On the Principal Ideal Theorem in Arithmetic Topology

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Samos, June 1 2007

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Some History

- At the beginning of the 60's D. Mumford and B. Mazur observed some analogies between the properties of 3-Manifolds and of Number Fields.
- Together with analogies developed by M. Morishita, N. Ramachadran, A. Reznikov and J.-L. Waldspurger, the foundations of "Arithmetic Topology" were set.
- The main tool for translating notions of one theory to the other is the MKR-dictionary (named after Mazur, Kapranov and Reznikov).
- It is not known why such a translation is possible. We will present a very basic version of this dictionary.

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The MKR Dictionary

- Closed, oriented, connected, smooth 3-manifolds correspond to the schemes $Spec\mathcal{O}_K$, where K is an algebraic number field.
- A link in M corresponds to an ideal in $\mathcal{O}_K$ and a knot in M corresponds to a prime ideal in $\mathcal{O}_K$.
- $\mathbb{Q}$ corresponds to the 3-sphere S^3.
- The class group $\mathrm{Cl}(K)$ corresponds to $H_1(M, \mathbb{Z})$.
- Finite extensions of number fields L/K correspond to finite branched coverings of 3-manifolds $\pi : M \to N$. A *branched cover* M of a 3-manifold N is given by a map π such that there is a link L of N with the following property: The restriction map $\pi : M \backslash \pi^{-1}(L) \to N \backslash L$ is a topological cover.

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Main Differences between the two Theories

- The group $\mathrm{Cl}(K)$ is always finite, while $H_1(M,\mathbb{Z}) = \mathbb{Z}^r \oplus H_1(M,\mathbb{Z})_{tor}$ is not.
- The Algebraic translation of the Poincare Conjecture is false!

  One should expect that $\mathbb{Q}$ would be the only number field with no unramified extensions. That is not true. Indeed,

Theorem

If $d < 0$ and the class group of $L = \mathbb{Q}(\sqrt{d})$ is trivial then L has no unramified extensions. There are precisely 9 such values of $d: -1, -2, -3, -7, -11, -19, -43, -67, -163$.

- Let $M_1 \to M$ a covering of 3-manifolds. A knot K in M does not necessarily lift to a knot in M_1, while every prime ideal $p \lhd \mathcal{O}_K$ gives rise to an ideal $p\mathcal{O}_L$. L/K is a Galois number fields extension.

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Let $G = Gal(\mathbb{L}/\mathbb{K})$ and $\mathfrak{q}$ be a prime ideal in $\mathcal{O}_{\mathbb{L}}$.

Definition

The decomposition group of $\mathfrak{q}$, $D_{\mathfrak{q}} \subset G$ is the subgroup of G preserving $\mathfrak{q}$,

$$D_{\mathfrak{q}} = \{q \in G : g(\mathfrak{q}) = \mathfrak{q}\}$$

- The quotient $\mathcal{O}_{\mathbb{L}}/\mathfrak{q}$ is a finite field.
- The image of the homomorphism $D_{\mathfrak{q}} \to Gal(\mathcal{O}_{\mathbb{L}/\mathfrak{q}})$ consists of exactly those automorphisms of $\mathcal{O}_{\mathbb{L}}/\mathfrak{q}$ which fix the subfield, $\mathcal{O}_{\mathbb{K}}/\mathfrak{p}$, $\mathfrak{p} = \mathcal{O}_{\mathbb{K}} \cap \mathfrak{q}$.
- The kernel of this homomorphism, $I_{\mathfrak{q}}$ is called the inertia group of $\mathfrak{q}$.
- We have the following exact sequence,

$$0 \to I_{\mathfrak{q}} \to D_{\mathfrak{q}} \to Gal(\mathcal{O}_{\mathbb{L}}/\mathfrak{q}/\mathcal{O}_{\mathbb{K}/\mathfrak{p}}) \to 0$$

Galois Extensions and Galois Branched Covers

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Galois Extensions and Galois Branched Covers

- The order of $I_{\mathfrak{q}}$, denoted by $e_{\mathfrak{q}}$ is called the ramification index.
- The order of $Gal(\mathcal{O}_{\mathbb{L}}/\mathfrak{q}/\mathcal{O}_{\mathbb{K}/\mathfrak{p}})$ will be denoted by $f_{\mathfrak{q}}$.
- The ideal $\mathfrak{p}\mathcal{O}_{\mathbb{L}}$ decomposes uniquely as a product of prime ideals, $\mathfrak{p}_1^{e_1} \ldots \mathfrak{p}_g^{e_g}$, where e_i is the ramification index of $\mathfrak{p}_i$.

