Alexios Terezakis

Deformation Theory

Master's Thesis

University of Athens, Department of Mathematics

Athens March 14, 2019

Advisor: Aristides Kontogeorgis

Master Thesis Committee

Ioannis Emmanouil

Aristides Kontogeorgis

Ioannis Dokas

To my parents

Contents

Εισαγωγή

Ένας moduli functor F από μια κατηγορία $\mathcal{C} \to \mathrm{Sets}$ είναι ένας συναρτητής από μια κατηγορία (schemes, sheaves, μορφισμών μεταξύ τους, αναπαραστάσεων) στην κατηγορία των συνόλων, ώστε ο F να στέλνει οικογένειες αντικειμένων της $\mathcal{C}$ πάνω από μια βάση B σε ένα στοιχείο της κλάσης ισοδυναμίας των αντικειμένων πάνω από το B. Ένα moduli πρόβλημα λέγεται fine όταν ο μοδυλι συναρτητής είναι representable, δηλαδή όταν υπάρχει ένα scheme X και ένας ισομορφισμός συναρτητών $h_X \cong F$. Όπου ο h_X είναι ο συναρτητής που στέλνει το αντικείμενο T στο σύνολο $\mathrm{Hom}(T, X)$, των μορφισμών $T \to X$ της κατηγορίας $\mathcal{C}$ και την απεικόνιση $f : T_1 \to T_2$ στην απεικόνιση

$$h_x(T_2) \ni h_2 \mapsto h_1 = h_2 \circ f \in h_X(T_1)$$

μέσω του διαγράμματος

$$\begin{array}{ccc} T_1 & \xrightarrow{h_1} & X \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle h_2} & \\ T_2 & & \end{array}$$

Η ύπαρξη ενός τέτοιου ισομορφισμού, σημαίνει ότι για οποιαδήποτε αντικείμενα T_i, T_j και συναρτήσεις $f_{ij} : T_i \to T_j$, υπάρχει μία συμβατή οικογένεια ισομορφισμών ϕ_i τέτοια ώστε το ακόλουθο διάγραμμα να μετατίθεται

$$\begin{CD} h_X(T_j) @>{\phi_j}>> F(T_j) \\ @V{h_X(f_{i,j})}VV @VV{F(f_{i,j})}V \\ h_X(T_i) @>{\phi_i}>> F(T_i) \end{CD}$$

Ένα από τα κλασσικά moduli προβλήματα είναι το moduli πρόβλημα των καμπυλών δεδομένου γένους g. Αυτός ο moduli συναρτητής απεικονίζει κάθε οικογένεια σχετικών καμπυλών $X \to T$ πάνω από ένα σχημε T, στην κλάση ισομορφίας τους, όπου δύο οικογένειες $\mathcal{X}_1, \mathcal{X}_2$ είναι ισόμορφες όταν υπάρχει ένας ισομορφισμός ϕ τέτοιος ώστε το παρακάτω διάγραμμα να μετατίθεται

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\phi} & \mathcal{X}_2 \\ & \searrow \quad \swarrow & \\ & T & \end{array}$$

Γεωμετρικά, οι καμπύλες πάνω από το Spec$k$, όπου το k είναι αλγεβρικά κλειστό σώμα, αντιστοιχούν στα σημεία του X, αφού η απεικόνιση $X \to \mathrm{Spec}k$ αντιστοιχεί σε στοιχείο του συνόλου $h_X(\mathrm{Spec}k) = \mathrm{Spec}k, X)$, δηλαδή σε ένα γεωμετρικό σημείο του X. Δυστυχώς, η ύπαρξη αυτομορφισμών καμπυλών, εμποδίζει τον μοδυλι συναρτητη από το να είναι representable. Για παράδειγμα αν C είναι μια αλγεβρική καμπύλη με κάποιον μη τετριμμένο αυτομορφισμό ϕ, τότε έχουμε το ακόλουθο διάγραμμα

$$\begin{array}{ccc} C & \xrightarrow{\phi} & C \\ \downarrow & & \downarrow \\ \mathrm{Spec}k & \xrightarrow{\mathrm{id}} & \mathrm{Spec}k \end{array}$$

Η σπουδαιότητα του παραπάνω παραδείγματος έγκειται στο ότι η απεικόνιση $\mathrm{id} : \mathrm{Spec}k \to \mathrm{Spec}k$ δεν περιγράφει την απεικόνιση των οικογενειών. Ως ένα ακόμα παράδειγμα θα δείξουμε ότι ο moduli συναρτητής των ελλειπτικών καμπυλών δεν είναι representable. Μια ελλειπτική καμπύλη πάνω από το $\mathbb{C}$ είναι μια λεία προβολική καμπύλη E, μαζί με ένα σταθεροποιημένο σημείο $e \in E$. Χρησιμοποιώντας το Θεώρημα Riemann-Roch σε συνδυασμό με την θεωρία διαμόρφωσης των τεσσάρων σημείων διακλάδωσης της διπλής επικάλυψης $E \to \mathbb{P}^1$, μπορούμε να δείξουμε ότι κάθε ελλειπτική καμπύλη μπορεί να περιγραφεί από τα σημεία μηδενισμού του ομογενούς πολυωνύμου

$$Y^2Z - X(X - Z)(X - \lambda Z),$$

Σε αυτό το μοντέλο το κλειστό σημείο e έχει προβολικές συντεταγμένες $e = [0 : 1 : 0]$, και $\lambda \in \mathbb{A}^1 - \{0, 1\}$. Το πολυώνυμο αυτό ορίζει μια οικογένεια

$$\mathcal{E} \to \mathbb{A}^1 - \{0, 1\},$$

πάνω από την τρυπημένη αφινική ευθεία, συνεπώς το $\mathbb{A}^1 - \{0, 1\}$ μπορεί να θεωρηθεί ως χώρος παραμέτρων για την οικογένεια. Η αναπαράσταση μια κλάσης ισομορφίας ως ίνα δεν είναι μοναδική, υπάρχει μια δράση της συμμετρικής ομάδας S_3 στο $\mathbb{A}^1 - \{0, 1\}$ η οποία παράγεται από τους αυτομορφισμούς $\lambda \mapsto 1/\lambda$, $\lambda \mapsto 1/(1-\lambda)$. Αν θέλουμε να παραμετρήσουμε ελλειπτικές καμπύλες χωρίς να κάποια προβολική εμφύτευση πρέπει να θεωρήσουμε το πηλίκο $\mathbb{A}^1 - \{0, 1\}$ προς την δράση αυτή της S_3. Ο χώρος που θα καταλήξουμε είναι ο δακτύλιος των αναλλοίωτων του $\mathbb{C}[\lambda]_{\lambda(\lambda-1)}$ που είναι η j-ευθεία, με

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

Υπάρχει μια αντιστοιχία μεταξύ των κλάσεων ισομορφισμού ελλειπτικών καμπυλών πάνω από το $\mathbb{C}$ και των μιγαδικών αριθμών $j \in \mathbb{C}$. Ωστόσο η αφινική ευθεία $\mathbb{A}^1$ δεν αποτελεί ένα fine moduli χώρο για τις ελλειπτικές καμπύλες. Πράγματι, έστω μια οικογένεια ελλειπτικών καμπυλών $\mathcal{E}_t$ ορισμένη πάνω από το $\mathbb{A}^1 - \{0\}$, η οποία δίνεται από την εξίσωση

$$Y^2Z = X^3 - tZ^3.$$

Για κάθε t όλες οι ίνες έχουν σταθερή j-αναλλοίωτη ίση με το 0. Έστω ότι το $\mathbb{A}^1$ αναπαριστά τις ελλειπτικές καμπύλες, τότε η παραπάνω οικογένεια θα πρέπει να αντιστοιχεί στον σταθερό μορφισμό $(\mathbb{A}^1 - \{0\}) \to \mathbb{A}^1_j$. Όμως η ελλειπτική καμπύλη $E_0 : Y^2Z = X^3 - Z^3$ έχει επίσης j-αναλλοίωτη 0. Συνεπώς η οικογένεια $\mathcal{E}_t$ θα είναι τετριμμένη και ίση με $E_0 \times (\mathbb{A}^1 - \{0\})$. Ωστόσο αυτό δεν είναι αληθές, πάνω από το function field $\mathbb{C}(t)$ οι οικογένειες $\mathcal{E}_t$ και $(\mathbb{A}^1 - \{0\})$ γίνονται ισόμορφες πάνω από την επέκταση $\mathbb{C}(t^{1/6})$.

Υπάρχουν διάφορες τεχνικές που μας επιτρέπουν να καταστήσουμε ένα μοδυλι πρόβλημα representable, όπως για παράδειγμα η εισαγωγή της έννοιας των αλγεβρικών χώρων και των stacks, ή αλλάζοντας την έννοια της ισομορφίας. Αν μας επιτρέπεται μια υπεραπλούστευση, μπορούμε να πούμε πως ένας τρόπος να οριστούν τα stacks ενός moduli χώρου καμπυλών είναι να ορίσουμε την κατηγορία με αντικείμενα τις proper smooth οικογένειες $\mathcal{X} \to S$, των οποίων οι ίνες είναι συνεκτικές καμπύλες δεδομένου γένους.

Η θεωρία παραμορφώσεων (Deformation theory) από την άλλη προέρχεται από την δουλεία των Kodaira και Spencer πάνω σε μιγαδικές πολλαπλότητες. Ο Grothendieck μετέφερε την θεωρία αυτή στην γλώσσα των Schemes. Μπορούμε να πούμε ότι η θεωρία παραμορφώσεων είναι η διαχείριση ενός moduli προβλήματος τοπικά, όπου μελετούνται οικογένειες πάνω από το φάσμα τοπικών Artin δακτυλίων. Ένας δακτύλιος του Artin είναι εξ ορισμού ένας δακτύλιος στον οποίο κάθε φθίνουσα ακολουθία ιδεωδών του τερματίζει ύστερα από πεπερασμένα βήματα. Ένα από τα πιο απλά παραδείγματα (που δεν είναι σώμα) είναι ο δακτύλιος $k[\epsilon]/\langle\epsilon^2\rangle$, όπου το ϵ αποτελεί ένα απειροστό βαθμού 2 με την έννοια ότι $\epsilon^2 = 0$. Στη δημοσίευση του ο Schlessinger μας παρέχει την γλώσσα και τα εργαλεία να χειριστούμε τα απειροστά σαν στοιχεία του 'εφαπτόμενου χώρου' και λύνει την αντίστοιχη συνήθη διαφορική εξίσωση μέσω τυπικών δυναμοσειρών δακτυλίων.

Συγκεκριμένα στο Κεφάλαιο 1 εισάγουμε τις κατηγορίες που θα χρησιμοποιήσουμε, τον Zariski εφαπτόμενο χώρο και πως να τον ορίσουμε για συναρτητές και τα διαφορικά Kähler. Στο Κεφάλαιο 2 ορίζουμε την έννοια της small extension (μικρής επεκτάσης), την έννοια smoothness και τέλος το κεντρικό αποτέλεσμα, το Θεώρημα του Schlessinger. Στο τελευταίο κεφάλαιο αποδεικνύουμε με την χρήση του Θεωρήματος του Schlessinger ότι ο Picard συναρτητής και ο συναρτητής παραμόρφωσης είναι pro-representable.

Αθήνα Μάρτιος 2019.

Introduction

A moduli functor F from a category $\mathcal{C} \to$ Sets is a functor from a category (schemes, sheaves,morphisms between them, representations) to the category of sets, so that F it sends families of objects of $\mathcal{C}$ over a base B to the element of equivalence class of objects over B. A moduli problem is called *fine* when the moduli functor is representable, that is there is a scheme X and an isomorphism of functors $h_X \cong F$. The functor h_X is the functor sending T to the set $\mathrm{Hom}(T, X)$, of morphisms of schemes $T \to X$, and the map $f : T_1 \to T_2$ to the map

$$h_X(T_2) \ni h_2 \mapsto h_1 = h_2 \circ f \in h_x(T_1)$$

by the diagram

$$\begin{array}{ccc} T_1 & \xrightarrow{h_1} & X \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle h_2} & \\ T_2 & & \end{array}$$

The existence of an isomorphism $h_X \cong F$, means that for every objects T_i, T_j and functions $f_{i,j} : T_i \to T_j$, there is a compatible set of isomorphisms ϕ_i so that the following diagram is commutative

$$\begin{array}{ccc} h_X(T_j) & \xrightarrow{\phi_j} & F(T_j) \\ {\scriptstyle h_X(f_{i,j})}\downarrow & & \downarrow{\scriptstyle F(f_{i,j})} \\ h_X(T_i) & \xrightarrow{\phi_i} & F(T_i) \end{array}$$

One of the classical moduli problems is the moduli problem of curves of genus g. This moduli functor to any family of relative curves $\mathcal{X} \to T$ over a scheme T assigns the isomorphy class of it, where two families $\mathcal{X}_1, \mathcal{X}_2$ are isomorphic when there is an isomorphism ϕ making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\phi} & \mathcal{X}_2 \\ & \searrow \quad \swarrow & \\ & T & \end{array}$$

Geometrically curves over Spec$k$, where k is an algebraically closed field k, correspond to points of X, since the structure map $X \to \mathrm{Spec}k$ corresponds to an element in the set $h_X(\mathrm{Spec}k) = \mathrm{Hom}(\mathrm{Spec}k, X)$ i.e. a geometric point of X.

