Weierstrass semigroups, Galois module structure of holomorphic differentials and applications

S. Karanikolopoulos A. Kontogeorgis

March 15, 2011

Aim of this talk

X is a projective nonsingular curve of genus $g \geq 2$ defined over an algebraicaly closed field of positive characteristic. We will denote the function field of X by F. The automorphism group of F will be denoted by G and it is a finite group.

1. Weierstrass semigroups
2. G-module structure of polydifferentials
3. Deformation theory of curves with automorphisms

Weierstrass semigroups and ramification filtration

Weierstrass semigroups

1. The Weierstrass semigroup Σ_P at the place P of F is the subsemigroup of the natural numbers that consists of all numbers $i \in \mathbb{N}$ such that there is an $f \in F$ with $(f)_\infty = iP$.
2. All numbers in the Weierstrass semigroup at P are called pole numbers. The set $\mathbb{N} - \Sigma_P$ is finite and consists of g elements. The elements of $\mathbb{N} - \Sigma_P$ are called gaps. All gaps are $\leq 2g - 1$.

Ramification Filtration

1.
$$G(P) = \{g \in G : g(P) = g\}.$$

2. The group G admits the following ramification filtration

$$G_0 \geq G_1 = \cdots = G_{i_1} > G_{i_1+1} = \cdots = G_{i_2} > G_{i_2+1} = \cdots = G_s > \{1\}.$$

Let t be a local uniformizer at P. The groups G_i are defined by $G_i = \{g \in G(P) : v_P(g(t) - t) \geq i + 1\}$.

Relations

Aim: Relate ramification filtration and the Weierstrass semigroup.

Relations

Aim: Relate ramification filtration and the Weierstrass semigroup.
Let m be the smaler pole number not divisible by p. Consider the space $L(mP)$, and fix a base. There is faithful representation of

$$\rho : G_1(P) \to L(mP).$$

Aim: Relate ramification filtration and the Weierstrass semigroup.
Let m be the smaler pole number not divisible by p. Consider the space $L(mP)$, and fix a base. There is faithful representation of

$$\rho : G_1(P) \to L(mP).$$

Moreover denote by f_m the function with $(f)_\infty = mP$. Assume that $f_m = \frac{1}{t^m}$. Consider the cocycle $a(\sigma) = \sigma(f_m) - f_m \in \langle f_0, \ldots, f_{m-1} \rangle_k$, $(f_i)_\infty = iP$. Then

$$\sigma(t) = t(1 + \alpha(\sigma)t^m)^{-1/m}$$

Relations

Aim: Relate ramification filtration and the Weierstrass semigroup.
Let m be the smaler pole number not divisible by p. Consider the space $L(mP)$, and fix a base. There is faithful representation of

$$\rho : G_1(P) \to L(mP).$$

Moreover denote by f_m the function with $(f)_\infty = mP$. Assume that $f_m = \frac{1}{t^m}$. Consider the cocycle $a(\sigma) = \sigma(f_m) - f_m \in \langle f_0, \ldots, f_{m-1} \rangle_k$, $(f_i)_\infty = iP$. Then

$$\begin{aligned} \sigma(t) &= t(1 + \alpha(\sigma)t^m)^{-1/m} \\ &= t - \frac{1}{m} t^{-|v_t(\alpha(\sigma)|+m+1} + \cdots \end{aligned}$$

$\sigma \in G_{m-|v_t(\alpha(\sigma)|}$.

Aim: Relate ramification filtration and the Weierstrass semigroup.
Let m be the smaler pole number not divisible by p. Consider the space $L(mP)$, and fix a base. There is faithful representation of

$$\rho : G_1(P) \to L(mP).$$

Moreover denote by f_m the function with $(f)_\infty = mP$. Assume that $f_m = \frac{1}{t^m}$. Consider the cocycle $a(\sigma) = \sigma(f_m) - f_m \in \langle f_0, \ldots, f_{m-1} \rangle_k$, $(f_i)_\infty = iP$. Then

$$\begin{aligned} \sigma(t) &= t(1 + \alpha(\sigma) t^m)^{-1/m} \\ &= t - \frac{1}{m} t^{-|v_t(\alpha(\sigma)|+m+1} + \cdots \end{aligned}$$

$\sigma \in G_{m-|v_t(\alpha(\sigma)|}$.
Every jump in the ramification filtration is the difference of two pole numbers.

