AUTOMORPHISMS OF CURVES AND WEIERSTRASS SEMIGROUPS

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Abstract. The relation of the Weierstrass semigroup with several invariants of a curve is studied. For Galois covers of curves with group $G$ we introduce a new filtration of the decomposition subgroup of $G$. The relation to the ramification filtration is investigated in the case of cyclic covers. We relate our results to invariants defined by Boseck and we study the one point ramification case. We also give applications to Hasse-Witt invariant and symmetric semigroups.

1. Introduction

Let $X$ be a projective nonsingular curve of genus $g \geq 2$ defined over an algebraic closed field $k$ of characteristic $p > 0$. Let $G$ be a subgroup of the automorphism group $\text{Aut}(X)$ of $X$ and let $G(P)$ be the subgroup of automorphisms stabilizing a point $P$ on $X$. Consider also the Weierstrass semigroup attached at the point $P$. In characteristic zero there are results [29] relating the structure of the Weierstrass semigroup at $P$ to the subgroup $G(P)$. In this paper we will try to prove analogous results in the positive characteristic case. It turns out that there is a close connection of the automorphism group, the Weierstrass semigroup and other invariants of the curve like the rank of the Hasse-Witt matrix. Aim of this article is to show how various invariants of a curve are encoded in the Weierstrass semigroup.

Section 2 is devoted to connections of the theory of decomposition groups $G_i(P)$ to the theory of Weierstrass semigroups. Both the theory of the Weierstrass semigroups and the decomposition groups $G_i(P)$ are more difficult when $p > 0$. In characteristic zero it is known that $G_i(P)$ is always a cyclic group, while when $p > 0$ and $p$ divides $|G_i(P)|$ the group $G_i(P)$ is no more cyclic and admits the following ramification filtration:

$$
G(P) = G_0(P) \supset G_1(P) \supset G_2(P) \supset \ldots ,
$$

Recall that the groups $G_i(P)$ are defined as $G_i(P) = \{ \sigma \in G(P) : v_P(\sigma(t) - t) \geq i + 1 \}$, for a local uniformizer $t$ at $P$ and $v_P$ is the corresponding valuation. Notice that $G_1(P)$ is the $p$-part of $G(P)$. For every point $P$ of the curve $X$ we consider the sequence of $k$-vector spaces

$$
L = L(0) = L(P) = \cdots = L((i - 1)P) < L(iP) \leq \cdots \leq L((2g - 1)P),
$$

where

$$
L(iP) := \{ f \in k(X)^* : \text{div}(f) + iP \geq 0 \} \cup \{0\}.
$$

We will write $\ell(D) = \dim_k L(D)$. An integer $i$ will be called a pole number if there is a function $f \in k(X)^*$ so that $\text{div}_\infty(f) = iP$ or equivalently $\ell((i - 1)P) + 1 = \ell(iP)$. The set of pole numbers at $P$ form a semigroup $H(P)$ which is called the Weierstrass semigroup at $P$. It is known that there are exactly $g$ pole numbers that are smaller or equal to $2g - 1$ and that every integer $i \geq 2g - 1$ is in the Weierstrass semigroup, see [40, I.6.7].
Remark 1. In characteristic zero for all but finite points (the so called Weierstass points), the generic situation for the gaps at $P$ is the set $\{1, \ldots, g\}$. This is not correct in positive characteristic, where the generic set of gaps might be different than the set $\{1, \ldots, g\}$.

Remark 2. The Weierstrass semigroup seems to be defined locally at a given point $P$. But the condition on the existence of function with only pole at $P$ is a global condition and lot of global invariants like the genus or the Hasse-Witt matrix [41] are encoded in it.

In section 2.1 we recall the relation of the representation of the $p$-part of the decomposition group into the Riemann-Roch spaces introduced in [20]. In section 2.2 we introduce the representation filtration that shares many properties with the ramification filtration and affects the structure of the Weierstrass semigroup. In propositions 21 and 22 we characterize the jumps of the representation filtration in terms of the Weierstrass semigroup and in terms of the intermediate field extensions. Valentini and Madan [46, lemma 2] did a similar computation for the gaps of the ramification filtration for cyclic groups using the theory of Witt vectors, see [45],[35].