Unfortunately, the existence of automorphisms of curves, prevents the moduli functor to be representable. For example if C is an algebraic curve which admits a non-trivial automorphism ϕ, then we have the diagram

$$\begin{array}{ccc} C & \xrightarrow{\phi} & C \\ \downarrow & & \downarrow \\ \mathrm{Spec}k & \xrightarrow{\mathrm{id}} & \mathrm{Spec}k \end{array}$$

The importance of the above example is that the map $\mathrm{id} : \mathrm{Spec}k \to \mathrm{Spec}k$ does not describe the map of the families.

As an other example we will show that the moduli space of elliptic curves is not representable. An elliptic curve over $\mathbb{C}$ is a smooth projective curve E, together with a selected closed point $e \in E$. As an application of Riemann-Roch theorem we can show that any elliptic curve can be described as the zero locus of the homogeneous polynomial

$$Y^2Z - X(X - Z)(X - \lambda Z),$$

using also the theory of configuration of the four ramification points of the two cover $E \to \mathbb{P}^1$. In this model the closed point e has projective coordinates $e = [0 : 1 : 0]$, and $\lambda \in \mathbb{A}^1 - \{0, 1\}$. This polynomial defines a family

$$\mathcal{E} \to \mathbb{A}^1 - \{0, 1\},$$

over the punctured affine line, so that $\mathbb{A}^1 - \{0, 1\}$ can be thought as a parameter space for the family. The representation of an isomorphy class as a fiber is not unique, there is an action of the symmetric group S_3 on $\mathbb{A}^1 - \{0, 1\}$ generated by the automorphisms $\lambda \mapsto 1/\lambda$, $\lambda \mapsto 1/(1 - \lambda)$. If we want to parametrize abstract elliptic curves without a projective embedding we have to consider the quotient of $\mathbb{A}^1 - \{0, 1\}$ modulo this S_3 action. The resulting space is the ring of invariants of $\mathbb{C}[\lambda]_{\lambda(\lambda-1)}$ which is the j-line, where

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

There is a bijection between isomorphism classes of elliptic curves over $\mathbb{C}$ and complex numbers $j \in \mathbb{C}$. However the affine line $\mathbb{A}^1$ is not a fine moduli space for elliptic curves.

Indeed, consider the family of elliptic curves $\mathcal{E}_t$ defined over the affine line $\mathbb{A}^1 - \{0\}$ given by equation

$$Y^2Z = X^3 - tZ^3.$$

For all t all fibers have constant j-invariant equal to 0. If $\mathbb{A}^1$ was a scheme representing elliptic curves, then the above given family should correspond to the constant morphism $(\mathbb{A}^1 - \{0\}) \to \mathbb{A}^1_j$. The elliptic curve $E_0 : Y^2Z = X^3 - Z^3$ also has j-invariant 0. So the family $\mathcal{E}_t$ should be trivial and equal to the fiber product $E_0 \times (\mathbb{A}^1 - \{0\})$. However this is not true, over the function field $\mathbb{C}(t)$ the families $\mathcal{E}_t$ and $(\mathbb{A}^1 - \{0\})$ become isomorphic over the field extension $\mathbb{C}(t^{1/6})$.

There are various techniques which allows us to represent moduli problems, for instance introducing algebraic spaces or stacks, or by altering the notion of equivalence in order to get rid of automorphisms of objects (introducing level structure) etc. If we are allowed an oversimplification we can say that one of the ways to define the stack of the moduli spaces of curves is to define a category whose objects are proper smooth families $\mathcal{X} \to S$, whose fibers are connected curves of given genus.

Deformation theory on the other hand originates from the work of Kodaira and Spencer on complex analytic manifolds. This work was formalized and translated into the language of schemes by Grothendieck. We can say that deformation theory is a local treatment of the moduli functor problem where families are considered only over spectra of local Artin rings. An Artin ring is by definition a ring where decreasing sequences of ideals are terminating after a finite number of steps. One of the easiest examples (which is not a field) is the ring of dual numbers $k[\epsilon]/\langle \epsilon^2 \rangle$. The quantity ϵ in the above example is an infinitesimal of order 2 in the sense that $\epsilon^2 = 0$. The seminal article of Schlessinger provides us with the language and tools to treat infinitesimals as elements in the "tangent vector space" and also solve the corresponding ordinary differential equations in terms of formal powerseries rings.

More precisely in Chapter 1 we introduce the categories that we will work, the Zariksi tangent space and how to define it for a functor and the Kähler differentials. In Chapter 2 we define the small extension, the notion of smoothness, and finally the main result, Schlessinger's Theorem. In the last Chapter we use Schlessinger's Theorem to prove that Picard functor and deformation functor are pro-representable.

Athens March 2019.

Chapter 1

Basic Definitions

1.1 Coefficient-Λ-algebras

Definition 1.1.1. *A* ***Coefficient-ring*** *is a complete, local, Noetherian ring A, with residue field $k \cong A/\mathfrak{m}_A$*

Definition 1.1.2. *A* ***Coefficient-ring homomorphism*** *is a continuous homomorphism $\phi : A' \to A$, such that $\phi^{-1}(\mathfrak{m}_A) = \mathfrak{m}_{A'}$ and $A/\mathfrak{m}_A \cong A'/\mathfrak{m}_{A'}$ $(\cong k)$, where A, A' are Coefficient-rings.*

Definition 1.1.3. *Fix Λ a coefficient-ring with residue field k of characteristic p.*

(*i*) *Denote by $\hat{\mathcal{C}}_\Lambda(A)$ the category whose objects are coefficient-Λ-algebras which are endowed with a coefficient-Λ-algebra homomorphism to A.*

(*ii*) *Denote by $\mathcal{C}_\Lambda(A)$ the full subcategory of $\hat{\mathcal{C}}_\Lambda(A)$ whose objects are artinian coefficient-Λ-algebras.*

(*iii*) *An A-**augmentation** is a coefficient-Λ-algebra homomorphism to A.*

Remark 1.1.4. *If A is the residue field k we write $\hat{\mathcal{C}}_\Lambda$ and $\mathcal{C}_\Lambda$ instead of $\hat{\mathcal{C}}_\Lambda(A)$ and $\mathcal{C}_\Lambda(A)$ respectively.*

The reason for the "ˆ" notation is that any coefficient-ring A may be written as the projective limit of Artinian rings.

$$A = \text{proj.lim.} A/(\mathfrak{m}_A)^n.$$

We call a functor F from and arbitrary category to sets, **representable** if there is an object X such that F is isomorphic to the functor $Y \to \text{Hom}(X, Y)$. If we knew that a given functor F on the larger category $\hat{\mathcal{C}}_\Lambda$ is representable, the representing coefficient-Λ-algebra, call it R is completely determined by the restriction of the functor to the smaller category $\mathcal{C}_\Lambda$. This is true because,

$$\text{Hom}(R, A) = \text{proj.lim. Hom}(R, A/(\mathfrak{m}_A)^n).$$

Definition 1.1.5. *We call a functor F* ***continuous*** *if:*

$$F(A) = \text{proj.lim. } F(A/(\mathfrak{m}_A)^n),$$

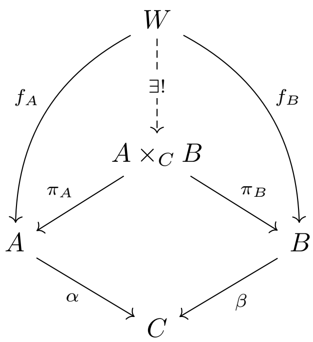
for all coefficient-Λ-algebras A.

We can see now that a continuous functor is determined by its restriction to $\mathcal{C}_\Lambda$.

Definition 1.1.6. *Schlessinger call a functor on the category $\mathcal{C}_\Lambda$* ***pro-representable****, when is represented by objects of the larger category $\hat{\mathcal{C}}_\Lambda$.*

But we will discuss about representability later.

Definition 1.1.7. *Let A, B, C be rings and let $\alpha : A \to C$ and $\beta : B \to C$ be ring maps. The fiber product is a ring denoted by $A \times_C B$ along with two morphisms $\pi_A : A \times_C B \to A$ and $\pi_B : A \times_C B \to B$, where $\alpha\pi_A = \beta\pi_B$, such that given any ring W with morphisms to $f_A : W \to A$ and $f_B : W \to B$, with $\alpha f_A = \beta f_B$, these morphisms factor through some unique $W \to A \times_C B$.*



This is the categorical definition, in the case of rings, the fiber product is the subset of $A \times B$

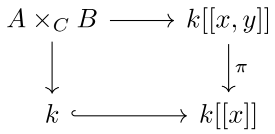
$$A \times_C b = \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}.$$

One of the reasons that we will use the "smaller" category $\mathcal{C}_\Lambda$, is that unlike the category $\hat{\mathcal{C}}_\Lambda$, fiber products always exists in $\mathcal{C}_\Lambda$, i.e. for A, A_1 and A_2 in $\mathcal{C}_\Lambda$ and morphisms $A_1 \to A$ and $A_2 \to A$ in $\mathcal{C}_\Lambda$, the fiber product $A_1 \times_A A_2$ lies in $\mathcal{C}_\Lambda$. Indeed, the ring $A_1 \times_A A_2$ is Λ-algebra via the map $\Lambda \to A_1 \times_A A_2$ induced by the maps $\Lambda \to A_1$ and $\Lambda \to A_1$. It is a local ring with maximal ideal

$$\mathfrak{m}_{A_1} \times_{\mathfrak{m}_A} \mathfrak{m}_{A_2} = \ker(A_1 \times_A A_2 \to k) \tag{1.1}$$

Note that, since the residue field of Λ is k, the map (1.1) is surjective. Finally, both A_1 and A_2 are Artin rings and so have finite length as Λ-modules. Hence the ring $A_1 \times A_2$ has finite length as Λ-module and this hold for the Λ-submodule $A_1 \times_A A_2$, i.e. $A_1 \times_A A_2$ is Artin ring.

Example 1.1.8. *If $A = k[[x, y]]$ and $B = k$ with morphisms to $C = k[[x]]$, then the fiber product $A \times_C B$ doesn't exist in $\hat{\mathcal{C}}_\Lambda$.*



Indeed we can check that the fiber product is given by the subring $k \oplus yk[[x, y]]$ in $k[[x, y]]$, the maximal ideal is $yk[[x, y]]$ and the Zariski tangent space (Definition

1.2.2) identified with the k-vector space $k[[x]]$ which is infinite dimensional, i.e. the $A \times_C B$ is not Noetherian.

Remark 1.1.9. *Furthermore, if we require that the morphisms of the fiber product are surjective, we can conclude that the fiber product exists in our category (i.e. the fiber product is Noetherian).*

Proposition 1.1.10. *If A, B are Noetherian rings, with surjective morphisms to ring C,*

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\pi_A} & A \\ \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \phi} \\ B & \xrightarrow{\psi} & C \end{array}$$

then the fiber product $A \times_C B$ is a Noetherian ring.

