Galois Module structure of spaces of polydifferentials

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RICAM Linz November 2013

Table of Contents

Curves -Differentials

- Curve X: projective nonsingular complete, defined over an algebraically closed field k, $\mathrm{char}(k) = p > 0$.
- Genus $g(X) \geq 2 \Rightarrow G = \mathrm{Aut}(X)$ is a finite group.
- $\Omega(n) = H^0(X, \Omega_X^{\otimes n})$ is a finite dimensional vector space of dimension $(2n-1)(g-1)$ which is a G-module.

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Study the G-module structure of $\Omega(n)$: Analyse $\Omega(n)$ into a direct sum of indecomposable $K[G]$-modules.

This is completely solved in characteristic 0 or when $p \nmid |G|$.

Applications

- Our motivation: Study the dimension of the tangent space to the deformation functor of curves with automorphisms

$$\begin{aligned}\dim_k \mathcal{T}_C(G) &= \dim_k H^0(G, \Omega^{\otimes 2})_G \\ &= \dim_k H^0(G, \Omega^{\otimes 2}) \otimes_{k[G]} k\end{aligned}$$

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- Other possible applications: decomposition of Jacobians, Arithmetic of fields generated by higher order Weierstrass points etc.

Modular Representation Theory

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Unsolved problem in characteristic p. Because for a p-group:

- Indecomposable is different than irreducible.
- Characters do not determine indecomposable modules.
- Unless G is a cyclic p-group it is almost impossible to determine the classes of indecomposable $K[G]$-modules up to isomorphism.
- Appearance of wild ramification.

Harbater-Katz-Gabber covers

- These are Galois covers $X \to \mathbb{P}^1$ with Galois group G a p-group ramified exactly above one point.

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Harbater-Katz-Gabber covers

- These are Galois covers $X \to \mathbb{P}^1$ with Galois group G a p-group ramified exactly above one point.
- If $|G| > m(2g-2)$ then the module $H^0(X, \Omega^{\otimes m})$ is indecomposable.
- In particular if G is a "big-action" $|G| > 2(g-1)$ the space of holomorphic differentials is indecomposable.

The cyclic case.

Assume that $G = \mathbb{Z}/p\mathbb{Z} = \langle\sigma\rangle$. Indecomposable $K[G]$-modules and the decomposition are described in terms of the Jordan-normal form of the generator σ.

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$$\sigma \mapsto \left.\begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}\right\} r \times r$$

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For every $0 \leq r \leq p-1$, there is a unique indecomposable module J_r for the cyclic group G.

Elementary Abelian groups

If $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ then there are infinitely many indecomposable $K[G]$-modules. Classifying them is considered impossible.

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Artin Schreier Extensions

$$y^{p^r} - y = f(x).$$

For these curves (which admit an elementary abelian group in their automorphism group) the problem of the determination of the Galois module structure of $\Omega(n)$ is solved. (Nakajima, Calderón, Salvador, Madan, Karanikolopoulos, etc)

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It is important that we know explicit bases for the spaces $\Omega(n)$ and that the generators σ_i of the elementary abelian groups involved have "similar" Jordan decomposition.

Mumford Curves

Over non-archimedean, complete, discretely valued fields K, D. Mumford has shown that curves whose stable reduction is split multiplicative, (*i.e.*, a union of rational curves intersecting in K-rational points with K-rational nodal tangents) are isomorphic to an analytic space of the form $X_\Gamma = \Gamma\backslash(\mathbb{P}^1 - \mathcal{L}_\Gamma)$.

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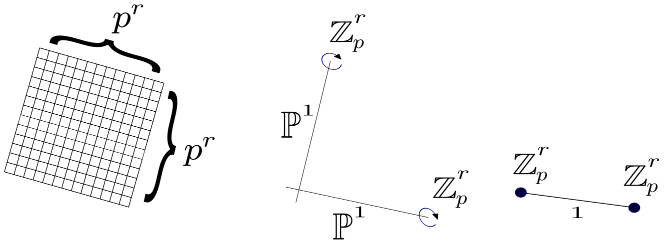
$$(y^{p^r} - y)(x^{p^r} - x) = c, \ |c| < 1 \Rightarrow c \equiv 0 \bmod m$$

Reduction:



Generalities on Mumford curves

Γ is a finitely generated, torsion free discrete subgroup of $\mathrm{PGL}(2, K)$, with $\mathcal{L}_\Gamma$ as set of limit points.

