, called the *Witt algebra*. The last relation in (1.46) means that these two subalgebras decouple from each other and thus, in order to take the overall local conformal algebra we must form the direct sum $\mathcal{W} \oplus \overline{\mathcal{W}}$. This in turn means that if we extend the Cartesian coordinates $(x, y) \in \mathbb{R}^2$ to the complex plane, i.e. $(x, y) \in \mathbb{C}$, then the variables z and $\bar{z}$ are independent and $\bar{z}$ is not the complex conjugate of z, but rather a complex coordinate. However, it should be kept in mind that the physical space is the two-dimensional submanifold defined by $z^* = \bar{z}$ on which we recover $(x, y) \in \mathbb{R}^2$. In the quantum case, the Witt algebra (1.46) will be replaced by the Virasoro algebra which has an additional term proportional to a central charge.

As mentioned above, in order to take the global conformal algebra, for which the global conformal transformation are invertible and everywhere well defined (i.e. they have no singularities) we need to mode out the subset of these local conformal transformations which do not have this property. First we note that holomorphic conformal transformations are generated by the vector fields

$$v(z) = -\sum_{n\in\mathbb{Z}} a_n \ell_n = \sum_{n\in\mathbb{Z}} a_n z^{n+1}\partial. \tag{1.47}$$

It is easy to see that in order for $v(z)$ to be well defined as $z \to 0$ and $a_n \neq 0$, we must take $n \geq -1$. To see what happens to $v(z)$ as $z \to \infty$ we make the transformation $z = -1/w$.

$$v(w) = \sum_{n\in\mathbb{Z}} a_n \left(-\frac{1}{w}\right)^{n+1} \left(\frac{dz}{dw}\right)^{-1} \partial_w = \sum_{n\in\mathbb{Z}} a_n \left(-\frac{1}{w}\right)^{n-1} \partial_w. \tag{1.48}$$

Similarly, non-singularity as $w \to 0$ means that $n \le 1$. Therefore, the infinitesimal transformations that are globally well defined are $\{\ell_{-1}, \ell_0, \ell_1\} \cup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$.

Exercise 1.12. Using (1.45) and table 1 show that ℓ_{-1} and $\bar{\ell}_{-1}$ are the generators of translations, $\ell_0 + \bar{\ell}_0$ are the generators of dilation, $i(\ell_0 - \bar{\ell}_0)$ are the generators of rotations and ℓ_1 and $\bar{\ell}_1$ are the generators of SCT's.

The group of global conformal transformations on the Riemann sphere is finite dimensional and consists only of Möbius transformations

$$z \mapsto \frac{az+b}{cz+d}, \qquad ad - bc = 1, \tag{1.49}$$

where $a, b, c, d \in \mathbb{C}$, analogously for $\bar{z}$. To each of these mappings we can associate the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}). \tag{1.50}$$

We easily see that the composition of two maps corresponds to matrix multiplication and the condition $ad - bc = 1$ to $\det A = 1$. Therefore, the global conformal group in two dimensions is isomorphic to the Lie group $PSL(2, \mathbb{C}) \equiv SL(2, \mathbb{C})/\mathbb{Z}_2$ and it is finite dimensional. The reason one quotients by $\mathbb{Z}_2$ is that A and $-A$ define the same transformation.

2 Correlation Functions

You have possibly seen the term "correlation function" many times and wonder what it really means. On the other hand, you are familiar with the uncertainty principle since your school years and from your quantum mechanics courses. A correlation function is the QFT analog of that principle. It is typical for correlation functions to diverge when the positions of two or more fields coincide. Quantum fields $\phi(x)$ are *operator valued distributions* rather than mere functions. This means that although they have a well defined vacuum expectation value (statistical average, or mean value), say within a given volume V

$$\langle 0|\phi(x)|0\rangle := \frac{1}{V}\int_V \mathrm{d}^3x \; \phi(x),$$

the fluctuations of the operator at a fixed point (i.e. its variance) $\langle 0|\phi(x)\phi(x)|0\rangle$ diverges as $V \to 0$. This reflects the infinite fluctuations of a quantum field measured at a precise position.

To the fields $\phi(z, \bar{z})$ in the theory we can associate a scaling dimension Δ and a spin s. Given such a field, we define the *holomorphic conformal dimension* h and its antiholomorphic counterpart $\bar{h}$ as

$$h = \frac{1}{2}(\Delta + s), \quad \bar{h} = \frac{1}{2}(\Delta - s). \tag{2.1}$$

Every conformal transformation $z \mapsto w(z)$ looks locally like a combined rescaling and rotation. The CFT will contain some fields, called *primary fields* which can only see this local behaviour, i.e. whose transformation properties depend only on the first derivative of w. To see this consider for example the metric

$$ds^2 = dz d\bar{z},$$

which under $z \mapsto w(z)$ and $\bar{z} \mapsto \bar{w}(\bar{z})$ transforms as

$$ds^2 \longrightarrow \partial w \bar{\partial} \bar{w} ds^2.$$

This is similar in form to the tensor transformation property. We would like to generalise this to include the conformal dimension of the fields.

A field $\phi(z, \bar{z})$ that under *any local conformal transformations* $z \mapsto w(z)$, $\bar{z} \mapsto \bar{w}(\bar{z})$, transforms as

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}), \tag{2.2}$$

it is called a *primary field* of conformal weight $(h, \bar{h})$. If $\phi(z, \bar{z})$, under *global conformal transformations*, transforms as in (2.2), then it is called a *quasi-primary field*. The fields that do not have this property are known as *secondary fields*.

The infinitesimal version of (2.2), under the conformal mapping $z \mapsto z + \epsilon(z)$ and $\bar{z} \mapsto \bar{z} + \bar{\epsilon}(\bar{z})$, is

$$\delta_{\epsilon,\bar{\epsilon}} \phi(z, \bar{z}) = \left(h\partial\epsilon + \epsilon\partial + \bar{h}\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}\right) \phi(z, \bar{z}). \tag{2.3}$$

We say the theory is covariant under the transformation (2.2) if the n-th correlation functions satisfy

$$\begin{aligned} G'^{(n)}(w_j, \bar{w}_j) &\equiv \langle \phi'_1(w_1, \bar{w}_1) \dots \phi'_n(w_n, \bar{w}_n) \rangle \\ &= \prod_{i=1}^{n} \left(\frac{dw_i}{dz_j}\right)^{-h_i} \left(\frac{d\bar{w}_i}{d\bar{z}_j}\right)^{-\bar{h}_i} \langle \phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle \\ &= \prod_{i=1}^{n} \left(\frac{dw_i}{dz_j}\right)^{-h_i} \left(\frac{d\bar{w}_i}{d\bar{z}_j}\right)^{-\bar{h}_i} G^{(n)}(z_j, \bar{z}_j). \end{aligned} \tag{2.4}$$

Example 2.1. If we act on the 2-point function $G^{(2)}(z_i, \bar{z}_i)$ with $\delta_{\epsilon,\bar{\epsilon}}$ from (2.3) we get

$$\delta_{\epsilon,\bar{\epsilon}} G^{(2)}(z_i, \bar{z}_i) = \langle \delta_{\epsilon,\bar{\epsilon}} \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \rangle + \langle \phi_1(z_1, \bar{z}_1) \delta_{\epsilon,\bar{\epsilon}} \phi_2(z_2, \bar{z}_2) \rangle = 0, \tag{2.5}$$

which gives the differential equation

$$\begin{aligned} &[(h_1\partial_1\epsilon(z_1) + \epsilon(z_1)\partial_1 + h_2\partial_2\epsilon(z_2) + \epsilon(z_2)\partial_2) \\ &\qquad + \left(\bar{h}_1\bar{\partial}_1\bar{\epsilon}(\bar{z}_1) + \bar{\epsilon}(\bar{z}_1)\bar{\partial}_1 + \bar{h}_2\bar{\partial}_2\bar{\epsilon}(\bar{z}_2) + \bar{\epsilon}(\bar{z}_2)\bar{\partial}_2\right)] \, G^{(2)}(z_i, \bar{z}_i) = 0. \end{aligned} \tag{2.6}$$

Setting $\epsilon(z) = 1 = \bar{\epsilon}(\bar{z})$, one can show that $G^{(2)}(z_i, \bar{z}_i)$, depends only on the distance

$$z_{ij} \equiv z_i - z_j, \quad \bar{z}_{ij} \equiv \bar{z}_i - \bar{z}_j. \tag{2.7}$$

Setting $\epsilon(z) = z$ and $\bar{\epsilon}(\bar{z}) = \bar{z}$, one can require that $G^{(2)}(z_i, \bar{z}_i) = C_{12}/z_{12}^{h_1+h_2}\bar{z}_{12}^{\bar{h}_1+\bar{h}_2}$, for some constant C_{12}. Finally, for $\epsilon(z) = z^2$ and $\bar{\epsilon}(\bar{z}) = \bar{z}^2$, one requires that $h_1 = h_2 = h$ and $\bar{h}_1 = \bar{h}_2 = \bar{h}$ to arrive to the result,

$$G^{(2)}(z_i, \bar{z}_i) = \frac{C_{12}}{z_{12}^{2h}\bar{z}_{12}^{2\bar{h}}}. \tag{2.8}$$

In two dimensional CFTs, we can always take a basis of quasi-primary ϕ_i with fixed conformal weight and one can normalize their 2-point functions as

$$\langle\phi_i(z,\bar{z})\phi_j(w,\bar{w})\rangle = \frac{\delta_{ij}}{(z-w)^{2h_i}(\bar{z}-\bar{w})^{2\bar{h}_i}}. \tag{2.9}$$

2.1 The Energy-Momentum Tensor

We would now like to explore the consequences of conformal invariance for correlation functions in a fixed domain (usually the entire complex plane). It is necessary to consider transformations which are not conformal everywhere, i.e. local conformal transformations. This brings in the *energy-momentum tensor* (or *stress-energy tensor*). The name energy-momentum tensor refers to Minkowski space-time while the name stress-energy tensor refers to the elastic properties of materials. In a slight abuse of notation we will use both names. In a classical field theory it is defined as the Noether current which is conserved and symmetric, in response of the action S to a general infinitesimal transformation $\epsilon^\mu(x)$,

$$\delta S = -\int \mathrm{d}^2x\; T^{\mu\nu}\partial_\mu\epsilon_\nu = -\int \mathrm{d}^2x\; T^{\mu\nu}\partial_{(\mu}\epsilon_{\nu)}. \tag{2.10}$$

This is valid even if the equations of motion are not satisfied. Then equations (1.8) and (1.9) imply that the corresponding variation of the action under an infinitesimal conformal transformation is

$$\delta S = \int \mathrm{d}^2x\; T^\mu{}_\mu(\Omega^2-1) = 0, \tag{2.11}$$

where $\Omega^2 = 1-\partial_\nu\epsilon^\nu$ is not an arbitrary function. The tracelessness of $T^{\mu\nu}$ then implies the invariance of the action under conformal transformations. Altogether, respectively in that order, the properties of the stress tensor which originate from invariance under, rotations, rescaling and its conservation law and when its position does not coincide with that of other fields, are

$$\boxed{T_{[\mu\nu]} = 0, \quad T^\mu{}_\mu = 0, \quad \partial_\mu T^{\mu\nu} = 0} \tag{2.12}$$

i.e. symmetric, traceless and conserved. There is a quantum version of the above relations demonstrated by the so-called Ward identities that we will see in the next subsection. The relations between its components $T_{\alpha\beta}$ in complex $(z,\bar{z})$ and $T_{\mu\nu}$ in Cartesian (x,y) coordinates are

$$\begin{array}{ccc} T_{zz} = \frac{1}{4}(T_{11}-2iT_{21}-T_{22}) & T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{11}+2iT_{21}-T_{22}) & T_{z\bar{z}} = T_{\bar{z}z} = 0 \\ T_{xx} = T_{zz}+T_{\bar{z}\bar{z}} & T_{yy} = -T_{xx} & T_{xy} = i(T_{zz}-T_{\bar{z}\bar{z}}) = T_{yx} \end{array} \tag{2.13}$$

Exercise 2.2. Show this using the transformation property of $T_{\mu\nu}$, as well as table 1.

The conservation law $g^{\alpha\gamma}\partial_\gamma T_{\alpha\beta} = 0$, implies that

$$\bar{\partial}T = \partial\bar{T} = 0. \tag{2.14}$$

Therefore, the energy-momentum tensor splits into a holomorphic and an antiholomorphic part and it is customary to write these parts as $T \equiv T(z) \equiv T_{zz}$ and $\bar{T} \equiv \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}$, respectively. It will also be useful to define a renormalized version thereof by

$$T(z) \equiv -2\pi T_{zz}, \quad \bar{T}(\bar{z}) \equiv -2\pi \mathrm{T}_{\bar{z}\bar{z}}. \tag{2.15}$$

2.2 Ward Identities

In this subsection we consider the consequences of a continuous symmetry transformation, explained in subsection 1.1, on the correlation functions

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{1}{Z} \int [\mathcal{D}\phi] \; \phi(x_1) \dots \phi(x_n) e^{-S[\phi]}, \tag{2.16}$$

where Z is the vacuum functional. Under a continuous symmetry transformation of the action and the integration measure the correlation functions of the theory are constrained via the so-called Ward identities. Since correlation functions are the main object of study in a quantum theory, one may say, that the Ward identities are the quantum analog of Noether's theorem.

The variation of the action under a symmetry transformation $\delta\phi(x) = \phi'(x) - \phi(x)$ is given by[1]

$$\begin{aligned} \delta S &= \int \mathrm{d}^2 x \; \delta\mathcal{L}(\phi, \partial_\mu \phi) \\ &= \int \mathrm{d}^2 x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \partial_\mu (\delta\phi) \right\} \\ &= \int \mathrm{d}^2 x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \right] \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \right) \right\} \end{aligned} \tag{2.17}$$

When the equations of motion are satisfied the term in the square brackets vanishes, so we are left with

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \right). \tag{2.18}$$

However, for the transformation $\delta\phi$ to be a symmetry, the Lagrangian must change by a total derivative $\delta\mathcal{L} = \partial_\mu F^\mu$. Equating this with (2.18) we get the conserved current

$$\partial_\mu j^\mu = 0, \tag{2.19}$$

with

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \delta\phi - F^\mu. \tag{2.20}$$

In particular, one may show that under (1.22) and (1.23) the action transforms as

$$\delta S = -\int \mathrm{d}^2 x \; \left(j^\mu\right)_a \partial_\mu \epsilon_a(x). \tag{2.21}$$

[1] The arguments presented here apply also in d dimensions and not just two that we will keep using just for the sake of argument.

Then the conservation law (2.19) simply follows from Noether's theorem, i.e. if the field configuration obeys the classical equations of motion, the action is invariant under any variation of the fields for any position dependent parameters $\epsilon_a(x)$.

