Deformations of Curves with automorphisms

representations on Riemann-Roch spaces

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Contents

The decomposition group $G(P)$ of a point on X admits the following ramification filtration

$$G(P) = G_0(P) \supset G_1(P) \supset G_2(P) \supset \ldots, \tag{1}$$

$G_i(P) = \{\sigma \in G(P) : v_P(\sigma(t)) - t \geq i+1\}$, and t is a local uniformizer at P and v_P is the corresponding valuation.

Faithful representation of $G_1(P)$ in $\mathrm{GL}(L(mP))$, where

$$L(iP) := \{f \in k(X)^* : \mathrm{div}(f) + iP \geq 0\} \cup \{0\}.$$

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Lemma

Let $1 \leq m \leq 2g-1$ be the smallest pole number not divisible by the characteristic. There is a faithful representation

$$\rho : G_1(P) \to \mathrm{GL}\big(L(mP)\big) \qquad (2)$$

Proof.

It is clear that the space $L(mP)$ is preserved by any automorphism in $G_1(P)$. Hence we have the desired representation ρ. Let f be a function with pole at P of order m. We can write f as $f = u/t^m$, where u is a unit in the local ring $\mathcal{O}_P$. Since $(m,p)=1$, Hensel's lemma implies that u is an m-th power so the local uniformizer can be selected so that $f = 1/t^m$. Let $\sigma \in G_1(P)$ be an element that acts trivially on $L(mP)$. Then $\sigma(1/t^m) = 1/t^m$ and $\sigma(t) = \zeta t$. Order of σ is p and $(p,m)=1$ we have that $\zeta = 1$, and σ is the identity element of $G_1(P)$. □

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1. The flag of vector spaces $L(iP)$ for $i \leq m$ is preserved, so the representation matrices are upper triangular, or in other words $G_1(P)$ is a subgroup of the Borel group of the flag.
2. We assume that $m = m_0 > m_1 > \cdots > m_r = 0$, are the pole numbers $\leq m$. Therefore, a basis for the vector space $L(mP)$ is given by

$$\left\{ 1, \frac{u_i}{t^{m_i}}, \frac{1}{t^m} : \text{ where } 1 < i < r, p \mid m_i \text{ and } u_i \text{ are certain units} \right\}$$

With respect to this basis, an element $\sigma \in G_1(P)$ acts on $1/t^m$ by

$$\sigma \frac{1}{t^m} = \frac{1}{t^m} + \sum_{i=1}^{r} c_i(\sigma) \frac{u_i}{t^{m_i}},$$

and then it maps the local uniformizer t to

$$\sigma(t) = \frac{\zeta t}{\left(1 + \sum_{i=1}^{r} c_i(\sigma) u_i t^{m - m_i}\right)^{1/m}} \tag{3}$$

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2. In our setting it seems that the simplest covers are the ones with 2-dimensional representations attached at wild ramification points.
3. Curves with 2-dimensional representations have only one jump in their ramification filtration at P, and that jump occurs at m, where m is the first non zero pole number.
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Remark

There are curves with two dimensional represenations attached to wild points:

$$F : \sum_{\nu=0}^{n} a_n y^{p^\nu} = \sum_{\mu=0}^{m} b_\mu x^\mu, \tag{4}$$

so that $m \not\equiv 0 \mod p$, $a_n, a_0, b_0 \neq 0$, $n \geq 1, m \geq 2$, studied by H. Stichtenoth. Let P_∞ be the unique place above the place p_∞ of the function field $k(x)$. Weierstrass semigroup at P_∞ is given by $m\mathbb{Z}_{\geq 0} + p^n\mathbb{Z}_{\geq 0}$. Select $m < p^n$ the first pole number is 0 and the second is m therefore $d = 2$. The ramification filtration of $G = \mathrm{Gal}(F/k(x))$ at P_∞ is given by:

$$G_0(P_\infty) = G_1(P_\infty) = \cdots = G_m(P_\infty) = \mathrm{Gal}(F/k(x)) > G_{m+1}(P_\infty) = \{1\}$$