Section 2.3 is devoted to the description of the Weierstrass semigroup when $G_1(P)$ is a cyclic group. This is an important case to understand, because of its simplicity and because of Oort conjecture regarding the deformation of a curve with automorphisms from characteristic $p$ to characteristic 0. These results should be seen as the characteristic $p$ analogon of the work of Morrison-Pinkham [29] on Weierstrass semigroups for Galois Weierstrass points.

In section 2.4 we still study the semigroups that can appear in the cyclic group case but using the Boseck invariants. These are invariants that were first introduced by Boseck in [4] in the study of basis of holomorphic differentials of curves that are cyclic extensions of the rational function fields. Boseck in his very important and interesting article studied the Weierstrass semigroups in this case. Unfortunately, he didn’t notice that there is one case that needs more attention, namely the case of a cover where only one point is ramified and the prime number is “small”. What we mean by small will be clear in the sequel. For example for the case of Artin-Schreier curves small means that the characteristic is smaller than the conductor. The same problem is now in several places in the literature like [6, p.235 remark (i)], [44, p. 170 in first 3 lines], [7, p. 3028, remark (i)]. For the special case of Artin-Schreier extension the correct treatment is the one in lemma 40.

The curves that appear as covers of the projective line with only one ramified point are very interesting. This is because curves with large automorphisms groups appear this way [38] and because of the Katz-Gabber compactification of local actions [18, Thm 1.4.1],[12, Cor. 1.9].

In section 3 we study the maximal gap in the semigroup and we give bounds for it, in terms of several invariants of the curve. When this maximal gap equals to $2g - 1$ at some point $P$, then the Weierstrass semigroup is called symmetric and that point is a Weierstrass point, see proposition 50. In proposition 50 we show that the existence of a single point in the curve that has a symmetric Weierstrass semigroup determines the genus of the curve. Many well known examples of curves enjoy that property. In particular, in lemmata 44 and 45, we determine the cyclic totally ramified coverings of the rational function field with this symmetric Weierstrass semigroups.

Section 4 is devoted to the dependence of the Cartier operator and the Hasse-Witt invariant to the theory of Weierstrass semigroups. We give some results about big actions, see lemma 18, corollary 56 and characterize some non classical curves with respect to the canonical linear series and nilpotent Cartier operator, see corollary 57. We also characterize all the maximal curves over $\mathbb{F}_{q^2}$ with two generators for the Weierstrass semigroup at a $\mathbb{F}_{q^2}$-rational point, as curves with symmetric Weierstrass semigroups at this point, see remark 59. Finally we study further the connection between symmetric Weierstrass semigroups and maximal curves.
2. Decomposition Groups $G_i(P)$

2.1. Ramification Groups.

**Proposition 3.** If $g \geq 2$ and $p \neq 2, 3$ then there is at least one pole number $m \leq 2g - 1$ not divisible by the characteristic $p$. Let $1 < m \leq 2g - 1$ be the smallest pole number not divisible by the characteristic. There is a faithful representation

$$\rho : G_1(P) \to \text{GL}(L(mP))$$

**Proof.** [20, lemmata 2.1.2.2] \[ \square \]

**Proposition 4.** A basis for the vector space $L(mP)$ is given by

$$\left\{1, \frac{u_i}{t^m}, \frac{1}{t^m} : \text{where } 1 < i < r, p \mid m_i \text{ and } u_i \text{ are certain units} \right\}.$$

With respect to this basis, an element $\sigma \in G_1(P)$ acts on $1/t^m$ by

$$\sigma \frac{1}{t^m} = \frac{1}{t^m} + \sum_{i=1}^{r} c_i(\sigma) \frac{u_i}{t^m},$$

The action on the local uniformizer is given by

$$\sigma(t) = t - \frac{1}{m} \sum_{i=1}^{r} c_i(\sigma) u_i t^{m_i-1} + \cdots,$$

and this implies

**Proposition 5.** Let $P$ be a wild ramified point on the curve $X$ and let

$$\rho : G_1(P) \to \text{GL}_r(k)$$

be the corresponding faithful representation we considered in Lemma 3. Let $m = m_r > m_{r-1} > \cdots > m_0 = 0$ be the pole numbers at $P$ that are $\leq m$. If $G_i(P) > G_{i+1}(P)$ then $i = m - m_k$, for some pole number $m_k$.