We consider now the infinitesimal symmetry transformation

$$\delta\phi(x) \equiv \phi'(x) - \phi(x) = -i\epsilon_a(x)T^a\phi(x), \quad (\epsilon_a(x) \ll 1), \tag{2.22}$$

acting on the correlation functions (2.16). Note that the positions are the same on both sides and that the parameters $\epsilon_a(x)$ depend now on the position. Under such a local transformation the action is not invariant and its variation $\delta S = S' - S$, is given by (2.21). Thus, one may write

$$\begin{aligned}
\langle\phi(x_1)\dots\phi(x_n)\rangle &\overset{(1)}{=} \frac{1}{Z}\int[\mathcal{D}\phi']\ \phi'(x_1)\dots\phi'(x_n)e^{-S'[\phi']}\\
&\overset{(2)}{=} \frac{1}{Z}\int[\mathcal{D}\phi']\ \{\phi(x_1)\dots\phi(x_n) + \delta(\phi(x_1)\dots\phi(x_n))\}e^{-(S[\phi]+\delta S[\phi])}\\
&\overset{(3)}{=} \frac{1}{Z}\int[\mathcal{D}\phi']\ \{\phi(x_1)\dots\phi(x_n) + \delta(\phi(x_1)\dots\phi(x_n))\}e^{-\left(S[\phi]+\int d^2x\ \partial_\mu(j^\mu)_a\epsilon_a(x)\right)}\\
&\overset{(4)}{=} \langle\phi'(x_1)\dots\phi'(x_n)\rangle - \int d^2x\ \partial_\mu\langle(j^\mu)_a\,\phi(x_1)\dots\phi(x_n)\rangle\epsilon_a(x)
\end{aligned} \tag{2.23}$$

In step (1) we did not perform a real change of integration variables, we just renamed the dummy integration variable $\phi \to \phi'$. In step (2) we performed a change of functional integration variables and we assumed that the functional integration measure is invariant under the local transformation (2.22), i.e. $[\mathcal{D}\phi'] = [\mathcal{D}\phi]$. In step (3) we integrated by parts (2.21) and substituted the result for δS. Finally, in step (4) we expanded to first order in ϵ and used $\delta\phi(x) = \phi'(x) - \phi(x)$ where necessary. In conclusion the above yields

$$\langle\delta\phi(x_1)\dots\phi(x_n)\rangle = \int d^2x\ \partial_\mu\langle(j^\mu)_a\,\phi(x_1)\dots\phi(x_n)\rangle\epsilon_a(x). \tag{2.24}$$

On the other hand one may write the variation explicitly as

$$\begin{aligned}
\langle\delta\phi(x_1)\dots\phi(x_n)\rangle &= -i\sum_{j=1}^{n}\langle\phi(x_1)\dots T_a\phi(x_j)\dots\phi(x_n)\rangle\epsilon_a(x_j)\\
&= -i\int d^2x\ \sum_{j=1}^{n}\delta(x-x_j)\langle\phi(x_1)\dots T_a\phi(x_j)\dots\phi(x_n)\rangle\epsilon_a(x)
\end{aligned} \tag{2.25}$$

Finally, since (2.24) holds for an arbitrary infinitesimal function $\epsilon_a(x)$, one arrives at the *Ward identity* for the current $(j^\mu)_a$

$$\boxed{\partial_\mu\langle(j^\mu)_a(x)\phi(x_1)\dots\phi(x_n)\rangle = -i\sum_{j=1}^{n}\delta(x-x_j)\langle\phi(x_1)\dots T_a\phi(x_j)\dots\phi(x_n)\rangle} \tag{2.26}$$

This identity says that the current $(j^\mu)_a$ is a conserved quantity, except when its position coincides with that of the other fields.

One can show a similar identity for the variation of the action with respect to the fields.

$$\boxed{\left\langle \frac{\delta S}{\delta\phi(x)}\phi(x_1)\dots\phi(x_n)\right\rangle = -\sum_{j=1}^{n}\delta(x-x_j)\langle\phi(x_1)\dots\phi(x_n)\rangle} \tag{2.27}$$

This is known as the *Schwinger-Dyson equation* which says that the classical equation of motion is satisfied by a quantum field inside a correlation function, as far as its space-time argument differs from those of all other fields. We will use this later on to derive the equation of motion for the propagator of the free boson and the free fermion.

Example 2.3. Consider the infinitesimal translation $x^\mu \to x^\mu - a^\mu$, then the field and the Lagrangian will change respectively as

$$\delta\phi = a^\mu\partial_\mu\phi, \qquad \delta\mathcal{L} = a^\mu\partial_\mu\mathcal{L}.$$

We thus get two conserved currents $(j^\mu)_\nu$, one for each of the translations a^ν

$$(j^\mu)_\nu = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\nu\phi - \delta^\mu{}_\nu\mathcal{L} =: T^\mu{}_\nu.$$

Finally, the generator of translations is given in (1.28), substituting into the Ward identity (2.26) the Ward identity associated with translations becomes

$$\boxed{\partial_\mu\langle T^\mu{}_\nu(x)\phi(x_1)\dots\phi(x_n)\rangle = -\sum_{j=1}^{n}\delta(x-x_j)\partial_\mu^j\langle\phi(x_1)\dots\phi(x_n)\rangle.} \tag{2.28}$$

☞ **Example 2.4.** Consider the Lorentz transformation (1.29) under which the fields transforms as in (1.33)

$$\phi'(x) = \phi\left(\Lambda^{-1}x\right) = \phi\left(x^\mu - \omega^\mu{}_\nu x^\mu\right)\right) = \phi(x) - \omega^\mu{}_\nu x^\nu\partial_\mu\phi(x).$$

Therefore, $\delta\phi(x) = -\omega^\mu{}_\nu x^\nu\partial_\mu\phi(x)$. Similarly, one may show that the Lagrangian changes by a total derivative

$$\delta\mathcal{L} = -\omega^\mu{}_\nu x^\nu\partial_\mu\mathcal{L} = -\partial_\mu\left(\omega^\mu{}_\nu x^\nu\mathcal{L}\right).$$

From this we take that $F^\mu = -\omega^\mu{}_\nu x^\nu\mathcal{L}$, to find that the conserved current is

$$(j^\mu)^{\nu\rho} = 2T^{\mu[\rho}x^{\nu]}.$$

The generator of the Lorentz transformation is given by (1.36), therefore, after using (2.28) the Ward identity associated with Lorentz transformations is

$$\boxed{\left\langle T^{[\mu\nu]}(x)\phi(x_1)\dots\phi(x_n)\right\rangle = -\frac{i}{2}\sum_{j=1}^{n}\delta(x-x_j)S_j^{\mu\nu}\langle\phi(x_1)\dots\phi(x_n)\rangle.} \tag{2.29}$$

Exercise 2.5. Assuming that the dilation conserved current is

$$(j^\mu)_D = T^\mu{}_\nu x^\nu ,$$

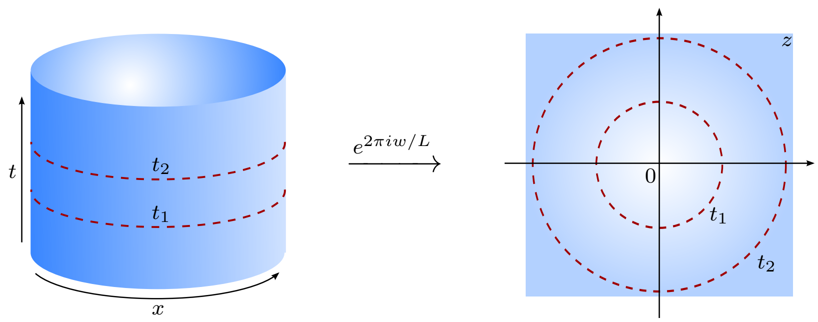
and knowing that the generator of dilations is as given by (1.38), find an expression for the associated Ward identity and by invoking (2.28) show that Ward identity for dilations can take the form

$$\boxed{\langle T^\mu{}_\mu(x)\phi(x_1)\dots\phi(x_n)\rangle = -\sum_{j=1}^{n}\delta(x-x_j)\Delta_j\langle\phi(x_1)\dots\phi(x_n)\rangle .} \tag{2.30}$$

Equations (2.28), (2.29) and (2.30) are the Ward identities associated with conformal invariance.

2.2.1 Holomorphic form of the Ward Identities

We now use *radial quantization*[2] on the complex plane, in order to derive Ward's identities in complex form. Consider an infinite cylinder of circumference L, with the time $t \in \mathbb{R}$, running along the "flat" direction of the cylinder and space being compactified with a coordinate $x \in [0, L]$, the points $(0, t)$ and (L, t) being identified. If we continue to Euclidean space, the cylinder is described by a single coordinate $w = x + it$ (or $\bar{w} = x - it$). We then "explode" the cylinder onto the complex plane (or rather, the Riemann sphere) via the mapping



(2.31)

The remote past ($t \to -\infty$) is situated at the origin $z = 0$, whereas the remote future ($t \to +\infty$) lies on the point at infinity on the Riemann sphere.

With the decomposition (2.14) of the energy-momentum tensor into holomorphic and antiholomorphic parts at hand, we can now define in radial quantization the *conserved charge*

$$Q = \frac{1}{2\pi i}\oint \left(\mathrm{d}z\, T(z)\epsilon(z) + \mathrm{d}\bar{z}\, \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\right), \tag{2.32}$$

from the *conserved current* $J(z,\bar{z}) \equiv T(z)\epsilon(z) + \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})$. The line integral is performed over some circle of fixed radius and our sign conventions are such that both the $\mathrm{d}z$ and the $\mathrm{d}\bar{z}$ integrations are taken in the counter-clockwise sense (hence the symbol $\oint$). Note that

[2]In the operator formalism of CFT one distinguishes a time direction from a space direction. This is natural in Minkowski space-time, but somewhat arbitrary in Euclidian space-time. This allows one to choose the space direction in more exotic ways, for instance along concentric circles centered at the origin. This choice of space and time leads to the so-called radial quantization of 2d-CFTs.

(2.32) is a formal expression that cannot be evaluated until we specify what other fields lie inside the contour.

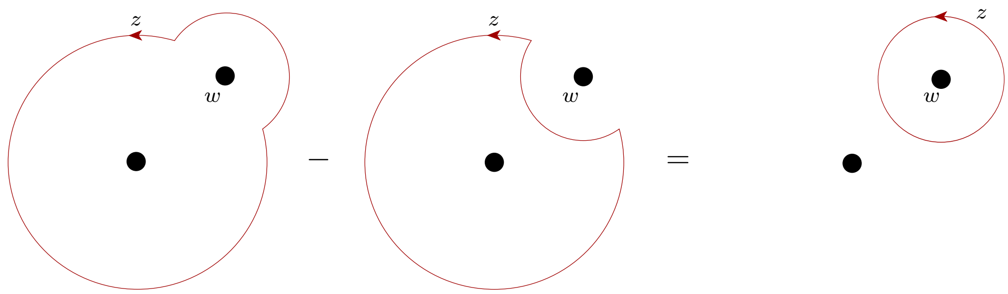
Within radial quantization time ordering inside the definition of correlation functions becomes *radial ordering*. Products of two operators $\mathcal{O}_1(z)\mathcal{O}_2(w)$, in Euclidean space quantization are only defined for $|z| > |w|$. Thus, we define the *radial-order operator*[3]

$$\mathcal{R}\left(\mathcal{O}_1(z)\mathcal{O}_2(w)\right) := \begin{cases} \mathcal{O}_1(z)\mathcal{O}_2(w), & |z| > |w| \\ \mathcal{O}_2(w)\mathcal{O}_1(z), & |z| < |w| \end{cases} . \tag{2.33}$$

Consider now two holomorphic fields $\phi(z)$ and $\psi(z)$ and then take the integral

$$I = \oint_w \mathrm{d}z\, \mathcal{R}\left(\phi(z)\psi(w)\right), \tag{2.34}$$

with the integration contour encircling counterclockwise the point w. We now split the integration contour into two fixed time circles:



, (2.35)

whose difference combines into a single integration about a contour drawn tightly around the point w, which is our initial contour. Therefore, (2.34) becomes

$$\oint_w \mathrm{d}z\, \mathcal{R}\left(\phi(z)\psi(w)\right) = \left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right) \mathrm{d}z\, \mathcal{R}\left(\phi(z)\psi(w)\right) = \left[\oint \mathrm{d}z\ \phi(z), \psi(w)\right]. \tag{2.36}$$

Note that whenever we write a contour integral without specifying the contour of integration it is understood that we integrate at a fixed time, i.e. along a circle centered at the origin. Integrating (2.36) over w we take

$$\left[\oint \mathrm{d}z\ \phi(z), \oint \mathrm{d}w\ \psi(w)\right] = \oint_0 \mathrm{d}w\ \oint_w \mathrm{d}z\, \mathcal{R}\left(\phi(z)\psi(w)\right). \tag{2.37}$$

The point for doing all this is that one can show (see exercise 2.6) that the variation of a primary field $\phi(w, \bar{w})$, is given by the equal time commutator of the field with the charge Q from (2.32)

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) &= [Q, \phi(w,\bar{w})] \\ &= \frac{1}{2\pi i}\left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right)\left\{\mathrm{d}z\, \mathcal{R}\left(T(z)\phi(w,\bar{w})\right)\epsilon(z) + \mathrm{d}\bar{z}\, \mathcal{R}\left(\bar{T}(\bar{z})\phi(w,\bar{w})\right)\bar{\epsilon}(\bar{z})\right\} \\ &= \frac{1}{2\pi i}\oint_w \left\{\mathrm{d}z\, \mathcal{R}\left(T(z)\phi(w,\bar{w})\right)\epsilon(z) + \mathrm{d}\bar{z}\, \mathcal{R}\left(\bar{T}(\bar{z})\phi(w,\bar{w})\right)\bar{\epsilon}(\bar{z})\right\} \\ &= \left(h\partial\epsilon + \epsilon\partial + \bar{h}\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}\right)\phi(w,\bar{w}), \end{aligned} \tag{2.38}$$

[3]The same definition holds but with a minus sign for fermionic operators.

where in the last line we have substituted the desired result, equation (2.3). This is the so-called *conformal Ward identity*. To summarize:

$$\boxed{\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) = [Q,\phi(w,\bar{w})] = \left(h\partial\epsilon + \epsilon\partial + \bar{h}\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}\right)\phi(w,\bar{w}).} \tag{2.39}$$

Inserting the holomorphic and antiholomorphic parts of (2.38), separately in a correlator and using Cauchy's formula one can deduce

$$\boxed{\begin{aligned}\langle T(z)\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle = \sum_{i=1}^{n}\left(\frac{h_i}{(z-w_i)^2}+\frac{\partial_i}{z-w_i}\right)& \\ \cdot\langle\phi_1(w_1,\bar{w}_1)...\phi_n(w_n,\bar{w}_n)\rangle + \mathrm{reg}(z)&\end{aligned},} \tag{2.40}$$

where $\mathrm{reg}(z)$ is a regular function on the complex plane. A similar relation holds for $\bar{T}(\bar{z})$.

Exercise 2.6. Given the conformal Ward identity

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) = \frac{1}{2\pi i}\left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right)\left\{\mathrm{d}z\ \mathcal{R}\left(T(z)\phi(w,\bar{w})\right)\epsilon(z) + \mathrm{d}\bar{z}\ \mathcal{R}\left(\bar{T}(\bar{z})\phi(w,\bar{w})\right)\bar{\epsilon}(\bar{z})\right\},$$

with the help of equations (2.36) and (2.37), convince yourself that it can be written as

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) = [Q,\phi(w,\bar{w})].$$

There is an easier way to derive the Ward identities (2.40) directly from equations (2.28), (2.29) and (2.30). This is the subject of the following exercise.

Exercise 2.7. Using the identity

$$\delta(x) = \frac{1}{\pi}\bar{\partial}\frac{1}{z} = \frac{1}{\pi}\partial\frac{1}{\bar{z}},$$

find explicit expressions for the Ward identities (2.28), (2.29) and (2.30) in complex form, with the n points x_i described now by the $2n$ complex coordinates $(w_i,\bar{w}_i)$. Also for (2.29) it is more convenient to multiply it by $\varepsilon_{\mu\nu}$ (with $\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0\end{pmatrix}$ totaly antisymmetric) and define $s_i \equiv \varepsilon_{\mu\nu}S_i^{\mu\nu}$, i.e. the spin of the field ϕ_i. Then by adding and subtracting the expressions you found for (2.29) and (2.30) and using (2.1) you must get

$$\begin{aligned}2\pi\langle T_{\bar{z}z}\phi(w_1,\bar{w}_1)\ldots\phi(w_n,\bar{w}_n)\rangle &= -\sum_{j=1}^{n}\bar{\partial}\frac{h_i}{z-w_i} \\ 2\pi\langle T_{z\bar{z}}\phi(w_1,\bar{w}_1)\ldots\phi(w_n,\bar{w}_n)\rangle &= -\sum_{j=1}^{n}\partial\frac{\bar{h}_i}{\bar{z}-\bar{w}_i}\end{aligned}.$$

Inserting these relations to the complex expressions that you found for (2.28) deduce (2.40).

