On the Principal Ideal Theorem in Arithmetic Topology

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- Together with analogies developed by M. Morishita, N. Ramachadran, A. Reznikov and J.-L. Waldspurger, the foundations of ”Arithmetic Topology” were set.

- The main tool for translating notions of one theory to the other is the MKR-dictionary (named after Mazur, Kapranov and Reznikov).

- It is not known why such a translation is possible. We will present a very basic version of this dictionary.
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Closed, oriented, connected, smooth 3-manifolds correspond to the schemes $\text{Spec} \mathcal{O}_K$, where $K$ is an algebraic number field.

A link in $M$ corresponds to an ideal in $\mathcal{O}_K$ and a knot in $M$ corresponds to a prime ideal in $\mathcal{O}_K$.

$\mathbb{Q}$ corresponds to the 3-sphere $S^3$.

The class group $\text{Cl}(K)$ corresponds to $H_1(M, \mathbb{Z})$.

Finite extensions of number fields $L/K$ correspond to finite branched coverings of 3-manifolds $\pi : M \to N$. A branched cover $M$ of a 3-manifold $N$ is given by a map $\pi$ such that there is a link $L$ of $N$ with the following property: The restriction map $\pi : M \setminus \pi^{-1}(L) \to N \setminus L$ is a topological cover.
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- The class group \( \text{Cl}(K) \) corresponds to \( H_1(M, \mathbb{Z}) \).
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Main Differences between the two Theories

- The group $\text{Cl}(K)$ is always finite, while $H_1(M, \mathbb{Z}) = \mathbb{Z}^r \oplus H_1(M, \mathbb{Z})_{\text{tor}}$ is not.

- The Algebraic translation of the Poincare Conjecture is false!

  One should expect that $\mathbb{Q}$ would be the only number field with no unramified extensions. That is not true. Indeed,

**Theorem**

If $d < 0$ and the class group of $L = \mathbb{Q}(\sqrt{d})$ is trivial then $L$ has no unramified extensions. There are precisely 9 such values of $d : -1, -2, -3, -7, -11, -19, -43, -67, -163$.

- Let $M_1 \rightarrow M$ a covering of 3-manifolds. A knot $K$ in $M$ does not necessarily lift to a knot in $M_1$, while every prime ideal $p \triangleleft \mathcal{O}_K$ gives rise to an ideal $p\mathcal{O}_L$. $L/K$ is a Galois number fields extension.
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Let $G = Gal(\mathbb{L}/\mathbb{K})$ and $q$ be a prime ideal in $\mathcal{O}_L$.

**Definition**

The decomposition group of $q$, $D_q \subset G$ is the subgroup of $G$ preserving $q$,

$$D_q = \{ q \in G : g(q) = q \}$$

- The quotient $\mathcal{O}_L/q$ is a finite field.
- The image of the homomorphism $D_q \rightarrow Gal(\mathcal{O}_L/q)$ consists of exactly those automorphisms of $\mathcal{O}_L/q$ which fix the subfield, $\mathcal{O}_K/p$, $p = \mathcal{O}_K \cap q$.
- The kernel of this homomorphism, $I_q$ is called the inertia group of $q$.
- We have the following exact sequence,

$$0 \rightarrow I_q \rightarrow D_q \rightarrow Gal(\mathcal{O}_L/q/\mathcal{O}_K/p) \rightarrow 0$$
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$$0 \rightarrow l_q \rightarrow D_q \rightarrow Gal(\mathcal{O}_L/q/\mathcal{O}_K/p) \rightarrow 0$$
The order of $I_q$, denoted by $e_q$, is called the ramification index.

The order of $\text{Gal}(O_L/q/O_K/p)$ will be denoted by $f_q$.

The ideal $pO_L$ decomposes uniquely as a product of prime ideals, $p_1^{e_1} \cdots p_g^{e_g}$, where $e_i$ is the ramification index of $p_i$.

**Theorem**

*Under the above assumptions,*

- $G$ acts transitively on $p_1 \cdots p_g$,
- $e_1 = \ldots = e_g := e$ and $f_{p_1} \cdots f_{p_g} := f$,
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Let $G$ be a group that acts on a 3-manifold $M$ and the map $p : M \to M/G$ is a branched covering.