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The relation to deformation theory

Problem:

$$D_P : \mathcal{C} \to \mathrm{Sets}, A \mapsto \left\{ \begin{array}{l} \text{lifts } G(P) \to \mathrm{Aut}(A[[t]]) \text{ of } \rho \text{ mod-} \\ \text{ulo conjugation with an element} \\ \text{of } \ker(\mathrm{Aut} A[[t]] \to k[[t]]) \end{array} \right\}$$

A is a local Artin Λ-algebra with residue field k. For the mixed characteristic case we consider Λ to be a complete Noetherian local ring with residue field k. Usually Λ is an algebraic extension of the ring of Witt vector $W(k)$. For the equicharacteristic case we take $\Lambda = k$.

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An easier problem

- Automorphism groups of formal power series is a difficult object to understand.
- For a k-algebra A with maximal ideal m_A, consider the multiplicative group $L_n(A) < GL_n(A)$, of invertible lower triangular matrices with entries in A, and invertible elements λ in the diagonal, such that $\lambda - 1 \in m_A$.
- Consider the following functor from the category $\mathcal{C}$ of local Artin k-algebras to the category of sets

$$F : A \in Ob(\mathcal{C}) \mapsto \left\{ \begin{array}{l} \text{liftings of } \rho : G(P) \to L_n(k) \\ \text{to } \rho_A : G(P) \to L_n(A) \text{ modulo} \\ \text{conjugation by an element} \\ \text{of } \ker(L_n(A) \to L_n(k)) \end{array} \right\}$$

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Global Deformation functors

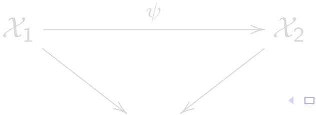
Deformation of the couple (X, G) over the local Artin ring A is a proper, smooth family of curves

$$\mathcal{X} \to \mathrm{Spec}(A)$$

together with a group homomorphism $G \to \mathrm{Aut}_A(\mathcal{X})$ together with a G-equivariant isomorphism ϕ from the fibre over the closed point of A to the original curve X:

$$\phi : \mathcal{X} \otimes_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \to X.$$

Two deformations $\mathcal{X}_1, \mathcal{X}_2$ are considered to be equivalent if there is a G-equivariant isomorphism ψ that reduces to the identity in the special fibre and making the following diagram commutative:



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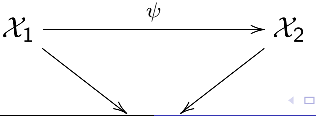

Global Deformation Functor

The global deformation functor is defined:

$$D_{\mathrm{gl}} : \mathcal{C} \longrightarrow \mathrm{Sets}, \mathrm{A} \mapsto \left\{ \begin{array}{l} \text{Equivalence classes} \\ \text{of deformations of} \\ \text{couples } (X, G) \text{ over } A \end{array} \right\}$$

Local Global Principle

- $$D_{\mathrm{loc}} = \prod D_{P_i}$$
  P_i runs over all wild ramified points.
- There is a smooth morphism $\phi : D_{\mathrm{gl}} \to D_{\mathrm{loc}}$
- The global deformation ring R_{gl} and the deformation rings R_i of the deformation functors D_{P_i} are related
  $$R_{\mathrm{gl}} = (R_1 \hat{\otimes} R_2 \hat{\otimes} \cdots \hat{\otimes} R_r)[[U_1, \ldots, U_N]],$$
  where $N = \dim_k H^1(X/G, \pi_*^G(\mathcal{T}_X))$, and R_i is the deformation ring of D_{P_i}.