2.3 Operator Product Expansion

In section 2 we introduced correlation functions which reflect the infinite fluctuations of a quantum field measured at a precise position. The operator product expansion (OPE) represents the product of two operators at different positions z and w respectively, by a sum of terms, each being a single operator, well defined as $z \to w$, multiplied by a ($\mathbb{k}$-valued, with $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$) function of $z - w$, possibly diverging as $z \to w$. This divergence embodies the infinite fluctuations as the two positions tent to each other. For example, consider the correlation function (2.9), then the OPE of two such fields will be of the form

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) \sim \sum_k C_{ij}{}^k (z-w)^{h_k - h_i - h_j} (\bar{z}-\bar{w})^{\bar{h}_k - \bar{h}_i - \bar{h}_j} \phi_k(w,\bar{w}), \tag{2.41}$$

here $C_{ij}{}^k$, are the *operator product coefficients* and are symmetric in i, j, k. In particular, using the conformal Ward identity (2.40) we see that the OPE of the stress tensor with a primary bulk field is[4]

$$\boxed{T(z)\phi(w,\bar{w}) = \left(\frac{h}{(z-w)^2} + \frac{\partial}{z-w}\right)\phi(w,\bar{w}) + \mathrm{reg}(z-w),} \tag{2.42}$$

with a similar expression for $\bar{T}(\bar{z})$. The most general OPE for T (similarly for $\bar{T}$), consistent with associativity is

$$\boxed{T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{\partial}{z-w}T(w) + \mathrm{reg}(z-w).} \tag{2.43}$$

The constant c is called the *central charge* and fixes the properties of the CFT. We also see that the conformal dimension of T is $h = 2$. The OPE of T with $\bar{T}$ has no poles. A consequence of (2.43) is the transformation behaviour of $T(z)$ under a conformal map $z \mapsto w(z)$

$$T(z) = \left(\frac{dw}{dz}\right)^2 T(w) + \frac{c}{12}\{w; z\}, \tag{2.44}$$

where

$$\{w; z\} := \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2, \tag{2.45}$$

is the *Schwarzian derivative*. Thus, we see that the energy momentum tensor is *not* a primary field. However, the Schwarzian derivative of (1.49) vanishes. This needs to be so, since $T(z)$ is a quasi-primary field.

2.3.1 The Free Boson

The simplest example of a CFT is that of the real free massless scalar field $\phi(x,t)$, usually called the *free boson*. In two dimensions its dynamics (in the massless case) are described by the action

$$S[\phi] = \frac{g}{2}\int \mathrm{d}^2x\ \partial_\mu\phi\partial^\mu\phi, \qquad g \in \mathbb{R}. \tag{2.46}$$

[4] Note that whenever we write OPEs it is understood that they make sense only inside a correlator, we thus drop $\langle - \rangle$.

We are interested in calculating the two point function (or propagator) $G^{(2)}(x,y) \equiv \langle\phi(x)\phi(y)\rangle$. From the Schwinger-Dyson equations (2.27) the propagator satisfies the differential equation

$$-\,g\partial_x^2 G^{(2)}(x,y) = \delta(x-y). \tag{2.47}$$

Exercise 2.8. Show this by integrating by parts (2.46), then find $\frac{\delta S}{\delta\phi(x)}$ and use the Schwinger-Dyson equations (2.27) to arrive to (2.47).

Because of rotational and translational invariance, the propagator will be a function of the distance $r = |x-y|$. Integrating within a disc of radius r centered around y we get the differential equation

$$1 = 2\pi g \int_0^r \mathrm{d}\rho\ \rho \left\{ -\frac{1}{\rho}\partial_\rho \left(\rho G'^{(2)}(\rho)\right) \right\} = -2\pi g r G'^{(2)}(r), \tag{2.48}$$

whose solution is

$$\langle\phi(x)\phi(y)\rangle = -\frac{1}{4\pi g}\ln^2(x-y), \tag{2.49}$$

or in complex coordinates

$$\langle\phi(z)\phi(w)\rangle = -\frac{1}{4\pi g}\ln(z-w). \tag{2.50}$$

Note that the field $\phi(z)$ is not itself a primary field because of the logarithm in (2.50), but its derivative has an OPE

$$\langle\partial\phi(z)\partial\phi(w)\rangle = -\frac{1}{4\pi g}\frac{1}{(z-w)^2}. \tag{2.51}$$

The associated energy momentum tensor is given by

$$T_{\mu\nu} = g\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right), \tag{2.52}$$

which, after using (2.15) and Table 1, it can be written in its quantum version as

$$\begin{aligned} T(w) &= -2\pi g\, {:}\partial\phi(w)\partial\phi(w){:} \\ &\overset{(1)}{=} -2\pi g \lim_{z\to w}\left(\partial\phi(z)\partial\phi(w) - \langle\partial\phi(z)\partial\phi(w)\rangle\right) \\ &\overset{(2)}{=} -2\pi g \lim_{z\to w}\left(\partial\phi(z)\partial\phi(w) + \frac{1}{4\pi g}\frac{1}{(z-w)^2}\right). \end{aligned} \tag{2.53}$$

In step (1) we used point splitting and Wick's theorem (see Appendix A.1) to rewrite ${:}\ast{:} = \mathcal{R}(\ast) - \langle\ast\rangle$, while in step (2) we used equation (2.51). It is also understood that whenever we write the product $A(z)B(w)$, of two operators, we mean the radially ordered product $\mathcal{R}(A(z)B(w))$. As expected, the normal ordering ${:}\ast{:}$ appears to ensure the vanishing of its vacuum expectation value. The OPE of $T(z)$ with itself can be calculated as follows

$$\begin{aligned} T(z)T(w) &= (2\pi g)^2 :\partial\phi(z)\partial\phi(z): :\partial\phi(w)\partial\phi(w): \\ &\overset{(1)}{=} (2\pi g)^2 \left(2\langle\partial\phi(z)\partial\phi(w)\rangle^2 + 4\langle\partial\phi(z)\partial\phi(w)\rangle :\partial\phi(z)\partial\phi(w):\right) \\ &\overset{(2)}{=} \frac{1/2}{(z-w)^4} - \frac{4\pi g}{(z-w)^2} :\partial\phi(z)\partial\phi(w): \\ &\overset{(3)}{=} \frac{1/2}{(z-w)^4} - \frac{4\pi g}{(z-w)^2} :\left(\partial\phi(w) + (z-w)\partial^2\phi(w) + \mathcal{O}\left((z-w)^2\right)\right)\partial\phi(w): \\ &\overset{(4)}{=} \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \end{aligned} \quad . \tag{2.54}$$

In step (1) we used Wick's theorem, while the factors of 2 and 4 arise as a result of counting all possible combinations of terms. In step (2) we used (2.51), in step (3) we Taylor expanded the z-dependent term in $:\partial\phi(z)\partial\phi(w):$ around the point w, and finally, in step (4) we used (2.51) to observe that $\partial T(w) = :\partial^2\phi(w)\partial\phi(w):$ and ignored the $\mathcal{O}\left((z-w)^2\right)$ terms since they are non singular. Comparing the result with (2.43) we observe that the central charge for the free boson is $c = 1$.

Another variation of the above is to consider a free boson with OPE

$$\phi(z)\phi(w) = -\frac{\epsilon}{4\pi g}\ln(z-w), \tag{2.55}$$

with energy momentum tensor and central charge given by

$$T(z) = -2\pi g\epsilon :\partial\phi(z)\partial\phi(z): + Q\partial^2\phi(z), \qquad c = 1 + 48\pi g\epsilon Q^2. \tag{2.56}$$

The extra term proportional to $Q \in \mathbb{R}$ in $T(z)$ above is a total derivative not affecting the energy momentum tensor being a conformal generator. The value of ϵ indicates whether the boson is spacelike ($\epsilon = 1$) or timelike ($\epsilon = -1$). The effect of the extra term in (2.56) is to shift $c > 1$ for $\epsilon = 1$ or $c < 1$ for $\epsilon = -1$. This is an important point because the value of the central charge indicates the unitarity of the theory, this will be briefly explained in subsection 3.2. We thus see that spacelike bosons always produce unitary representations. As for Q we will interpret it as a background charge at infinity later on when we will talk about vertex operators. It is for specific values of Q at $c < 1$ that fit in the Kac table that the theory is unitary.

Exercise 2.9. Show that for $T(z)$ as given in (2.56) and using the OPE (2.55) that the central charge is indeed as given in (2.56). What can you say about unitarity if $Q \to iQ$?

2.3.2 The Free Fermion

The action for a free massless Majorana fermion in two Euclidean dimensions ($\eta_{\mu\nu} = \delta_{\mu\nu}$) is given by

$$S[\Psi] = \frac{g}{2}\int \mathrm{d}^2x\ \bar{\Psi}\partial\!\!\!/\Psi, \tag{2.57}$$

where we have used Dirac's slash notation with $\partial\!\!\!/ \equiv \gamma^\mu \partial_\mu$ and have defined the Dirac adjoint $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$. We recall that a Majorana spinor is a real spinor $\Psi = \Psi^*$. The gamma matrices satisfy the Clifford algebra relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{2.58}$$

In two Euclidean dimensions a representation of the gamma matrices is given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.59}$$

Therefore, one can calculate

$$\gamma^0 \partial\!\!\!/ = \gamma^0 \left(\gamma^0 \partial_0 + \gamma^1 \partial_1\right) = \begin{pmatrix} 0 & \partial_x - i\partial_y \\ \partial_x + i\partial_y & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & \partial \\ \bar{\partial} & 0 \end{pmatrix}. \tag{2.60}$$

Thus, if we write $\Psi = \left(\psi, \bar{\psi}\right)^T$, the action (2.57) can be written in complex form as

$$S[\psi, \bar{\psi}] = g \int \mathrm{d}^2 x \, \left(\bar{\psi} \partial \bar{\psi} + \psi \bar{\partial} \psi\right), \tag{2.61}$$

whose equations of motion read $\partial \bar{\psi} = 0 = \bar{\partial} \psi$ (i.e. the Cauchy-Riemann equations (1.42)). Once more, we are interested in finding the propagator $G^{(2)}_{ij} \equiv \langle \psi_i(z, \bar{z}) \psi_j(w, \bar{w}) \rangle$, where here $i, j = 1, 2$. From the Schwinger-Dyson equations (2.27) one can show that the propagator satisfies the equation of motion

$$g \delta(x - y) \left(\gamma^0 \gamma^\mu\right)_{ik} \partial_\mu G^{(2)}_{kj}(x, y) = \delta_{ij} \delta(x - y), \tag{2.62}$$

or in complex form

$$2g \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} G^{(2)}_{11} & G^{(2)}_{1\bar{2}} \\ G^{(2)}_{\bar{2}1} & G^{(2)}_{\bar{2}\bar{2}} \end{pmatrix} = \frac{1}{\pi} \begin{pmatrix} \bar{\partial} \frac{1}{z - w} & 0 \\ 0 & \partial \frac{1}{\bar{z} - \bar{w}} \end{pmatrix}, \tag{2.63}$$

where the factor of $1/\pi$ comes from the identity $\delta(x) = \frac{1}{\pi} \bar{\partial} \frac{1}{z} = \frac{1}{\pi} \partial \frac{1}{\bar{z}}$. From the equations of motion (2.63) one can read off the solution

$$\langle \psi(z) \psi(w) \rangle = \frac{1}{2\pi g} \frac{1}{z - w}, \quad \langle \bar{\psi}(\bar{z}) \bar{\psi}(\bar{w}) \rangle = \frac{1}{2\pi g} \frac{1}{\bar{z} - \bar{w}}, \quad \langle \psi(z) \bar{\psi}(\bar{w}) \rangle = 0 = \langle \bar{\psi}(\bar{z}) \psi(w) \rangle. \tag{2.64}$$

Comparing this to (2.9) we see that the conformal dimension of the fermions is indeed $h_\psi = \frac{1}{2}$.

Exercise 2.10. Consider now two real fermions ψ_i, $i = 1, 2$ from which we form the complex combinations

$$\psi_\pm = \frac{\psi_1 \pm i\psi_2}{\sqrt{2}}.$$

Show that the OPE of the complex fermion with itself is

$$\psi_+(z) \psi_-(w) = \frac{1}{2\pi g} \frac{1}{z - w}.$$

To calculate the energy momentum tensor we use the Lagrangian from (2.61) and employ the canonical form

$$T^{\alpha\beta} = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha \psi_\delta)} g^{\gamma\beta}\partial_\gamma\psi_\delta - g^{\alpha\beta}\mathcal{L}, \tag{2.65}$$

where $\alpha, \beta, \gamma, \delta = z, \bar{z}$ and ψ_δ are the components of $\Psi = \left(\psi, \bar{\psi}\right)$. The above expression for $T^{\alpha\beta}$ for fermions can simplify even further if we impose the equations of motion which are first order, a trick which we cannot use in the case of a scalar field whose equations of motion are second order in derivatives. This means that we can set $\mathcal{L} = 0$ in $T^{\alpha\beta}$, we are thus left with

$$T^{\alpha\beta} = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha \psi_\delta)} g^{\gamma\beta}\partial_\gamma\psi_\delta. \tag{2.66}$$

Then one may show (see exercise 2.11) that the holomorphic part (similarly the antiholomorphic part) of the energy momentum tensor is given by

$$T(z) = -\pi g :\psi(z)\partial\psi(z): . \tag{2.67}$$

Exercise 2.11. Using equation (2.66) and the Lagrangian from the action (2.61) calculate the components T^{zz}, $T^{\bar{z}\bar{z}}$, $T^{z\bar{z}}$. Then from (2.15) show equation (2.67).

As in the case of the free boson one can perform a similar calculation for the free fermion for the OPE of $T(z)$ with itself (see expertise 2.12) to find

$$T(z)T(w) = \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{2.68}$$

which satisfies (2.43) for $c = 1/2$.

Exercise 2.12. Show equation (2.68) using

$$\langle\partial\psi(z)\psi(w)\rangle = -\frac{1}{2\pi g}\frac{1}{(z-w)^2}, \ \langle\psi(z)\partial\psi(w)\rangle = \frac{1}{2\pi g}\frac{1}{(z-w)^2}, \ \langle\partial\psi(z)\partial\psi(w)\rangle = -\frac{1}{\pi g}\frac{1}{(z-w)^3}.$$

Exercise 2.13. Show that for the complex free fermion considered in exercise 2.10 the energy momentum tensor is given by

$$T(z) = -\pi g \sum_i :\psi_i(z)\partial\psi_i(z): = -\pi g\left(:\psi_+(z)\partial\psi_-(z): + :\psi_-(z)\partial\psi_+(z): \right)$$

and has central charge $c = 1$.

3 The Operator Formalism

In the operator formalism, in a nutshell, a 2d CFT is determined by the following data:

♣ A *space of states*[5] $\mathcal{H}$, a $\mathbb{C}$-vector space, as well as, a *space of fields* $\mathscr{F}$, an $\mathcal{S}$-graded vector space $\mathscr{F} = \bigoplus_{\Delta\in\mathcal{S}} \mathscr{F}^{(\Delta)}$, with $\mathcal{S}$, the *spectrum*, a discrete subset of $\mathbb{R}$ and $0 < \dim \mathscr{F}^{(\Delta)} < \infty$.