Motivation

Deformation Theory

Deformation of the couple (X, G) over the local Artin ring A is a proper, smooth family of curves

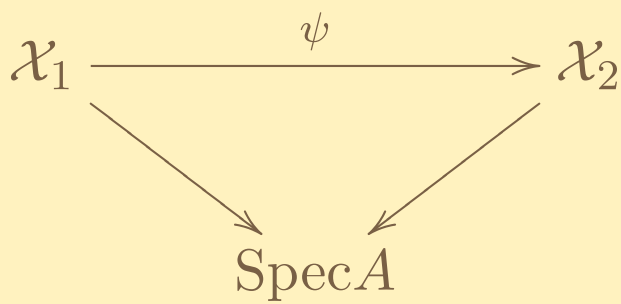
$$\mathcal{X} \to \mathrm{Spec}(A)$$

together with a group homomorphism $G \to \mathrm{Aut}_A(\mathcal{X})$ and a G-equivariant isomorphism ϕ from the fibre over the closed point of A to the original curve X:

$$\phi : \mathcal{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \to X.$$

Deformation Theory

Two deformations $\mathcal{X}_1, \mathcal{X}_2$ are considered to be equivalent if there is a G-equivariant isomorphism ψ that reduces to the identity in the special fibre and making the following diagram commutative:



Deformation Theory

The problem of determining deformations reduces to a "configuration problem of points" and a local problem.

Deformation Theory

The problem of determining deformations reduces to a "configuration problem of points" and a local problem.
Local problem: For every wild ramified point P fix a represetnation

$$\rho : G(P) \to \mathrm{Aut} k[[t]].$$

Deformation Theory

The problem of determining deformations reduces to a "configuration problem of points" and a local problem.

Local problem: For every wild ramified point P fix a represetnation

$$\rho : G(P) \to \mathrm{Aut} k[[t]].$$

Then define the folowing deformation functor:

$$D_P : \mathcal{C} \to \mathrm{Sets},$$

$$A \mapsto \left\{ \begin{array}{l} \text{lifts } G(P) \to \mathrm{Aut}(A[[t]]) \text{ of } \rho \text{ modulo conjugation} \\ \text{by an element of } \ker(\mathrm{Aut} A[[t]] \to k[[t]]) \end{array} \right\}$$

The problem of determining deformations reduces to a “configuration problem of points” and a local problem.
Local problem: For every wild ramified point P fix a represetnation

$$\rho : G(P) \rightarrow \mathrm{Aut} k[[t]].$$

Then define the folowing deformation functor:

$$D_P : \mathcal{C} \rightarrow \mathrm{Sets},$$

$$A \mapsto \left\{ \begin{array}{l} \text{lifts } G(P) \rightarrow \mathrm{Aut}(A[[t]]) \text{ of } \rho \text{ modulo conjugation} \\ \text{by an element of } \ker(\mathrm{Aut} A[[t]] \rightarrow k[[t]]) \end{array} \right\}$$

The group $\mathrm{Aut} k[[t]]$ is ugly. Someone prefers to work with linear representations.

The deformation functor sattisfies certain assumptions (Schlessinger criteria) such that

$$D(k[\epsilon]/\epsilon^2) \text{ is a vector space.}$$

Problem: Compute the dimension of $D(k[\epsilon])$.

$$D(k[\epsilon]/\epsilon^2) = H^1(X, G, \mathcal{T}_X)$$

The deformation functor sattisfies certain assumptions (Schlessinger criteria) such that

$$D(k[\epsilon]/\epsilon^2) \text{ is a vector space.}$$

Problem: Compute the dimension of $D(k[\epsilon])$.

$$\begin{aligned} D(k[\epsilon]/\epsilon^2) &= H^1(X, G, \mathcal{T}_X) \\ &= H^1(X/G, \pi_*^G(\mathcal{T}_X)) \oplus H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) \text{ Bertin-Mézard 2000} \end{aligned}$$