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Relative Ramification

Lemma

Let $\mathcal{X} \to \mathrm{Spec} A$ be an A-curve, admitting a fibrewise action of the finite group G, where A is a Noetherian local ring. Let $S = \mathrm{Spec} A$, and $\Omega_{\mathcal{X}/S}$, $\Omega_{\mathcal{Y}/S}$ be the sheaves of relative differentials of $\mathcal{X}$ over S and $\mathcal{Y}$ over S, respectively. Let $\pi : \mathcal{X} \to \mathcal{Y}$ be the quotient map. The sheaf

$$\mathcal{L}(-D_{\mathcal{X}/\mathcal{Y}}) = \Omega^{-1}_{\mathcal{X}/S} \otimes_S \pi^* \Omega_{\mathcal{Y}/S}.$$

is the ideal sheaf the horizontal Cartier divisor $D_{\mathcal{X}/\mathcal{Y}}$. The intersection of $D_{\mathcal{X}/\mathcal{Y}}$ with the special and generic fibre of $\mathcal{X}$ gives the ordinary branch divisors for curves.

Lifting matrix representations

Assumptions:

- R is a complete local regular integer domain.
- $\mathcal{X} \rightarrow \mathrm{Spec} R$ be a deformation of the couple (X, G).
- P is a wild ramified point of the special fibre X.
- There is a a 2-dimensional representation $\rho : G_1(P) \rightarrow \mathrm{GL}_k(H^0(X, \mathcal{L}(mP)))$ attached to P.
- There is a G-invariant horizontal divisor that intersects the special fibre at mP. **Not always possible!**

Then:

- there is a free R-module M of rank 2 generated by $1, \tilde{f}$ so that $M := \langle 1, \tilde{f} \rangle_R \subset H^0((\mathcal{X}, \mathcal{L}(\alpha D)))$, where $1 \leq \alpha \in \mathbb{N}$ and $M \otimes_R k = H^0(X, \mathcal{L}(mP))$.
- the representation ρ can be lifted to a representation

$$\tilde{\rho} : G_1(P) \rightarrow \mathrm{GL}_R(\langle 1, \tilde{f} \rangle_R).$$

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- There is a a 2-dimensional representation $\rho : G_1(P) \to \mathrm{GL}_k(H^0(X, \mathcal{L}(mP)))$ attached to P.
- There is a G-invariant horizontal divisor that intersects the special fibre at mP. **Not always possible!**

Then:

- there is a free R-module M of rank 2 generated by $1, \tilde{f}$ so that $M := \langle 1, \tilde{f} \rangle_R \subset H^0((\mathcal{X}, \mathcal{L}(\alpha D)))$, where $1 \leq \alpha \in \mathbb{N}$ and $M \otimes_R k = H^0(X, \mathcal{L}(mP))$.
- the representation ρ can be lifted to a representation

$$\tilde{\rho} : G_1(P) \to \mathrm{GL}_R(\langle 1, \tilde{f} \rangle_R).$$

Lifting matrix representations

Assumptions:

- R is a complete local regular integer domain.
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Lifting Matrix Representations

The basis element $\tilde{f}$ is of the form

$$\tilde{f} = \frac{1}{(T^m + a_{m-1}T^{m-1} + \cdots + a_1 T_1 + a_0)} u(T), \qquad (5)$$

where $a_0, \ldots, a_{m-1} \in m_R$ and $u(T)$ is a unit in $R[[T]]$ reducing to $1 \mod m_R$.

Finding the horizontal divisor D

Problem: Two horizontal divisors can collapse to the same point in the special fibre There are conditions:

- For a curve X and a branch point P of X we will

$$\mathrm{ar}_P(\sigma) = \sum_{i=1}^{s} \mathrm{ar}_{P_i}(\sigma).$$

  denote by $i_{G,P}$ the order function of the filtration of G at P. The Artin representation of the group G is defined by $\mathrm{ar}_P(\sigma) = -f_P i_{G,P}(\sigma)$ for $\sigma \neq 1$ and $\mathrm{ar}_P(1) = f_P \sum_{\sigma \neq 1} i_{G,P}(\sigma)$

- The integer $i_{G,P}(\sigma)$ is equal to the multiplicity of $P \times P$ in the intersection of $\Delta.\Gamma_\sigma$ in the relative A-surface $\mathcal{X} \times_{\mathrm{Spec} A} \mathcal{X}$, where Δ is the diagonal and Γ_σ is the graph of σ.