[5] May or may not be a Hilbert space. There are examples where the space of states is not a Hilbert space, as the inner product is not positive-definite.

♠ Its *correlation functions*, which are defined for collections of vectors in $\mathscr{F}$, together with an isomorphism $\iota\colon \mathscr{F} \to \mathcal{H}$, the *state-field correspondence*, in the sense that a field inserted at a point can be thought of as a state and vice versa.

As we have seen, two-dimensional CFTs contain an infinite variety of coordinate transformations that although not everywhere well defined, are locally conformal and they are holomorphic mappings from the complex plane to itself. The corresponding infinite-dimensional symmetry algebra of the CFT is related to a preferred subspace $\mathscr{F}_0$ of $\mathscr{F}$, that is characterised by the property that it only allows holomorphic dependance of the coordinates for the correlation functions.

The OPE is associative and if we consider the case of two holomorphic fields $\phi_1, \phi_2 \in \mathscr{F}_0$, then the associativity of the OPE implies that the states in $\mathscr{F}_0$ form a *representation* of the so-called vertex operator algebra $\mathcal{V}$ (to be defined later on). The same also holds for the vertex operator algebra associated to the anti-holomorphic fields and one can decompose the whole space $\mathscr{F}$ (or $\mathcal{H}$) as

$$\mathcal{H} = \bigoplus_{i,\bar{\jmath}\in\mathcal{I}} \left(R_i \otimes_{\mathbb{C}} \bar{R}_{\bar{\jmath}}\right)^{\oplus M_{i\bar{\jmath}}}, \tag{3.1}$$

where $\mathcal{I}$ denotes the set indexing the irreducible representations of $\mathcal{V}$, $\{R_i \mid i \in \mathcal{I}\}$ the corresponding representations and $M_{i\bar{\jmath}} \in \mathbb{N}$ denotes the multiplicity with which the tensor product $R_i \otimes_{\mathbb{C}} \bar{R}_{\bar{\jmath}}$ occurs in $\mathcal{H}$. These statements will make more sense later on.

We must also assume the existence of a vacuum state $|0\rangle \in \mathcal{H}$ upon which the space of states is constructed. In free field theories, the vacuum may be defined as the state annihilated by the positive frequency part of the field [BYB, Sect. 2.1 & 6.1.1].

The $\mathfrak{sl}(2)$-invariant vacuum

To be precise we should call $|0\rangle$ the $\mathfrak{sl}(2)$-invariant vacuum, since e.g. for a non-unitary theory on a cylinder, it is not the state of lowest energy and thus not the real vacuum. It will always be clear from the context whether "vacuum" refers to the state of lowest energy or the $\mathfrak{sl}(2)$-invariant state $|0\rangle$. Moreover, the expressions, *correlation function*, *n-point function*, *amplitude* and *vacuum-expectation value* all refer to the (radially ordered) vacuum-expectation value $\langle 0|\ldots|0\rangle$ with respect to the $\mathfrak{sl}(2)$-invariant vacuum.

3.1 The Virasoro Algebra

We can now define the action of the stress tensor T and its antiholomorphic counterpart $\bar{T}$ on the space of states $\mathcal{H}$, via their mode expansion. In general, a holomorphic (similarly an antiholomorphic) field $\phi(z)$ of conformal dimension $(h, 0)$ can be mode expanded as follows

$$\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-h}\phi_n, \quad \phi_n = \frac{1}{2\pi i}\oint \mathrm{d}z \; z^{n+h-1}\phi(z). \tag{3.2}$$

From (2.43) we know that the stress tensor has conformal dimension $h = 2$, we thus take the mode expansion

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n, \quad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n. \tag{3.3}$$

We see that under the scale change $z \to z/\lambda$, for which $T(z) \to \lambda^2 T(z/\lambda)$, we have $L_{-n} \to \lambda^n L_{-n}$. The operators $L_{-n}, \bar{L}_{-n}$, thus have scaling dimension n. Equation (3.3) is formally inverted by the relations

$$L_n = \frac{1}{2\pi i}\oint \mathrm{d}z\; z^{n+1} T(z), \quad \bar{L}_n = \frac{1}{2\pi i}\oint \mathrm{d}\bar{z}\; \bar{z}^{n+1} \bar{T}(\bar{z}), \qquad n \in \mathbb{Z}. \tag{3.4}$$

From (2.43) one can deduce that the modes fulfil the *Virasoro algebra*

$$\boxed{\begin{array}{l} [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\left(n^3 - n\right)\delta_{n+m,0} \\ \left[L_n, \bar{L}_m\right] = 0 \\ \left[\bar{L}_n, \bar{L}_m\right] = (n-m)\bar{L}_{n+m} + \frac{c}{12}\left(n^3 - n\right)\delta_{n+m,0} \end{array}}\;. \tag{3.5}$$

Note that the Virasoro algebra decomposes into holomorphic and antiholomorphic parts. These are denoted by **Vir** and $\overline{\textbf{Vir}}$, which are generated by the holomorphic and antiholomorphic modes respectively[6].

In these notes we will assume $c = \bar{c}$. In the case where $c = 0$ we retrieve the Witt algebra (1.46). One can identify $L_{-1} + \bar{L}_{-1}$ and $i\left(L_{-1} - \bar{L}_{-1}\right)$ as generators of translations, $L_0 + \bar{L}_0$ and $i\left(L_0 - \bar{L}_0\right)$ as generators of dilations and rotations respectively, while $L_1 + \bar{L}_1$ and $i\left(L_1 - \bar{L}_1\right)$ are generators of special conformal transformations.

The Virasoro algebra is infinite dimensional and it was originally discovered in the context of string theory. To see how one can obtain equations (3.5), one needs to employ the procedure for making contact between OPEs and commutators of operator modes discussed in subsection 2.2. The commutator of two contour integrations $\left[\oint \mathrm{d}z, \oint \mathrm{d}w\right]$ is evaluated by first fixing w and deforming the difference between the two z integrations into a single z contour drawn tightly around the point w, as in (2.35) and (2.37). In evaluating the z contour integration, we may perform operator product expansions to identify the leading behavior as z approaches w. The w integration is then performed without further subtlety. For the modes of the stress-energy tensor, this procedure gives

$$\begin{aligned} [L_n, L_m] &= \frac{1}{(2\pi i)^2}\left[\oint \mathrm{d}z\;, \oint \mathrm{d}w\right] z^{n+1} T(z) w^{n+1} T(w) \\ &= \frac{1}{(2\pi i)^2}\oint\oint \mathrm{d}z\mathrm{d}w\; z^{n+1} w^{n+1}\left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2}\right. \\ &\qquad\qquad \left. + \frac{\partial T(w)}{z-w} + \mathrm{reg}(z-w)\right) \\ &= \frac{1}{2\pi i}\oint \mathrm{d}w\;\left(\tfrac{c}{12}(n+1)n(n-1)w^{n-2}w^{m+1}\right. \\ &\qquad\qquad \left. + 2(n+1)w^n w^{m+1} T(w) + w^{n+1} w^{m+1} \partial T(w)\right). \end{aligned} \tag{3.6}$$

[6] Some times in the literature these are called *chiral* and *antichiral* or *left* and *right moving* parts.

Integrating the last term by parts and combining with the second term gives $(n-m)w^{n+m+1}T(w)$, so performing the w integration, produces the required result.

Exercise 3.1. Do the missing steps described above to show the commutator of **Vir**.

The vacuum state $|0\rangle \in \mathcal{H}$ must be invariant under global conformal transformations. This means that it must be annihilated by $L_{-1,0,1}$ and $\bar{L}_{-1,0,1}$. This, however, can be recovered by the condition that $T(z)|0\rangle$ and $\bar{T}(\bar{z})|0\rangle$ are well defined as $z, \bar{z} \to 0$, which implies

$$L_n|0\rangle = 0, \quad \bar{L}_n|0\rangle = 0, \qquad n \geq -1. \tag{3.7}$$

Performing the corresponding contour integral with (2.42), we get the commutation relations

$$\begin{aligned} [L_n, \phi(w,\bar{w})] &= h(n+1)w^n\phi(w,\bar{w}) + w^{n+1}\partial\phi(w,\bar{w}) \\ \left[\bar{L}_n, \phi(w,\bar{w})\right] &= \bar{h}(n+1)\bar{w}^n\phi(w,\bar{w}) + \bar{w}^{n+1}\bar{\partial}\phi(w,\bar{w}) \end{aligned}. \tag{3.8}$$

Exercise 3.2. Show this using (2.36) and the OPE (2.42).

From the state-field correspondence ♠ we see that when primary fields act on the vacuum, create asymptotic states

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle. \tag{3.9}$$

After applying the relations (3.8) to the asymptotic state

$$|h,\bar{h}\rangle \equiv \phi(0,0)|0\rangle, \tag{3.10}$$

we take

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle, \quad \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle. \tag{3.11}$$

Thus, $|h,\bar{h}\rangle$ is an eigenstate of the Hamiltonian[7]. Similarly,

$$L_n|h,\bar{h}\rangle = \bar{L}_n|h,\bar{h}\rangle = 0, \qquad n > 0. \tag{3.12}$$

Exercise 3.3. Show equations (3.11) and (3.12) by direct application of (3.8) on the vacuum state $|0\rangle$.

3.2 Highest Weight Representations

Highest weight representation are familiar to physicists through the theory of angular momentum. Just as the energy eigenstates of the Hamiltonian of a rotationally invariant system fall into irreducible representations of $\mathfrak{su}(2)$, in a conformaly invariant theory the energy eigenstates of the Hamiltonian fall into representation of the Virasoro algebra (the local conformal algebra). The way one may construct these representations is similar to the $\mathfrak{su}(2)$ case. The only difference here is that the Virasoro algebra is infinite dimensional and thus we are dealing with infinite dimensional representations. However, one may overcome this by

[7]As will be seen later, the Hamiltonian is proportional to $L_0 + \bar{L}_0 - \frac{c}{12}$.

passing to a finite dimensional subspace spanned by *highest weight vectors*, called a *Verma module*. The name module is just another name for *representation space*. Then the associated representations (which can be seen as vectors in the Verma module) are called *highest weight representations*.

From the Virasoro algebra (3.5) we see that no pair of generators commute, so one can choose L_0 to be diagonal in the Verma module. From the defining relations of **Vir** it is easy to see that

$$[L_0, L_{\pm n}] = \mp n L_n, \qquad n > 0. \tag{3.13}$$

Thus, L_n is a *lowering operator* and L_{-n} is a *raising operator*. For a state $|h\rangle$ to be a *highest weight state* one has[8]

$$L_n|h\rangle = \bar{L}_n|\bar{h}\rangle = 0, \qquad n > 0, \tag{3.14}$$

which is compatible with (3.12). This state is, of course, the asymptotic state (3.10) created by applying a primary field $\phi(0)$ of dimension h on the vacuum $|0\rangle$. One can construct more states in the Verma module by applying the raising operators L_{-n} in all possible ways

$$\prod_{i=1}^{n} L_{-k_i}|h\rangle, \qquad 1 \leq k_1 \leq \ldots \leq k_n. \tag{3.15}$$

Recall that since $L_0|h\rangle = h|h\rangle$, then the above state has an L_0 eigenvalue

$$h' = h + \sum_{i=1}^{n} k_i \equiv h + N, \tag{3.16}$$

where $N = \sum_{i=1}^{n} k_i$ is called the *level* of the state. The states in (3.15) are called *descendant states* of the asymptotic state $|h\rangle$ and (3.15) constitutes a basis for the Verma module at level N. Table 2 shows the lowest states of a Verma module.

Level	**Dimension**	**State**
0	h	$\|h\rangle$
1	$h+1$	$L_{-1}\|h\rangle$
2	$h+2$	$L_{-2}\|h\rangle,\ L_{-1}^2\|h\rangle$
3	$h+3$	$L_{-3}\|h\rangle,\ L_{-1}L_{-2}\|h\rangle,\ L_{-1}^3\|h\rangle$
4	$h+4$	$L_{-4}\|h\rangle,\ L_{-1}L_{-3}\|h\rangle,\ L_{-1}^2L_{-2}\|h\rangle,\ L_{-2}^2\|h\rangle,\ L_{-1}^4\|h\rangle$
$\vdots$	$\vdots$	$\vdots$
N	$h+N$	$p(N)$ states

Table 2: Lowest states of the Verma module.

In the table $p(N)$ denotes the partition of the integer N generated by the function

$$\frac{1}{\varphi(q)} := \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} = \sum_{n=0}^{\infty} p(n) q^n, \qquad \left(q \equiv e^{2\pi i \tau}\right), \tag{3.17}$$

[8] This is rather a "lowest" weight state because it is annihilated by L_n rather than L_{-n} but it is customary in many textbooks to be called as a "highest" weight state.

where $\varphi(q)$ is the Euler function and $\tau \in \mathbb{C}$.

Exercise 3.4. Show that the L_0 eigenvalue of (3.15) is indeed as given by (3.16) by acting with L_0 and using (3.13) to commute it past the L_{-k_i}'s.

The inner product of two highest weight states $|i\rangle$ and $|j\rangle$, simply is

$$\langle i|j\rangle = \delta_{ij}. \tag{3.18}$$

If we Hermitian conjugate T and $\bar{T}$ and restricting to the real surface $\bar{z} = z^*$, we get

$$L_n^\dagger = L_{-n}, \quad \bar{L}_n^\dagger = \bar{L}_{-n}. \tag{3.19}$$

This relation together with the Virasoro algebra (3.5) and highest weight condition (3.14) can be used to write the inner product of an arbitrary pair of fields in terms of the inner product of primary fields.

Exercise 3.5. Show (3.19) by first Hermitian conjugating (3.3) on the real surface $\bar{z} = z^*$ and then using the fact that

$$\phi(z)^\dagger = \bar{z}^{-2h}\phi\left(\frac{1}{\bar{z}}\right).$$

One can define an inner product on the Verma module using the Hermitian conjugate (3.19). If we consider the states

$$\prod_{i=1}^{m} L_{-k_i}|h\rangle, \qquad \prod_{i=1}^{n} L_{-l_i}|h\rangle, \tag{3.20}$$

then their inner product simply is

$$\langle h|\prod_{i=m}^{1} L_{k_i}\prod_{i=1}^{n} L_{-l_i}|h\rangle. \tag{3.21}$$

A similar analysis can be done for the Verma modules associated with the antiholomorphic generator $\bar{L}_n$ of $\overline{\mathbf{Vir}}$. Thus, we have seen that the set of modes of the holomorphic part of the stress tensor $\{L_n \mid n \in \mathbb{Z}\}$ generate the holomorphic representations $\{R_i \mid i \in \mathcal{I}\}$, while the set of modes of the antiholomorphic part of the stress tensor $\{\bar{L}_n \mid n \in \mathbb{Z}\}$ generate the antiholomorphic representations $\{\bar{R}_{\bar{\jmath}} \mid \bar{\jmath} \in \mathcal{I}\}$. However, since the two parts decouple, in order to take the physical space of states one needs to take tensor products of the above representations. Thus the space of states decomposes into *highest weight representations* of $\mathbf{Vir} \oplus \overline{\mathbf{Vir}}$ of the form (3.1).

We saw that each module is spanned by a *highest weight state* $|h, \bar{h}\rangle$ and an infinite set of descendent states of the form $L_{m_1} \ldots \bar{L}_{n_1} \ldots |h, \bar{h}\rangle$, with all $m, n < 0$. Once we know the central charge c, of the theory and the conformal weights $\left(h, \bar{h}\right)$, of all primary fields, we can construct the space of states. However, some care has to be taken in the construction of a basis, since not all products of L's and $\bar{L}$'s are linearly independent.