Theorem

Under the above assumptions,

- *G acts transitively on $\mathfrak{p}_1 \ldots \mathfrak{p}_g$,*
- *$e_1 = \ldots = e_g := e$ and $f_{\mathfrak{p}_1} \ldots f_{\mathfrak{p}_g} := f$,*
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- Let G be a group that acts on a 3-manifold M and the map $p : M \to M/G$ is a branched covering.
- The subgroup $D_K \subset G$ contains all the elements which map a knot $K \subset M$ to itself and is called the decomposition group of K.
- We assume that the action of D_K on K is orientation preserving.
- The image of the natural homomorphism $D_K \to Homeo(K)$ is exactly the group of deck transformations, $Gal(K/K')$, of the covering $K \to K' = K/D_K$.
- The kernel of this homomorphism, I_K, is called the inertia group of K.
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$p^{-1}(K)$ is a link in M whose components we denote by $K_1, \ldots K_g$,

$$p^{-1}(K) = K_1 \cup \ldots \cup K_g.$$

Theorem (Sikora)

Under the above assumptions

- G *acts transitively on* $K_1, \ldots K_g$,
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Split, Ramified and Inert Primes and Knots

Let C_p be a cyclic group and $G = C_p$. Then each prime $\mathfrak{q} \lhd \mathcal{O}_{\mathbb{L}}$ and each knot $K \subset M$ is either split, inert or ramified. if $\mathfrak{p} = \mathfrak{q} \cap \mathcal{O}_{\mathbb{K}}$ the $\mathfrak{q}$ is

- split if $\mathfrak{p}\mathcal{O}_{\mathbb{L}} = \mathfrak{q}_1 \ldots \mathfrak{q}_p$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_p$, are different prime, one of which is $\mathfrak{p}$. In this situation C_p permutes these primes.
- ramified if $\mathfrak{p}\mathcal{O}_{\mathbb{L}} = \mathfrak{q}^p$. Here C_p fixes the elements of $\mathfrak{q}$.
- inert if $\mathfrak{p}\mathcal{O}_{\mathbb{L}} = \mathfrak{q}$. In this situation C_p acts non-trivially on $\mathfrak{q}$.

Let $G = C_p$, then $p : M \to M/G$ is a branched covering. If $K \subset M$ satisfies the previous assumptions, then K is

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- ramified id K/G is a component of the branching set. Here C_p fixes $p^{-1}(K/G) = K$.
- inert if $p^{-1}(K/G) = K$ and the C_p−action on K is non-trivial.

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The Number Fields Case

Theorem (The Principal Ideal Theorem)

Let K be a number field and let $K^{(1)}$ be the Hilbert class field of K. Let $\mathcal{O}_K$, $\mathcal{O}_{K^{(1)}}$ be the rings of integers of K and $K^{(1)}$ respectively. Let P be a prime ideal of $\mathcal{O}_{K^{(1)}}$. We consider the prime ideal

$$\mathcal{O}_K \triangleright p = P \cap \mathcal{O}_K$$

and let

$$p\mathcal{O}_{K^{(1)}} = (PP_2 \dots P_r)^e = \prod_{g \in \mathrm{CL}(K)} g(P)$$

be the decomposition of $p\mathcal{O}_{K^{(1)}}$ in $\mathcal{O}_{K^{(1)}}$ into prime ideals. The ideal $p\mathcal{O}_{K^{(1)}}$ is principal in $K^{(1)}$.

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The 3-Manifold Case

Definition

We define the Hilbert Manifold $M^{(1)}$ of M as the universal covering space $\widetilde{M}$ of M modulo the commutator group,

$$M^{(1)} = M / [\pi_1(M), \pi_1(M)]$$

Theorem (The Principal Ideal Theorem for Knots)

1. *Let K_1 be a knot in $M^{(1)}$. Denote by $G(K_1)$ the subgroup of $G = \pi(M)/[\pi_1(M), \pi_1(M)]$ fixing K_1. Consider the link $L = \bigcup_{g \in G/G(K_1)} gK_1$. Then L is zero in $H_1(M^{(1)}, \mathbb{Z})$.*
2. *Let L be a link in M that is a homologically trivial. Then there is a family of tame knots K_ϵ in M with $\epsilon > 0$, that are boundaries of embedded surfaces E_ϵ so that $\lim_{\epsilon \to 0} K_\epsilon = L$ and $E = \lim_{\epsilon \to 0} E_\epsilon$ is an embedded surface with $\partial E = L$.*

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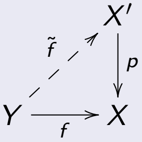
Useful Theorems

Theorem (Path Lifting Property)

Let (Y, y_0), (X, x_0) be topological spaces (arcwise connected, semilocally simply connected), let $p : (X', x'_0) \to (X, x_0)$ be a topological covering with $p(x'_0) = x_0$ and let $f : (Y, y_0) \to (X, x_0)$ be a continuous map. Then, there is a lift $\tilde{f} : Y \to X'$ of f,



making the above diagram commutative if and only if

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(X', x'_0)),$$

where f_*, p_* are the induced maps of fundamental groups.