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Numerical conditions

Corollary

Assume that $V = G_1(P)$ is an elementary abelian group with more than one $\mathbb{Z}/p\mathbb{Z}$ components. If V can be lifted to characteristic zero, then $\frac{|V|}{p} \mid m+1$.

Proof.

The group V acts on the generic fibre, where the possible stabilizers of points are cyclic groups. Since V is not cyclic it can not fix any point P_i in the intersection of the branch locus with the generic fibre. Only a cyclic component of V can fix a point P_i. Since V act on the set of points P_i, each orbit has $|V|/p$ elements. For any element $\sigma \in V$ the Artin representation $\mathrm{ar}_{P_i}(\sigma) = 1$ (no wild ramification at the generic fibre). The number of $\{P_i\}$ is $m+1$ and the desired result follows. □

Conditions for the existence of D

We are looking for a $G_1(P)$-invariant divisor intersecting the special fibre at mP

Let $T = \{\bar{P}_i\}_{i=1,\ldots,s}$ be the set of horizontal branch divisors that restricts to P in the special fibre of X. This space is acted on by $G_1(P)$, since $\bar{P}_i$ are all components of the branch divisor. Each of the $\bar{P}_i$ is fixed by some element of G but not necessarily by the whole group $G_1(P)$, unless $G_1(P)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Let $O(T)$ be the set of orbits of T under the action of the group $G_1(P)$, on T. A horizontal divisor D supported on T, is invariant under the action of $G_1(P)$ if and only if, the divisor D is of the form:

$$D = \sum_{C \in O(T)} n_C \sum_{P \in C} P,$$

i.e., horizontal Cartier divisors that are in the same orbit of the action of $G_1(P)$ must appear with the same weight in D.

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Easy numerical conditions

- If one orbit of $G_1(P)$ acting on T is a singleton, i.e., there is a $\bar{P}_i$ fixed by the whole group $G_1(P)$, then the semigroup

$$\sum_{C \in O(T)} n_C \# C, \quad n_C \in \mathbb{N},$$

is the semigroup of natural numbers, and we are done. This is the case when the group $G_1(P)$ is cyclic.

- If $\#T \not\equiv 0 \mod p$ then there is at least one orbit that is a singleton. Indeed, if all orbits have more than one element then all orbits must have cardinality divisible by p, and since the set T is the disjoint union of orbits it must also have cardinality divisible by p.

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Lemma

If m is the first pole number that is not divisible by the characteristic, and $p \nmid m+1$ then there is an orbit that consists of only one element.

Proof.

Since $G_1(P)$ is abelian P_i in the same orbit are fixed by the same subgroup $H \subset G_1(P)$. Fix a P_i. Its orbit has p^a elements $0 \leq a$. If $a = 0$ then P_i is fixed by the whole group $G_1(P)$. Consider all P_j's fixed by $\langle \sigma \rangle$. If their orbits have more than one element then the union of their orbits has order a power of p. This implies that the sum of the Artin representations at the generic fibre is divisible by p while on the special fibre it is $m+1$, a contradiction. □

Limited application in lifting from positive characteristic

If the elementary abelian group $G_1(P)$ has more than two cyclic components, then there is no horizontal $G_1(P)$-invariant divisor D contained in the branch locus and intersecting the special fibre at P with multiplicity m.

Indeed, since the stabilizers of elements in the generic fibre are cyclic groups of order p, all orbits of elements are divisible by p. Therefore, a $G_1(P)$-invariant divisor should have degree divisible by p. This, can not happen since $(m, p) = 1$.