Furthermore, there will be states $|\chi\rangle$ in the Verma module which are of the form (3.15) and which are also annihilated by L_n for all $n > 0$

$$L_n|\chi\rangle = 0, \qquad (n > 0). \tag{3.22}$$

A state other than the highest weight state that is annihilated by L_n for all $n > 0$ is called a *null state*. Null states are orthogonal to all the other states in the Verma module and thus they form a submodule. In particular for a null state we have $\langle\chi|\chi\rangle = 0$. A Verma module which contains one or more null states is reducible. One can construct an irreducible Verma module by quotienting out this null submodule.

Example 3.6. There are many such states, to construct an example just consider the following state at level 2

$$|\chi\rangle = \left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)|h\rangle, \quad \text{with} \quad c = \frac{2h(5-8h)}{2h+1}$$

and $|h\rangle$ a highest weight state, then for $n \geq 0$ we have

$$\begin{aligned} L_n|\chi\rangle &= \left([L_n, L_{-2}] - \frac{3}{2(2h+1)}[L_n, L_{-1}^2]\right)|h\rangle \\ &= \left([L_n, L_{-2}] - \frac{3}{2(2h+1)}\left(L_{-1}[L_n, L_{-1}] + [L_n, L_{-1}]L_{-1}\right)\right)|h\rangle \end{aligned}$$

This can only be non-zero for $n = 0$. Thus we find that $L_0|\chi\rangle = 2|\chi\rangle$. Next we see that $\langle\chi|h\rangle = \langle h|\left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)^\dagger|h\rangle = 0$.

From the above example we see that if we calculate some amplitude between two physical states $\langle h'|h\rangle$ we can shift $|h\rangle \to |h\rangle + |\chi\rangle$. The new state is still physical but the amplitude will remain the same - for any other choice of physical state $|h'\rangle$. In string theory, this is a stringy gauge symmetry whereby two physical states are equivalent if their difference is a null state. This turns out to be the origin of Yang-Mills and other gauge symmetries within string theory.

Exercise 3.7. Consider the following state at level 2

$$|\chi\rangle = \left(L_{-2} + \eta L_{-1}^2\right)|h\rangle.$$

Tune η and h so that $|\chi\rangle$ is a null state. Hint: The conditions $L_1|\chi\rangle = L_2|\chi\rangle = 0$ are sufficient for this, since it then follows from the Virasoro algebra that $L_n|\chi\rangle = 0$, for $n > 2$.

Finally, to conclude this subsection, a representation of **Vir** (similarly of $\overline{\textbf{Vir}}$) is said to be *unitary* if it contains no negative norm states (known as ghosts in string theory). For instance, one can find a simple bound on the values of the central charge c and on the highest weight h in order for the representations to be unitary by considering the norm

$$\langle h|L_n L_{-n}|h\rangle = \left(2nh + \frac{c}{12}n(n^2-1)\right)\langle h|h\rangle. \tag{3.23}$$

We see that if $c < 0$ this becomes negative for n sufficiently large. Therefore, all representations with negative central charge are nonunitary. Furthermore, if $n = 1$ we see that all representations with $h < 0$ are also nonunitary.

Exercise 3.8. Show equation (3.23).

There is a general formula for one to decide whether or not a representation is unitary due to Kac, known as the *Kac determinant*, however, it is beyond the scope of these notes to go into more details so we will not reproduce it here. We will simply mention that whenever this determinant is negative then the representation is nonunitary. The big success of CFT in the study of two dimensional systems is due in great part to the knowledge of the Kac determinant. This formula is of central importance in the theory of *minimal models* but lets not go further into these matters. What is important for our purposes is that one can show that all representations with $c \geq 1$ and $h \geq 0$ are unitary. One can also find unitary representations in the regions where $c \in (0,1)$ and $h > 0$, but this is not always the case. There is a formula for c, h for one to decide which representation are unitary known as *Kac table* but we will not need it here. For those interested see [BYB, Sec. 7.2].

3.3 The Free Boson

In this subsection we will use string theory notation for the coordinates, i.e. $(\sigma^0, \sigma^1) = (\tau, \sigma)$. Consider now a free boson on a cylinder of circumference L, i.e. we demand that $\phi(\tau, \sigma + L) = \phi(\tau, \sigma)$. The field can be Fourier expanded as

$$\phi(\tau,\sigma) = \sum_{n\in\mathbb{Z}} e^{2\pi i n\sigma/L}\phi_n(\tau), \qquad \phi_n(\tau) = \frac{1}{L}\int \mathrm{d}\sigma\ e^{-2\pi i n\sigma/L}\phi(\tau,\sigma). \tag{3.24}$$

After varying the action (2.46) and integrating by parts we get

$$\delta S = \frac{g}{2}\int \mathrm{d}^2\sigma\ \delta\phi\partial^2\phi + \text{boundary terms}. \tag{3.25}$$

Thus we have to simply solve the wave equation whose general solution, after using conformal coordinates $z = e^{2\pi(\sigma^1 + i\sigma^0)/L}$ so that time runs radially, is

$$\phi(z,\bar{z}) = \phi_0 - \frac{i\pi_0}{4\pi g}\ln(z\bar{z}) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n z^{-n} + \bar{a}_n \bar{z}^{-n}\right), \tag{3.26}$$

where ϕ_0 is the zeroth mode of the Fourier coefficient ϕ_n and π_n is the momentum conjugate to ϕ_n. Also the a_n's satisfy

$$[a_n, a_m] = n\delta_{n+m}, \quad [a_n, \bar{a}_m] = 0, \quad [\bar{a}_n, \bar{a}_m] = n\delta_{n+m}. \tag{3.27}$$

This mode expansion can be split into two independent sets of left and right moving oscillators[9]

$$\begin{aligned}\phi(z) &= \frac{1}{2}\phi_0 + \frac{i}{\sqrt{4\pi g}}\left(-a_0\ln z + \sum_{n\neq 0}\frac{a_n}{nz^n}\right)\\ \bar{\phi}(\bar{z}) &= \frac{1}{2}\phi_0 + \frac{i}{\sqrt{4\pi g}}\left(-\bar{a}_0\ln \bar{z} + \sum_{n\neq 0}\frac{\bar{a}_n}{n\bar{z}^n}\right)\end{aligned}, \qquad a_0 = \bar{a}_0 \equiv \frac{\pi_0}{\sqrt{4\pi g}}. \tag{3.28}$$

[9]Note however, that strictly speaking, we cannot write $\phi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$ because the zero mode ϕ_0 would be duplicated in the process. Thus we must keep in mind that $\phi(z)$ is not purely holomorphic.

The operators a_n are creation or annihilation operators associated with the right movers and $\bar{a}_n$ with the left movers respectively. This is a good point to pause for a while and make a remark. If instead of the scalar field $\phi(z,\bar{z})$ one had considered vector fields $X^\mu(z,\bar{z})$, $\mu = 0,\ldots,d$ with $L = 2\pi$ and $g = 1/2\pi\alpha'$ then the action (2.46) would become the action of a string in conformal gauge and the associated mode expansion (3.26) would have described a closed string. Then, in this set up, ϕ_0 is a component of the center of mass of the string and π_0 is the string's total momentum.

Since π_0 commutes with all the a_n and $\bar{a}_n$ the Fock space is built upon a one parameter family of vacua which we will denote by $|\alpha\rangle$, with α the continuous eigenvalue of a_0. Furthermore, the a_n and $\bar{a}_n$ are creation ($n < 0$) and annihilation ($n > 0$) operators, that is

$$\begin{aligned} a_n|\alpha\rangle &= \bar{a}_n|\alpha\rangle = 0 \\ a_0|\alpha\rangle &= \bar{a}_0|\alpha\rangle = \alpha|\alpha\rangle \end{aligned} \qquad (n > 0). \tag{3.29}$$

As we have seen in subsection 2.3.1 the field ϕ is not a primary field but its derivative is

$$i\partial\phi(z) = \frac{1}{\sqrt{4\pi g}} \sum_{n\in\mathbb{Z}} a_n z^{-n-1}. \tag{3.30}$$

Note, we have included the zeroth mode a_0 in the sum. Using the expression for the energy momentum tensor (2.53) we see that

$$T(z) = \frac{1}{2} \sum_{n,m\in\mathbb{Z}} z^{-n-m-2} :a_n a_m: . \tag{3.31}$$

Therefore, we can construct the Virasoro generators (note we do not use normal ordering just yet)

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} a_{n-m}a_m, \quad \bar{L}_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} \bar{a}_{n-m}\bar{a}_m \qquad (n \neq 0). \tag{3.32}$$

Similarly,

$$L_0 = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n. \tag{3.33}$$

Following standard QFT practice of canonical quantization (see exercise 3.9) one can also compute the Hamiltonian and it is seen to take the form

$$H(L) = \frac{2\pi}{L}(L_0 + \bar{L}_0). \tag{3.34}$$

Exercise 3.9. **(a)** By reexpressing the free field Lagrangian (2.46) in terms of the Fourier modes (3.24) find an expression for the Hamiltonian using the momentum conjugate to ϕ_n, defined as $\pi_n = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_n}$ and $[\phi_n, \pi_m] = i\delta_{nm}$. **(b)** Then by defining annihilation and creation operators respectively by

$$\begin{aligned} a_n &= \frac{-i}{\sqrt{4\pi g}}(2\pi g n\phi_n + i\pi_{-n}), \qquad (n > 0) \\ a^\dagger_{-n} &= \frac{i}{\sqrt{4\pi g}}(-2\pi g n\phi_{-n} + i\pi_n), \qquad (n < 0) \end{aligned},$$

with similar expressions for $\bar{a}_n$ and $\bar{a}^\dagger_n$, write the Hamiltonian that you found in terms of these operators. Show also that these operators satisfy the commutation relations (3.27). **(c)** Finally, deduce equation (3.34).

Of course, the $:L_n:$'s satisfy the Virasoro algebra. One can perform a direct calculation but it is notoriously complicated and messy. We will only sketch the proof of how one may proceed in an alternative way. First one calculates the commutator $[L_n, L_m]$ without worrying about normal orderings to find that it obeys the Witt algebra (1.46). When considering normal ordering we must generalize the commutator to

$$[:L_n:, :L_m:] = (n-m):L_{n+m}: + C(n)\delta_{n+m,0}. \tag{3.35}$$

The easiest way to determine the $C(n)$ is to note the following. First one imposes the Jacobi identity

$$[:L_k:, [:L_m:, :L_n:]] + [:L_m:, [:L_n:, :L_k:]] + [:L_n:, [:L_k:, :L_m:]] = 0. \tag{3.36}$$

If we impose that $k+m+n=0$ with $k, m, n \neq 0$ (so that no pair of them adds up to zero) then this reduces to

$$(m-n)C(k) + (n-k)C(m) + (k-m)C(n) = 0.$$

Picking $k=1$ and $m=-n-1$ and noting that by definition $C(n)$ is odd, we learn that $C(0)=0$ and

$$C(n+1) = \frac{(n+2)C(n) - (2n+1)C(1)}{n-1}. \tag{3.37}$$

This is just a difference equation and given $C(2)$ it will determine $C(n)$ for $n>1$ (note that it can't determine $C(2)$ given $C(1)$). We can look for a solution to this by considering polynomials. Since it must be odd in n the simplest guess is

$$C(n) = c_1 n^3 + c_2 n, \qquad c_1, c_2 \in \mathbb{R}. \tag{3.38}$$

Note that if we shift L_0 by a constant l then $C(n)$ is shifted by $2nl$. This means that we can change the value of c_2. Therefore we will fix it to be $c_1 = -c_2$. Finally we must calculate c_1. To do this we consider the ground state

$$\begin{aligned}\langle 0| :L_2: :L_{-2}: |0\rangle &= \langle 0|[:L_2:, :L_{-2}:]|0\rangle \\ &= 4\langle 0| :L_0: |0\rangle + 6c_1\langle 0|0\rangle \\ &= 6c_1\end{aligned}$$

Of course we know that had we used the Virasoro algebra (3.5) the last calculation would have given $\langle 0| :L_2: :L_{-2}: |0\rangle = \frac{c}{2}$ and thus we conclude that $c_1 = c/12$.

3.4 Vertex Operators

The correlation functions of the theory determine the OPE of the conformal fields, as one can see from (2.9) and (2.41) for example. In turn, the OPE of two conformal fields is given in terms of a sum of single fields as in (2.41). Thus, we see that the OPE defines a certain product on the fields via the operator product coefficients ${C_{ij}}^k$, which are the only non-trivial input in the OPE. It is, therefore, the operator product coefficients that force the product to involve the complex parameters z_i in a non-trivial way and hence, the OPE does not directly define an algebra (in the appropriate sense); the resulting structure is a *vertex (operator) algebra* $\mathcal{V}$.

We are not going to give a full mathematical definition of what a vertex operator algebra (VOA) is, but instead a bit of a history will be sufficient for our purpose. Vertex operators appeared in the early days of string theory as local operators describing propagation of string states. In the mean time, Belavin, Polyakov and Zamolodchikov [BPZ] initiated the study of $2d$-CFT. Vertex algebras can be seen in retrospect as the mathematical equivalent of the chiral symmetry algebras of CFT. Moreover, the key property of associativity of vertex algebras is equivalent to the property of OPE in CFT, which goes back to the pioneering works of Polyakov and Wilson. Thus, vertex algebras may be thought of as the mathematical language of $2d$-CFT.

In a nutshell, a vertex algebra is a vector space $\mathcal{V}$ equipped with a vector $|0\rangle$ and an operation $Y(\bullet, z)$, assigning to each $A \in \mathcal{V}$ a vertex operator (or formal power series or formal distribution)

$$Y(A,z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}, \tag{3.39}$$

where each A_n is a linear operator on $\mathcal{V}$, so that for any $v \in \mathcal{V}$ we have $A_n v = 0$ for large enough n. These data are subject to a list of axioms which we are not going to discuss here since it is not in our interest for these notes. For those interested see [FBZ] and [Fr] for example. We are already familiar with such operators, recall for example (3.30).