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$$\mathrm{Aut}(X_\Gamma) = \mathrm{Norm}_\Gamma \mathrm{PGL}(2, K)/\Gamma,$$

where $\mathrm{Norm}_\Gamma \mathrm{PGL}(2, K)$ is the normalizer of Γ in $\mathrm{PGL}(2, K)$.

- Mumford curves are ordinary
- $$\mathrm{Aut}(X_\Gamma) \leq \min\{12(g-1), 2\sqrt{2}(\sqrt{g}+1)^2\}$$

Holomorphic Polydifferentials

- Γ is a free subgroup of $\mathrm{PGL}(2, K)$.
- K be a field, non-archimedean valued and complete, of characteristic $p > 0$.
- $P_{2(n-1)}$ is the K-vector space of polynomials of degree $\leq 2(n-1)$.
- $\mathrm{PGL}(2, K)$ acts on $P_{2(n-1)}$ from the right: $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2, K)$ and $F \in P_{2(n-1)}$,

$$F^{\phi}(T) := \frac{(cT+d)^{2(n-1)}}{(ad-bc)^{n-1}} F\left(\frac{aT+b}{cT+d}\right) \in K[T].$$

Polydifferentials

Theorem (P. Schneider, J. Teitelbaum)

$$\Omega(n) = \mathrm{H}^1(\Gamma, P_{2(n-1)}).$$

For an Γ-module P

$$\mathrm{Der}(\Gamma, P) = \{d : \Gamma \to P : d(\gamma_1\gamma_2) = d(\gamma_1) + d(\gamma_2)^{\gamma_1}\}$$

$$\mathrm{PrinDer}(\Gamma, P) = \{d_m : \gamma \mapsto m^\gamma - m\}$$

$$\mathrm{H}^1(\Gamma, P_{2(n-1)}) = \frac{\mathrm{Der}(\Gamma, P)}{\mathrm{PrinDer}(\Gamma, P)}$$

Dimension Verification

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- $\dim \mathrm{PrinDer}(\Gamma, P) = \begin{cases} 2n-1 & \text{if } n > 1 \\ 0 & \text{if } n = 1. \end{cases}$

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- $\dim \mathrm{H}^1(\Gamma, P_{2(n-1)}) = \begin{cases} (2n-1)(g-1) & \text{for } n > 1 \\ g & \text{for } n = 1. \end{cases}$

Action of N/Γ on $\Omega(n)$:

For $\phi \in N$ and $d \in \mathrm{Der}(\Gamma, P)$, we define the action d^ϕ of N/Γ on a derivation d as follows:

$$(d^\phi)(\gamma) := [d(\phi\gamma\phi^{-1})]^\phi.$$

This is the usual action of N/Γ on group cohomology.

Holomorphic Differentials $n = 1$

$$\begin{aligned} \mathrm{H}^0(X,\Omega) &= \mathrm{H}^1(\Gamma,P_0) = \mathrm{H}^1(\Gamma,K) = \mathrm{Hom}(\Gamma,K) = \mathrm{Hom}(\Gamma,\mathbb{Z})\otimes K \\ &= \mathrm{Hom}(\Gamma^{\mathrm{ab}},\mathbb{Z})\otimes K, \end{aligned} \tag{1}$$

Theorem (B. Köck)

The integral representation:

$$\rho : N/\Gamma \to \mathrm{GL}(g,\mathbb{Z})$$

on holomorphic differentials is faithful, unless the cover $X \to X/G = Y$ is not tamely ramified, the characteristic $p = 2$ and $g_Y = 0$.

Remark 1

Previous theorem shows that holomorphic differentials on Mumford curves are, in some sense, similar to holomorphic differentials on Riemann surfaces; for a Riemann surface Y there is a faithful action of its automorphism group on $\mathrm{H}^1(Y, \mathbb{Z})$, which induces a faithful representation of a subgroup of the automorphism group on the symplectic matrices $\mathrm{Sp}(2g, \mathbb{Z})$.

Remark 2

The group Γ can be interpreted as the fundamental group of the curve X_Γ.

Application to order of the group

Corollary

If the order of any $g \in N/\Gamma$ is a prime number q, then $q \leq g + 1$.