Useful Theorems

Theorem (Dehn Lemma)

Let M be a 3-manifold and $f : D^2 \to M$ be a map such that for some neighborhood A of ∂D^2 in D^2 $f\mid_A$ is an embedding and $f^{-1}f(A) = A$. Then $f\mid_{\partial D^2}$ extends to an embedding $g : D^2 \to M$.

Corollary

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Theorem (Part I)

Let K_1 be a knot in $M^{(1)}$. Denote by $G(K_1)$ the subgroup of G fixing K_1. Consider the link $L = \bigcup_{g \in G/G(K_1)} gK_1$. Then L is zero in $H_1(M^{(1)}, \mathbb{Z})$.

Proof.

Since the diagram

$$\begin{array}{ccccc} & & K_1 & \longrightarrow & M^{(1)} \\ & \overset{\tilde{f}}{\nearrow} & \downarrow p & & \downarrow p \\ S^1 & \xrightarrow{f} & p(K_1) & \longrightarrow & M \end{array}$$

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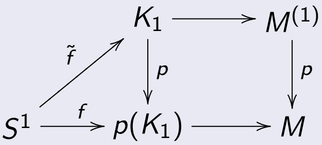

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Therefore $f_*(\pi_1(S^1)) = 0$ as an element in $H_1(M, \mathbb{Z})$, hence there is a topological (possibly singular) surface $\phi : E \to M$ so that

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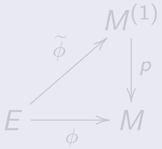
The surface E is homotopically trivial therefore the Dehn Lemma implies that there is a map $\widetilde{\phi}$ making the following diagram commutative:



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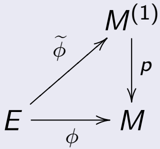

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We have seen that when a knot (resp. link) K lifts to a knot (resp. link) in the Hilbert Manifold, then it is homologically trivial.

What remains is to show that there exists an embedding of a surface E in $M^{(1)}$ such that $\partial E = K$.

Theorem (Part II)

Let L be a link in M that is a homologically trivial. Then there is a family of tame knots K_ϵ in M with $\epsilon > 0$, that are boundaries of embedded surfaces E_ϵ so that $\lim_{\epsilon \to 0} K_\epsilon = L$ and $E = \lim_{\epsilon \to 0} E_\epsilon$ is an embedded surface with $\partial E = L$.

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- K_i is given by the embedding $f_i : S^1 \to M$.
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$$= \gamma_1(I) + \gamma_2(I) + \delta_1(I) + \delta_2(I) + \alpha(I) - \delta_2(I) + \beta(I) - \delta_1(I) =$$

$$= \gamma_1(I) + \alpha(I) + \gamma_2(I) + \beta(I).$$

- The tame knot $\gamma_1 \alpha \gamma_2 \beta$ is the boundary of a topological surface.

Proof of The Main Result

Proof (Continued).

- Let $I = [0,1] \subset \mathbb{R}$.
- Every path in M, *i.e.* every function $f : I \to M$, defines a cycle in $H_1(M, \mathbb{Z})$.
- We will abuse the notation and we will denote by $f(I)$ the homology class of the path $f(I)$.
- We compute in $H_1(M, \mathbb{Z})$:

$$0 = f_1(S^1) + f_2(S^1) = \gamma_1(I) + \gamma_2(I) + \delta_1(I) + \delta_2(I) + 0 =$$

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Proof (Continued).

- By the Corollary it is the boundary of an embedded surface E_ϵ.
- Choose an orientation on E_ϵ so that on $P \in \partial E_\epsilon$ one vector of the oriented basis of $T_P E_\epsilon$ is the tangent vector of the curves ∂E_ϵ and the other one is pointing inwards of E.
- Denote the second vector by N_P.
- We choose the same orientation on all surfaces E_ϵ in the same way, *i.e.* the induced orientation on the common curves of the boundary is the same.
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The Seifert Theorem

As a corollary of the principal ideal theorem for knots we state the following:

Theorem (*Seifert*)

Every link in a simply connected 3 manifold is the boundary of an embedded surface.

Proof.

Let M be simply connected. The Hilbert manifold of M coincides with M and the result follows. □

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