Proof. First we will prove that both π_A and π_B are surjectives. Indeed if $a_0 \in A$ then $\phi(a_0) \in C$ and because ψ is surjective there is $b_0 \in B$ such that $\phi(a_0) = \psi(b_0)$. Hence $(a_0, b_0) \in A \times_C B$ and $\pi_A(a_0, b_0) = a_0$. Now we can easily check that

$$\ker\pi_A \cap \ker\pi_b = \{0\}.$$

Finally we claim that if R is ring and $I_1, \dots, I_n \subseteq R$ are ideals such that

$$I_1 \cap \cdots \cap I_n = \{0\}, \tag{1.2}$$

and R/I_i is Noetherian for all $i = 1, \dots, n$, then R is Noetherian too. Indeed, each R/I_i is Noetherian R-module and so $R/I_1 \times \cdots \times R/I_n$ is Notherian R-module. But the morphism

$$R \to R/I_1 \times \cdots \times R/I_n,$$

is injective because of the (1.2) and so the R is Noetherian R-module, i.e. R is Noetherian ring. □

1.2 Zariski Tangent Space

Definition 1.2.1. *Fix Λ a coefficient-ring and R a coefficient-Λ-algebra. We define $t^*_R = t^*_{R/\Lambda}$ the* ***Zariski cotangent space****,*

$$t^*_R := \mathfrak{m}_R/((\mathfrak{m}_R)^2 + \mathfrak{m}_\Lambda \cdot R).$$

Definition 1.2.2. *So now we define the* ***Zariski tangent space*** *as,*

$$t_R := \mathrm{Hom}_{k-v.s}(\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_\Lambda \cdot R), k).$$

Remark 1.2.3. *Since R is Noetherian, t^*_R is a finite dimensional k-vector space.*

Remark 1.2.4. *By $k[\epsilon]$ we mean the ring in which $\epsilon^2 = 0$. So it is obvious that,*

$$k[\epsilon] \cong k \oplus \epsilon k.$$

Proposition 1.2.5. *There is a natural isomorphism of k-vector spaces,*

$$\mathrm{Hom}_{k-v.s.}\left(\mathfrak{m}_R/\left(\mathfrak{m}_R^2+\mathfrak{m}_\Lambda\cdot R\right),k\right)\cong\mathrm{Hom}_{\Lambda-alg.}\left(R,k[\epsilon]\right).$$

Proof. Since the maximal ideal of $k[\epsilon]$ has square zero, there is a bijection

$$\mathrm{Hom}_{\Lambda-alg.}\left(R/\left(\mathfrak{m}_R^2+\mathfrak{m}_\Lambda\cdot R\right),k[\epsilon]\right)\simeq\mathrm{Hom}_{\Lambda-alg.}\left(R,k[\epsilon]\right). \tag{1.3}$$

The short exact sequence

$$0\to\mathfrak{m}_R\to R\to R/\mathfrak{m}_R\to 0,$$

induces the short exact sequence

$$0\to\mathfrak{m}_R/\left(\mathfrak{m}_R^2+\mathfrak{m}_\Lambda\cdot R\right)\to R/\left(\mathfrak{m}_R^2+\mathfrak{m}_\Lambda\cdot R\right)\to R/\mathfrak{m}_R\to 0$$

Since these are k-vector spaces, the sequence splits, and we have a decomposition of Λ-algebras,

$$R/\left(\mathfrak{m}_R^2+\mathfrak{m}_\Lambda\cdot R\right)=k\oplus\mathfrak{m}_R/\left(\mathfrak{m}_R^2+\mathfrak{m}_\Lambda\cdot R\right).$$

Hence

$$\begin{aligned}&\mathrm{Hom}_{\Lambda-alg.}(k\oplus\mathfrak{m}_R/\left(\mathfrak{m}_R^2+\mathfrak{m}_\Lambda\cdot R\right),k\oplus\epsilon k)\cong\\&\mathrm{Hom}_{k-v.s.}\left(\mathfrak{m}_R/\left(\mathfrak{m}_R^2+\mathfrak{m}_\Lambda\cdot R\right),k\right).\end{aligned} \tag{1.4}$$

(1.3) and (1.4), gives the result. □

Definition 1.2.6. *Let $F:\mathcal{C}_\Lambda\to\mathrm{Sets}$ be any covariant functor such that, $F(k)$ consists of a single element. Then the Zariski tangent space of F, (denoted t_F), is the* **set** $F(k[\epsilon])$**.**

In this generality, we can not have a natural k-vector space structure.

Remark 1.2.7. *The idea is that we have an "addition" on,*

$$\begin{gathered}k[\epsilon]\times_k k[\epsilon]\xrightarrow{+}k[\epsilon]\\(x\oplus y_1\cdot\epsilon,x\oplus y_2\cdot\epsilon)\longrightarrow x\oplus(y_1+y_2)\cdot\epsilon.\end{gathered}$$

Definition 1.2.8. *We say that F satisfies the "Tangent space Hypothesis" (or just (T_k)) when the mapping,*

$$h:F(k[\epsilon]\times_k k[\epsilon])\to F(k[\epsilon])\times F(k[\epsilon])$$

is a bijection (1-1).

Remark 1.2.9. *If F satisfies the (T_k) we define "vector-addition" in Zariski tangent space t_F,*

$$\begin{array}{ccccc}F(k[\epsilon])\times F(k[\epsilon]) & \xrightarrow{h^{-1}} & F(k[\epsilon]\times k[\epsilon]) & \xrightarrow{F(+)} & F(k[\epsilon])\\ \big\downarrow{=} & & & & \big\downarrow{=}\\ t_F\times t_F & & \longrightarrow & & t_F\end{array}$$

Now we will define more generally the *Zariski tangent A-module*. We can make the analogous definitions but this time we will have A-module instead of k-vector space.

Definition 1.2.10. *Let* $F : \mathcal{C}_\Lambda(A) \to \text{Sets}$ *be any contravariant functor such that,* $F(A)$ *consists of a single element. Then we define the* ***Zariski tangent space*** *as,*

$$t_{F,A} := F(A[\epsilon]).$$

($A[\epsilon] = A \oplus \epsilon A$ *is as previously a free* A*-module of rank 2, where* $\epsilon^2 = 0$.)

In this generality the previous "Tangent space Hypothesis" is now,

Definition 1.2.11. *(Tangent space Hypothesis)*
We say that D *satisfies the "Tangent space Hypothesis" (or just* (T_A)*) when the mapping,*

$$h : F(A[\epsilon] \times_A A[\epsilon]) \to F(A[\epsilon]) \times F(A[\epsilon])$$

is a bijection $(1-1)$.

1.3 Kähler Differentials

Definition 1.3.1. *(Kähler differentials)*
Consider the homomorphism

$$\phi : R \otimes_\Lambda R \to R$$
$$\sum_i (r_i \otimes s_i) \to \sum_i r_i s_i,$$

and $I = \ker\phi$. *The Kähler differentials is the pair* $(\Omega_{R/\Lambda}, d)$, *where* $\Omega_{R/\Lambda} = I/I^2$ *and a map*

$$d : R \to \Omega_{R/\Lambda}$$
$$r \to (1 \otimes r) - (r \otimes 1).$$

There is a second definition for the Kähler differentials, that will be very useful.

Second Definition 1.3.2. *We define the module* $\Omega_{R/\Lambda}$ *to be the free* R*-module* F *generated by the symbols* $\{dr,\ r \in R\}$, *quotient with the* R*-submodule generated by all expressions of the form:*

(i) $d\lambda$, *for* $\lambda \in \Lambda$.

(ii) $d(r_1 + r_2) - dr_1 - dr_2$, *for* $r_1, r_2 \in R$,

(iii) $d(r_1 r_2) - r_1 dr_2 - r_2 dr_1$, *for* $r_1, r_2 \in R$,

The derivation

$$d : R \to \Omega_{R/\Lambda},$$

is defined by sending r *to* dr.

Definition 1.3.3. *If* M *is an* R*-module, a* Λ*-**derivation** is an* Λ*-module homomorphism* $d : R \to M$ *which satisfies the following conditions,* *(i)* $d(\lambda) = 0$, *for* $\lambda \in \Lambda$

(ii) $d(r_1 r_2) = r_1 d(r_2) - r_2 d(r_1)$, *for* $r_1, r_2 \in R$

The collection of Λ*-derivations of* R *into an* R*-module* M *is denoted by* $\mathrm{Der}_\Lambda(R, M)$.

It is easy to check that $\mathrm{Der}_\Lambda(R, M)$ is an R-submodule of $\mathrm{Hom}_{\Lambda-mod}(R, M)$

Proposition 1.3.4. *The module of Kähler differentials of* R *over* Λ *has the following universal property. For any* R*-module* M*, and for any* Λ*-derivation,* $d' : R \to M$*, there exists a unique* R*-module homomorphism* $f : \Omega_{R/\Lambda} \to M$*, that makes the above diagram commutative,*

$$\begin{array}{ccc} R & \xrightarrow{d} & \Omega_{R/\Lambda} \\ & \searrow^{d'} & \downarrow f \\ & & M \end{array}$$

The proof of the Proposition 1.3.4 by using the Second Definition 1.3.2 is left to the reader. For a proof with the Definition 1.3.1 and hence the equivalence of two definitions, see [5].

Proposition 1.3.5. *There is a canonical* R*-module isomorphism*

$$\mathrm{Hom}_{R-mod}(\Omega_{R/\Lambda}, M) \cong \mathrm{Der}_\Lambda(R, M),$$

Proof. The isomorphism, is the map,

$$(\phi : \Omega_{R/\Lambda} \to M) \mapsto (\phi \circ d : R \to M),$$

with inverse, the map who sending a $\psi \in \mathrm{Der}_\Lambda(R, M)$ to the unique R-module homomorphism, that given from the universal property of the Kähler differentials (Proposition 1.3.4). □

Corollary 1.3.6. *The functor from the category of modules over* R *to the category of sets, which maps every* R*-module* M *to* $\mathrm{Der}_\Lambda(R, M)$ *is representable by the module of Kähler differentials*

Example 1.3.7. *For* $P = \Lambda[[x_1, \ldots, x_n]]$*, the module of Kähler differentials is given by*

$$\Omega_{P/\Lambda} \cong \oplus_{i=1}^n P \cdot dx_i.$$

Using the Second Definition 1.3.2, if $\delta : P \to M$ *is a* Λ*-derivation, it is easy to see that the unique homomorphism is the* $f : \Omega_{P/\Lambda} \to M$ *with* $f(dx_i) = \delta(x_i)$.

Example 1.3.8. *Let* $I \subseteq P$ *be an ideal of* $P = \Lambda[[x_1, \ldots, x_n]]$*. The ring* P *is Noetherian and hence there are* $f_1, \ldots, f_m \in P$ *such that* $I = (f_1, \ldots, f_m)$*. Now the Kähler differentials of* $R = P/I$ *is,*

$$\Omega_{R/\Lambda} \cong (\oplus_{i=1}^n R \cdot dx_i) / (df_1, \ldots, df_m).$$

We can check this easily, by using the universal property of the Káhler differentials.

Proposition 1.3.9. *If R is a coefficient-Λ-algebra, we will show that $\Omega_{R/\Lambda} \otimes_R k$ is isomorphic as k-vector space with the*

$$t^*_{R/\Lambda} := \mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_\Lambda \cdot R).$$

Proof. We define $\phi : \mathfrak{m}_R \to \Omega_{R/\Lambda} \otimes k$, the morphsim that sends $m \in \mathfrak{m}_R$ to $d(m) \otimes 1_k$, where $d : R \to \Omega_{R/\Lambda}$ is the universal Λ-derivation). It suffices to prove that the next sequence is a short exact sequence of k-vector spaces.

$$0 \to \mathfrak{m}_R^2 + \mathfrak{m}_\Lambda \cdot R \overset{\imath}{\hookrightarrow} \mathfrak{m}_R \overset{\phi}{\to} \Omega_{R/\Lambda} \otimes_R k \to 0.$$

ϕ is surjective. Indeed, for an arbitrary $(d(r) \otimes \kappa) \in \Omega_{R/\Lambda} \otimes_R k$, with $r \in R$ and $\kappa \in k$, there are $\lambda_1, \lambda_2 \in \Lambda$ and $m \in \mathfrak{m}_R$ such that $r = \lambda_1 1_R + m,\ \kappa = \lambda_2 1_k$.

$$d(r) \otimes \kappa = (\lambda_1 d(1_R) + d(m)) \otimes \lambda_2 1_k = \lambda_2 d(m) \otimes 1_k = d(\lambda_2 m) \otimes 1_k,$$

and $\lambda_2 m \in \mathfrak{m}_R$.
ϕ is 1-1. Indeed, for $m_1, m_2 \in \mathfrak{m}_R$ $\mu \in \mathfrak{m}_\Lambda$ and $r \in R$, $d(m_1 m_2) = d(m_1)m_2 + d(m_2)m_1$ and $d(\mu r) = d(r)\mu$. Hence

$$\begin{aligned}\phi(m_1 m_2) &= (d(m_1)m_2 + d(m_2)m_1) \otimes 1_k = \\ &= d(m_1) \otimes m_2 1_k + d(m_2) \otimes m_1 1_k = 0,\end{aligned}$$

and

$$\phi(\mu r) = d(r)\mu \otimes 1_k = d(r) \otimes \mu 1_k = 0.$$

Finally for any $x \in \mathfrak{m}_R^2 + \mathfrak{m}_\Lambda \cdot R$, there are $m_{i,1}, m_{1,2} \in \mathfrak{m}_R,\ i = 1, \ldots, n$, $\mu \in \mathfrak{m}_\Lambda$ and $r \in R$ such that

$$x = \sum m_{i,1} m_{i,2} + \mu r,$$

Thus $\phi(x) = \sum \phi(m_{i,1} m_{i,2}) + \phi(\mu r) = 0$, i.e. $\mathfrak{m}_R^2 + \mathfrak{m}_\Lambda \cdot R \subseteq \ker\phi$. The next lemma completes the proof. □