The deformation functor sattisfies certain assumptions (Schlessinger criteria) such that

$$D(k[\epsilon]/\epsilon^2) \text{ is a vector space.}$$

Problem: Compute the dimension of $D(k[\epsilon])$.

$$\begin{aligned} D(k[\epsilon]/\epsilon^2) &= H^1(X, G, \mathcal{T}_X) \\ &= H^1(X/G, \pi_*^G(\mathcal{T}_X)) \oplus H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) \text{ Bertin-Mézard 2000} \\ &= H^0(X, \Omega_X^{\otimes 2})_G \text{ K. 2007} \end{aligned}$$

The deformation functor sattisfies certain assumptions (Schlessinger criteria) such that

$$D(k[\epsilon]/\epsilon^2) \text{ is a vector space.}$$

Problem: Compute the dimension of $D(k[\epsilon])$.

$$\begin{aligned} D(k[\epsilon]/\epsilon^2) &= H^1(X,G,\mathcal{T}_X) \\ &= H^1(X/G,\pi_*^G(\mathcal{T}_X)) \oplus H^0(X/G,R^1\pi_*^G(\mathcal{T}_X)) \text{ Bertin-Mézard 2000} \\ &= H^0(X,\Omega_X^{\otimes 2})_G \text{ K. 2007} \end{aligned}$$

$$\dim_k H^1(X/G,\pi_*^G(\mathcal{T}_X)) = 3g_{X/G} - 3 + \sum_{\mu=1}^{r} \left\lceil \sum_{i=0}^{n_\mu} \frac{(e_i^{(\mu)}-1)}{e_0^{(\mu)}} \right\rceil .$$

Tangent space to the local problem

$$H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) = \bigoplus_{\nu=1}^{r} H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$$

Tangent space to the local problem

$$H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) = \bigoplus_{\nu=1}^{r} H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$$

$\hat{\mathcal{T}}_{X,P_i}$ is the $\hat{\mathcal{O}}_{X,x}$ module generated by $\frac{d}{dt}$ with the adjoint action.

Tangent space to the local problem

$$H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) = \bigoplus_{\nu=1}^{r} H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$$

$\hat{\mathcal{T}}_{X,P_i}$ is the $\hat{\mathcal{O}}_{X,x}$ module generated by $\frac{d}{dt}$ with the adjoint action.
$H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$ is just ordinary group cohomology.

Tangent space to the local problem

$$H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) = \bigoplus_{\nu=1}^{r} H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$$

$\hat{\mathcal{T}}_{X,P_i}$ is the $\hat{\mathcal{O}}_{X,x}$ module generated by $\frac{d}{dt}$ with the adjoint action.
$H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$ is just ordinary group cohomology.

- Bertin-Mézard 2000 $\mathbb{Z}/p\mathbb{Z}$
- Cornelissen-Kato 2002: Ordinary curves

Tangent space to the local problem

$$H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) = \bigoplus_{\nu=1}^{r} H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$$

$\hat{\mathcal{T}}_{X,P_i}$ is the $\hat{\mathcal{O}}_{X,x}$ module generated by $\frac{d}{dt}$ with the adjoint action.
$H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$ is just ordinary group cohomology.

- Bertin-Mézard 2000 $\mathbb{Z}/p\mathbb{Z}$
- Cornelissen-Kato 2002: Ordinary curves

Artin-Schreier extensions are easy to manage.

Tangent space to the local problem

$$H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) = \bigoplus_{\nu=1}^{r} H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$$

$\hat{\mathcal{T}}_{X,P_i}$ is the $\hat{\mathcal{O}}_{X,x}$ module generated by $\frac{d}{dt}$ with the adjoint action.
$H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$ is just ordinary group cohomology.

- Bertin-Mézard 2000 $\mathbb{Z}/p\mathbb{Z}$
- Cornelissen-Kato 2002: Ordinary curves

Artin-Schreier extensions are easy to manage. One is tempted to go up the whole p-extensions step by step using Artin-Schreier p-extensions and LHS-spectral sequence to relate cohomology of $G/H, H, G$.