**Remark 6.** Notice that in this article we enumerate the pole numbers in terms of an increasing function $i \mapsto m_i$. In [20] the enumeration is in terms of a decreasing function.

2.2. Jumps in the ramification filtration and divisibility of the Weierstrass semigroup.

Consider a Galois cover $\pi : X \to Y = X/G$ of algebraic curves, and let $P$ be a fully ramified point of $X$. How are the Weierstrass semigroup sequences of $P$, and $\pi(P)$ related?

**Lemma 7.** Let $k(X), k(Y) = k(X)^G$ denote the function fields of the curves $X$ and $Y$ respectively. The morphisms

$$N_G : k(X) \to k(Y) \text{ and } \pi^* : k(Y) \to k(X),$$

sending $f \in k(X)$ to $N_G(f) = \prod_{g \in G} f$ and $g \in k(Y)$ to $\pi^* g \in k(X)$ respectively, induce injections

$$N_G : H(P) \to H(Q) \text{ and } \pi^* : H(Q) \xrightarrow{\times |G|} H(P),$$

where $Q := \pi(P)$.

**Proof.** For every element $f \in k(X)$ such that $(f)_\infty = mP$, the element $N_G(f)$ is a $G$-invariant element, so it is in $k(Y)$. Moreover, the pole order of $N_G(f)$ seen as a function on $k(X)$ is $|G| \cdot m$. But since $P$ is fully ramified the valuation of $N_G(f)$ expressed in terms of the local uniformizer at $\pi(P)$ is just $-m$.

On the other hand side an element $g \in k(Y)$ seen as an element of $k(X)$ by considering the pullback $\pi^*(g)$ has for the same reason valuation at $P$ multiplied by the order of $G$. \[ \square \]
Remark 8. The condition of fully ramification is necessary in the above lemma. Indeed, if a point \( Q \in Y \) has more than one elements in \( \pi^{-1}(Q) \) then the pullback of \( g \), such that \( (g)_\infty = mQ \), is supported on \( \pi^{-1}(Q) \) and gives no information for the Weierstrass semigroup of any of the points \( P \in \pi^{-1}(Q) \).

We will prove the following:

Lemma 9. If an element \( f \) such that \( (f)_\infty = aP \) is invariant under the action of a subgroup \( H < G_1(P) \), then \( |H| \) divides \( a \).

Proof. Write \( f = u/t^a \) in terms of a local uniformizer \( t \) at \( P \) and a unit \( u \). Since \( f \) is invariant it is the pullback of a function \( g \in k(X/H) \). If \( t' \) is a local uniformizer at \( Q = \pi(P) \) then \( g \) is expressed as \( g = u'/t^b \). Since \( t' = t^{H|v} \) for some unit \( v \), \[29\] 2.2c, the desired result follows. \( \square \)

Definition 10. Let \( 0 = m_0 < m_1 < m_2 < \ldots < m_r \) be the sequence of pole numbers up to \( m_r \). For each \( 0 \leq i \leq r \) we consider the representations

\[ \rho_i : G_1(P) \to GL(L(m_iP)) \]

We form the decreasing sequence of groups:

\[ G_1(P) = \ker \rho_0 \supset \ker \rho_1 \supset \ker \rho_2 \supset \cdots \supset \ker \rho_r = \{1\}. \]

We will call this sequence of groups “the representation” filtration.