- The subgroup $D_K \subset G$ contains all the elements which map a knot $K \subset M$ to itself and is called the decomposition group of $K$.
- We assume that the action of $D_K$ on $K$ is orientation preserving.
- The image of the natural homomorphism $D_K \to \text{Homeo}(K)$ is exactly the group of deck transformations, $\text{Gal}(K/K')$, of the covering $K \to K' = K/D_K$.
- The kernel of this homomorphism, $I_K$, is called the inertia group of $K$.
- $0 \to I_K \to D_K \to \text{Gal}(K/K') \to 0$.
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Galois Extensions and Galois Branched Covers
Let G be a group that acts on a 3-manifold M and the map \( p : M \rightarrow M/G \) is a branched covering.

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\[ |I_K| = e_K \text{ and } |\text{Gal}(K/K')| = f_K. \]
Consider again the case of a group $G$ acting on a manifold $M$, such that $p : M \rightarrow M/G$ is a branched covering. Let $K$ be a knot in $M$, such that $K/G$ is a knot in $M/G$.

$p^{-1}(K)$ is a link in $M$ whose components we denote by $K_1, \ldots K_g$,

$$p^{-1}(K) = K_1 \cup \ldots \cup K_g.$$ 

**Theorem (Sikora)**

Under the above assumptions

- $G$ acts transitively on $K_1, \ldots K_g$,
- $e_{K_1} \ldots e_{K_g} := e$ and $f_{K_1} = \ldots f_{K_g} := f$
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Let $C_p$ be a cyclic group and $G = C_p$. Then each prime $q \triangleleft \mathcal{O}_L$ and each knot $K \subset M$ is either split, inert or ramified. If $p = q \cap \mathcal{O}_K$ the $q$ is

- split if $p\mathcal{O}_L = q_1 \ldots q_p$, where $q_1, \ldots, q_p$ are different prime, one of which is $p$. In this situation $C_p$ permutes these primes.
- ramified if $p\mathcal{O}_L = q^p$. Here $C_p$ fixes the elements of $q$.
- inert if $p\mathcal{O}_L = q$. In this situation $C_p$ acts non-trivially on $q$.

Let $G = C_p$, then $p : M \to M/G$ is a branched covering. If $K \subset M$ satisfies the previous assumptions, then $K$ is

- split if $p^{-1}(K) = K_1 \cup \ldots \cup K_p$, where $K_1, \ldots, K_p$ are different knots. Here $C_p$ cyclicly permutes these knots
- ramified if $K/G$ is a component of the branching set. Here $C_p$ fixes $p^{-1}(K/G) = K$.
- inert if $p^{-1}(K/G) = K$ and the $C_p$—action on $K$ is non-trivial.
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Let $C_p$ be a cyclic group and $G = C_p$. Then each prime $q < \mathcal{O}_L$ and each knot $K \subset M$ is either split, inert or ramified. If $p = q \cap \mathcal{O}_K$ the $q$ is

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Split, Ramified and Inert Primes and Knots

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The Number Fields Case

**Theorem (The Principal Ideal Theorem)**

Let $K$ be a number field and let $K^{(1)}$ be the Hilbert class field of $K$. Let $\mathcal{O}_K$, $\mathcal{O}_{K^{(1)}}$ be the rings of integers of $K$ and $K^{(1)}$ respectively. Let $P$ be a prime ideal of $\mathcal{O}_{K^{(1)}}$. We consider the prime ideal

$$\mathcal{O}_K \triangleright p = P \cap \mathcal{O}_K$$

and let

$$p\mathcal{O}_{K^{(1)}} = (PP_2 \ldots P_r)^e = \prod_{g \in \text{CL}(K)} g(P)$$

be the decomposition of $p\mathcal{O}_{K^{(1)}}$ in $\mathcal{O}_{K^{(1)}}$ into prime ideals. The ideal $p\mathcal{O}_{K^{(1)}}$ is principal in $K^{(1)}$.

This theorem was conjectured by Hilbert and the proof was reduced to a purely group theoretic problem by E. Artin.
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The 3-Manifold Case

Definition

We define the Hilbert Manifold $M^{(1)}$ of $M$ as the universal covering space $\tilde{M}$ of $M$ modulo the commutator group,

$$M^{(1)} = M / [\pi_1(M), \pi_1(M)].$$

Theorem (The Principal Ideal Theorem for Knots)

1. Let $K_1$ be a knot in $M^{(1)}$. Denote by $G(K_1)$ the subgroup of $G = \pi(M) / [\pi_1(M), \pi_1(M)]$ fixing $K_1$. Consider the link $L = \bigcup_{g \in G / G(K_1)} gK_1$. Then $L$ is zero in $H_1(M^{(1)}, \mathbb{Z})$.