Tangent space to the local problem

$$H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) = \bigoplus_{\nu=1}^{r} H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$$

$\hat{\mathcal{T}}_{X,P_i}$ is the $\hat{\mathcal{O}}_{X,x}$ module generated by $\frac{d}{dt}$ with the adjoint action.
$H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$ is just ordinary group cohomology.

- Bertin-Mézard 2000 $\mathbb{Z}/p\mathbb{Z}$
- Cornelissen-Kato 2002: Ordinary curves

Artin-Schreier extensions are easy to manage. One is tempted to go up the whole p-extensions step by step using Artin-Schreier p-extensions and LHS-spectral sequence to relate cohomology of $G/H, H, G$. The difficult part to handle using this approach is the computation of the kernel of the transgretion map.

Tangent space to the local problem

$$H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) = \bigoplus_{\nu=1}^{r} H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$$

$\hat{\mathcal{T}}_{X,P_i}$ is the $\hat{\mathcal{O}}_{X,x}$ module generated by $\frac{d}{dt}$ with the adjoint action.
$H^1(G(P_i), \hat{\mathcal{T}}_{X,P_i})$ is just ordinary group cohomology.

- Bertin-Mézard 2000 $\mathbb{Z}/p\mathbb{Z}$
- Cornelissen-Kato 2002: Ordinary curves

Artin-Schreier extensions are easy to manage. One is tempted to go up the whole p-extensions step by step using Artin-Schreier p-extensions and LHS-spectral sequence to relate cohomology of $G/H, H, G$. The difficult part to handle using this approach is the computation of the kernel of the transgretion map. We have been able to use this approach only for very special cases of groups.
(K 2007, elementary abelian covers)

An other Unsolved problem

Determine the G-module structure of $H^0(X, \Omega^{\otimes m})$ in positive characteristic when G is a p-group.

An other Unsolved problem

Determine the G-module structure of $H^0(X, \Omega^{\otimes m})$ in positive characteristic when G is a p-group.
Only partial results known: Several authors put restrictions on

1. Group Structure (cyclic, elementary abelian)
2. Ramification filtration (weakly ramification, i.e. $G_2(P) = \{1\}$)

An other Unsolved problem

Determine the G-module structure of $H^0(X, \Omega^{\otimes m})$ in positive characteristic when G is a p-group.
Only partial results known: Several authors put restrictions on

1. Group Structure (cyclic, elementary abelian)
2. Ramification filtration (weakly ramification, i.e. $G_2(P) = \{1\}$)
3. **This Lecture:** One point covers.

An other Unsolved problem

Determine the G-module structure of $H^0(X, \Omega^{\otimes m})$ in positive characteristic when G is a p-group.
Only partial results known: Several authors put restrictions on

1. Group Structure (cyclic, elementary abelian)
2. Ramification filtration (weakly ramification, i.e. $G_2(P) = \{1\}$)
3. **This Lecture:** One point covers.

Knowledge of the G-module structure of $H^0(X, \Omega^{\otimes m})$ implies the $H^0(X, \Omega^{\otimes m})_G$.

An other Unsolved problem

Determine the G-module structure of $H^0(X, \Omega^{\otimes m})$ in positive characteristic when G is a p-group.
Only partial results known: Several authors put restrictions on

1. Group Structure (cyclic, elementary abelian)
2. Ramification filtration (weakly ramification, i.e. $G_2(P) = \{1\}$)
3. **This Lecture:** One point covers.

Knowledge of the G-module structure of $H^0(X, \Omega^{\otimes m})$ implies the $H^0(X, \Omega^{\otimes m})_G$. This is linear algebra and can be done by a computer but it is not easy to do by hand for certain groups (more on this later).

- K. 2008 $\mathbb{Z}/p^n\mathbb{Z}$ using results from S. Nakajima
- K-Köck 2010 Weakly ramified curves

Katz-Gabber covers

Every p-Galois extension of $k[[t]]$ can be globalized:

$$\begin{array}{ccc} L & & X \\ G \Big| & & \Big\downarrow G \\ \mathrm{Quot}(k[[t]]) & & \mathbb{P}^1_k \end{array}$$

The cover $X \to \mathbb{P}^1_k$ has only one ramified point P such that $G(P) = G$ and the localization of $X \to \mathbb{P}^1_k$ gives rize to the extension $L/k((t))$.