Lemma 1.3.10. *For an arbitrary $m_0 \in \mathfrak{m}_R \setminus \mathfrak{m}_R^2 + \mathfrak{m}_\Lambda \cdot R$ there is a Λ-derivation $D : R \to k$, such that $D(m_0)$ is non zero.*

Proof. First we will see that it suffices to define the derivation, for the elements of the maximal ideal $\mathfrak{m}_R$ of R. Indeed, for an arbitrary $r \in R$ there is $\lambda \in \Lambda$ such that $\pi_R(r) = \pi_\Lambda(\lambda)$, and the next diagram commutes,

$$\begin{array}{ccc} \Lambda & \xrightarrow{\imath} & R \\ & \searrow^{\pi_\Lambda} & \downarrow{\scriptstyle \pi_R} \\ & & k \end{array}$$

i.e. $\pi_R(r) = \pi_R(\lambda 1_R)$. It follows that there is $m \in \mathfrak{m}_R$, such that $r = \lambda 1_R + m$, and hence for any Λ-derivation δ,

$$\delta(r) = \delta(\lambda 1_R) + \delta(m) = \delta(m).$$

Since R is Artinian there are $x_1, \ldots, x_s \in \mathfrak{m}_R$, such that

$$\mathfrak{m}_R/\left(\mathfrak{m}_R^2 + \mathfrak{m}_\Lambda \cdot R\right) = (\overline{x}_1, \ldots, \overline{x}_s),$$

and $\overline{x}_1, \ldots, \overline{x}_s$ are independent over k. There are also $y_1, \ldots, y_\ell \in \mathfrak{m}_R^2 + \mathfrak{m}_\Lambda$, such that

$$\mathfrak{m}_R = (x_1, \ldots, x_s, y_1, \ldots, y_\ell)\,.$$

Hence there are $a_i, b_i \in R$ such that $m_0 = \sum a_i x_i + \sum b_i y_i$, and since $m_0 \notin \mathfrak{m}_R^2 + \mathfrak{m}_\Lambda$, there is i_0, such that a_{i_0} is invertible.

$$\begin{aligned} D(m_0) &= \sum D(a_i x_i) + \sum D(b_i y_i) = \sum D(a_i x_i) \\ &= \sum D(a_i)\overline{x}_i + D(x_i)\overline{a}_i = \sum \overline{a}_i D(x_i). \end{aligned}$$

It is obvious, that it suffices to define the derivation D only for the x_i, and from the independence of the x_i we can choose arbitrary image for them.
We define $D(x_{i_0}) = 1$ and $D(x_j) = 0$ for $j \neq i_0$, and $D(m_0) = \overline{a}_{i_0}$ which is non zero, since a_{i_0} is invertible. The proof is complete. □

Remark 1.3.11. *Note that any R in $\hat{\mathcal{C}}_\Lambda$ is generated, as Λ-module, by the Λ and the maximal ideal $\mathfrak{m}_R$. We have already prove this in the last proof.*

Remark 1.3.12. *We can easily check that the tangent space $t_{F,A}$ is just the subset of $\mathrm{Hom}_{\Lambda-alg.}(R, A[\epsilon])$ consisting of those Λ-algebra homomorphisms whose composition with the projection $A[\epsilon] \to A$ is equal to ρ.*

Proposition 1.3.13. *Let F be a pro-representable functor. Let $A \in \hat{\mathcal{C}}_\Lambda$ and $\rho : R \to A$ a coefficient-Λ-algebra homomorphism which induces to A a structure of R-algebra. Then we have a natural isomorphism of A-modules,*

$$\mathrm{Hom}_{A-mod.}(\Omega_{R/\Lambda}\hat{\otimes}_R A, A) \cong t_{F,A}.$$

Proof. We have the ismorphisms,

$$\begin{gathered} \mathrm{Hom}_{A-mod.}(\Omega_{R/\Lambda}\hat{\otimes}_R A, A) \cong \mathrm{Hom}_{R-mod.}(\Omega_{R/\Lambda}, A) \\ \mathrm{Hom}_{R-mod.}(\Omega_{R/\Lambda}, A) \cong \mathrm{Der}_\Lambda(R, A). \end{gathered} \tag{1.5}$$

Moreover there is a natural injection

$$\imath : \mathrm{Der}_\Lambda(R, A) \hookrightarrow \mathrm{Hom}_{\Lambda-alg.}(R, A[\epsilon]),$$

which sends a derivation δ to the homomorphism

$$\begin{aligned} \rho_\delta : R &\to A[\epsilon] \\ r &\to \rho(r) \oplus \epsilon \cdot \delta(r) \end{aligned}$$

The injection $\imath$ identifies $\mathrm{Der}_\Lambda(R, A)$ with the subset of $\mathrm{Hom}_{\Lambda-alg.}(R, A[\epsilon])$, consisting of the homomorphisms such that composition with the projection $A[\epsilon] \to A$ yields $\rho : R \to A$. □

Remark 1.3.14. *The proposition 1.3.9 is clearly a special case of the last proposition.*

Chapter 2

Schlessinger's Representability Theorem

2.1 Small Extensions

We start with a useful lemma, and then we will introduce the very useful notion and some properties of the small extension.

Lemma 2.1.1. *A morphism $B \to A$ in $\hat{\mathcal{C}}_\Lambda$ is surjective if and only if the induced map $t_B^* \to t_A^*$ is surjective.*

Proof. ($\Leftarrow$) If the morphism $B \to A$ is surjection, then obviously the induced map on cotangent spaces is surjective.

($\Rightarrow$) Conversely, we have the commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}_\Lambda \cdot A/\left(\mathfrak{m}_A^2 \cap \mathfrak{m}_\Lambda \cdot A\right) & \longrightarrow & \mathfrak{m}_A/\mathfrak{m}_A^2 & \longrightarrow & t_A^* & \longrightarrow & 0 \\ & & \alpha \big\uparrow & & \beta \big\uparrow & & \gamma \big\uparrow & & \\ 0 & \longrightarrow & \mathfrak{m}_\Lambda \cdot B/\left(\mathfrak{m}_B^2 \cap \mathfrak{m}_\Lambda \cdot B\right) & \longrightarrow & \mathfrak{m}_B/\mathfrak{m}_B^2 & \longrightarrow & t_B^* & \longrightarrow & 0 \end{array}$$

Since α and γ are surjections, β is also surjection. We need to prove that $\mathrm{Im}\,(B \to A) = A$, by the *Nakayama's Lemma* (since B is Artinian $\mathfrak{m}_B$ is nilpotent) it suffices to show that

$$A = \mathrm{Im}\,(B \to A) + \mathfrak{m}_B \cdot A.$$

From the Remark 1.3.11, it suffices to prove that the map $\mathfrak{m}_B \cdot A \to \mathfrak{m}_A$ is surjection. Using once more *Nakayama's Lemma* we have to show that the map

$$\mathfrak{m}_B A/\mathfrak{m}_B \mathfrak{m}_A \to \mathfrak{m}_A/\mathfrak{m}_A^2$$

is surjection. We know that the next diagram is commutative,

$$\underbrace{\mathfrak{m}_B/\mathfrak{m}_B^2 \longrightarrow \mathfrak{m}_B \cdot A/\mathfrak{m}_B\mathfrak{m}_A \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2}_{\beta}.$$

Since β is surjection, the proof is complete. $\square$

Definition 2.1.2. *Let $p : B \to A$ be a surjection in $\mathcal{C}_\Lambda$,*

- *p is a* ***small extension*** *if* $\ker p$ *is a nonzero principal ideal (t) such that $\mathfrak{m}_B \cdot t = (0)$.*
- *p is* ***essential*** *if for any morphism $q : C \to B$ in $\mathcal{C}_\Lambda$ such that pq is surjective, it follows that q is surjective.*

Lemma 2.1.3. *Let $f : B \to A$ be a surjective morphism in $\mathcal{C}_\Lambda$. Then f can be factored as a composition of small extensions.*

Proof. The maximal ideal $\mathfrak{m}_B$ of B is nilpotent since B is Artin ring, say $\mathfrak{m}_B^n = 0$. First we get a factorization

$$B \to B/\ker(f)\mathfrak{m}_B^{n-1} \to \cdots \to B/\ker(f) \cong A$$

of f into a composition of surjections whose kernels are annihilated by the maximal ideal. Thus it suffices to prove the lemma when f itself is such a map. In this case $\ker(f)$ is a k-vektor space, which has finite dimension. Take a basis $t_1, \ldots, t_r$ of $\ker(f)$ as a k-vector space to get a factorization

$$B \to B/(t_1) \to \cdots \to B/(t_1, \ldots, t_r) \cong A$$

of f into a composition of small extensions. $\square$

Lemma 2.1.4. *Let $p : B \to A$ be a surjection in $\mathcal{C}_\Lambda$. Then*

(i) p is essential if and only if the induced map $p_* : t_B^* \to t_A^*$ is an isomorphism.

(ii) If p is a small extension, the p is not essential if and only if p has a section, i.e. a homomorphism

$$s : A \to B, \ \textit{with } ps = 1_A$$

Proof. (*i*) If p_* is an isomorphism, then by Lemma 2.1.1, p is essential. Conversely let $\bar{t}_1, \ldots, \bar{t}_r$ be a basis of t_A^*, and lift the $\bar{t}_i$, back to elements t_i in B. Set

$$C = \Lambda[t_1, \ldots, t_r] \subseteq B.$$

Then p induces a surjection from C to A, since p is essential, $C = B$. Thus $\dim_k t_B^* \leq r = \dim_k t_A^*$, and hence $t_B^* \cong t_A^*$.

(*ii*) If p has a section s, then s is not surjective, an so p is not essential. If p is not essential, then the subring C constructed above, is proper subring of B. Since length(B) = length $(A) + 1$, C is isomorphic to A. The isomorphism $C \simeq A$ yields the section.

$\square$

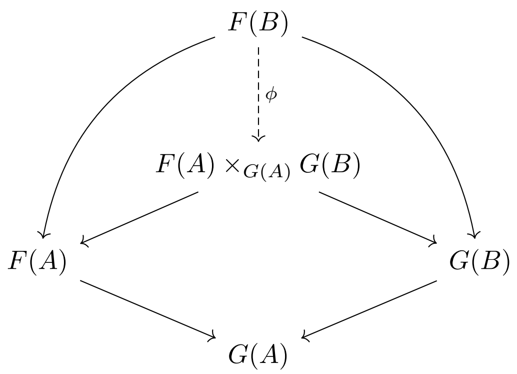
2.2 Functors of Artin rings

A functor of Artin rings is a covariant functor

$$F : \mathcal{C}_\Lambda \to Sets.$$

Definition 2.2.1. *Assuming that F is a covariant functor, a couple for F is a pair (A, ξ), where $A \in \mathcal{C}_\Lambda$ and $\xi \in F(A)$. A morphism of couples $u : (A, \xi) \to (A', \xi')$ is a morphism $u : A \to A'$ in $\mathcal{C}_\Lambda$ such that $F(u)(\xi) = \xi'$.*

We are trying to find lifts, so suppose that we have a surjection $B \to A$ and a functor $F : \mathcal{C}_\Lambda \to Sets$. Let α be an element of $F(A)$ and suppose that we want a lift of α in $F(B)$. If there is a morphism of functors $u : F \to G$, we have the next commutative diagram.



If in addition, ϕ is surjection, it suffices to have lifts from $G(A)$ to $G(B)$. Indeed, we first map α to $u(A)(\alpha) \in G(A)$, then lift the $u(A)(\alpha)$ to an element $\beta \in G(B)$. Now, the pair (α, β) is in $F(A) \times_{G(A)} G(B)$, and by using that the ϕ is surjection, we get an element $\zeta \in F(B)$ such that, $\phi(\zeta) = (\alpha, \beta)$. Clearly ζ is a lift of α.

Definition 2.2.2. *A morphism of functors $F \to G$ is* ***smooth*** *if for any surjection $B \to A$ in $\mathcal{C}_\Lambda$, the morphism*

$$F(B) \to F(A) \times_{G(A)} G(B),$$

is surjective.

Remark 2.2.3. *If $F \to G$ is a smooth morphism of functors, and a surjection $B \to A$, for a lift from $F(A)$ to $F(B)$ it suffices to have lifts from $G(A)$ to $G(B)$.*

We remind that a **pro-representable** functor, is a functor $F : \mathcal{C}_\Lambda \to k$ such that, there exists a ring R in $\hat{\mathcal{C}}_\Lambda$ with

$$F(A) \cong \mathrm{Hom}_{\Lambda-alg.}(R, A).$$

Clearly any representable functor is a trivial example of pro-representable.