Let \( \sigma \in \ker \rho_i \). Then \( \rho_{i+1}(\sigma) \) has the following form

\[ \rho_{i+1}(\sigma) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ a_{i+1,1}(\sigma) & a_{i+1,2}(\sigma) & \cdots & a_{i+1,\nu}(\sigma) & 1 \end{pmatrix} \]

Observe also that all functions \( a_{i+1,\nu} : \ker \rho_i \to k \) are group homomorphisms into the additive group of the field \( k \). Notice that

\[ \ker \rho_{i+1} = \ker \rho_i \cap \bigcap_{\nu=i+1}^r \ker a_{i+1,\nu}. \]

Lemma 11. The linear series \( |m_1P| = \mathbb{P}(L(m_1P)) \) defines a map \( X \to \mathbb{P}^1 \) of degree \( m_1 \). This gives rise to an algebraic extension \( F/k(f_1) \) of degree \( m_1 \), where \( (f_1)_\infty = m_1P \). Let \( F \) denote the function field of the curve \( X \). The extension \( F/k(f_1) \) is Galois if and only if \( m_1 = |\ker \rho_1| \), and in this case the Galois group is \( \ker \rho_1 \).

Proof. Observe that \( F \supseteq F^{\ker \rho_1} \supseteq k(f_1) \). Notice that the degree of the extension \( F/k(f_1) \) is \( m_1 \) therefore if \( |\ker \rho_1| = m_1 \) then \( k(f_1) = F^{\ker \rho_1} \).

Assume now that the extension \( F/k(f_1) \) is Galois with Galois group \( A \). Since \( F^{\ker \rho_1} \supseteq k(f_1) = F^A \) we have that \( A \supseteq \ker \rho_1 \). By definition \( A = \{ \sigma \in G_1(P) : \sigma(f_1) = f_1 \} \subseteq \ker \rho_1 \). \( \square \)

Remark 12. In characteristic zero a point \( P \) such that the covering map \( X \to \mathbb{P}^1 \) introduced by the linear system \( |m_1P| \) is Galois, is called Galois Weierstrass point. The Weierstrass semigroup at a Galois Weierstrass point in characteristic zero is now well understood, \[29\]. Notice that in positive characteristic the condition \( m_1 \leq g \) does not imply that \( P \) is a Weierstrass point, since the canonical linear system might be non classical. In section 2.3 we will restrict ourselves to cyclic totally ramified extensions of order \( p^n \), \( n \geq 2 \) and then \( P \) is a Weierstrass point according to theorem 2 in \[45\].

Lemma 9 can be generalized as follows:
Proposition 13. The order $|\ker \rho_i|$ of the group $\ker \rho_i$ divides $m_\nu$, for all $\nu \leq i$.

Proof. This comes out from lemma 9 since the elements $f_i$ such that $(f_i)_\infty = m_1P$ are by definition $\ker \rho_i$-invariant.

We recall the following definition:

Definition 14. Given a finite group $G$ the Frattini subgroup $\Phi(G)$ is defined to be the intersection of all proper maximal subgroups of $G$. If $G$ is a $p$-group then the Frattini group is $\Phi(G) = G^p[G,G]$, and is the minimal normal subgroup $N$ of $G$ such that $G/N$ is elementary abelian.

Consider the first pole number $m_1$ in the Weierstrass semigroup and a function $f_1$ such that $(f_1)_\infty = m_1P$. The image of the representation

$$\rho : G_\infty(P) \to \text{GL}(L(m_1P))$$

is an elementary abelian group since the representation is $2$-dimensional, therefore the commutator and the Frattini subgroup of $G_\infty(P)$ are in the kernel of $\rho$. This combined with lemma 9 gives us the following

Lemma 15. The order $|\Phi(G)|$ of $\Phi(G)$ divides the first pole number.

Example 16. In the work of C. Lehr, M. Matignon [23] a notion of big action is defined. Curves having a big action were studied further by M. Rocher and M. Matignon [28],[33].

A curve $X$ together with a subgroup $G$ of the automorphism group of $X$ is called a big-action if $G$ is a $p$-group and

$$\frac{|G|}{g} > \frac{2p}{p-1}.$$ 

All big actions have the following property [23]:

Proposition 17. Assume that $(X, G)$ is a big action. There is a unique point $P$ of $X$ such that $G_\infty(P) = G$, the group $G_\infty(P)$ is not trivial and strictly contained in $G_\infty$ and the quotient $X/G_\infty(P) \cong \mathbb{P}^1$. Moreover, the group $G$ is an extension of groups

$$0 \to G_\infty(P) \to G = G_\infty(P) \xrightarrow{\pi} (\mathbb{Z}/p\mathbb{Z})^1 \to 0.$$ 

Since the curve $X/G_\infty(P)$ is of genus zero, the Weierstrass semigroup equals the semigroup of natural numbers, therefore $|G_\infty(P)| \cdot \mathbb{Z}_+$ is a subsemigroup of the Weierstrass semigroup at the unique fixed point $P$.