Katz-Gabber covers

Let $X \to \mathbb{P}^1(k)$ be a cover with only one full ramified point and Galois group $G = \mathrm{Gal}(X/\mathbb{P}^1)$ a p-group.

Katz-Gabber covers

Let $X \to \mathbb{P}^1(k)$ be a cover with only one full ramified point and Galois group $G = \mathrm{Gal}(X/\mathbb{P}^1)$ a p-group.

Construction of a basis for the m-holomorphic polydifferentials of X

Katz-Gabber covers

Let $X \to \mathbb{P}^1(k)$ be a cover with only one full ramified point and Galois group $G = \mathrm{Gal}(X/\mathbb{P}^1)$ a p-group.

Construction of a basis for the m-holomorphic polydifferentials of X

$$\mathrm{div}(df_1^{\otimes m}) = \left(-2mp^h + m \sum_{i=0}^{\infty} (e_i - 1) \right) P,$$

which in turn is equal to $m(2g_X - 2)P$ by Riemann-Hurwitz formula.

Katz-Gabber covers

Proposition:

1. The set $f_i df_1, \deg \operatorname{div}(f_i) \leq 2g_X - 2$ is a basis for the space of holomorphic differentials for X.

Katz-Gabber covers

Proposition:

1. The set $f_i df_1$, $\deg \mathrm{div}(f_i) \leq 2g_X - 2$ is a basis for the space of holomorphic differentials for X.
2. The set $f_i df_1^{\otimes m}$, $\deg \mathrm{div}(f_i) < m(2g_X - 2)$ is a basis for the space of holomorphic differentials for X.

Katz-Gabber covers

Proposition:

1. The set $f_i df_1$, $\deg \operatorname{div}(f_i) \leq 2g_X - 2$ is a basis for the space of holomorphic differentials for X.
2. The set $f_i df_1^{\otimes m}$, $\deg \operatorname{div}(f_i) < m(2g_X - 2)$ is a basis for the space of holomorphic differentials for X.

Proposition:

1. The set $f_i df_1$, $\deg \operatorname{div}(f_i) \leq 2g_X - 2$ is a basis for the space of holomorphic differentials for X.
2. The set $f_i df_1^{\otimes m}$, $\deg \operatorname{div}(f_i) < m(2g_X - 2)$ is a basis for the space of holomorphic differentials for X.

Proof: All m-holomorphic differentials are of the form $f df_1^{\otimes m}$. Therefore the condition for being holomorphic is translated to the condition $f \in L(m(2g_X - 2)P)$. Therefore the linear independent elements $f_i df_1^{\otimes m}$ with $\deg \operatorname{div} f_i = m_i \leq m(2g_X - 2)$ form a basis for $\Omega_X(m)$.

Remarks

Remark: For the $m > 1$ case we fist observe that the space of m-holomorphic differentials has dimension

$$\dim L(mW) = m(2g-2) + 1 - g = (2m-1)g - 2m + 1.$$

On the other hand side the number of f_i such that $\deg \operatorname{div}(f_i) \leq m(2g-2)$ can be computed as follows:
In the interval $[0, 2g-1]$ there are g such elements. In the interval $(2g-1, m(2g-2)]$ there are $m(2g-2) - (2g-1) = 2mg - 2m - 2g + 1$ elements. In total there are $2mg - 2m - 2g + 1 + g = (2m-1)g - 2m + 1$ and this coincides with the dimension of the space of m-holomorphic differentials.

Symmetric Semigroup

Corrolary: The Weierstrass semigroup at P is symmetric, i.e., $2g-1$ is a gap.
Proof: Recall that the set of gaps (elements in the natural numbers that are not in the Weierstrass semigroup) has g elements $\{i_1 = 1, \ldots, i_g \leq 2g-1\}$.
If $2g-1$ is a gap then then there are g-pole numbers $i\mu$ with $i_\mu \leq 2g-2$ and the number of pole numbers $\leq 2g-2$ equals g (0 is always a pole number).
If $2g-1$ is a pole number then we can form only $g-1$ holomorphic differentials of the form $f_i df_1$.