Example 2.2.4. *For any ring R in $\mathrm{ob}(\hat{\mathcal{C}}_\Lambda)$, we define the (pro-representable) functor of Arting rings,*

$$h_R(A) = \mathrm{Hom}(R, A).$$

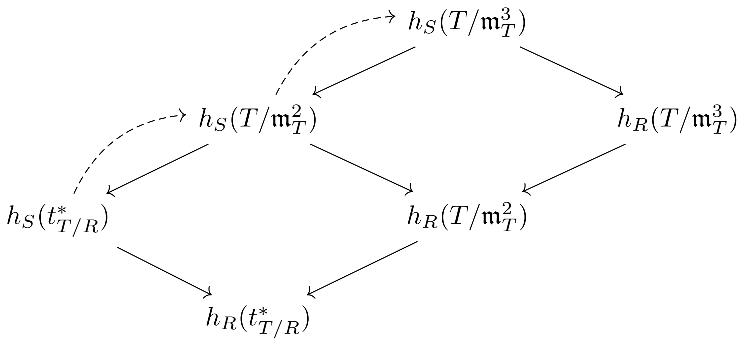
When Λ is fixed, we will write h_R instead of $h_{R/\Lambda}$.

Proposition 2.2.5. *Let $R \to S$ be a morphism in $\hat{\mathcal{C}}_\Lambda$. Then $h_S \to h_R$ is smooth if and only if S is a power series ring over R.*

Proof. ($\Rightarrow$) Suppose $h_S \to h_R$ is smooth, pick $x_1, \ldots, x_n \in S$, which induce a basis of $t^*_{S/R}$. If we set $T = R[[X_1, \ldots, X_n]]$, we get a morphism $u_1 : S \to T/(\mathfrak{m}_T^2 + \mathfrak{m}_R \cdot T)$ of local R-algebras, obtained by mapping x_i on the residue class of X_i. By smoothness u_1 lifts to $u_2 : S \to T/\mathfrak{m}_T^2$. Indeed, we can map the u_1 in to an element $\tilde{u}_1 \in h_R(\mathfrak{m}_T^2 + \mathfrak{m}_R \cdot T)$, obviously there is a lift $v_1 \in h_R(T/\mathfrak{m}_T^2)$ and finally by the smoothness, a lift of u_1 in $h_S(T/\mathfrak{m}_T^2)$. Thence u_2 lifts to $u_3 : S \to T/\mathfrak{m}_T^3$ (Figure 2.2.6), ... etc. Thus we get a $u : S \to T$ which, by choice of u_1, induce an isomorphism of $t^*_{S/R}$ with $t^*_{T/R}$, so that u is surjection by the Lemma 2.1.1. Furthermore, if we choose $y_i \in S$ such that $u(y_i) = X_i$ and produce a local morphism $v : T \to S$ of R-algebras such that $uv = 1_T$; in particular v is an injection. Clearly v induces a bijection on the cotangent spaces, so again by the Lemma 2.1.1 v is surjection. It follows that v is an isomorphism of $T = R[[X_1, \ldots, X_n]]$ with S.

($\Leftarrow$) If S is a power series ring over R, then it is clear that $h_R \to h_S$ is smooth. $\square$

Figure 2.2.6.



Proposition 2.2.7. (*i*) *If $F \to G$ and $G \to H$ are smooth, then the composition $F \to H$ is smooth.*

(*ii*) *If $u : F \to G$ and $v : G \to H$ are morphisms such that u is surjective and vu is smooth, then v is smooth.*

(*iii*) *If $F \to G$ and $H \to G$ are morphism such that $F \to G$ is smooth, then $F \times_G H \to H$ is smooth.*

The proof of this proposition is completely formal and left to the reader.

2.3 Universal Elements

Assume we have an R in $\mathrm{ob}(\hat{\mathcal{C}}_\Lambda)$. An element $\hat{u} \in \hat{F}(R)$, is called a formal element of F. By definition $\hat{u}$ can be represented as a system of elements $u_{n+1} \in F(R/\mathfrak{m}^{n+1})$, such that for every $n \geq 1$, the map

$$F(R/\mathfrak{m}^{n+1}) \to F(R/\mathfrak{m}^n)$$

induced by the projection $R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}$, sends $u_{n+1} \mapsto u_n$.

Lemma 2.3.1. *Let $R \in \mathrm{ob}(\hat{\mathcal{C}}_\Lambda)$. There is a $1-1$ correspondence between $\hat{F}(R)$ and the set of morphism of functors $\{h_R \to F\}$.*

Proof. Each formal element $\hat{u} \in \hat{F}(R)$ defined as, $\hat{u} = \text{proj.lim.} u_n$, where $u_n \in F(R/\mathfrak{m}^n)$. Yoneda's Lemma gives a morphism of functors

$$h_{R/\mathfrak{m}^n} \to F,$$

for each u_n. The next commutative diagram

$$\begin{array}{ccc} F(R/\mathfrak{m}^n) & \longrightarrow & \{h_{R/\mathfrak{m}^n} \to F\} \\ \downarrow & & \downarrow \\ F(R/\mathfrak{m}^{n+1}) & \longrightarrow & \{h_{R/\mathfrak{m}^{n+1}} \to F\} \end{array}$$

induce a new commutative diagram

$$\begin{array}{ccc} h_{R/\mathfrak{m}^n} & \longrightarrow & h_{R/\mathfrak{m}^{n+1}} \\ & \searrow & \downarrow \\ & & F \end{array}$$

Since for each $A \in \mathrm{ob}(\mathcal{C}_\Lambda)$

$$h_{R/\mathfrak{m}^n}(A) \to h_{R/\mathfrak{m}^{n+1}}(A),$$

is a bijection for all but finitely many n, we may define $h_R(A) \to F(A)$ as,

$$\lim_{n\to+\infty} [h_{R/\mathfrak{m}^n}(A) \to F(A)].$$

Conversely, each morphism $h_R \to F$ defines a formal element $\hat{u} \in \hat{F}(R)$, where $u_n \in F(R/\mathfrak{m}^n)$ is the image of the projection $R \to R/\mathfrak{m}^n$ via the map

$$h_R(R/\mathfrak{m}^n) \to F(R/\mathfrak{m}^{n-1})$$

□

Definition 2.3.2. *If R is in $\mathrm{ob}(\mathcal{C}_\Lambda)$ and $\hat{u} \in \hat{F}(R)$, we call $(R, \hat{u})$ a **formal couple** for F.*

Definition 2.3.3. *The differential*

$$t_{R/\Lambda} \to t_F$$

*of the morphism $h_R \to F$ defined by $\hat{u}$ is called the **characteristic map** of $\hat{u}$ (or of the formal couple $(R, \hat{u})$) and denoted by $d\hat{u}$.*

Definition 2.3.4. *If $(R, \hat{u})$ is such that the induced morphism*

$$h_R \to F$$

*is an isomorphism, then F is pro-representable, and we also say that F is pro-represented by the formal couple $(R, \hat{u})$. In this case $\hat{u}$ is called a **universal** formal element for F, and $(R, \hat{u})$ is a universal formal couple.*

A universal formal couple seldom exists, we will therefore need to introduce some weaker properties of a formal couple. We will now introduce the notions of "verslity" and "semiversality", which are slightly weaker that universality, based on the notion of smooth functor.

Definition 2.3.5. *Let F be a functor of Artin rings and R in* $\mathrm{ob}(\hat{\mathcal{C}}_\Lambda)$. *A formal element $\hat{u} \in \hat{F}(R)$ is called* ***versal****, if the morphism $h_R \to F$ defined by $\hat{u}$, is smooth.*

Definition 2.3.6. *The formal element $\hat{u}$ is called* ***semiuniversal*** *if it is versal and moreover, the differential $t_{R/\Lambda} \to t_F$ is bijective. Schlessinger calls the formal couple $(R, \hat{u})$ a (pro-representable) hull of F.*

It is clear by the definitions that

$$\hat{u} \ \text{ universal} \Rightarrow \ \hat{u} \ \text{ semiuniversal} \Rightarrow \ \hat{u} \ \text{ versal}$$

2.4 Schlessinger's Theorem

Suppose now that we have F a functor of Artin rings, $A, A', A'' \in \mathrm{ob}(\mathcal{C}_\Lambda)$ and a diagram

$$\begin{array}{ccc} A' & & A'' \\ & \searrow \quad \swarrow & \\ & A & \end{array} \tag{2.1}$$

This diagram induces a new diagram

$$\begin{array}{ccccc} & & F(A' \times_A A'') & & \\ & \swarrow & \downarrow{\scriptstyle \alpha} & \searrow & \\ & & F(A') \times_{F(A)} F(A'') & & \\ & \swarrow & & \searrow & \\ F(A') & & & & F(A'') \\ & \searrow & & \swarrow & \\ & & F(A) & & \end{array} \tag{2.2}$$

Finally we get a map

$$\alpha : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{2.3}$$

So we have some properties that we seek for the map α. Namely,

(H_0) For $k = R/\mathfrak{m}_R$, $F(k)$ consists of one element.

(H_1) For every diagram (2.1), where $A'' \to A$ is a small extension, the morphism α in (2.3) is a surjection.

(H_2) For every diagram (2.1), where $A = k$, $A'' = k[\epsilon]$, the morphism α in (2.3) is bijection.

(H_3) The set $F(k[\epsilon])$ has a structure of finite dimensional k-vector space.

(H_4) For every diagram (2.1), where $A' = A''$ and $A' \to A$ and $A'' \to A$ are equal and small extensions, the morphism α in (2.3) is a bijection.

(H_ℓ) For every diagram (2.1), the morphism α in (2.3) is bijection.

Proposition 2.4.1. *A pro-representable functor F satisfies the conditions (H_0), (H_3) and (H_ℓ).*

Proof. (H_0) The set $\mathrm{Hom}(R, R/\mathfrak{m}_R)$ contains only the caconical quotient map $R \to R/\mathfrak{m}_R$.

(H_3) In the first chapter we have seen that

$$F(k[\epsilon]) = \mathrm{Hom}_{\Lambda-alg.}(R, k[\epsilon]) = Der_\Lambda(R, k) = t_{R/\Lambda}.$$

So is the relative tangent space of R over Λ. Since R is Noetherian ring, the tangent space is finite dimensional.

(H_ℓ) The proof is simple and left to the reader.

□

Remark 2.4.2. *Note that (H_1), (H_2) and (H_4) are special cases of the condition (H_ℓ).*

Corollary 2.4.3. *A pro-representable functor $F = h_R$ satisfies the conditions (H_1), (H_2) and (H_4).*

The next Lemma is just the Remark 1.2.9.

Lemma 2.4.4. *If F is a functor of Artin rings satisfying (H_0) and (H_2) then the set $F(k[\epsilon])$ has a structure of k-vector space in a factorial way.*

Grothendieck's Theorem 2.4.5. *Let $F : \mathcal{C}_\Lambda \to \mathrm{Sets}$ be a covariant functor. Then F is pro-representable if and only if F satisfies the conditions (H_0), (H_3) and (H_ℓ).*

A proof can be found in [6].

In contrast to this theorem, which requires the property (H_ℓ) i.e. check all diagrams of the form (2.1), the theorem of Schlessinger artfully cuts down the number of diagrams for which one must check.

Lemma 2.4.6. *Let F is a functor of Artin rings satisfying (H_0), (H_1) and (H_2) and $\pi : A' \to A$ a small extension with $\ker\pi = (t)$. Then the map*

$$\beta : t_F \times F(A') \xrightarrow{\alpha^{-1}} F(k[\epsilon] \times_k A') \xrightarrow{F(\gamma)} F(A' \times_A A') \longrightarrow F(A') \times_{F(A)} F(A')$$

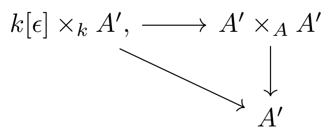
induced by the map

$$\begin{aligned} \gamma : k[\epsilon] \times_k A' &\to A' \times_A A' \\ (x + y\epsilon, a') &\mapsto (a' + yt, a') \end{aligned}$$

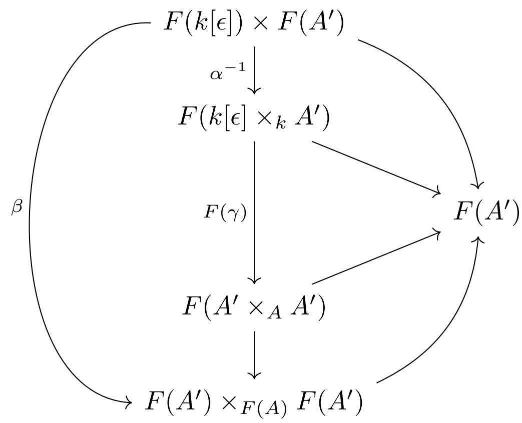
is surjective. If in addition F satisfying (H_4), β is bijection.