Lemma 18. Let $(X, G)$ be a big action. The smallest non trivial pole number $m_1$ in $H(P)$ is $|G_\infty(P)|$.

Proof. Let $f_1$ be the function in the function field $k(X)$ such that $(f_1)_\infty = m_1P$. By [28] $\ker \rho_1 = G_\infty(P)$ and the result follows by lemma 11.

Theorem 19. Let $F_0$ be an algebraic function field of genus $g$, defined over an algebraically closed field $k$ of characteristic $p > 0$. Select a place $P_0$ of $F_0$ and consider an Artin-Schreier extension $F/F_0$ given by $F = F_0(y)$, $y^p - y = G$, where $v_{P_0}(G) = \{h_1 < h_2 < \cdots < h_{g_{P_0}} \leq 2g_{P_0} - 1\}$, if $g_{P_0} > 1$ $\phi = g_{P_0} = 0$.

Then the gaps $\mathcal{G}(P_0)$ at $P_0$ are given by

$$\mathcal{G}(P_0) = \left\{ \begin{array}{ll}
\{1 = h_1 < h_2 < \cdots < h_{g_{P_0}} \leq 2g_{P_0} - 1\}, & \text{if } g_{P_0} > 1 \\
\emptyset, & \text{if } g_{P_0} = 0
\end{array} \right.$$ 

The gaps $\mathcal{G}(P)$ are given by

$$\mathcal{G}(P) = \bigcup_{i=1}^{p^h-1} A_i,$$
with \[ A_i = \left\{ mi - pj^h : 1 \leq j \leq \left\lfloor \frac{mi}{p^k} \right\rfloor \right\} \cup \{ mi + pj^h : j = 1, 2, \ldots, g_{F_i} \}. \]

**Proof.** This is combination of theorems 1 and 5 of Lewittes, in [25]. Keep in mind that we are working over an algebraically closed field, therefore all places have degree 1. Also Lewittes proves his theorem for the more difficult case \( k = \mathbb{F}_q \) but the proof over an algebraically closed field is the same and even simpler. \( \square \)

**Remark 20.** The numbers \( m_i, p^h \) are always pole numbers, since it is not possible to express them as a gap given by theorem 19. This means that \( m\mathbb{Z}_+ + p^h\mathbb{Z}_+ \) is always included in \( H(P) \). If the genus of \( F_0 \) is not 0, then the inclusion \( m\mathbb{Z}_+ + p^h\mathbb{Z}_+ \subseteq H(P) \) can be a strict inclusion, see remark 25.

**Proposition 21.** If \( i \) is a jump for the representation filtration, i.e., \( \ker \rho_i \supsetneq \ker \rho_{i+1} \), then \( m_{i+1} \notin \langle m_1, \ldots, m_i \rangle_{\mathbb{Z}_+} \). Also the number of jumps in the representation filtration equals the minimal number of generators of \( H(P) \) up to \( m_r \).

**Proof.** Let \( f_i \) be elements in the function field of the curve such that \( \text{div}_\infty(f_i) = m_i P \).

Let \( f_1 \) be the first such function with a unique pole at \( P \) of order \( m_1 \).

Suppose first that \( m_2 = 2m_1, m_3 = 3m_1, \ldots, m_{\nu m_1} \) are the first \( \nu \) pole numbers. Then a basis for the space \( L(m_i P) \) consists of \( \{ f_1, f_1^2, \ldots, f_1^\nu \} \) and it is clear that \( \ker \rho_1 = \ker \rho_2 = \cdots = \ker \rho_{\nu m_1} \).