Understanding the representation

Let f_1 be the function corresponding to the first not trivial pole number m_1. The action of $\sigma \in G$ on f_1 is given by $\sigma(f_1) = f_1 + \alpha_1(\sigma)$. Notice that the multiples of νm_1 correspond to powers f_1^ν that are acted on by σ by

$$\sigma f_1^\nu = (f_1 + a_1)^\nu = f_1^\nu + \sum_{\mu=0}^{\nu-1} \binom{\nu}{\mu} a_1(\sigma)^{\nu-\mu} f_1^\mu.$$

Understanding the representation

Let f_1 be the function corresponding to the first not trivial pole number m_1. The action of $\sigma \in G$ on f_1 is given by $\sigma(f_1) = f_1 + \alpha_1(\sigma)$. Notice that the multiples of νm_1 correspond to powers f_1^ν that are acted on by σ by

$$\sigma f_1^\nu = (f_1 + a_1)^\nu = f_1^\nu + \sum_{\mu=0}^{\nu-1} \binom{\nu}{\mu} a_1(\sigma)^{\nu-\mu} f_1^\mu.$$

Let m_2 be the next pole number not divisible by m_1 that corresponds to a new function f_2. The polynomial ring $k[f_1, f_2]$ contains all functions that correspond to the subsemigroup generated by the pole numbers m_1, m_2. Moreover if $\sigma(f_2) = f_2 + a_2(\sigma)$ then we can extend this action to $k[f_1, f_2]$.

Understanding the representation

Let f_1 be the function corresponding to the first not trivial pole number m_1. The action of $\sigma \in G$ on f_1 is given by $\sigma(f_1) = f_1 + \alpha_1(\sigma)$. Notice that the multiples of νm_1 correspond to powers f_1^ν that are acted on by σ by

$$\sigma f_1^\nu = (f_1 + a_1)^\nu = f_1^\nu + \sum_{\mu=0}^{\nu-1} \binom{\nu}{\mu} a_1(\sigma)^{\nu-\mu} f_1^\mu.$$

Let m_2 be the next pole number not divisible by m_1 that corresponds to a new function f_2. The polynomial ring $k[f_1, f_2]$ contains all functions that correspond to the subsemigroup generated by the pole numbers m_1, m_2. Moreover if $\sigma(f_2) = f_2 + a_2(\sigma)$ then we can extend this action to $k[f_1, f_2]$. We can go on by defining the action on the generators f_ν of the semigroup up to the element f_m.

Understanding the representation

Remark: The elements $a_\nu(\sigma) = (\sigma - 1)f_\nu$ are (not necessarily trivial) cocycles in $H^1(G, k[f_1, \ldots, f_{\nu-1}])$.

Remark: The elements $a_\nu(\sigma) = (\sigma - 1) f_\nu$ are (not necessarily trivial) cocycles in $H^1(G, k[f_1, \ldots, f_{\nu-1}])$.
Definition: Define a weighted degree function on $k[f_1, \ldots, f_\nu]$ by $\deg(f_\nu) = m_\nu$.

Remark: The elements $a_\nu(\sigma) = (\sigma - 1)f_\nu$ are (not necessarily trivial) cocycles in $H^1(G, k[f_1, \ldots, f_{\nu-1}])$.

Definition: Define a weighted degree function on $k[f_1, \ldots, f_\nu]$ by $\deg(f_\nu) = m_\nu$.

There are finite many generators of the Weierstrass semigroup. Let r be their number. Define $k_\ell[f_1, \ldots, f_r]$ to be the vector space generated by elements of the degree $\leq \ell$.

Remark: The space of (m)- holomorphic differentials is isomorphic to $k_{m(2g_X-2)}[f_1, \ldots, f_r]$.

Remark: The elements $a_\nu(\sigma) = (\sigma - 1)f_\nu$ are (not necessarily trivial) cocycles in $H^1(G, k[f_1, \ldots, f_{\nu-1}])$.