2. Let $L$ be a link in $M$ that is a homologically trivial. Then there is a family of tame knots $K_\epsilon$ in $M$ with $\epsilon > 0$, that are boundaries of embedded surfaces $E_\epsilon$ so that $\lim_{\epsilon \to 0} K_\epsilon = L$ and $E = \lim_{\epsilon \to 0} E_\epsilon$ is an embedded surface with $\partial E = L$. 
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Theorem (Path Lifting Property)

Let $(Y, y_0)$, $(X, x_0)$ be topological spaces (arcwise connected, semilocally simply connected), let $p : (X', x'_0) \to (X, x_0)$ be a topological covering with $p(x'_0) = x_0$ and let $f : (Y, y_0) \to (X, x_0)$ be a continuous map. Then, there is a lift $\tilde{f} : Y \to X'$ of $f$,

$$\begin{array}{ccc}
X' & \xrightarrow{\tilde{f}} & X \\
\downarrow{p} & & \downarrow{p} \\
Y & \xrightarrow{f} & X
\end{array}$$

making the above diagram commutative if and only if

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(X', x'_0)),$$

where $f_*$, $p_*$ are the induced maps of fundamental groups.
Theorem (Dehn Lemma)

Let $M$ be a 3-manifold and $f : D^2 \to M$ be a map such that for some neighborhood $A$ of $\partial D^2$ in $D^2$ $f|_A$ is an embedding and $f^{-1}f(A) = A$. Then $f|_{\partial D^2}$ extends to an embedding $g : D^2 \to M$.

Corollary

If a tame knot is the boundary of a topological and possibly singular surface then the knot is the boundary of an embedded surface.
Useful Theorems

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Proof of The Main Result

Theorem (Part I)

Let $K_1$ be a knot in $M^{(1)}$. Denote by $G(K_1)$ the subgroup of $G$ fixing $K_1$. Consider the link $L = \bigcup_{g \in G/G(K_1)} gK_1$. Then $L$ is zero in $H_1(M^{(1)}, \mathbb{Z})$.

Proof.

Since the diagram

\[
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K_1 & \rightarrow & M^{(1)} \\
\downarrow \tilde{f} & & \downarrow p \\
S^1 & \rightarrow & p(K_1) \\
\downarrow f & & \downarrow \\
& & M
\end{array}
\]

commutes we have that

\[f_*(\pi_1(S^1)) \subset p_*(\pi_1(K_1)) \subset p_*(\pi_1(M^{(1)})) = p_*(\langle \pi_1(M), \pi_1(M) \rangle)\]
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Proof (Continued).

Therefore \( f_*(\pi_1(S^1)) = 0 \) as an element in \( H_1(M, \mathbb{Z}) \), hence there is a topological (possibly singular) surface \( \phi : E \to M \) so that

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f(S^1) = p(K^1) = \partial \phi(E).
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The surface \( E \) is homotopically trivial therefore the Dehn Lemma implies that there is a map \( \tilde{\phi} \) making the following diagram commutative:

\[
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M(1) & \xrightarrow{\tilde{\phi}} & E \\
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with the additional property \( \partial \tilde{\phi}(E) = p^{-1}(\partial \phi(E)) = L \).
Proof of The Main Result

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We have seen that when a knot (resp. link) \( K \) lifts to a knot (resp. link) in the Hilbert Manifold, then it is homologically trivial.

What remains is to show that there exists an embedding of a surface \( E \) in \( M^{(1)} \) such that \( \partial E = K \).

**Theorem (Part II)**

Let \( L \) be a link in \( M \) that is a homologically trivial. Then there is a family of tame knots \( K_\epsilon \) in \( M \) with \( \epsilon > 0 \), that are boundaries of embedded surfaces \( E_\epsilon \) so that \( \lim_{\epsilon \to 0} K_\epsilon = L \) and \( E = \lim_{\epsilon \to 0} E_\epsilon \) is an embedded surface with \( \partial E = L \).
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Proof.