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2-dimensional representations

$$\sigma\left(\frac{1}{t^m}\right) = \frac{1}{t^m} + c(\sigma),$$

Define the following representation of V to automorphisms of formal powerseries rings:

$$\rho : V \to \mathrm{Aut}(k[[t]]),$$

$$\sigma \mapsto \rho_\sigma,$$

where

$$\rho_\sigma(t) = \frac{t}{(1+c(\sigma)t^m)^{1/m}} = t\left(1+\sum_{\nu=1}^{\infty}\binom{-1/m}{\nu}c(\sigma)^\nu t^{\nu m}\right).$$

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Small extensions

Let

$$0 \to \ker\pi \to A' \to A \to 0$$

be a small extension, i.e. $\ker\pi \cdot m_{A'} = 0$, where $m_{A'}, m_A$ are the maximal ideals of A, A' respectively. Assume we have the following data:

- $C(\sigma) = c(\sigma) + \delta(\sigma)$
- $\lambda(\sigma) = 1 + \lambda_1(\sigma)$,
- $\tilde{f} = f + \Delta$.

Where $\delta(\sigma), \lambda_1(\sigma) \in m_{A'}$, $\Delta \in m_{A'}((t))$.

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Where $\delta(\sigma), \lambda_1(\sigma) \in m_{A'}$, $\Delta \in m_{A'}((t))$.

Then we have:

$$\tilde{\rho}_\sigma\left(f+\Delta)\right)=\lambda(\sigma)(f+\Delta)+c(\sigma)+\delta(\sigma).$$

This implies that ($f=1/t^m$):

$$\tilde{\rho}_\sigma\left(\frac{1}{t^m}\right)=\frac{\lambda(\sigma)}{t^m}+c(\sigma)+\left(\delta(\sigma)+\lambda(\sigma)\Delta-\tilde{\rho}_\sigma\Delta\right),$$

or equivalently:

$$\tilde{\rho}_\sigma(t)=\rho_\sigma(t)+t\left(\sum_{\nu=0}^{\infty}\binom{-1/m}{\nu}\sum_{k=1}^{\nu}\binom{\nu}{k}E^k c(\sigma)^{\nu-k}t^{m\nu}\right), \quad (6)$$

where

$$E=\delta(\sigma)+\lambda(\sigma)\Delta-\tilde{\rho}_\sigma\Delta+\frac{\lambda_1(\sigma)}{t^m}\in m_{A'}((t)).$$

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Suppose that we can extend $\rho_\sigma(t)$ to a homomorphism $\tilde{\rho}_{\sigma,A} \in \mathrm{Aut} A[[t]]$. A further extension of ρ_σ over A' is then given by

$$\tilde{\rho}_{\sigma,A'}(t) = \tilde{\rho}_{\sigma,A}(t) + \rho'_\sigma(t),$$

where $\rho'_\sigma(t) \in \ker\pi[[t]]$. Since $\Delta \in m_{A'}((t))$ and since $\ker\pi \cdot m_{A'} = 0$

$$\tilde{\rho}_{\sigma,A'}(\Delta) = \tilde{\rho}_{\sigma,A}(\Delta).$$

Thus, equation (6) allows us to compute the value of $\tilde{\rho}_{\sigma,A'}(t)$ from the value of $\tilde{\rho}_{\sigma,A}(t)$.