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If the order of any $g \in N/\Gamma$ is a prime number q, then $q \leq g+1$.

$$M = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathrm{GL}(q-1, \mathbb{Z}). \qquad (2)$$

which has characteristic polynomial $\frac{x^q-1}{x-1} = 1 + x + \cdots x^{q-1}$ (it is the companion matrix of this polynomial), and is the prototype for an integral representation of a cyclic group of order q with minimal degree $q-1$, i.e., there are no integral representations of a cyclic group of order q in $r \times r$ matrices for $r < q-1$.

Curves of the form $(x^p - x)(y^p - y) = c$.

Let $A, B \subset \mathrm{PGL}(2, K)$ be the finite subgroups of order p generated respectively by

$$\epsilon_A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \epsilon_B = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},$$

where $s \in K^*$.

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where $s \in K^*$.

For a general choice of s, the groups A and B generate a discrete subgroup N isomorphic to the free product $A * B$. The group $\Gamma := [A, B]$ is

(i) a normal subgroup of N such that $N/\Gamma \cong A \times B$ and

(ii) a free group of rank $(p-1)^2$. A basis of Γ is given by $[a, b]$ for $a \in A \backslash \{1\}$ and $b \in B \backslash \{1\}$.

Action on differentials

$$1 \to [A, B] := \Gamma \to A * B := N \to A \times B \to 1.$$

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The elements $e_{i,j} = [\epsilon_A^i, \epsilon_B^j]$, $1 \leq i, j \leq p - 1$ form a basis of the free group Γ.

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$$a[\epsilon_A, \epsilon_B]a^{-1} = [a\epsilon_A, \epsilon_B][a, \epsilon_B]^{-1} \quad (3)$$

$$b[\epsilon_A, \epsilon_B]b^{-1} = [\epsilon_A, b]^{-1}[\epsilon_A, b\epsilon_B], \quad (4)$$

for every $a \in A$ and $b \in B$.

Action on differentials

$$e_{i,j}^{\epsilon_A} = [\epsilon_A \cdot \epsilon_A^i, \epsilon_B^j][\epsilon_A, \epsilon_B^j]^{-1} \text{ for } 1 \leq i, j \leq p-1.$$

Since the group $\Gamma^{\mathrm{ab}} \cong \mathbb{Z}^g$ is a $\mathbb{Z}$-module and usually when we consider $\mathbb{Z}$-modules we use additive notation, we rewrite the equation above as:

$$e_{i,j}^{\epsilon_A} = e_{i+1,j} - e_{1,j} \text{ for } 1 \leq i \leq p-2, 1 \leq j \leq p-1$$

and

$$e_{p-1,j}^{\epsilon_A} = -e_{1,j} \text{ for } 1 \leq j \leq p-1.$$

Action on differentials

In terms of the above given basis, the action can be expressed by the following matrix given in block diagonal form

$$\begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix}, \tag{5}$$

where there are $p-1$ blocks

$$M = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathrm{GL}(p-1, \mathbb{Z}). \tag{6}$$

Invariants and indecomposables

Proposition

Let G be a finite cyclic p-group and let V be a $K[G]$-module. The number of indecomposable $K[G]$-summands of V, which are $K[G]$-modules equals the dimension of the space of invariants V^G.

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Remark

The assumption that G is cyclic is necessary. There is an example of an indecomposable $K[\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}]$-module (Heller and Reiner) with space of invariants has dimension > 1.

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Lemma

If H is an abelian p-group acting on a non-trivial K-vector space M, then $M^H \neq \{0\}$.

Criterion for idecomposability

Proposition

Let H be a group such that for every non-trivial $K[H]$-module M, $M^H \neq \{0\}$. Suppose that for a $K[H]$-module V the space V^H is one-dimensional, then V is indecomposable.

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Proof.

Every non-trivial indecomposable summand of V contributes a non-trivial invariant subspace to V^H. Therefore, if $\dim V^H = 1$, then there could be only one indecomposable summand. □

Application

Proposition

The space of holomorphic differentials on the Subrao curves is a $K[A \times B]$-indecomposable module.

Action

1. Give a description of the conjugation action of N on Γ. fix a set of representatives $\{n_i \in N\}$ for N/Γ, $1 \leq i \leq \#N/\Gamma$. Set

$$\Gamma \ni n_i \gamma_j n_i^{-1} = w_{ij} \quad 1 \leq i \leq \#N/\Gamma, 1 \leq j \leq g,$$

where w_{ij} are words in $\gamma_1, \ldots, \gamma_g$.