We will be interested in an exponential version of the vertex operator of a field $\phi(z,\bar{z})$, which is easier to use for computations. If we consider the free boson (3.26) one can construct a family of vertex operators

$$V_\alpha(z,\bar{z}) := \,:\!e^{i\sqrt{2}\alpha\phi(z,\bar{z})}\!:. \tag{3.40}$$

The normal ordering simply means that within the exponential the different operators commute. The family of vertex operators (3.40) are primary fields with conformal dimension

$$h_\alpha = \bar{h}_\alpha = \frac{\alpha^2}{4\pi g}. \tag{3.41}$$

The vertex operators can be decomposed into left and right chiral vertex operators as[10]

$$V_\alpha(z,\bar{z}) = V_\alpha(z) \otimes \bar{V}_\alpha(\bar{z}), \tag{3.42}$$

with

$$V_\alpha(z) = \,:\!e^{i\sqrt{2}\alpha\phi(z)}\!: \tag{3.43}$$

and $\phi(z)$ as given in (3.28) with a similar expression for the antiholomorphic part. To calculate the OPE of two vertex operators one can use the following relation for a single harmonic oscillator

$$:\!e^{\alpha\phi_1}\!:\,:\!e^{\beta\phi_2}\!: \,=\, :\!e^{\alpha\phi_1+\beta\phi_2}\!:\, e^{\alpha\beta\langle\phi_1\phi_2\rangle}, \tag{3.44}$$

[10]The reason we write the tensor product of the two vertex operators is essentially explained in footnote 9 and thus we cannot have a product form $V_\alpha(z,\bar{z}) = V_\alpha(z)\bar{V}_\alpha(\bar{z})$.

where

$$\phi_i = p_i a + q_i a^\dagger, \tag{3.45}$$

is some linear combination of annihilation and creation operators. Using the above and (2.50) the OPE of two vertex operators takes the form

$$V_\alpha(z)V_\beta(w) = e^{\ln|z-w|^{\frac{2\alpha\beta}{4\pi g}}} V_{\alpha+\beta}(w) + \ldots = |z-w|^{\frac{\alpha\beta}{2\pi g}} V_{\alpha+\beta}(w) + \ldots \tag{3.46}$$

However, from (2.9) we see that under global conformal transformations the fields in the two point function must have the same conformal dimension. Furthermore, the correlator must not grow with distance which means that either $\alpha\beta < 0$ or $g < 0$. We want this to be independent of the choice of the coupling constant so the only possibility is $\alpha = -\beta$, i.e.

$$V_\alpha(z)V_{-\alpha}(w) = |z-w|^{-2h_\alpha} + \ldots \tag{3.47}$$

In general one may argue that the n-point correlator of vertex operators is given by

$$\left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \right\rangle = \exp\left\{ \sum_{i<j} \ln|z_{ij}|^{2\alpha_i\alpha_j} \right\} = \prod_{i<j} |z_{ij}|^{2\alpha_i\alpha_j}, \tag{3.48}$$

which is nonzero provided the neutrality condition

$$\sum_i \alpha_i = 0, \tag{3.49}$$

is satisfied. However, if we consider the modified energy momentum tensor (2.56) then the extra term is interpreted as the presence of a background charge $-2Q$ at infinity. This is created by the vertex operator V_{-2Q}. Thus in this case the only non-vanishing correlation functions are those with $\sum_i \alpha_i = 2Q$.

Example 3.10. We can use the vertex operators to bozonize the fermions from exercises 2.10 and 2.13. First we define the vertex operators to be

$$\psi_\pm(z) := V_{\pm\frac{1}{\sqrt{2}}} = :e^{\pm i\phi(z)}:,$$

where $\phi(z)$ is a spacelike boson. It is not difficult to see that the conformal dimension (3.41) (we choose $g = 1/4\pi$ for convenience) in this case becomes $h_1 = 1/2$. Next we note that there is a $U(1)$ symmetry generated by the current $J(z) = :\psi_+(z)\psi_-(z):$. To see this we use point splitting and Wick's theorem

$$\begin{aligned} J(z) &\overset{(1)}{=} \lim_{w\to z}\left(\psi_+(w)\psi_-(z) - \frac{1}{w-z}\right) \\ &\overset{(2)}{=} \lim_{w\to z}\left(\frac{:e^{i(\phi(w)-\phi(z))}: - 1}{w-z}\right) \\ &\overset{(3)}{=} i\lim_{w\to z}\frac{:\phi(w)-\phi(z):}{w-z} + \mathcal{O}\left(i^2\right) \\ &= i:\partial\phi(z): \end{aligned}$$

In step (1) we used the OPE (3.47), in step (2) we used equation (3.44) and in step (3) we expanded the exponential to leading order. It is not difficult to see from the OPE (2.51) that $J(z)$ is a primary field of conformal dimension $h_J = 1$.

Exercise 3.11. Show that the OPE of J with the vertex operators $\psi_\pm$ is given by

$$J(z)\psi_\pm(w) = \mp\frac{\psi_\pm(w)}{z-w}.$$

Exercise 3.12. Show that

$$\psi_\pm\partial\psi_\mp = \frac{1}{2}(\partial\phi)^2 \pm \frac{i}{2}\partial^2\phi.$$

Then by using the expression for the energy momentum tensor from exercise 2.13 show that the energy momentum tensor of the bosonized theory is

$$T = \frac{1}{2}J^2.$$

4 CFT on the Torus

The CFT on the full complex plane we formulated up to now, decouples into holomorphic and antiholomorphic sectors. In fact, the two sectors may describe two distinct theories since they do not interfere. However, this situation is very unphysical.

The decoupling exists only at the fixed point in parameter space (the conformally invariant point) and in the infinite plane geometry. One, therefore, can solve this problem by coupling the holomorphic and antiholomorphic sectors of the theory, through the geometry of space on which the theory is defined. In this way, one imposes physical constraints on the holomorphic-antiholomorphic content of a CFT without leaving the fixed point. The infinite plane is topologically equivalent to the Riemann sphere, i.e. the Riemann surface of genus $g = 0$. One may study CFTs on Riemann surfaces of arbitrary genus g. The simplest non-spherical case is that of genus $g = 1$, i.e. a torus, which is equivalent to a plane with periodic boundary conditions, in two directions as in figure 2.

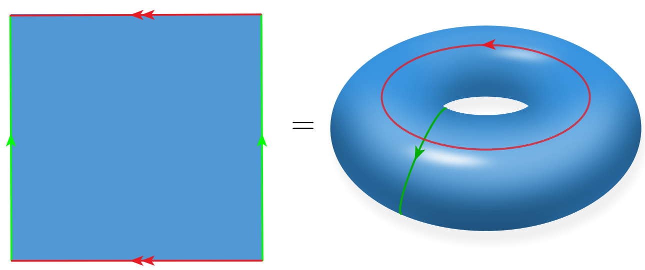


Figure 2: First we identify the edges indicated with the green arrows and then those with the red to obtan the torus from the plane.

Recall the map (2.31), from the cylinder to the complex plane. We now want the inverse procedure, i.e. to go back to the infinite cylinder from which we can construct a torus of length R, by cutting a segment of the cylinder and by gluing the two boundaries of the segment together. More precisely, we need to consider the map $z \mapsto w(z) = \frac{L}{2\pi i}\ln z$, from the complex plane parameterized by z, to the infinite cylinder of circumference L, parameterized by w (with $w = w + L$). On the cylinder, time translations are movements in the imaginary direction, generated by the Hamiltonian

$$H(L) = \frac{1}{2\pi}\int_0^L T_{tt}\,\mathrm{d}x = -\frac{1}{2\pi}\oint \left(\mathrm{d}w\; T(w) + \mathrm{d}\bar{w}\;\bar{T}(\bar{w})\right). \tag{4.1}$$

Using the transformation law (2.44) of the stress tensor and performing a change of integration variables, one can obtain the action of $H(L)$ in the space of states of the complex plane (see exercise 4.1)

$$H(L) = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right), \tag{4.2}$$

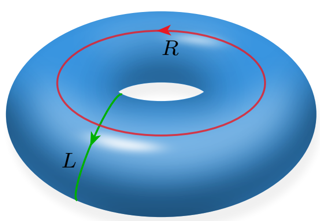
The constant term ensures the vanishing of the vacuum energy density in the limit $L \to \infty$. As one can see, for the Hamiltonian (4.2) to be bounded below, the space of states must consist of highest weight representations of $\mathbf{Vir} \oplus \overline{\mathbf{Vir}}$, i.e. to posses a decomposition of the form (3.1).

Exercise 4.1. Show equation (4.2) following the steps described above by finding an expression for $T(w)$ and plugging it into (4.1). Hint: Use also (3.4) and the fact that $\oint \mathrm{d}z\ \frac{1}{z} = 2\pi i$.

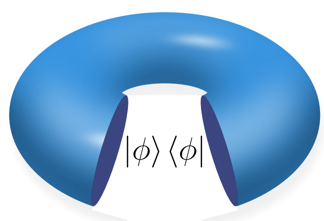
In a similar calculation, one can also show that the total momentum operator $P(L)$, which generates translations along the circumference of the cylinder is

$$P(L) = \frac{2\pi i}{L}(L_0 - \bar{L}_0). \tag{4.3}$$

The action of twisting the cylinder corresponds to a finite translation around its circumference, while gluing the ends together corresponds to taking the trace. In terms of CFT this means that one has to *sum over intermediate states* on a circle:



$$= \sum_{\phi}$$



(4.4)

The Hamiltonian and the momentum operators then propagate states along different directions of the torus and the spectrum of the theory is encoded in the *partition function*. If we define the torus *modular parameter* $\tau \equiv \frac{iR}{L}$ and combine the above ideas, the torus partition function can be written as

$$\begin{aligned} Z(\tau,\bar{\tau}) &= \mathrm{Tr}\, e^{-(H(L)\,\mathrm{Im}\,\tau - iP(L)\,\mathrm{Re}\,\tau)} \\ &\overset{(1)}{=} \mathrm{Tr}\, e^{\pi i\left((\tau+\bar{\tau})\left(L_0+\bar{L}_0-\frac{c}{12}\right)-(\tau-\bar{\tau})(L_0-\bar{L}_0)\right)} \\ &= \mathrm{Tr}\, e^{2\pi i\left(\tau\left(L_0-\frac{c}{24}\right)-\bar{\tau}\left(\bar{L}_0-\frac{c}{24}\right)\right)} \\ &\overset{(2)}{=} \mathrm{Tr}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\right) \end{aligned} \quad . \tag{4.5}$$

In step (1) we used $\mathrm{Re}\,\tau = \frac{1}{2}(\tau+\bar{\tau})$ and $\mathrm{Im}\,\tau = \frac{1}{2i}(\tau-\bar{\tau})$, while in step (2) we have set

$$q \equiv e^{2\pi i\tau}, \quad \bar{q} \equiv e^{-2\pi i\bar{\tau}}. \tag{4.6}$$

Note that the partition function (4.5) depends on R, L only through their ratio τ. An important feature of this parametrisation of the torus is that it is not unique and this is the subject of the next subsection. Now, if we decompose the space of states into irreducible representations of $\mathbf{Vir} \oplus \overline{\mathbf{Vir}}$ as in (3.1) the partition function can be rewritten as

$$Z(\tau, \bar{\tau}) = \sum_{i,\bar{\jmath} \in \mathcal{I}} M_{i\bar{\jmath}} \chi_i(\tau) \bar{\chi}_{\bar{\jmath}}(\bar{\tau}), \tag{4.7}$$

where

$$\chi_i(\tau) = \mathrm{Tr}_{R_i} q^{L_0 - \frac{c}{24}}, \quad \bar{\chi}_{\bar{\jmath}}(\bar{\tau}) = \mathrm{Tr}_{\bar{R}_{\bar{\jmath}}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}, \tag{4.8}$$

are the *characters* of the irreducible representations R_i and $\bar{R}_{\bar{\jmath}}$ respectively and they are the generating functions of the (irreducible if Z is of the form (4.7)) Verma module. The **Vir**-character (4.8) of a generic Verma module can be written as

$$\chi_i(\tau) = q^{h_i - c/24} \sum_{n=0}^{\infty} \dim(h_i + n) q^n, \tag{4.9}$$

where $\dim(h+n)$ is the number of linearly independent states at level n in the Verma module. Since $\dim(h+n) \leq p(n)$, where $p(n)$ we recall is the number of (possibly dependent) states at level n, the above series uniformly converge if $|q| < 1$, (that is for τ in the upper half plane) since $|q| < 1$ is the domain of convergence of the series (3.17). Thus the character of a generic Verma module may take the form

$$\chi_i(\tau) = q^{h_i - c/24} \sum_{n=0}^{\infty} p(n) q^n = \frac{q^{h_i - c/24}}{\varphi(q)}. \tag{4.10}$$

The Euler function is related to the Dedekind η function through a Ramanujan identity as

$$\varphi(q) = q^{-\frac{1}{24}} \eta(\tau). \tag{4.11}$$

We therefore arrive to the generic **Vir**-character

$$\chi_i(\tau) = \frac{q^{h_i + \frac{1-c}{24}}}{\eta(\tau)}, \tag{4.12}$$

with a similar expression for $\bar{\chi}_{\bar{\jmath}}(\bar{\tau})$. Then the partition function of an irreducible Verma module simply reads

$$Z(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{i,\bar{\jmath} \in \mathcal{I}} M_{i\bar{\jmath}}\, q^{h_i + \frac{1-c}{24}} \bar{q}^{\bar{h}_{\bar{\jmath}} + \frac{1-c}{24}}. \tag{4.13}$$

It is a known theorem of the representation theory of the Virasoro algebra that whenever $Z(\tau)$ is of the form (4.7) then the associated Verma module is irreducible and the characters are given by (4.12).

4.1 Modular Invariance

As mentioned in the previous section, the parametrization of the torus we used is not unique. More generally, one characterises a torus by its two periods $\omega_1, \omega_2 \in \mathbb{C}$. Then a torus is defined by specifying two linearly independent lattice vectors on the plane and identifying points that differ by an integer combination of these vectors. Thus, if $\omega = (\omega_1, \omega_2)$ is one vector and $\omega' = (\omega_1', \omega_2')$ is another one, then expressing the last one as an integer combination of the first we should have

$$\omega' = A\omega, \qquad A \in SL(2,\mathbb{Z}). \tag{4.14}$$

One can show that the modular parameter τ under the change of period (4.14) transforms as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1. \tag{4.15}$$

Exercise 4.2. Defining the modular parameter as $\tau \equiv \omega_1/\omega_2$ and letting $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ show the transformation property (4.15) by direct application of (4.14).

It is easy to check that a simultaneous sign reversal in all parameters a, b, c, d leaves τ unaffected. This amounts to taking A to $-A$ and thus the underlying symmetry group is $PSL(2,\mathbb{Z})$ which is isomorphic to the modular group Γ. The modular group is generated by the two transformations

$$\mathcal{T}\colon \tau \mapsto \tau + 1, \quad \mathcal{S}\colon \tau \mapsto -\frac{1}{\tau}, \tag{4.16}$$

which act on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$. In particular τ is an element of the so-called *fundamental domain* F_0, see figure 3. A fundamental domain of Γ is a domain of $\mathbb{H}$ such that no pair of points within F_0 can be reached by any modular transformation and any point outside can be reached from a unique point inside, by some modular transformation. The diagram in figure 3 shows part of the construction of F_0 for the action of Γ on $\mathbb{H}$. We see that $\mathcal{T}$ and $\mathcal{S}$ when acting on elements of F_0 generate some other domains of $\mathbb{H}$.

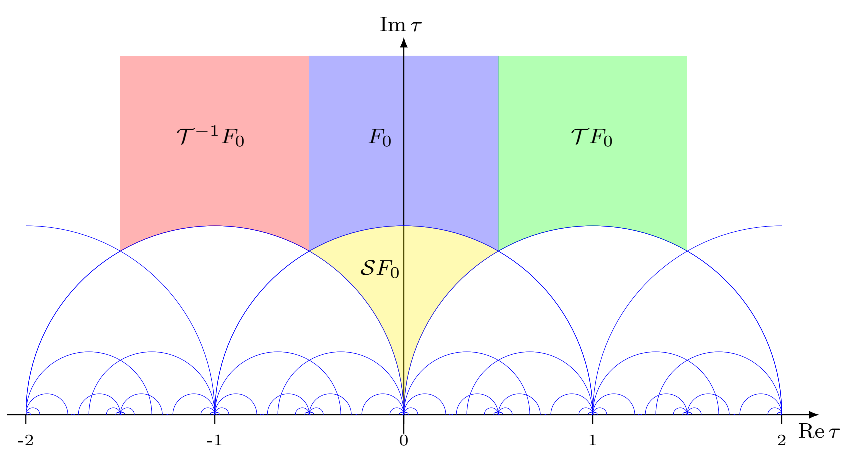
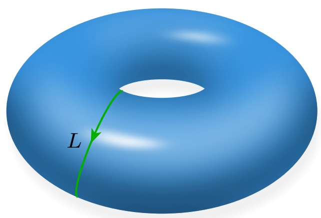


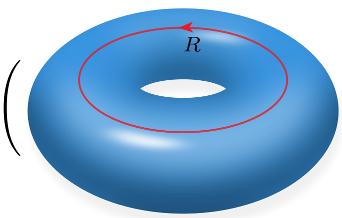
Figure 3: The fundamental domain F_0 of Γ. The other domains can be obtained from F_0 by applying $\mathcal{T}$ and $\mathcal{S}$ as shown in the figure.