Definition: Define a weighted degree function on $k[f_1, \ldots, f_\nu]$ by $\deg(f_\nu) = m_\nu$.

There are finite many generators of the Weierstrass semigroup. Let r be their number. Define $k_\ell[f_1, \ldots, f_r]$ to be the vector space generated by elements of the degree $\leq \ell$.

Remark: The space of (m)- holomorphic differentials is isomorphic to $k_{m(2g_X-2)}[f_1, \ldots, f_r]$.

This allows us to compute the dimension of $H^0(X, \Omega^{\otimes 2})_G$ which is equal to the dimension of the complement of the vector space generated by $a_i(g), i = 1, \ldots, r \;\; g \in G$.

How many indecomposable summands there are?

Understanding the structure for the indecomposable summands is complicated but all the information is inside the representation matrix.

How many indecomposable summands there are?

Understanding the structure for the indecomposable summands is complicated but all the information is inside the representation matrix.
We have a cover $X \to \mathbb{P}^1$, with only one fully ramified point $Q \mapsto P$. The semigroup of P at $\mathbb{P}^1$ is just the semigroup of natural numbers. If x is in the rational function field with $(x)_\infty = P$ then $(x^n)_\infty = nP$.

How many indecomposable summands there are?

Understanding the structure for the indecomposable summands is complicated but all the information is inside the representation matrix.
We have a cover $X \to \mathbb{P}^1$, with only one fully ramified point $Q \mapsto P$. The semigroup of P at $\mathbb{P}^1$ is just the semigroup of natural numbers. If x is in the rational function field with $(x)_\infty = P$ then $(x^n)_\infty = nP$.
The elements x^n seen as elements in the function field of X are in the Weierstrass semigroup of Q and $(x^n)_{\infty,X} = p^s n$. The G-invariant functions in $L(kP)$ have pole numbers divisible by p^n.

How many indecomposable summands there are?

Understanding the structure for the indecomposable summands is complicated but all the information is inside the representation matrix.
We have a cover $X \to \mathbb{P}^1$, with only one fully ramified point $Q \mapsto P$. The semigroup of P at $\mathbb{P}^1$ is just the semigroup of natural numbers. If x is in the rational function field with $(x)_\infty = P$ then $(x^n)_\infty = nP$.
The elements x^n seen as elements in the function field of X are in the Weierstrass semigroup of Q and $(x^n)_{\infty,X} = p^s n$. The G-invariant functions in $L(kP)$ have pole numbers divisible by p^n.
Every indecomposable summand has exactly one invariant element, so the number of indecomposable summands in $L(nP)$ is just $\left\lfloor \frac{n}{p^n} \right\rfloor$.

Compute the tangent space to the local deformation functor

One point ramified curve:

$$H^0(X, G, \mathcal{T}_C) = H^0(X/G, \pi_*^G(\mathcal{X}) \oplus H^1(G, \hat{\mathcal{T}}_X).$$

Compute the tangent space to the local deformation functor

One point ramified curve:

$$H^0(X,G,\mathcal{T}_C)=H^0(X/G,\pi_*^G(\mathcal{X})\oplus H^1(G,\hat{\mathcal{T}}_X).$$

$$\dim H^0(X/G,\pi_*^G(\mathcal{T}_X))=-3+\left\lceil \sum_{i=0}^{n_\mu}\frac{(e_i^{(\mu)}-1)}{e_0^{(\mu)}}\right\rceil$$

And

$$\dim H^0(X,G,\mathcal{T}_C)=\dim k_{4g_X-4}[f_1,\ldots,f_r]_G.$$

Compute the tangent space to the local deformation functor

One point ramified curve:

$$H^0(X, G, \mathcal{T}_C) = H^0(X/G, \pi_*^G(\mathcal{X}) \oplus H^1(G, \hat{\mathcal{T}}_X).$$

$$\dim H^0(X/G, \pi_*^G(\mathcal{T}_X)) = -3 + \left\lceil \sum_{i=0}^{n_\mu} \frac{(e_i^{(\mu)} - 1)}{e_0^{(\mu)}} \right\rceil$$

And

$$\dim H^0(X, G, \mathcal{T}_C) = \dim k_{4g_X - 4}[f_1, \ldots, f_r]_G.$$

We can now compute $H^1(G, \hat{\mathcal{T}}_X)$.