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   where w_{ij} are words in $\gamma_1, \ldots, \gamma_g$.
2. Compute $d_{i,\ell}(n_i \gamma_j n_i^{-1})^{n_i}$
3. Consider the effect of taking the quotient by principal derivations.

Derivations

For an integer k, will denote by

$$\binom{T}{k} = \frac{T(T-1)(T-2)\cdots(T-k+1)}{k!} \in K[T],$$

which is a polynomial of degree k.

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which is a polynomial of degree k.
Since

$$\binom{T+1}{k} = \binom{T}{k} + \binom{T}{k-1},$$

the automorphism $\sigma : T \mapsto T+1$ acts on

$$\sigma : \binom{T}{k} \mapsto \binom{T}{k} + \binom{T}{k-1}.$$

A natural basis for the space of polynomials of degree $2n-1$

- First attempt

$$\{1, T, T^2, \ldots, T^{2n-1}\}.$$

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- Better basis

$$\left\{ (T^p - T)^{i(k)} \binom{T}{j(k)} : 0 \le k \le 2n-1 \right\},$$

$k = i(k) \cdot p + j(k)$ and $0 \le j(k) < p$

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$k = i(k) \cdot p + j(k)$ and $0 \leq j(k) < p$
Get Jordan decomposition.

Basis of derivations

Consider the derivation $d^{(k)}_{[\alpha,\beta]}$ for $k = 0, \dots, 2(n-1)$ and $\alpha \in A \backslash \{1\}, \beta \in B \backslash \{1\}$, which is characterized by

$$d^{(k)}_{[\alpha,\beta]}([\alpha', \beta']) = \begin{cases} \left[(T^p - T)^i \cdot \binom{T}{j} \right]^{\beta^{-1}} & \text{if } \alpha = \alpha' \text{ and } \beta = \beta', \\ 0 & \text{otherwise,} \end{cases}$$

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where i and j are determined by $k = i \cdot p + j$ and $0 \le j < p$. For $\delta \in A$

$$\left(d^{(k)}_{[\alpha,\beta]}\right)^{\delta} = \begin{cases} d^{(k)}_{[\delta^{-1}\alpha,\beta]} + d^{(k-1)}_{[\delta^{-1}\alpha,\beta]} & \text{if } j > 0 \text{ and } \alpha \neq \delta, \\ d^{(k)}_{[\delta^{-1}\alpha,\beta]} & \text{if } j = 0 \text{ and } \alpha \neq \delta, \\ -\sum_{\alpha' \neq 1} \left(d^{(k)}_{[\alpha',\beta]} + d^{(k-1)}_{[\alpha',\beta]}\right) & \text{if } j > 0 \text{ and } \alpha = \delta, \\ -\sum_{\alpha' \neq 1} \left(d^{(k)}_{[\alpha',\beta]}\right) & \text{if } j = 0 \text{ and } \alpha = \delta, \end{cases}$$

Basis of derivations

$$d_{ab}^k := d^{(k)}_{[\epsilon_A^a, \epsilon_B^b]} \text{ for } 1 \leq a, b \leq p-1, 0 \leq k \leq 2n-2$$

order them by lexicographical order with respect to (k, a, b); that is,

$$d_{11}^0, d_{11}^1, \ldots, d_{11}^{2(n-1)}, d_{21}^0, d_{21}^1, \ldots, d_{21}^{2(n-1)}, \ldots,$$
$$d_{(p-1),1}^0, d_{(p-1),1}^1, \ldots, d_{(p-1),1}^{2(n-1)}, \ldots$$

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$$d_{(p-1),1}^0, d_{(p-1),1}^1, \ldots, d_{(p-1),1}^{2(n-1)}, \ldots$$

The square matrix Q of degree $(2n-1)(p-1)^2$ of the action by $\delta = \epsilon_A$ is then decomposed into $p-1$ blocks like

$$Q = \begin{pmatrix} M & & & \\ & M & & \\ & & \ddots & \\ & & & M \end{pmatrix},$$

where M is a square matrix of degree $(2n-1)(p-1)$.

The M-matrix

$$M = \begin{pmatrix} -N & -N & -N & \cdots & -N & -N \\ N & & & & & \\ & N & & & & \\ & & N & & & \\ & & & \ddots & & \\ & & & & N & 0 \end{pmatrix}$$

where N is a square matrix of degree $2n-1$, which is of the form

$$N = \begin{pmatrix} J_p & & & & \\ & J_p & & & \\ & & \ddots & & \\ & & & J_p & \\ & & & & J_r \end{pmatrix},$$

where J_ℓ denotes the $\ell \times \ell$-Jordan block with diagonal entries equal to 1.