Note also that $\mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3 = -1$. One way to see this is to represent $\mathcal{T}, \mathcal{S}$ by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.17}$$

The above discussion concludes that had we taken an other parametrization of the torus then we would have ended up with an equivalent torus to the one we considered in section 4. This in turn means that the physics is unchanged regardless of the parametrization and thus the partition function of the equivalent tori must be the same. In other words, had we started with the Hamiltonian $\tilde{H}(R) = \int_0^R \mathrm{d}x\, T_{xx}$ and the map $z \mapsto \tilde{w}(z) = \frac{R}{2\pi} \ln z$, this would amount to a second representation of the *same partition function* with time running perpendicular to our fist choice. This is best depicted in the following pictorial equation.

$Z\Big($ 

$\Big) \stackrel{!}{=} Z\Big($ 

$\Big)$ (4.18)

This means that the torus partition function is *modular invariant*. This places severe constraints on the possible bulk field content of the theory. To see this a bit more work needs to be done. First we note (without a proof) that the characters transform among themselves under the modular transformations (4.16) as

$$\chi_i(\tau+1) = \sum_{j\in\mathcal{I}} T_{ij}\chi_j(\tau), \quad \chi_i\left(-\frac{1}{\tau}\right) = \sum_{j\in\mathcal{I}} S_{ij}\chi_j(\tau). \tag{4.19}$$

Finding explicit expressions for the matrices T and S (these are constant matrices, i.e. independent of τ) for a particular model is not a trivial task at all (at least for S it is not). For the Virasoro characters (4.12) the $\mathcal{T}$ transformation is just a phase and one may show that

$$T_{ij} = \delta_{ij} e^{2\pi i\left(h_i - \frac{c}{24}\right)}. \tag{4.20}$$

To see this we first note that the Dedekind η function is a modular (cusp) form of weight 1/2, this means that under $\mathcal{T}$ and $\mathcal{S}$ it transforms respectively as

$$\eta\,(\tau+1) = e^{\frac{\pi i}{12}}\,\eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau). \tag{4.21}$$

Exercise 4.3. Show equation (4.20) for the Virasoro characters (4.12) using (4.21).

Thus if the space of states decomposes as in (3.1) then (4.7) under $\mathcal{T}$ and $\mathcal{S}$ implies that[11]

$$\begin{aligned} Z(\tau+1) &= \sum_{i,k,\bar{j},\bar{l}\in\mathcal{I}} T_{ik} M_{i\bar{j}} \bar{T}_{\bar{j}\bar{l}} \chi_k(\tau)\bar{\chi}_{\bar{l}}(\bar{\tau}) = \sum_{k,\bar{l}\in\mathcal{I}} M_{k\bar{l}}\chi_k(\tau)\bar{\chi}_{\bar{l}}(\bar{\tau}) = Z(\tau) \\ Z\left(-\frac{1}{\tau}\right) &= \sum_{i,k,\bar{j},\bar{l}\in\mathcal{I}} \mathrm{S}_{ik} M_{i\bar{j}} \bar{S}_{\bar{j}\bar{l}} \chi_k(\tau)\bar{\chi}_{\bar{l}}(\bar{\tau}) = \sum_{k,\bar{l}\in\mathcal{I}} M_{k\bar{l}}\chi_k(\tau)\bar{\chi}_{\bar{l}}(\bar{\tau}) = Z(\tau) \end{aligned} \tag{4.22}$$

[11] From now on we drop the $\bar{\tau}$ dependence of the partition function since τ and $\bar{\tau}$ are not independent variables but complex conjugates of each other.

provided that

$$\sum_{i,\bar{\jmath}\in\mathcal{I}} T_{ik} M_{i\bar{\jmath}} \bar{T}_{\bar{\jmath}\bar{l}} = \sum_{i,\bar{\jmath}\in\mathcal{I}} S_{ik} M_{i\bar{\jmath}} \bar{S}_{\bar{\jmath}\bar{l}} = M_{k\bar{l}}, \tag{4.23}$$

and $\bar{T}$ and $\bar{S}$ are the matrices defined as in (4.19) for the antiholomorphic representations. This provides very powerful constraints for the multiplicities $M_{i\bar{\jmath}}$. The case where $M_{i\bar{\jmath}} = \delta_{i\bar{\jmath}}$ is known as the *Cardy case*, for which (4.23) implies that S is unitary and the partition function (4.7) takes the simple form

$$Z(\tau) = \sum_{i\in\mathcal{I}} |\chi_i(\tau)|^2. \tag{4.24}$$

Exercise 4.4. Show that the partition function (4.13) is invariant under $\tau \mapsto \tau + 1$ provided that $h_i = \bar{h}_{\bar{\jmath}}$.

4.2 The Free Boson on the Torus

From the discussion in subsection 2.3.1 we know that the free boson has central charge $c = 1$, while from the discussion on vertex operators we know from equation (3.41) that its conformal weight is $h_\alpha = \frac{\alpha^2}{4\pi g}$. Plugging these into the character formula (4.12) we get the character

$$\chi_\alpha(\tau) = \frac{q^{\frac{\alpha^2}{4\pi g}}}{\eta(\tau)}. \tag{4.25}$$

First thing to note here is that $\alpha \in \mathbb{R}$, so there are infinitely many characters. The diagonal partition function (4.24) for the free boson on the torus is not a sum, but rather an integral

$$Z_{\text{bos}}(\tau) = \frac{2}{|\eta(\tau)|^2} \int_{-\infty}^{\infty} \mathrm{d}\alpha \; (q\bar{q})^{\frac{\alpha^2}{4\pi g}} = \frac{2}{|\eta(\tau)|^2} \int_{-\infty}^{\infty} \mathrm{d}\alpha \; e^{-\frac{\mathrm{Im}\,\tau}{g}\alpha^2} = \frac{\sqrt{4\pi g}}{\sqrt{\mathrm{Im}\,\tau}|\eta(\tau)|^2}. \tag{4.26}$$

The proper derivation requires of course a discussion of the measure and the normalization, but the result is correct. This factor appears in the partition function of the bosonic string, which is described by a tensor product of 26 free bosonic theories (plus ghosts).

Exercise 4.5. Verify that (4.26) is invariant under the **(a)** $\mathcal{T}$-transformation and **(b)** $\mathcal{S}$-transformation (i.e. $\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$) by using Poisson's resummation formula

$$\sum_{n\in\mathbb{Z}} f(nr) = \frac{1}{r} \sum_{m\in\mathbb{Z}} \tilde{f}\left(\frac{m}{r}\right), \tag{4.27}$$

where the Fourier transform $\tilde{f}$ is defined as

$$\tilde{f}(y) = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{2\pi i x y} f(x), \tag{4.28}$$

and the sum form of the Dedekind η-function

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3}{2}\left(n-\frac{1}{6}\right)^2}. \tag{4.29}$$

4.3 Compactified Boson and *T*-Duality

Consider the free boson CFT from section 3.3 but with ϕ, now compactified on a circle of radius R. This means that the field is invariant under rotations $z \to e^{2\pi i} z$, up to identifications $\phi \sim \phi + 2\pi R n$, for $n \in \mathbb{Z}$, i.e.

$$\phi(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = \phi(z, \bar{z}) + 2\pi R n, \qquad (n \in \mathbb{Z}). \tag{4.30}$$

The mode expansion (3.26) now reads[12]

$$\phi(z, \bar{z}) = \phi_0 - i\left(a_0 \ln z + \bar{a}_0 \ln \bar{z}\right) + i \sum_{n \neq 0} \frac{1}{n}\left(a_n z^{-n} + \bar{a}_n \bar{z}^{-n}\right), \tag{4.31}$$

First thing to note is that now $a_0 \neq \bar{a}_0$. Using (4.30) and the mode expansion (4.31) gives

$$a_0 - \bar{a}_0 = Rn. \tag{4.32}$$

Therefore, equations (3.29) become

$$a_0|\alpha\rangle = \alpha|\alpha\rangle \qquad \bar{a}_0|\alpha\rangle = (\alpha - Rn)|\alpha\rangle. \tag{4.33}$$

Thus, the diagonal partition function reads

$$Z_{\text{bos}}(R) = \frac{1}{|\eta(\tau)|^2} \sum_{\alpha \in \mathbb{Z}} q^{h_\alpha} \bar{q}^{\bar{h}_\alpha} = \frac{1}{|\eta(\tau)|^2} \sum_{\alpha, n \in \mathbb{Z}} q^{\frac{1}{2}\alpha^2} \bar{q}^{\frac{1}{2}(\alpha - Rn)^2}. \tag{4.34}$$

Exercise 4.6. Show that under the modular $\mathcal{T}$-transformation the argument of this sum picks up an additional factor of $\exp\left[2\pi i n\left(\alpha R - \frac{R^2 n}{2}\right)\right]$ and then by demanding invariance under $\mathcal{T}$ get

$$\alpha_{m,n}(R) = \frac{m}{R} + \frac{Rn}{2}, \qquad (m, n \in \mathbb{Z}). \tag{4.35}$$

From the previous exercise we see that the action of $a_0, \bar{a}_0$ on a highest weight state $|m, n\rangle$ (4.33) becomes

$$a_0|m, n\rangle = \left(\frac{m}{R} + \frac{Rn}{2}\right)|m, n\rangle, \qquad \bar{a}_0|m, n\rangle = \left(\frac{m}{R} - \frac{Rn}{2}\right)|m, n\rangle. \tag{4.36}$$

Thus, the bulk spectrum can be written as a direct sum

$$\mathcal{H}_{\text{bos}}(R) = \bigoplus_{m,n \in \mathbb{Z}} R_{\alpha_{m,n}(R)} \otimes \bar{R}_{\bar{\alpha}_{m,n}(R)}, \tag{4.37}$$

of highest weight representations of the $\mathfrak{u}(1)$ current algebra. The integer n is the winding number and m is related to the total momentum p via $p = \frac{1}{2}(\alpha_{m,n} + \bar{\alpha}_{m,n}) = \frac{m}{R}$.

[12] Setting $g = 1/4\pi$ for convenience.

Exercise 4.7. Using Poisson's resummation formula (4.27) show that

$$\sum_{n\in\mathbb{Z}} e^{-\pi a n^2 + bn} = \frac{1}{\sqrt{a}} \sum_{m\in\mathbb{Z}} e^{-\frac{\pi}{a}\left(m+\frac{b}{2\pi i}\right)^2}, \tag{4.38}$$

and use this to deduce the invariance of the partition function under the modular $\mathcal{S}$-transformation.

It is not difficult to see that the bulk spectrum (4.37) (or equivalently the associated partition function) is invariant under the substitution $R \mapsto 2/R$, i.e.

$$Z_{\text{bos}}(2/R) = Z_{\text{bos}}(R). \tag{4.39}$$

This is the usual T-duality relation for the compactified free boson. In string theory, this simply states that closed strings propagating around a circle cannot distinguish if the size of the circle is R or $2/R$. The self-dual radius $R = \sqrt{2}$ is the minimal length scale that the strings resolve.

4.4 Fusion Algebra and Verlinde's Formula

The action of the Virasoro generators on the product of two primary fields, preserves the Virasoro algebra and endows the tensor product of the representations with the structure of a representation. This leads to a natural product on representations, called the *fusion product*, which constrains the fields that appear in the OPE. The consistency of the OPE (2.41) with the existence of null vectors leads to the *fusion algebra* of the CFT [Ca04][Ca89].

$$R_i \otimes R_j = \sum_{k\in\mathcal{I}} \mathcal{N}_{ij}{}^{k} R_k, \tag{4.40}$$

where $\mathcal{N}_{ij}{}^{k} \in \mathbb{N}_0$ are the *fusion numbers*. This applies separately to the holomorphic and antiholomorphic sectors and determines how many copies of R_k occur in the fusion of R_i with R_j. The fusion algebra is commutative, associative and contains an identity given by the vacuum representation R_0.

Consistency of the CFT on the torus implies that the fusion numbers are given in terms of particular products of matrix elements of the modular matrix

$$\mathcal{N}_{ij}{}^{k} = \sum_{l\in\mathcal{I}} \frac{S_{il} S_{jl} \bar{S}_{kl}}{S_{0l}}. \tag{4.41}$$

This is the so called *Verlinde formula* [Ver88]. In these notes, we will make the simplifying assumption that $\mathcal{N}_{ij}{}^{k} \in \{0, 1\}$. In full generality, the fusion numbers may be larger than one, but it is not so for the *Virasoro minimal models* as well as the WZW models that we will study later in these notes. This reflects the absence of multiplicity greater than one in ordinary tensor products of representations of $\mathfrak{su}(2)$.

4.5 Rational Conformal Field Theory

In Section 4.4, we gave an explicit relation between the modular transformation $\mathcal{S}$ of the characters and the fusion numbers $\mathcal{N}$ which proves to be a very general fact. This naturally leads to the concept of *rational conformal field theory* (RCFT).

RCFT: a definition

Definition 4.8. A CFT is said to be *rational* if its Hilbert space contains only a finite number of irreducible highest weight representations R_i of the chiral algebra $\mathcal{V}$.

RCFTs may contain an infinite number of Virasoro representations, however, these can be reorganised into a finite set of irreducible representations by linearly transforming one into another under the action of the modular group. Thus, the underlying chiral algebra is *extended* due to the existence of additional symmetries.

The term "rational" is because if there are only a finite number of primary fields then the conformal weights are all rational numbers [Va88, AM88].

The only theories that contain only a finite number of Virasoro irreducible representations, are the Virasoro minimal models, however, we will not discuss them in these notes, for those interested, see [BYB] for a concise exposition. Another prime example of RCFTs which are completely solvable, are the WZW models which will discuss in the next Chapter. For a condensed panoramic view of the development of two-dimensional RCFT in the last twenty-five years see [FRS10], or for a lightning review of RCFT see [GW03, Sect. 2].

Consider now a RCFT whose Hilbert space $\mathcal{H}$ decomposes into a finite number of irreducible representations

$$\mathcal{H} = \bigoplus_{i,j\in\mathcal{I}} (R_i \otimes \bar{R}_j)^{M_{ij}}, \tag{4.42}$$

of a chiral algebra $\mathcal{V}$, such that $\mathbf{Vir} \subset \mathcal{V}$. On the set $\mathcal{I}$, indexing the representations R_i, we assume there is the *charge conjugation* $(i^\vee)^\vee = i$, which preserves the conformal weights and the fusion numbers

$$h_i = h_{i^\vee}, \qquad \mathcal{N}_{ij}{}^{k} = \mathcal{N}_{i^\vee j^\vee}{}^{k^\vee}. \tag{4.43}$$

From this we define the *charge conjugation matrix* as

$$C_{ij} = \delta_{ij^\vee}. \tag{4.44}$$

The charge conjugation matrix can be used to raise and lower indices (just like the metric tensor). The modular matrix satisfies

$$S^2 = C, \qquad S_{ij^\vee} = \bar{S}_{ij}. \tag{4.45}$$

This requires that the characters, under modular transformations must transform as

$$\chi_i(q) = \sum_{j\in\mathcal{I}} S_{ij}\chi_j(\tilde{q}), \qquad \chi_i(\tilde{q}) = \sum_{j\in\mathcal{I}} S_{ij^\vee}\chi_j(q) = \sum_{j\in\mathcal{I}} \bar{S}_{ij}\chi_j(q), \tag{4.46}$$

where $q = e^{2\pi i \tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$. The fusion numbers also satisfy the following identities

$$\mathcal{N}_{ij}{}^{k} = \mathcal{N}_{ji}{}^{k}, \quad \sum_{k\in\mathcal{I}} \mathcal{N}_{ij}{}^{k}\mathcal{N}_{kl}{}^{r} = \sum_{k\in\mathcal{I}} \mathcal{N}_{il}{}^{k}\mathcal{N}_{kj}{}^{r}, \quad \mathcal{N}_{0i}{}^{j} = \delta_{ij}, \quad \mathcal{N}_{ij}{}^{0} = \delta_{ij^\vee}. \tag{4.47}$$

Note, that the commutativity and associativity of the fusion rules is reflected in the first and second identities respectively.