Proof. (H_2) gives that $\alpha : F(k[\epsilon] \times_k A') \to t_F \times F(A')$ is bijection and hence we have the inverse map α^{-1}. Since π is small extension, γ is bijection and so is the $F(\gamma)$. Finally, $F(A' \times_A A') \to F(A' \times_A A')$ is surjection by the (H_1). For the case that (H_4) is satisfied just notice that $F(A' \times_A A') \to F(A') \times_{F(A)} F(A')$ is bijection. □

Remark 2.4.7. *Furthermore, β induces a transitive group action of the vector space t_F on the set $F(\pi)^{-1}(\eta)$, where $\eta \in F(A)$. First notice the commutative diagram*



where the vertical arrow is the "right" projection. Hence the above diagram is commutative too.



i.e. if $v \in t_F$ and $\eta' \in F(A')$ then

$$\beta(v, \eta') = (\tau(v, \eta'), \eta').$$

The action is given by the map τ and it is transitive by the surjectivity of β. If in addition F satisfyies (H_4) the action is free.

Schlessinger's Theorem 2.4.8. *Let $F : \mathcal{C}_\Lambda \to Sets$ be a functor of Artin rings satisfying condition (H_0) (i.e. $F(k)$ is singleton). Let $A' \to A$ and $A'' \to A$ be homomorphisms in $\mathcal{C}_\Lambda$ and let*

$$\alpha : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'') \tag{2.4}$$

be the natural map. Then

- *(i) F has a semiuniversal formal element if and only if it satisfies the conditions: (H_1), (H_2) and (H_3)*
- *(ii) F has a universal element if and only if it also satisfies the additional condition (H_4).*

Proof. (*i*) Let's assume that F has a semiuniversal formal element $(R, \hat{u})$. Consider a homomorphism $f : A' \to A$ and a small extension $\pi : A'' \to A$, both in $\mathcal{C}_\Lambda$, and let

$$(\xi', \xi'') \in F(A') \times_{F(A)} F(A'')$$

such that

$$\xi := F(f)(\xi') = F(\pi)(\xi''),\ \xi \in F(A).$$

By the versality of $(R, \hat{u})$ the maps

$$h_R(A') \to F(A')$$

$$h_R(A'') \to h_R(A) \times_{F(A)} F(A'') \tag{2.5}$$

are surjections. Therefore there are

$$g' \in h_R(A') \text{ and } g'' \in h_R(A'')$$

such that $\hat{F}(g')(\hat{u}) = \xi'$ and $g'' \mapsto (fg', \xi'')$ under the map (2.5), i.e.

$$\pi g'' = fg' \text{ and } \hat{F}(g'')(\hat{u}) = \xi''.$$

Consequently

$$\begin{array}{ccc} h_R(A') & \longrightarrow & F(A') \\ \downarrow & & \downarrow \\ h_R(A) & \longrightarrow & F(A) \end{array} \qquad \begin{array}{ccc} g' & \longrightarrow & \xi' \\ \downarrow & & \downarrow \\ \pi g'' = fg' & \longrightarrow & \hat{F}(fg')(\hat{u}) = \xi \end{array}$$

$\hat{F}(\pi g'')(\hat{u}) = \xi$. Using now the morphism

$$g' \times g'' : R \to A' \times_A A''$$

we obtain an element $\zeta := \hat{F}(g' \times g'') \in F(A' \times_A A'')$, which by construction is $\alpha(\zeta) = (\xi', \xi'')$, where α is the map (2.4). This proves that the map α in (2.4) is surjection, i.e. $(R, \hat{u})$ satisfying (H_1).
If $A'' = k[\epsilon]$ and $A = k$, obviously $A'' \to A$ is a small extension and α in (2.4) surjective. Let $\zeta_1, \zeta_2 \in F(A' \times_k k[\epsilon])$ such that

$$\alpha(\zeta_1) = \alpha(\zeta_2) = (\xi', \xi''). \tag{2.6}$$

Since $(R, \hat{u})$ is semiuniversal

$$t_{R/\Lambda} = h_R(k[\epsilon]) \to F(k[\epsilon]) = t_F \tag{2.7}$$

is bijective. By smoothness applied to the projection $A' \times_k k[\epsilon] \to A'$,

$$h_R(A' \times_k k[\epsilon]) \to h_R(A') \times_{F(A')} F(A' \times_k k[\epsilon])$$

is surjective. Choose now g' as before, $\hat{F}(g')(\hat{u}) = \xi'$, since (2.6) both $(g', \zeta_1), (g', \zeta_2)$ belong to $h_R(A') \times_{F(A')} F(A' \times_k k[\epsilon])$. Hence we obtain two morphisms

$$g' \times g_i : R \to A' \times_k k[\epsilon], i = 1, 2$$

such that $\hat{F}(g' \times g_i)(\hat{u}) = \zeta_i$, $i = 1, 2$. It follows that $\hat{F}(g_i)(\hat{u}) = \xi''$, $i = 1, 2$. By the bijectivity of (2.7), $g_1 = g_2$ and hence $\zeta_1 = \zeta_2$, i.e. the map α in 2.4 is bijective, and so $(R, \hat{u})$ satisfying (H_2).
Condition (H_3) satisfied because the differential $t_{R/\Lambda} \to t_F$ is linear and is a bijection by semiuniversality.

Conversely, let's assume that F satisfies (H_1), (H_2), (H_3). We will construct a couple $(R, \hat{u})$ by a projective system

$$\{R_n \mid pr_{n+1} : R_{n+1} \to R_n\}_{n \geq 0}$$

of Λ-algebras in $\mathcal{C}_\Lambda$, a sequence $\{u_n \in F(R_n)\}_{n \geq 0}$ such that

$$F(pr_{n+1})(u_{n+1}) = u_n, \ n \geq 0,$$

and we will show that is a semiuniversal formal couple.
We take $R_0 = k$ and $u_0 \in F(k)$ the unique element. Let $r = \dim_k(t_F)$, $\{t_1, \dots, t_r\}$ a basis of t_F and $S = \Lambda[[T_1, \dots, T_r]]$ with maximal ideal $\mathfrak{m}_S$, we set

$$R_1 = S/(\mathfrak{m}_S^2 + \mathfrak{m}_\Lambda S).$$

Since we have $R_1 \cong k[\epsilon] \times_k \cdots \times_k k[\epsilon]$, r times, by (H_2) we deduce that $F(R_1) = t_F \times \cdots \times t_F$, (r times), hence there exists $u_1 \in F(R_1)$, which induces a bijection between $t_{R_2/\Lambda}$ and t_F. Suppose we have found (R_{q-1}, u_{q-1}), where $R_{q-1} = S/J_{q-1}$. In order to construct (R_q, u_q), we consider the family $\mathcal{I}$ of all ideal $J \subseteq S$ such that,

(a) $(\mathfrak{m}_S)J_{q-1} \subseteq J \subseteq J_{q-1}$
(b) there is $u \in F(S/J)$ with $u \mapsto u_{q-1}$ via the map $F(S/J) \to F(R_{q-1})$.

$\mathcal{I}$ is nonempty because $J_{q-1} \in \mathcal{I}$. We will choose J_q to be the minimal element of $\mathcal{I}$, therefore we need to prove that $\mathcal{I}$ has a minimal element. Since the set $\mathcal{I}$ corresponds to a collection of finite vector subspaces of $J_{q-1}/((\mathfrak{m}_S)J_{q-1})$, it suffices to show that $\mathcal{I}$ is closed with respect to finite intersections. Let $J, K \in \mathcal{I}$ and $K = I \cap J$. Clearly $J \cap K$ satisfies the condition (a). We may enlarge J, if necessary, so that $J + K = J_{q-1}$, without changing the intersection $J \cap K$. Then

$$S/(J \cap K) \to S/J \times_{R_q} S/K,$$

is an isomorphism. By (H_1) the map

$$\alpha : F(S/K) \to F(S/I) \times_{F(R_{q-1})} F(S/J)$$

is surjective (see Remark 2.1.3), therefore there exists $u \in F(S/K)$ such that $u \mapsto u_{q-1}$, i.e. $J \cap K$ satisfies condition (b) as well, hence $J \cap K \in \mathcal{I}$. We take $R_q = S/J_q$ and $u_q \in F(R_q)$ an element which is mapped to u_{q-1}. By induction we have constructed a formal couple $(R, \hat{u})$. We now show that is a semiuniversal formal couple for F. First notice that $t_F \cong t_R$ by choice of R_1. Therefore we only have to prove versality. If $\pi : A' \to A$ is a small extension, we will show that the map

$$\hat{u}_\pi : h_R(A') \to h_R(A) \times_{F(A)} F(A'),$$

is surjective. Let $(f, \xi') \in h_R(A) \times_{F(A)} F(A')$, i.e. $\hat{F}(f)(\hat{u}) = F(\pi)(\xi')$. We must find $f' \in h_R(A')$ such that $\hat{u}_\pi(f') = (f, \xi')$. Let's consider the commutative diagram

$$\begin{array}{ccc} h_R(k[\epsilon]) \times h_R(A') & \longrightarrow & t_F \times F(A') \\ \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} \\ h_R(A') \times_{h_R(A)} h_R(A') & \longrightarrow & F(A') \times_{F(A)} F(A') \end{array} \tag{2.8}$$

where β_1 is bijection and β_2 surjection by the lemma (2.4.6). Assume that we have f' satisfying condition $\pi f' = f$, then f' and ξ' have the same image in $F(A)$ and so, if $\eta' := \hat{F}(f')(\hat{u})$,

$$(\xi', \eta') \in F(A') \times_{F(A)} F(A').$$

Hence there is $v \in t_F$ such that $\beta_2(v, \eta') = (\xi', \eta')$. It follows that for some $f'' \in h_R(A')$.

$$\begin{array}{ccc} (v, f') & \longrightarrow & (v, \eta') \\ \downarrow & & \downarrow \\ (f'', f') & \longrightarrow & (\xi', \eta') \end{array}$$

Clearly $\pi f'' = f$, since f' and f'' have the same image in $h_R(A)$, hence

$$\hat{u}_\pi(f'') = (f, \xi').$$

It follows that it suffices to find $f' \in h_R(A')$ with $\pi f' = f$. Since A is Artin ring there is q such that f factor as

$$R \longrightarrow R_q \xrightarrow{f_q} A, \qquad f : R \to A$$

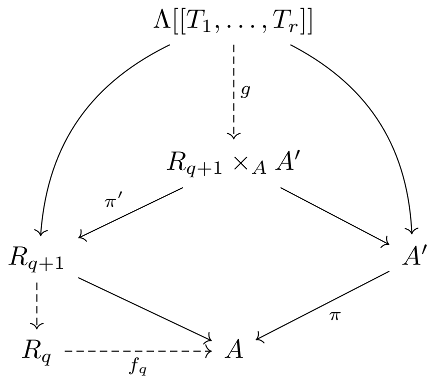
Then f' exists if and only if there exists ϕ which makes the following diagram commutative,

$$\begin{array}{ccc} R_{q+1} & \xrightarrow{\phi} & A' \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ R_q & \xrightarrow{f_q} & A \end{array} \tag{2.9}$$

In order to create a morphism $\Lambda[[x]] \to A'$, choose arbitrary $y_i \in \pi^{-1}(\rho(T_i))$ for each $i = 1, \ldots, r$, where ρ is the composition of maps

$$\Lambda[[T_1, \ldots, T_R]] \to R_{q+1} \text{ and } R_{q+1} \to A.$$

We get a morphism given by $T_i \mapsto y_i$. This morphism induce the commutative diagram



First notice that, since π is small extension, π' is small extension too. If π' is essential, then g must be surjective and hence

$$R_{q+1} \times_A A' \cong \Lambda[[T_1, \dots, T_r]]/I,$$

for some $I \subseteq \Lambda[[T_1, \dots, T_r]]$. Obviously $I \subseteq J_{q+1}$ and since π' is small extension $\mathfrak{m}_S J_{q+1} \subseteq I$. Moreover the map

$$F(R_{q+1} \times_A A') \to F(R_{q+1}) \times_{F(A)} F(A'),$$

is surjective by (H_1) and hence there is $u \in F(R_{q+1} \times_A A')$ inducing $u_{q+1} \in F(R_{q+1})$, which inducing $u_q \in F(R_q)$. it follows that $I \in \mathcal{I}$, and by the minimality of J_{q+1} in $\mathcal{I}$, $J_{q+1} \subseteq I$. But π' is a small extension and has non zero kernel, hence $I \subset J_{q+1}$ which is a contradiction.
So π' is not essential and by the lemma 2.1.4 (ii), π' has a section

$$s : R_{q+1} \to R_{q+1} \times_A A'.$$

It follows that the map $R_{q+1} \xrightarrow{s} R_{q+1} \times_A A' \to A'$ makes the diagram (2.9) commutative and proves that the $(R, \hat{u})$ is semiuniversal.