First Order infinitesimals

V is the elementary abelian group $G_1(P)$.

$$V = \oplus_{i=1}^{s} V_i$$

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First Order infinitesimals

Proposition

Let $m+1=\sum_{i\geq 0} b_i p^i$ *be the p-adic expansion of m. If* $\left\lfloor \frac{2b_0}{p} \right\rfloor = \left\lfloor \frac{b_0+b_{\nu-1}}{p} \right\rfloor$ *for all* $2 \leq \nu \leq s$*, then the map*

$$\Psi : H^1(V,\mathcal{T}_{\mathcal{O}}) \rightarrow \bigoplus_{\nu=1}^{s} H^1(V_\nu,\mathcal{T}_{\mathcal{O}}), \tag{7}$$

sending $v \mapsto \sum_{\nu=1}^{s} \mathrm{res}_{V\rightarrow V_i} v$ *is an isomorphism. Moreover*

$$H^1(V,\mathcal{T}_{\mathcal{O}}) \cong \bigoplus_{i=2,\binom{i/m}{p-1}=0}^{m+1} b_i \frac{1}{t^i}, \tag{8}$$

where $b_i \in \mathrm{Hom}(V,k)$.

Comparison

The map

$$\begin{aligned} \mathcal{T}_{\mathcal{O}} &\rightarrow k[[t]]/t^{m+1} \\ f(t)\frac{d}{dt} &\rightarrow f(t)/t^{m+1} \end{aligned} \tag{9}$$

is a V-equivariant isomorphism.

Proposition

Assume that P is a wild ramified point of X with a two dimensional representation attached to it. An extension $\tilde{\rho}_\sigma$ gives rise to the following cocyle in $H^1(V, \frac{1}{t^{m+1}}k[[t]])$:

$$\alpha(\sigma) = \frac{1}{m}\left(\frac{\lambda_1(\sigma)}{t^m} + \lambda_1(\sigma)c(\sigma) - \delta(\sigma) + \sum_{\mu=0}^{m-1} \frac{2m-\mu}{m} \frac{a_{\mu,1}c(\sigma)}{t^{m-\mu}} \right),$$

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$$\tilde{\rho}_\sigma(t) = \frac{\lambda(\sigma)^{-\frac{1}{m}} t}{\left(1 + t^m c(\sigma)\right)^{1/m} \left(1 + \frac{E_1 t^m}{1 + c(\sigma) t^m}\right)^{1/m}}$$

Also

$$\left.\frac{d}{d\epsilon} \tilde{\rho}_\sigma \circ \rho_\sigma^{-1}\right|_{\epsilon=0}$$

is the image of $\tilde{\rho}$ in $H^1(V, \frac{1}{t^{m+1}} k[[t]])$.

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Some conclusions

Corollary

If $G_1(P) = \mathbb{Z}/p\mathbb{Z}$ or if the assumptions of proposition 3.1 hold then the tangent vector corresponding to $0 \neq \frac{d}{dt} \in H^1(V, \mathcal{T}_{\mathcal{O}})$ is an obstructed deformation.

Proof.

The element $\frac{d}{dt}$ corresponds to $\frac{1}{t^{m+1}} \in H^1(V, \frac{1}{t^{m+1}}k[[t]])$. It is impossible to obtain a vector in the direction of $\frac{1}{t^{m+1}}$ using a matrix representation, i.e. an element in $F(\cdot)$.
Notice that since we have assumed that the representation attached to P is two dimensional we have that $m > 1$. □

More obstructions

Corollary

Assume that $p \nmid m+1$ *and the assumptions for computing* $H^1(V, \mathcal{T}_{\mathcal{O}})$ *hold. Assume also that* V *is an elementary abelian group with more than one component. Unbstructed deformations should satisfy* $b_i(\sigma) = \lambda_i c(\sigma)$ *for some element* $\lambda_i \in k$.

Proof.

Condition $p \nmid m+1$ implies that every deformation is coming from a matrix representation and condition follows by using the image of deformations comming from matrix representations. □

Representing Matrix Deformations

Let H be a p-group with identity e and let $\rho : H \to L_n(k)$ be a faithful representation of H. Let $\Lambda[H, n]$ be the commutative Λ-algebra generated by X_{ij}^g for $g \in H, 1 \leq j \leq i \leq n$, such that

$$X_{ij}^e = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$X_{ij}^{gh} = \sum_{l=1}^{n} X_{il}^g X_{lj}^h \text{ for } g, h \in H \text{ and } 1 \leq i, j \leq n. \tag{10}$$

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Representing Matrix representations

For every Λ-algebra A we have a canonical bijection

$$\mathrm{Hom}_{\Lambda-\mathrm{Alg}}(\Lambda[H,n],A) \cong \mathrm{Hom}(H, L_n(A)),$$

where a Λ-algebra homomorphism $f : \Lambda[H,n] \to A$ corresponds to the group homomorphism ρ_f that sends $g \in H$ to the matrix $(f(X_{ij}^g))$.