The classification of all RCFTs is still an open problem and as mentioned in [BYB] it will probably remain for a while. A possible way to achieve this would be to first classify all possible fusion rules and use the information provided by Verlinde's formula (4.41) to extract information about the operator content of the theory.

A Normal ordering

Given a set of field operators $\{\phi_1(z_1),\ldots,\phi_n(z_n)\}$, we have two types of special orderings, namely, normal and radial (time) ordering. Normal ordering places all the annihilation operators on the right and *a fortiori* all creation operators on the left.

$$:\prod_{i=1}^{n}\phi_i(z_i): \tag{A.1}$$

No specifications are required since all annihilation operators commute with one another as do all the creation ones. If we take the vacuum expectation value (VEV) of the normal order product it therefore vanishes by definition

$$\left\langle :\prod_{i=1}^{n}\phi_i(z_i): \right\rangle = 0. \tag{A.2}$$

What's wrong?

Given the equation $[a,a^\dagger]=1$, if we "normal order" both sides we get

$$:[a,a^\dagger]: = 1 \implies :aa^\dagger: - :a^\dagger a: = 1 \implies 0=1$$

Answer: We never "normal order" equations. Normal ordering is not a derived notion, i.e. it is not derived from the ordinary product any more than the cross product is not derived from the scalar product.

The other special ordering is the radial (time) ordering (2.33) which places the field operators in radial (chronological) order, we reproduce it here for convenience

$$\mathcal{R}\left(\phi_1(z_1)\phi_2(z_2)\right) := \begin{cases} \phi_1(z_1)\phi_2(z_2), & |z_1|>|z_2| \\ \phi_2(z_2)\phi_1(z_1), & |z_1|<|z_2| \end{cases}. \tag{A.3}$$

A.1 Wick's Theorem

Normal ordering ensures the vanishing of the vacuum expectation value on one hand and on the other hand, the radial (time) ordering expresses correlation functions in terms of a vacuum expectation value. Wick's theorem relates these two orderings in the case of free fields. Before we state the theorem, we define the *contraction* of two field operators $\phi_i(z_i)$ with $\phi_j(z_j)$, within the normal ordered product (A.1) to simply be the omission of these two operators from (A.1) and their replacement by the two-point function $\langle\phi_i(z_i)\phi_j(z_j)\rangle$. The contraction is denoted by brackets and we write

$$:\phi_1(z_1)\ldots\overbracket{\phi_j(z_j)\ldots\phi_k}(z_k)\ldots\phi_n(z_n): \;=\; :\prod_{\substack{i=1\\ i\neq j,k}}^{n}\phi_i(z_i):\langle\phi_j(z_j)\phi_k(z_k)\rangle. \tag{A.4}$$

Example A.1. Consider the four fields $\{\phi_1, \phi_2, \phi_3, \phi_4\}$, then a contraction of ϕ_1 with ϕ_3, for example, simply is

$$:\overline{\phi_1\phi_2\phi_3}\phi_4: \;=\; :\phi_2\phi_4:\langle\phi_1\phi_3\rangle.$$

Exercise A.2. Calculate the contraction

$$:\phi_1\phi_2\phi_3\phi_4:.$$

The normal ordered product differs from $\phi_1(z_1)\phi_2(z_2)$ by the VEV, i.e.

$$\phi_1(z_1)\phi_2(z_2) = :\phi_1(z_1)\phi_2(z_2): + \langle\phi_1(z_1)\phi_2(z_2)\rangle \tag{A.5}$$

We now combine normal ordered products with radial (time) ordered ones. The radial ordered product $\mathcal{R}\left(\phi_1(z_1)\phi_2(z_2)\right)$ is given by

$$\begin{aligned}
\mathcal{R}\left(\phi_1(z_1)\phi_2(z_2)\right) &\overset{(1)}{=} \phi_1(z_1)\phi_2(z_2)\theta(|z_1|-|z_2|) + \phi_2(z_2)\phi_1(z_1)\theta(|z_2|-|z_1|) \\
&\overset{(2)}{=} :\phi_1(z_1)\phi_2(z_2):\big(\theta(|z_1|-|z_2|)+\theta(|z_2|-|z_1|)\big) \\
&\quad + \Big\langle \left(\phi_1(z_1)\phi_2(z_2)\theta(|z_1|-|z_2|) + \phi_2(z_2)\phi_1(z_1)\theta(|z_2|-|z_1|)\right)\Big\rangle \\
&\overset{(3)}{=} :\phi_1(z_1)\phi_2(z_2): + \langle\mathcal{R}\left(\phi_1(z_1)\phi_2(z_2)\right)\rangle \\
&\overset{(4)}{=} :\phi_1(z_1)\phi_2(z_2): + \langle\phi_1(z_1)\phi_2(z_2)\rangle.
\end{aligned} \tag{A.6}$$

In step (1) we rewrote definition (A.3) using the Heaviside function, in step (2) we used (A.5) and the important observation that $:[\phi_1(z_1), \phi_2(z_2)]: = 0$, which means (3) that the normal ordered products are automatically radially ordered[13], while in step (4) we took the VEV in order to find that $\langle\mathcal{R}\left(\phi_1(z_1)\phi_2(z_2)\right)\rangle = \langle\phi_1(z_1)\phi_2(z_2)\rangle$. Equation (A.6) is Wick's theorem for two fields and it can also be written as:

$$\mathcal{R}\left(\phi_1(z_1)\phi_2(z_2)\right) = :\phi_1(z_1)\phi_2(z_2): + :\overline{\phi_1(z_1)\phi_2}(z_2): \tag{A.7}$$

For the case of three fields, the above steps (1)-(4) as in (A.6) give

$$\begin{aligned}
\mathcal{R}\left(\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\right) = {} & :\phi_1(z_1)\phi_2(z_2)\phi_3(z_3): + :\phi_1(z_1):\langle\phi_2(z_2)\phi_3(z_3)\rangle \\
& + :\phi_2(z_2):\langle\phi_1(z_1)\phi_3(z_3)\rangle + :\phi_3(z_3):\langle\phi_1(z_1)\phi_2(z_2)\rangle.
\end{aligned} \tag{A.8}$$

Exercise A.3. Do this.

We can therefore see a pattern:

Wick's Theorem

Theorem A.4. *The radially (time) ordered product is equal to the normal ordered*

[13]The converse is not true.

product, plus all possible ways of contracting pairs of fields with in it, i.e.

$$\mathcal{R}\left(\phi_1(z_1)\ldots\phi_n(z_n)\right) = \,:\!\phi_1(z_1)\ldots\phi_n(z_n)\!:\, + \langle \mathcal{R}\left(\phi_1(z_1)\ldots\phi_n(z_n)\right)\rangle. \tag{A.9}$$

The notation $\langle \mathcal{R}\left(\phi_1(z_1)\ldots\phi_n(z_n)\right)\rangle$ means we sum over all possible contractions.

Example A.5. Consider the four fields $\{\phi_1, \phi_2, \phi_3, \phi_4\}$, then by direct application of Wick's theorem we get

$$\begin{aligned}\mathcal{R}\left(\phi_1\phi_2\phi_3\phi_4\right) = \,&:\!\phi_1\phi_2\phi_3\phi_4\!: + :\!\phi_1\phi_2\phi_3\phi_4\!: + :\!\phi_1\phi_2\phi_3\phi_4\!: + :\!\phi_1\phi_2\phi_3\phi_4\!:\\ &+ :\!\phi_1\phi_2\phi_3\phi_4\!: + :\!\phi_1\phi_2\phi_3\phi_4\!: + :\!\phi_1\phi_2\phi_3\phi_4\!:\\ &+ :\!\phi_1\phi_2\phi_3\phi_4\!: + :\!\phi_1\phi_2\phi_3\phi_4\!:\\ &+ :\!\phi_1\phi_2\phi_3\phi_4\!:\end{aligned} \tag{A.10}$$

Taking the VEV of this expression and using (A.4) and (A.2) we see that only the fully contracted terms in the first line survive, we thus get

$$\mathcal{R}\left(\phi_1\phi_2\phi_3\phi_4\right) = \,:\!\phi_1\phi_2\phi_3\phi_4\!: + \langle\phi_1\phi_2\rangle\langle\phi_3\phi_4\rangle + \langle\phi_1\phi_3\rangle\langle\phi_2\phi_4\rangle + \langle\phi_1\phi_4\rangle\langle\phi_2\phi_3\rangle. \tag{A.11}$$

Exercise A.6. Use Wick's theorem to calculate $\mathcal{R}\left(\phi_1\phi_2 :\!\phi_3\phi_4\!:\right)$.

B Generalized normal ordering

The normal ordered products introduced previously are only useful when considering free fields whose OPE with themselves contains only one singular term, see for example the OPEs (2.51) and (2.64). One regularizes the product of two such fields by simply subtracting the corresponding VEV by virtue of Wick's theorem. However, this is no longer true for fields which are not free in the above sense. What happens when we try to regularize $T(z)T(w)$ by subtracting the VEV $\langle T(z)T(w)\rangle$ from the product as $z \to w$? It will eliminate the singular term proportional to the central charge with the remaining subleading singularities remaining. In order to avoid this we need to generalise the procedure by subtracting all the singular terms from the OPE. We generalize the normal ordered product of two operators $A(z)B(z)$ to

$$:\!A(z)B(z)\!: \,\to (AB)(z). \tag{B.1}$$

In the following, the OPE of two operators A and B will be written as

$$A(z)B(w) = \sum_{n=-\infty}^{N}\frac{\{AB\}_n(w)}{(z-w)^n} = \sum_{n=1}^{N}\frac{\{AB\}_n(w)}{(z-w)^n} + \mathrm{reg}(A(z)B(w)), \tag{B.2}$$

where $N \in \mathbb{Z}_+$, and the composite fields $\{AB\}_n(w)$, are non-singular at $w = z$. The regular part of the expansion is the normal order product, that is

$$(AB)(w) = \frac{1}{2\pi i}\oint_w \mathrm{d}z\,\frac{A(z)B(w)}{z-w} = \{AB\}_0(w)\ , \tag{B.3}$$

whereas the contraction of the two operators is the singular part of the OPE

$$\overbracket{A(z)B}(w) = \sum_{n=1}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \ . \tag{B.4}$$

Exercise B.1. Verify equation (B.3) by substituting (B.2) in the integrant.

Hence we may now write

$$(AB)(w) = \lim_{z\to w} \Big(A(z)B(w) - \overbracket{A(z)B}(w) \Big) \,, \tag{B.5}$$

while the OPE is expressed as

$$A(z)B(w) = \overbracket{A(z)B}(w) + \mathrm{reg}(A(z)B(w)). \tag{B.6}$$

The regular terms are

$$\mathrm{reg}(A(z)B(w)) = \sum_{n=-\infty}^{0} \frac{\{AB\}_n(w)}{(z-w)^n}. \tag{B.7}$$

We can rewrite this as a sum of normal ordered pairs if we Taylor expand $A(z)$ around w

$$\mathrm{reg}(A(z)B(w)) = \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} \big(A^{(n)}B\big)(w), \tag{B.8}$$

we thus see that $\big(A^{(n)}B\big)(w) = \{AB\}_{-n}(w)$.

B.1 Generalized Wick's Theorem

Now that we have a generalized notion for normal ordering, which accommodates for interactive fields, we wish to reformulate Wick's theorem using the above results. We should mention, however, that we are not interested in the most general form of Wick's theorem as in (A.9), this cannot be achieved for interacting fields. We proceed as in [BBSS88].

We may prove the following properties:

- The contraction with the a normal ordered product

$$\overbracket{A(z)(B}C)(w) = \frac{1}{2\pi i} \oint_w \frac{\mathrm{d}x}{x-w} \left\{ \overbracket{A(z)B}(x)C(w) + B(x)\overbracket{A(z)C}(w) \right\} \tag{B.9}$$

The second term is simply $(B(w)\overbracket{A(z)C}(w))$. In the first term, if $A(z)B(x)$ have a poles with $1/(z-x)^n$ singularities we should use

$$\frac{1}{(z-x)^n} = \sum_{r=0} \left(\begin{array}{c} n+r-1 \\ r \end{array} \right) \frac{(x-w)^r}{(z-w)^{n+r}} \tag{B.10}$$

and combine with poles $1/(x-w)^m$. We obtain

$$(A(BC))_r = (B(AC)_r) + \sum_{q=0}^{r-1} \binom{r-1}{q} ((AB)_{r-q}C)_q \ , \qquad r > 0 \ . \tag{B.11}$$

For later use note the expressions of the lowest modes

$$\begin{aligned}
(A(BC))_1 &= (B(AC)_1) + ((AB)_1 C) \ , \\
(A(BC))_2 &= (B(AC)_2) + ((AB)_2 C) + ((AB)_1 C)_1 \ .
\end{aligned} \tag{B.12}$$

For $r = 0$ there is an additional term as below.

- The generalized *Wick contraction* reads

$$(A(BC)) - (B(AC)) = ((AB)C) - ((BA)C) \ , \tag{B.13}$$

or

$$(A(BC)) = ((AB)C) + (B(AC)) - ((BA)C) \ . \tag{B.14}$$

From the second expression above we see the usual contraction of free fields has the additional last term. Note also that the normal ordering is not associative.
To compute the opposite

$$(B\overbracket{C)(z)A}(w) = \overbracket{A(w)(B}C)(z) \ , \tag{B.15}$$

we use the above formula and we expand the result around $z = w$.

- The derivative of a normal ordered product

$$(AB)' = (A'B) + (AB') \ . \tag{B.16}$$

In the proof we use that $(A'B) = (AB)_{-1}$.

- The normal order of the commutator

$$([A, B]) = \sum_{r>0} \frac{(-1)^{r+1}}{r!} \ \partial^r \{AB\}_r \ . \tag{B.17}$$

Note that setting $B = A$ we obtain a consistency relation for the various terms of the OPE of A with itself.
In general

$$(BA)_n = \sum_{r=0} \frac{(-1)^{r+n}}{r!} \ \partial^r \{AB\}_{r+n} \ . \tag{B.18}$$

The normal order form of the commutator comes by setting $n = 0$ and separating the $r = 0$ from the sum.

- Note that in general $A(z)B(w) \neq B(w)A(z)$ but $\overbracket{A(z)B}(w) = \overbracket{B(w)A}(z)$. An example is

$$\begin{aligned}
T(z)T(w) &= \overbracket{T(z)T}(w) + [TT](w) + \mathcal{O}(z-w) \ , \\
\overbracket{T(z)T}(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} \ .
\end{aligned} \tag{B.19}$$

Clearly $\overbracket{T(z)T}(w) = \overbracket{T(w)T}(z)$. However,

$$\begin{aligned} T(z)T(w) &= \overbracket{T(z)T}(w) + [TT](w) + \mathcal{O}(z-w) \\ &= \overbracket{T(w)T}(z) - T''[z] + [TT](z) + \mathcal{O}(w-z) \neq T(w)T(z) \ , \end{aligned} \tag{B.20}$$

which shows that there is the additional term $-T''[z]$ in the finite part.

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