(ii) If F is pro-representable then, as already proved, satisfies conditions (H_1), (H_2), (H_3) and (H_4).
Conversely, suppose F satisfies (H_1) through (H_4). By the first part of the theorem we have that $(R, \hat{u})$ is a semiuniversal formal couple of F. We will prove that is universal by showing that for every A in $\mathcal{C}_\Lambda$ the map

$$h_R \to F(A)$$

induced by $\hat{u}$ is bijective. We will proceed by induction on $\dim_k(A)$. Let $\pi : A' \to A$ be a small extension. The inductive hypothesis gives, $h_R(A) \cong F(A)$. By the veraslity, the map

$$\hat{u}_\pi : h_R(A') \to h_R(A) \times_{F(A)} F(A') \cong F(A'),$$

is surjective. Assume $u'_1, u'_2 \in h_R$ such that

$$\hat{u}_\pi(u'_i) = \eta' \in F(A'), \ i = 1, 2. \tag{2.10}$$

and we will prove that $u'_1 = u'_2$. By (2.10) and the commutative diagram

$$\begin{array}{ccc} h_R(A') & \longrightarrow & F(A') \\ \downarrow & & \downarrow \\ h_R(A) & \xrightarrow{\cong} & F(A) \end{array}$$

it follows that both u'_1, u'_2 have the same image via the map $h_R(A') \to h_R(A)$. Hence there is $x \in t_R$ such that $\tau(x, u'_1) = u'_2$ (see 2.4.7) and clearly there is $y \in t_R$ such that $\tau(y, u'_1) = u'_1$. The pairs (x, u'_1) and (y, u'_2) fit in diagram (2.8) as follows,

$$\begin{array}{ccc} (x, u'_1) & \longrightarrow & (x, \eta') \\ \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} \\ (u'_2, u'_1) & \longrightarrow & (\eta', \eta') \end{array} \qquad \begin{array}{ccc} (y, u'_1) & \longrightarrow & (y, \eta') \\ \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} \\ (u'_1, u'_1) & \longrightarrow & (\eta', \eta') \end{array}$$

Note that $t_R \cong t_F$ by semiuniversality. The maps β_1, β_2 are both bijective by the lemma 2.4.6, consequently $x = y$ and finally $u'_1 = u'_2$.

□

Remark 2.4.9. *In other words, in Schlessinger's language, a functor of Artin rings such that $F(k)$ consists of a single element, has a hull if and only if satisfying (H_1), (H_2), (H_3). Furthermore F is pro-representable if and only if it also satisfying (H_4).*

Chapter 3

Examples

3.1 The Picard functor

We remind that, for a scheme X, $\text{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ the group of isomorphism classes of invertible sheaves on X. Now suppose X is a scheme over $\text{Spec}\Lambda$. For an A in $\mathcal{C}_\Lambda$ we define,

$$X_A := X \times_{\text{Spec}\Lambda} \text{Spec}A.$$

We fix $\eta_0 \in \text{Pic}(X_k)$ and let $\mathcal{P}(A)$ be the set of of those η in $\text{Pic}(A)$ such that $\eta \otimes_A k = \eta_0$. We claim that $\mathcal{P}$ is pro-representable under suitable conditions. We will first prove two lemmas on flatness, following Schlessinger.

Lemma 3.1.1. *Let A be a ring, J a nilpotent ideal in A and*

$$u : M \to N,$$

a homomorphism of A-modules, with N flat over A. If $\overline{u} : M/JM \xrightarrow{\cong} N/JN$ is an isomorphism, then f is also an isomorphsim.

Proof. Let $K = \text{coker } u$ and tensor the exact sequence

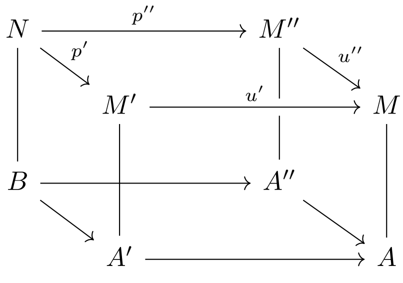
$$M \to N \to K \to 0,$$

with A/J. Then $K/JK = 0$ and Nakayama's lemma for nilpotent ideals implies that $K = 0$. Now let $K' = \ker u$ and tensor the exact sequence

$$0 \to K' \to M \to N \to 0$$

with A/J. By the flatness of N we get $K' = 0$, so that u is an isomorphism. □

Lemma 3.1.2. *Consider a commutative diagram*



of compatible ring and module homomorphsims, where

$$B = A' \times_A A'' \text{ and } N = M' \times_M M''.$$

Suppose

(i) M' *is free over* A' *and* M'' *is free over* A'',

(ii) $A''/J \xrightarrow{\cong} A$ *is an isomorphism, where* J *is a nilpotent ideal of* A'',

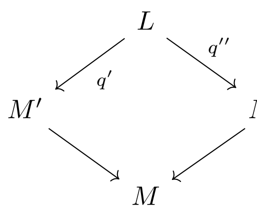
(iii) u' *induces an isomorphism* $M' \otimes_{A'} A \xrightarrow{\cong} M$, *and similarly for* u''.

Then N *is flat over* B *and* p' *induces an isomorphsim* $N \otimes_B A' \xrightarrow{\cong} M'$, *respectively* p'' *an isomorphism* $N \otimes_B A'' \xrightarrow{\cong} M''$.

Remark 3.1.3. *Over an Artin local ring, flat modules are free (Lemma A.0.5), so the lemma above it suffices for our purposes.*

Proof. First choose a basis $\{x_i\}_{i\in I}$ for M'. We can now see, using the (iii), that $\{u'(x_i)\}_{i\in I}$ form a basis for M, and so is free. Choosing $x_i'' \in (u'')^{-1}(u'(x_i))$, i.e. $u''(x_i'') = u'(x_i')$, we get a homomorphism $\sum A''x_i'' \to M''$ of A''-modules, whose reduction modulo the ideal J is an isomorphism. By the Lemma 3.1.1 it follows that M'' is free on generators x_i''. Finally it is easily to check that N is free on generators $x_i' \times x_i''$, and that the projections on the factors induce isomorphsims. ☐

Corollary 3.1.4. *With the same notations as above, let* L *be a* B*-module with a commutative diagram*



where q' *induces* $L \otimes_B A' \xrightarrow{\cong} M'$. *Then the canonical morphism* $q' \times q''$ *is an isomorphism.*

Let A and B be two rings, we call a homomorphism $\phi : A \to B$ flat, if ϕ makes the B a flat A-module. Let X and Y be two schemes and $f : X \to Y$ a morphism of schemes. We say that f is flat if the induced homomorphism on every stalk is a flat homomorphism. If X is a scheme over $\mathrm{Spec}\Lambda$ we say that X is flat over Λ if the morphism between them is flat.

Proposition 3.1.5. *Let* X *be a scheme over* $\mathrm{Spec}\Lambda$ *and assume that*

(i) X *is flat over* Λ,

(ii) $A \xrightarrow{\cong} H^0(X_A, \mathcal{O}_{X_A})$ *is isomorphism for each* A *in* $\mathcal{C}_\Lambda$,

(iii) $\dim_k H^1(X_k, \mathcal{O}_{X_k}) < \infty$.

Then $\mathcal{P}$ is pro-representable by a pro-couple (R, ξ).

Proof. Let $u' : (A', \eta') \to (A, \eta)$, $u'' : (A'', \eta'') \to (A, \eta)$ be morphisms of couples, where u'' is a surjection. Let L, L', L'' be corresponding invertible sheaves on $Y = X_A, X' = X_A$ and $X'' = X_A$. Then we have morphisms

$$p' : L' \to L \text{ and } p'' : L'' \to L, \tag{3.1}$$

of sheaves on the sp(X_0), compatible with $\mathcal{O}_{X'} \to \mathcal{O}_Y$, $\mathcal{O}_{X''} \to \mathcal{O}_Y$. The morphsims in (3.1) induce isomorphisms

$$L' \otimes_{A'} A \xrightarrow{\cong} L \text{ and } L'' \otimes_{A'} A \xrightarrow{\cong} L.$$

Let $B = A' \times_A A''$ and let $Z = X_B$, then we have a commutative diagram

$$\begin{array}{ccccc} & & \mathcal{O}_Z & & \\ & \swarrow & & \searrow & \\ \mathcal{O}_{X'} & & & & \mathcal{O}_{X''} \\ & \searrow & & \swarrow & \\ & & \mathcal{O}_Y & & \end{array}$$

of sheaves on sp(X_0). Thus there is a canonical isomorphism

$$\mathcal{O}_Z \xrightarrow{\cong} \mathcal{O}_{X'} \times_{\mathcal{O}_X} \mathcal{O}''_X.$$

Hence $N = L' \times_L L''$ is a sheaf on Z which is invertible, and by the Lemma 3.1.2, the projections of N on L' and L'' induce isomorphisms

$$N \otimes_B A' \xrightarrow{\cong} L' \text{ and } N \otimes_B A'' \xrightarrow{\cong} L''.$$

If now M is another invertible sheaf on Z for which there exist isomorphisms

$$M \otimes_B A' \xrightarrow{\cong} L' \text{ and } M \otimes_B A'' \xrightarrow{\cong} L'',$$

we have morphisms $q' : M \to L'$ and $q'' : M \to L''$, which induce the isomorphisms and thus a commutative diagram

$$\begin{array}{ccccccc} & & & M & & & \\ & & {}^{q'}\swarrow & & \searrow^{q''} & & \\ L' & & & & & & L'' \\ & {}_{p'}\searrow & & & & \swarrow_{p''} & \\ & & L & \xrightarrow{\theta} & L & & \end{array}$$

Where θ is the automorphism of L given by the composition

$$L \xrightarrow{\cong} L' \otimes_{A'} A \xrightarrow{\cong} M \otimes_B A \xrightarrow{\cong} L'' \otimes_{A'} A \xrightarrow{\cong} L.$$

By hypothesis (ii), θ is multiplication by some unit $a \in A$. Lifting a back to a'' in A'', we can take $a''q''$ instead of q'', so the diagram changing to

$$\begin{array}{ccccc} & & M & & \\ & \swarrow & & \searrow & \\ L' & & & & L'' \\ & \searrow & & \swarrow & \\ & & L & & \end{array}$$

It follows that $M \xrightarrow{\cong} N$ is an isomorphism. We have therefore proved that $\mathcal{P}(A' \times_A A'') \cong \mathcal{P}(A') \times_{\mathcal{P}(A)} \mathcal{P}(A'')$, for any surjection $A'' \to A$ in $\mathcal{C}$.
Finally, if $Y = X_{k[\epsilon]}$, we have $\mathcal{O}_Y = \mathcal{O}_{X_0} \oplus \epsilon\mathcal{O}_{X_0}$, so there is an exact sequence

$$0 \to \mathcal{O}_{X_0} \to \mathcal{O}_Y^* \to \mathcal{O}_{X_0}^* \to 1$$

where the morphism $\mathcal{O}_{X_0} \to \mathcal{O}_Y^*$ maps $f \mapsto 1 + \epsilon f$. Hence

$$\mathcal{P}(k[\epsilon]) \cong \ker\left(H^1(X_0, \mathcal{O}_Y^*) \to H^1(X_0, \mathcal{O}_{X_0}^*)\right) \cong H^1(X_0, \mathcal{O}_{X_0})$$

which has finite dimension by (iii). □

3.2 Deformations of curves

A deformation of a smooth curve X over the spectrum of a local ring $\mathrm{Spec}(R)$ is a proper flat morphism $\phi : \mathcal{X} \to \mathrm{Spec}(R)$ together with an isomorphism of X with the scheme theoretic fiber of $\mathcal{X}$ over the maximal ideal $\mathfrak{m}$ of R, that is

$$X \cong \mathcal{X}_0 = \mathcal{X} \otimes_{\mathrm{Spec}R} \mathrm{Spec}(R/\mathfrak{m}).$$

Definition 3.2.1. *A morphism of finite type $\phi : X \to S$ between Noetherian schemes is **proper** when for every discrete valuation ring R with fraction field k and every square of morphisms*

$$\begin{array}{ccc} \mathrm{Spec}(k) & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ \mathrm{Spec}(R) & \longrightarrow & S \end{array}$$

there is a unique morphism $\mathrm{Spec}(R) \to X$ *fitting into the diagram.*

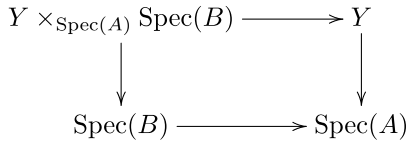
We can consider the deformation functor Def_X of curves with automorphisms from the category of local Artin algebras to the category of sets:

$$\mathrm{Def}_X(A) = \{\text{deformations of } X \text{ over } A/\text{isomorphisms}\},$$

where two deformations $\mathcal{X}_i \to \mathrm{Spec}A$, $i = 1, 2$ are considered to be isomorphic if they fit in a commutative diagramm

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow[\cong]{\phi} & \mathcal{X}_2 \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}A & \end{array}$$

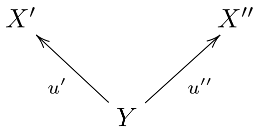
Given a deformation $Y \to \mathrm{Spec}(A)$ and a morphism $A \to B$, then we can define the induced deformation $Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \to \mathrm{Spec}(B)$ in terms of the commutative diagram



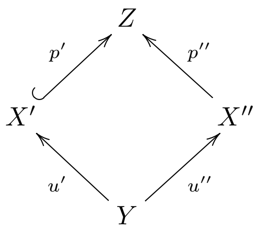
In this way a notion of morphisms of deformations can be defined.