Examples

The first non zero pole number is prime to p.

This implies that there is a faithfull representation of G on $L(mP)$ that has dimension 2.

The first non zero pole number is prime to p.

This implies that there is a faithfull representation of G on $L(mP)$ that has dimension 2.
The group G is elementary abelian, generated by elements σ_i.

The first non zero pole number is prime to p.

This implies that there is a faithfull representation of G on $L(mP)$ that has dimension 2.
The group G is elementary abelian, generated by elements σ_i.
The space of “local holomorphic differentials” is $k_{m(2g_X-2)}[f]$ with action

$$\sigma_i f = f + c(\sigma_i)$$

The first non zero pole number is prime to p.

This implies that there is a faithfull representation of G on $L(mP)$ that has dimension 2.
The group G is elementary abelian, generated by elements σ_i.
The space of “local holomorphic differentials” is $k_{m(2g_X-2)}[f]$ with action

$$\sigma_i f = f + c(\sigma_i)$$

$$\sigma_i f^k = f^k + \sum_{\nu=0}^{k-1} \binom{k}{\nu} f^\nu c(\sigma_i)^{k-\nu}$$

The first non zero pole number is prime to p.

This implies that there is a faithfull representation of G on $L(mP)$ that has dimension 2.
The group G is elementary abelian, generated by elements σ_i.
The space of "local holomorphic differentials" is $k_{m(2g_X-2)}[f]$ with action

$$\sigma_i f = f + c(\sigma_i)$$

$$\sigma_i f^k = f^k + \sum_{\nu=0}^{k-1} \binom{k}{\nu} f^\nu c(\sigma_i)^{k-\nu}$$

We have to know when a binomial coefficient is zero mod p.

Dimension of covariant elements

For a module $k_\ell[f]$ with the above action consider the p-adic expansion of $\ell = \sum_{i=1}^{n} a_i p^i$. Let $\chi : \{0, \ldots, p-1\} \to \{0, 1\}$ be the function defined by:

$$\chi(a) := \begin{cases} 1, & \text{if } a \neq 0 \\ 0, & \text{if } a = 0 \end{cases}.$$

Then

$$\dim k_\ell[f]_G = \sum_{i=1}^{n} \chi(a_i).$$

Cyclic groups

Let G be a cyclic group of order p^n. We need n-jumps in the ramification filtration so we need n generators for the semigroup. Let f_m be function corresponding to the first pole number prime to p. The values for the cocylce $a(g^k) = g^k f_m - f_m$ is determined by the cocycle condition

$$a(g^k) = (1 + g + g^2 + \cdots + g^{k-1})a(g).$$

Cyclic groups

Let G be a cyclic group of order p^n. We need n-jumps in the ramification filtration so we need n generators for the semigroup. Let f_m be function corresponding to the first pole number prime to p. The values for the cocylce $a(g^k) = g^k f_m - f_m$ is determined by the cocycle condition

$$a(g^k) = (1 + g + g^2 + \cdots + g^{k-1})a(g).$$

The action of g^ν on $a(g)$ can be described as follows:

Cyclic groups

Select the basis $(g-1)^k f_m$.

Select the basis $(g-1)^k f_m$.
Then on this basis the action is given by the Jordan matrix

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \ddots & & \vdots \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

Cyclic groups

Select the basis $(g-1)^k f_m$.
Then on this basis the action is given by the Jordan matrix

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \ddots & & \vdots \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

This matrix has k power

$$a_{ij} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ \binom{k}{j-i} & \text{if } j > i \end{cases}$$

In the above formula $\binom{k}{\mu} = 0$ if $\mu > k$.

Cyclic groups

The jumps for the ramification filtration are given by

$$m - v_t(a(g^{p^i})).$$

Cyclic groups

The jumps for the ramification filtration are given by

$$m - v_t(a(g^{p^i})).$$

Question: Can one prove the Hasse-Arf theorem this way?