On Deformations of Curves with Automorphisms

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Introduction

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- If $g \geq 2$, $p \nmid |\mathrm{Aut}(X)|$ then $|\mathrm{Aut}(X)| \leq 84(g-1)$.
- If $p \mid |\mathrm{Aut}(X)|$ the above bound is wrong,

$$F : x^{p^h+1} + y^{p^h+1} + z^{p^h+1} = 0$$

$\mathrm{Aut}(F) = PGU(3, p^{2h})$, $|\mathrm{Aut}(F)| = f(g)$, f polynomial in g of degree 4.

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- $$M_g = \left\{ \begin{array}{c} \text{isomorphisms classes of} \\ \text{curves of genus } g \end{array} \right\}.$$

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- Families of curves X with given base T:

$$X \to T \Longleftrightarrow T \to M_g$$

Determine the maximum dimension of the base.

Partial Results

- Cornelissen-Kato *Equivariant deformation of Mumford curves and of ordinary curves in positive characteristic*
  Duke Math. J. 116 2003
  Ordinary Curves

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- Bertin, José and Mézard, Ariane *Déformations formelles des revêtements sauvagement ramifies de courbes algébriques*
  Inventiones Math. 2000, 141
  Cyclic groups

Functorial Statement of the problem:

A deformation of the couple (X, G) over a local ring A is a proper, smooth family of curves

$$\mathcal{X} \to \mathrm{Spec}(A)$$

together with a group homomorphism $G \to \mathrm{Aut}_A(\mathcal{X})$ such that there is a G-equivariant isomorphism ϕ from the fibre over the closed point of A to the original curve X.

Two deformations $\mathcal{X}_1, \mathcal{X}_2$ are considered to be equivalent if there is a G-equivariant isomorphism ψ, making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\psi} & \mathcal{X}_2 \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec} A & \end{array}$$

The global deformation functor is defined:

$$D_{\mathrm{gl}} : \mathcal{C} \to \mathrm{Sets}, \mathrm{A} \mapsto \left\{ \begin{array}{l} \text{Equivalence classes} \\ \text{of deformations of} \\ \text{couples } (X, G) \text{ over } A \end{array} \right\}$$

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- Deformations with automorphisms:
  Compute Grothendieck's equivariant cohomology

$$\dim_k H^1(X, G, \mathcal{T}_X) = \dim_k D_{gl}(k[\epsilon]/\epsilon^2).$$

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- Characteristic $p > 0$. The group $G(x)$ is solvable and admits a series:

$$G(x) = G_0(x) \rhd G_1(x) \rhd \cdots \rhd G_i(x) \rhd \cdots G_n(x) \rhd \{1\},$$

$G_0(x)/G_1(x)$ cyclic of order prime to the characteristic and G_i/G_{i+1} is elementary abelian.

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- We define the local deformation functor at x_i, from the category $\mathcal{C}$ of local Artin algebras over k, to the category of sets, by:

$$D_i : \mathcal{C} \longrightarrow \mathrm{Sets},\, A \mapsto \left\{ \begin{array}{l} \text{lifts } G_{x_i} \longrightarrow \mathrm{Aut}(A[[t]]) \text{ of } \rho_i \\ \text{modulo conjugation with} \\ \text{an element of } \Pi_{A,k} \end{array} \right.$$

- $$0 \to H^1(X/G, \pi_*^G(\mathcal{T}_X)) \to H^1(X, G, \mathcal{T}_X) \to$$
  $$\to H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) \to 0$$

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- $D_i(k[\epsilon]) = H^1(G_{x_i}, \hat{\mathcal{T}}_{X,x_i})$
- $\dim_k H^1(X/G, \pi_*^G(\mathcal{T}_X)) =$
  $3g_{X/G} - 3 + \sum_{k=1}^r \left\lceil \sum_{i=0}^{n_k} \frac{(e_i^{(k)}-1)}{e_0^{(k)}} \right\rceil.$

Final Step

Given the short exact sequence of groups

$$1 \to K \to G \to G/K \to 1,$$

and a G-module A, how are the cohomology groups

$$H^i(G, A), H^i(K, A) \text{ and } H^i(G/K, A^K)$$

related? The answer is given in terms of Lyndon Hochschild Serre spectral sequence

- For small values of i the LHS spectral sequence gives us the low degree terms exact sequence:

$$0 \to H^1(G/K, A^K) \stackrel{\text{inf}}{\to} H^1(G, A) \stackrel{\text{res}}{\to} H^1(K, A)^{G/K} \stackrel{\text{tg}}{\to}$$

$$\stackrel{\text{tg}}{\to} H^2(G/K, A^K) \stackrel{\text{inf}}{\to} H^2(G, K).$$

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$$\xrightarrow{\text{tg}} H^2(G/K, A^K) \xrightarrow{\text{inf}} H^2(G, K).$$

- The above sequence allows us to reduce the problem to easier computations with cyclic groups.

Examples

- Let p be a prime number, $p > 3$ let X be the Fermat curve

$$x_0^{1+p} + x_1^{1+p} + x_2^{1+p} = 0.$$

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$$C_f : w^p - w = \sum_{i=1,(i,p)=1}^{m-1} a_i x^i + x^m$$

$$\dim_k H^1(C_f, G, \mathcal{T}_{C_f}) = m + \left\lceil \frac{m}{p} - \frac{2+m}{p^{m+1}} \right\rceil.$$