Theorem 3.2.2. *For any curve X the functor Def_X satisfy H_1, H_2, H_3, H_4 of Schlessinger theorem.*

Proof. Consider the morphisms of couples $(A', \eta') \to (A, \eta)$ and $(A'', \eta'') \to (A, \eta)$, where $A'' \to A$ is a surjection. Let X', Y, X'' be deformations in the equivalence class of η', η, η'' respectively and consider the diagram



Then there is a prescheme Z, flat over $A' \times_A A''$, the sum of X' and X'' under Y, in the category of preschemes. The closed immersions $X \to Y \to Z$ give Z a structure of deformation of X over $A' \times_A A''$ such that the following commutative diagram of deformations

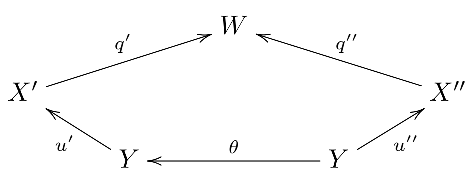


This proves that

$$\mathrm{Def}_X(A' \times_A A'') \to \mathrm{Def}_X(A') \times_{\mathrm{Def}_X(A)} \mathrm{Def}_X(A'')$$

is surjective, for every surjection $A'' \to A$. Therefore the condition H_1 is satisfied.

Suppose that W is a deformation over B, inducing the deformations X' and X''. There is a commutative diagram of deformations,



where θ is the composition

$$\begin{array}{ccccc} Y & & \xrightarrow{\quad\quad\quad\theta\quad\quad\quad} & & Y \\ \downarrow\cong & & & & \cong\uparrow \\ X' \otimes_{\mathrm{Spec}(A')} \mathrm{Spec}(A) & \xrightarrow{\cong} & W \otimes_{\mathrm{Spec}(B)} \mathrm{Spec}(A) & \longrightarrow & X'' \otimes_{\mathrm{Spec}(A')} \mathrm{Spec}(A) \end{array}$$

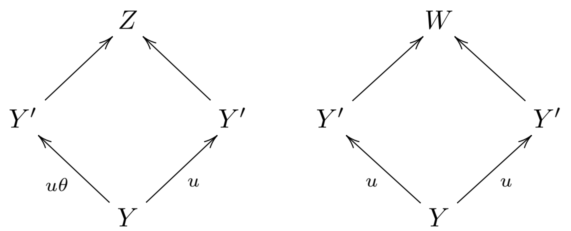
If θ can be lifted to an automorphism of θ' of X', such that $\theta' u' = u'\theta$ then q' can be replaced with $q'\theta'$ and then $W \overset{\cong}{\to} Z$. For the special case $A = k$, $Y = X$, $\theta = \mathrm{Id}$ this lifting θ' exists and condition (H_2) is satisfied.

For the condition H_4: consider a morphism of couples $p : (A', \eta') \to (A, \eta)$, where p is a small extension. For a morphism $B \to A$, let $\mathrm{Def}_X^{\eta}(B)$ denote the set of $\zeta \in \mathrm{Def}_X(B)$ such that $\zeta \otimes_B A = \eta$. Select a deformation Y' in the class of η'. We will prove that the following are equivalent:

(i) $\mathrm{Def}_X^{\eta}(A' \times_A A') \xrightarrow{\cong} \mathrm{Def}_X^{\eta}(A') \times \mathrm{Def}_X^{\eta}(A')$

(ii) Every automorphism of the deformation $Y = Y' \otimes_{A'} A$ is induced by an automorphism of the deformation Y'.

We first prove that $(i) \Rightarrow (ii)$. Consider the induced morphism of deformations $u : Y \to Y'$. If θ is an automorphism of Y, then we can construct deformations Z, W over $A' \times_A A'$ to give "sum diagrams" of deformations.



The deformations Z, W have isomorphic projections on both factors, there is an isomorphism $\rho : Z \xrightarrow{\cong} W$, which induces automorphisms θ_1, θ_2 of Y' and an automorphism ϕ of Y such that

$$\theta_1 u\theta = u\phi, \;\; \theta_2 u = u\phi.$$

Therefore, $u\theta = \theta_1^{-1}\theta_2 u$ and $\theta_1^{-1}\theta_2$ induces θ.

Now we will prove $(ii) \Rightarrow (i)$. From (ii) for $I = \ker p$ follows that $t_F \otimes I$ acts freely on η', that is $(\eta')^{\sigma} = \eta'$ implies $\sigma = 0$. Since the action of $t_F \otimes I$ on $\mathrm{Def}_X^{\eta}(A')$ is transitive, the space $\mathrm{Def}_X^{\eta}(A')$ is a principal homogeneous space under $t_F \otimes I$, which is equivalent to (i).

We will now prove the finiteness condition H_3. Since X is smooth over k one can prove using Chech cohomology [4] that

$$t_{\mathrm{Def}_X} \cong H^1(X, \Theta),$$

where Θ is the tangent sheaf of the curve X and by Serre-Duality and Riemann-Roch theorem has dimension equal to

$$\dim_k H^1(X, \Theta) = \dim_k H^0(X, \Omega^{\otimes 2}) = 3(g-1).$$

□

Appendix A

Flat Modules

Definition A.0.1. *An R-module M is called* ***flat*** *if for every short exact sequece*

$$0 \to N_1 \to N_2 \to N_3 \to 0,$$

the induced sequence

$$0 \to M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0,$$

is also exact.

We call a functor F between categories of modules **left exact** if for every exact sequence $0 \to N_1 \to N_2 \to N_3$ the sequence $0 \to F(N_1) \to F(N_2) \to F(N_3)$, similarly right exact and **exact** when is both right and left exact. So the above definition is that the functor F_M which sends an R-module N to the R-module $F_M(N) = M \otimes_R N$ is exact. Since the F_M is always right exact (for a proof see [1]), flatness is actually that F_M is left exact, i.e. a way to say that for every injection $N_1 \to N_2$ the induced homomorphism $M \otimes_R N_1 \to M \otimes_R N_2$ is an injection.

Remark A.0.2. (*i*) *Note that if M, N are two R-modules and S an R-submodule of N, in general $M \otimes_R S$ is not a submodule of $M \otimes_R N$. You can check this for example by taking $M = \mathbb{Z}/2\mathbb{Z}$, $N = \mathbb{Q}$ and $S = \mathbb{Z}$.*

(*ii*) *Similarly, if $\phi : N \to N'$ is an R-module homomorphism, we can guarantee a surjection*

$$\text{Im Id} \otimes_R \text{Im}\phi \to \text{Im}(\text{Id} \otimes \psi) \subseteq M' \otimes_R N'$$

but not always a bijection. So in general we cannot identify $\text{Im Id} \otimes_R \text{Im}\phi$ *with* $\text{Im}(\text{Id} \otimes \phi)$.

Nevertheless if we require that F_M maps injections to injections, i.e. $M \otimes_R S \to M \otimes_R N$ is an injection whenever $S \to N$ is an injection, it is immediate that we do not have "strange" situations as above.

Proposition A.0.3. *An R-module M is flat if and only if for every injection $N \to N'$ the map $M \otimes_R N \to M \otimes_R N'$ is an injection.*

Proof. The "only if" is obvious. For the "if" let

$$0 \longrightarrow N_1 \xrightarrow{u_1} N_2 \xrightarrow{u_2} N_3,$$

be an exact sequence. First notice that u_1 is monomorphism and hence the induced homomorphism $\mathrm{Id} \otimes u_1 : M \otimes_R N_1 \to M \otimes_R N_2$ is an injection. Thus it remains to show that $\mathrm{Im}(\mathrm{Id} \otimes u_1) = \ker(\mathrm{Id} \otimes u_2)$. Let $m \otimes u_1(a)$ be an arbitrary element of $\mathrm{Im}(\mathrm{Id} \otimes u_1)$, where $a \in N_1$, then

$$(\mathrm{Id} \otimes u_2)(m \otimes u_2(a)) = m \otimes u_2(u_1(a)) = m \otimes 0,$$

i.e. $m \otimes u_1(a)$ is in $\ker(\mathrm{Id} \otimes u_2)$. Using again the hypothesis we conclude that

$$M \otimes_R (N_2/\ker(u_2)) \to M \otimes_R N_3,$$

is an injection. Since $M \otimes_R (N_2/\ker(u_2)) \cong (M \otimes_R N_2) / (M \otimes_R \ker(u_2))$, it follows that $\ker(\mathrm{Id} \otimes u_2) \subseteq M \otimes_R \ker(u_2) = M \otimes_R \mathrm{Im}(u_1) = \mathrm{Im}(\mathrm{Id} \otimes u_1)$. □

Example A.0.4. (*i*) *Free modules are flat. Indeed suppose M is a free R-module, i.e. $M \cong \bigoplus_{i \in I} R$. Let $N \to N'$ be a monomorphism of R-modules and we want to prove that $M \otimes_R N \to M \otimes_R N'$ Note first that $M \otimes N = \bigoplus_{i \in I} R \otimes_R N = \bigoplus_{i \in I} N$, so we want to prove that*

$$\bigoplus_{i \in I} N \to \bigoplus_{i \in I} N'$$

is monomorphism, but this is clear when $N \to N'$ is monomorphism.

(*ii*) *Projective modules are flat. Indeed let P be a projective module and recall that tensor products commute with direct sums. It follows that a module is flat if and only if each summand is flat. Since any projective module is a direct summand of a free module (you can check this immediate using the universal property of projective modules) every projective module is flat.*

Lemma A.0.5. *Let R be an Artin local ring and let M be an R-module. Then M is flat over R if and only if R is a free R-module.*

Proof. Assume that M is a flat module. Since $M/\mathfrak{m}M$ is an $R/\mathfrak{m}$-module, i.e. a vector space, we can choose $m_i \in M$ for all $i \in I$, such that the elements $\overline{m}_i \in M/\mathfrak{m}M$ forms a basis over the residue field. Let $F = \bigoplus_{i \in I} R$ a free R-module. It is clear that the induced homomorphism $M/\mathfrak{m}M \to F/\mathfrak{m}F$ is a bijection. Finally using that R is Artin ring we conclude that $\mathfrak{m}$ is nilpotent and Lemma 3.1.1 completes the proof. □

Bibliography

[1] M. Atiyah, I. Macdonald, *Introduction to Commutative Algebra*, Addison Wesley, 1969.

[2] G. Cornell, J. H. Silverman, G. Stevens, *Modular Forms and Fermatb••s Last Theorem*, Springer-Verlag, New York, 1997.

[3] R. Hartshorne, *Deformation Theory*, Springer-Verlag, New York, 2010.

[4] J. Harris Ian Morisson, *Moduli of curves*, Springer Verlag 1998 GTM 187.

[5] H. Matsumura, *Commutative Algebra*, W. A. Benjamin Co., New York, 1970.

[6] N. Nitsure, *Notes on Deformation Theory*, Guanajuato, 2006.

[7] M. Schlessinger, Functors of Artin rings, *Transactions AMS* 130 (1968), 208-222.

[8] E. Sernesi, *Deformations of Algebraic Schemes*, Springer-Verlag, Berlin Heidelberg, 2006.