- Consider a link with two components. Let $L = K_1 \cup K_2$.
  - $K_i$ is given by the embedding $f_i : S^1 \to M$.
  - The passage from two components to $n > 2$ follows by induction.
  - Select two points $P_i, Q_i$ on $f_i(S^1)$, such that $d(P_i, Q_i) = \epsilon, i = 1, 2$.
  - The embedding of the two curves can be seen as the union of two curves $\gamma_i : [0, 1] \to M, \delta_i : [0, 1] \to M$, so that $\gamma_i(0) = \delta_i(1) = P_i$, $\gamma_i(1) = \delta_i(1) = Q_i$. This means that the ”small” curve is $\delta_i$.
  - Since $M$ is tamely path connected we can find two paths $\alpha, \beta : [0, 1] \to M$ such that $\alpha(0) = P_1, \alpha(1) = Q_2, \beta(0) = P_2, \beta(1) = Q_1$, that are close enough so that the rectangle $\alpha(-\delta_2)\beta(-\delta_1)$ is homotopically trivial.
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Proof (Continued).

- Let \( I = [0, 1] \subset \mathbb{R} \).
- Every path in \( M \), i.e. every function \( f : I \to M \), defines a cycle in \( H_1(M, \mathbb{Z}) \).
- We will abuse the notation and we will denote by \( f(I) \) the homology class of the path \( f(I) \).
- We compute in \( H_1(M, \mathbb{Z}) \):

\[
0 = f_1(S^1) + f_2(S^1) = \gamma_1(I) + \gamma_2(I) + \delta_1(I) + \delta_2(I) + 0 = \gamma_1(I) + \gamma_2(I) + \delta_1(I) + \delta_2(I) + \alpha(I) - \delta_2(I) + \beta(I) - \delta_1(I) = \gamma_1(I) + \alpha(I) + \gamma_2(I) + \beta(I).
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- The tame knot \( \gamma_1 \alpha \gamma_2 \beta \) is the boundary of a topological surface.
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- We will abuse the notation and we will denote by $f(I)$ the homology class of the path $f(I)$.
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\[ 0 = f_1(S^1) + f_2(S^1) = \gamma_1(I) + \gamma_2(I) + \delta_1(I) + \delta_2(I) + 0 = \]
\[ = \gamma_1(I) + \gamma_2(I) + \delta_1(I) + \delta_2(I) + \alpha(I) - \delta_2(I) + \beta(I) - \delta_1(I) = \]
\[ = \gamma_1(I) + \alpha(I) + \gamma_2(I) + \beta(I). \]
- The tame knot $\gamma_1 \alpha \gamma_2 \beta$ is the boundary of a topological surface.
Proof of The Main Result

Proof (Continued).

- By the Corollary it is the boundary of an embedded surface $E_\epsilon$.
- Choose an orientation on $E_\epsilon$ so that on $P \in \partial E_\epsilon$ one vector of the oriented basis of $T_P E_\epsilon$ is the tangent vector of the curves $\partial E_\epsilon$ and the other one is pointing inwards of $E$.
- Denote the second vector by $N_P$.
- We choose the same orientation on all surfaces $E_\epsilon$ in the same way, i.e. the induced orientation on the common curves of the boundary is the same.
- We take the limit surface for $\epsilon \to 0$. 
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Two Cases

We have to distinguish the following two cases

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The Seifert Theorem

As a corollary of the principal ideal theorem for knots we state the following:

\textbf{Theorem (Seifert)}

Every link in a simply connected 3 manifold is the boundary of an embedded surface.

\textbf{Proof.}

Let $M$ be simply connected. The Hilbert manifold of $M$ coincides with $M$ and the result follows.
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**Theorem (Seifert)**

*Every link in a simply connected 3 manifold is the boundary of an embedded surface.*

**Proof.**

Let $M$ be simply connected. The Hilbert manifold of $M$ coincides with $M$ and the result follows.
As a corollary of the principal ideal theorem for knots we state the following:

**Theorem (Seifert)**

*Every link in a simply connected 3-manifold is the boundary of an embedded surface.*

**Proof.**

Let $M$ be simply connected. The Hilbert manifold of $M$ coincides with $M$ and the result follows.
Bibliography


Bibliography II