Now we treat the general case. If \( m_{i+1} \) is in the semigroup \( \langle m_1, \ldots, m_i \rangle_{\mathbb{Z}_+} \) generated by all \( m_1, \ldots, m_i \), then \( m_{i+1} = \sum_{\nu \in \mathbb{Z}_+} \nu m_i \) and

\[ f_{i+1} = \prod_{\nu \in \mathbb{Z}_+} f_1^{\nu \alpha}. \]

This implies that \( \ker \rho_{i+1} = \ker \rho_i \). \( \square \)

The next proposition characterize exactly the jumps of the representation filtration:

**Proposition 22.** Denote by \( X_i \) the curve \( X/\ker \rho_i \) and by \( F_i \) the corresponding function field. Suppose that \( X_1 = \mathbb{P}^1 \) and \( F_1 = k(f_1) \). The jumps at the representation filtration appear exactly at \( i \) such that \( F_i \neq F_{i+1} \). Moreover if \( i_1, i_2, \ldots, i_n \) are the jumps of the representation filtration then for all \( \ell \), with \( 1 \leq \ell \leq n \) we have \( F_{i_{\ell}} = k(f_0, f_{i_1}, \ldots, f_{i_{\ell}}) \), i.e. the jumps appear exactly at the integers \( i \) such that the function \( f_{i} \) corresponding to the pole number \( m_i \) gives rise to an element that can not be expressed as a rational function of \( f_0, \nu < i \).

**Proof.** We have the following picture

\[ G_1(P) = \ker \rho_0 \supsetneq \ker \rho_1 \supsetneq \cdots \supsetneq \ker \rho_{i-n} = \{ 1 \} \]

for the representation filtration and the strict inclusion holds for a subscript \( i \) only when \( i \in \{ i_1, \ldots, i_n \} \). In general \( r + 1 \geq n \) with equality holds if all pole numbers up to \( m_r \) are generators, or equivalently when all the inclusions above are strict. We also have

\[ \begin{array}{cccccc}
X/G_1(P) & \longrightarrow & X/\ker \rho_1 & \longrightarrow & \cdots & X/\ker \rho_i & \longrightarrow & \cdots & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_0 := F_i^{G_1(P)} & \subseteq & F_1 := k(f_1) & \subseteq & F_i & \subseteq & F_r \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q_0 & \longrightarrow & Q_1 & \longrightarrow & \cdots & Q_i & \longrightarrow & \cdots & \longrightarrow & P 
\end{array} \]

where the first line corresponds to coverings, the second to function fields and the last to their totally ramified places.
Notice that \( \ker \rho_r = \ker \rho = \{1\} \) therefore \( F_r \) is the function field of the curve \( X \). Notice also that since \( F_1 \) is rational \( F_0 = F_r^{G_r(P)} \subseteq F_1 \) is also rational.

Let \( Q_i \) be the restriction at the intermediate fields \( F_i \) of the place \( P \) of \( F_r \). We will denote be \( H_{F_r}(Q_i) \) the Weierstrass semigroup of the field \( F_i \) at the place \( Q_i \). We have assumed that \( F_1 = k(f_1) \) is a rational function field and thus the semigroup of \( P_1 \) is \( H_{F_r}(Q_1) = \mathbb{Z}_+ \). This gives us that \( \ker \rho_1 \| \mathbb{Z}_+ \subseteq H_{F_r}(P) \).

There are now two cases: Either the group \( G_1(P) \) acts trivially on \( f_1 \) therefore \( \ker \rho_1 = G_1(P) \), or the group \( G_1(P) \) might act on \( f_1 \) by translations (this is the only possible Galois action of a \( p \)-group on a rational function field see [44]). The function field generator of \( F_0 \) gives rise to pole numbers that are multiples of \( |G_1(P)| \). Since \( |\ker \rho_1| \) divides \( |G_1(P)| \) all pole numbers coming from the function field generator of \( F_0 \) are already in \( |\ker \rho_1| \mathbb{Z}_+ \subseteq H_{F_r}(P) \). Notice that \( 0 \) is a jump in the representation filtration only if \( f_1 \) is not \( G_1(P) \)-invariant. In that case \( F_0(f_1) = F_1 \).