The representation $\rho : H \to L_n(k)$ corresponds to a homomorphism $\Lambda[H,n] \to k$. Its kernel is a maximal ideal, which we denote by m_ρ. We take the completion $R(H)$ of $\Lambda[H,n]$ at m_ρ. The canonical map $\Lambda[H,n] \to R(H)$, gives rise to a map $\rho_{R(H)} : H \to L_n(R(H))$, such that the diagram:

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Elementary Abelian Groups

- Equicharacteristic Deformations. Set $x_i = X_{21}^{g_i} - c(g_i)$, g_i generates the group as a $\mathbb{Z}$-module. $R(H) := k[[x_1, \ldots, x_n]]$.
- Liftings to characteristic zero. Set $y_i = X_{22}^{g_i} - 1$. Condition $\sum_{\nu=0}^{p-1} (X_{22}^{g})^{\nu} = 0$. and commuting relation $X_{21}^{g} - X_{21}^{h} + X_{22}^{g} X_{21}^{h} - X_{22}^{h} X_{21}^{g} = 0$. Interesting case is that with one component. After computing the conjugation classes we obtain:

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Let $\tilde{\rho}_{\sigma,A} = \{\tilde{\rho}_{\sigma,A}(t)\}_{\sigma \in V}$ be a representation of $V \to \mathrm{Aut}A[[t]]$, and consider the corresponding element in $F(A)$. If this element in $F(A)$ can be lifted to an element in $F(A')$ then $\tilde{\rho}_{\sigma,A}$ can be lifted to a representation $V \to \mathrm{Aut}A'[[t]]$.

Proof.

Consider extensions of the homomorphisms $\tilde{\rho}_{\sigma,A'} \in \mathrm{Aut}A'[[t]]$ for every $\sigma \in V$. The element $\tilde{\rho}_{\sigma,A'}\tilde{\rho}_{\tau,A'}\tilde{\rho}^{-1}_{\sigma\tau,A'}$ is a 2-cocycle and gives rise to a cohomology class in $H^2(V, \mathcal{T}_{\mathcal{O}})$. If the liftings are comming from liftings of matrix representations then there is no group theoretic obstruction in lifting $\tilde{\rho}_{\sigma,A}$ to $\tilde{\rho}_{\sigma,A'}$ since a simple computation shows that the lifts defined by satisfy the relations

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Proposition

Assume that the hypotheses of proposition relating matrix representations to local representation hold. Consider the ring R_1 defined by

$$R_1 = \begin{cases} k & \text{in the equicharacteristic case} \\ R' & \text{in the mixed characteristic case} \end{cases}$$

Let $b = 1$ if $p \mid m+1$ and $b = 2$ if $p \nmid m+1$. Let Σ be the subset of numbers $b \leq i \leq m$ so that $\binom{\frac{i}{m}}{p-1} = 0$. Consider the ring $\bar{R} := R_1[[X_i : i \in \Sigma]]$ and the k-vector space $W \subset H^1(V, \mathcal{T}_{\mathcal{O}})/\langle d/dt \rangle$ generated by elements $\lambda_i c(\sigma) t^{m+1-i} \frac{d}{dt}$. There is a surjection $R_P \to \bar{R}$ that induces an isomorphism $W \cong \mathrm{Hom}(\bar{R}, k[\epsilon]/\epsilon^2)$. The Krull dimension of R_P is equal to $\#\Sigma$.