Kronecker product description of M

$$W = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 0 \end{pmatrix}.$$

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Proposition

The $K[A]$-module structure of $\mathrm{Der}(\Gamma, P_{2(n-1)})$ is given by:

$$\mathrm{Der}(\Gamma, P_{2(n-1)}) = \begin{cases} \left(J_p^{(p-1)\left\lfloor \frac{2n-1}{p} \right\rfloor} \oplus J_p^{r-1} \oplus J_{p-r} \right)^{p-1} & \text{if } p \nmid 2n-1, \\ \left(J_p^{(p-1)\frac{2n-1}{p}} \right)^{p-1} & \text{if } p \mid 2n-1 \end{cases}.$$

Projective modules injective hulls

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_{2n-2} & \xrightarrow{i_1} & K[A] \oplus K[A]^{\left\lfloor \frac{2n-1}{p} \right\rfloor} & \xrightarrow{\pi_1} & J_{p-r} & \longrightarrow & 0 \\
 & & \downarrow \psi & & \phi \downarrow & & \downarrow \chi & & \\
0 & \longrightarrow & \mathrm{Der}(\Gamma, P_{2n-2}) & \xrightarrow{i_2} & K[A]^{p-1} \oplus K[A]^{\left\lfloor \frac{(2n-1)(p-1)}{p} \right\rfloor (p-1)} & \xrightarrow{\pi_2} & J_r^{p-1} & \longrightarrow & 0
\end{array}$$

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\end{array}$$

$$\begin{aligned}
H^1(\Gamma, P_{2n-2}) &\cong \mathrm{Der}(\Gamma, P_{2n-2})/\mathrm{PrinDer}(\Gamma, P_{2n-2}) \\
&\cong K[A]^{(p-1)\left\lfloor \frac{(2n-1)(p-1)}{p} \right\rfloor - 1 - \left\lfloor \frac{2n-1}{p} \right\rfloor} \oplus K[A]/J_r \oplus J_{p-r}^{p-1} \\
&\cong K[A]^{(p-1)(2n-1) - p\left\lceil \frac{2n-1}{p} \right\rceil} \oplus J_{p-r}^{p}.
\end{aligned}$$

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$$0 \to \Omega_X^{\otimes n} \to \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big) \to \Sigma \to 0.$$

The $K[A \times B]$-structure

$$0 \to H^0(X, \Omega_X^{\otimes n}) \to H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big) \to H^0(X, \Sigma)$$

$$\to H^1(X, \Omega_X^{\otimes n}) = 0.$$

Theorem

The $K[G]$-module $H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big)$ is a free $K[G]$-module of rank $(2n-1)(g_Y - 1 + r_0)$, where r_0 denotes the cardinality of $X^G_{\mathrm{ram}} = \{P \in X/G : e(P) > 1\}$, and g_Y denotes the genus of the quotient curve $Y = X/G$.

Proof.

Uses a criterion of B. Köck, on characterizing projective modules on curves. □

Final result

$$0 \longrightarrow H^0(X, \Omega_X^{\otimes n}) \longrightarrow K[G]^{2n-1} \longrightarrow H^0(X, \Sigma) \longrightarrow 0$$

Final Results

Theorem

For $n > 1$ we write $2n-1 = q \cdot p + r$ with $0 \leq r < p$.

1. *As a $K[A]$-module the following decomposition holds*

$$H^0(X, \Omega_X^{\otimes n}) = K[A]^{(p-1)(2n-1)-p\left\lceil \frac{2n-1}{p} \right\rceil} \bigoplus \left(K[A]/(\epsilon_A - 1)^{p-r}\right)^p .$$

   A similar result holds for the group B.

2. *As a $K[G]$-module ($G = A \times B$) the following decomposition holds:*

$$\begin{aligned} H^0(X, \Omega_X^{\otimes n}) &= K[G]^{2n-1-2\left\lceil \frac{2n-1}{p} \right\rceil} \bigoplus \\ & K[G]/(\epsilon_A - 1)^{p-r} \bigoplus K[G]/(\epsilon_B - 1)^{p-r}. \end{aligned}$$