Suppose that \( 1 \) is the first jump of the representation filtration, i.e. take \( i_1 = 1 \). We proceed to the first \( m_i \) such that \( m_i \not\in |\ker \rho_1| \mathbb{Z}_+ \), i.e. we choose the minimal \( m_i \) with the property that \( |\ker \rho_1| \nmid m_i \) for \( 1 < i \). This means that the element \( f_1 \) is not \( \ker \rho_1 \)-invariant and \( \ker \rho_0 \supsetneq \ker \rho_1 \), i.e. the first jump in the representation filtration appears at the first \( m_i \) that is not in \( |\ker \rho_1| \mathbb{Z}_+ \). The extension \( F_{i_1}/F_1 \) is elementary abelian with Galois group \( \ker \rho_1/\ker \rho_{i_1} \). The element \( f_{i_1} \) is not an element in \( F_1 \) since every element in \( F_1 \) is \( \ker \rho_1 \)-invariant.

Suppose now that \( i_1 \neq 1 \). Then \( \ker \rho_0 = \ker \rho_1 = \cdots = \ker \rho_{i_1} \), with \( m_\nu \in |\ker \rho_1| \mathbb{Z}_+ \) for all \( \nu \leq i_1 \). We choose \( m_{i_2} \) minimally such that \( m_{i_2} \not\in |\ker \rho_1| \mathbb{Z}_+ \), i.e. \( |\ker \rho_1| \nmid m_{i_2} \). This means that the corresponding function \( f_{i_2} \) is not \( \ker \rho_{i_1} \)-invariant and \( \ker \rho_{i_1} \supsetneq \ker \rho_{i_2} \). Notice that \( F_0 = F_1 = \cdots = F_{i_1} \). The extension \( F_{i_1}/F_{i_2} = F_{i_2}/F_1 \) is elementary abelian with Galois group \( \ker \rho_1/\ker \rho_{i_2} \), but the element \( f_{i_2} \) is not an element in \( F_{i_2} \) since it is not \( \ker \rho_{i_2} \)-invariant.

The result is proved the same way going up step by step in terms of Artin-Schreier extensions.

\[ \square \]

**Remark 23.** The extensions \( F_{i_1+1}/F_{i_2} \) can be described as an Artin-Schreier extensions. For example if \( \nu = 1 \), with \( i_1 \neq 0 \), we have \( \text{Gal}(F_{i_1+1}/F_1) = \oplus_1 \mathbb{Z}/p\mathbb{Z} \) for each cyclic direct summand of \( \text{Gal}(F_{i_1+1}/F_1) \). Notice that \( F_0 = F_1 = \cdots = F_{i_1} \). Then, for the action of \( \sigma \in \oplus_1 \mathbb{Z}/p\mathbb{Z} \) on \( f_{i_1+1} \) we have:

\[ \sigma^\ell(f_{i_1+1}) = f_{i_1+1} + \sum_{0 \leq \mu \leq i_1} c_\mu(\sigma)f_{\mu}, \]

and \( c_\mu(\sigma) \in k \). We have that \( \sum_{0 \leq \mu \leq i_1} c_\mu(\sigma)f_{\mu} \) is \( \ker \rho_1 \)-invariant. Thus, we set

\[ Y_{i_1+1} = \frac{f_{i_1+1}}{\sum_{0 \leq \mu \leq i_1} c_\mu(\sigma)f_{\mu}}, \]

and observe that

\[ \sigma^\ell(Y_{i_1+1}) = Y_{i_1+1} + \ell. \]

This gives us that \( Y_{i_1+1} \) is a set of Artin-Schreier generators for the extension \( F_{i_1+1}/F_1 \) and the elements \( Y_{i_1+1} \) satisfy equations of the form \( Y_{i_1+1}^P - Y_{i_1+1} = a_{i_1} \) for some element in \( a_{i_1} \in F_1 = F_{i_1} \). But actually we need only one extra generator \( f_{i_1+1} \) in order to express \( F_{i_1+1} = k(f_1, f_{i_1+1}) \).

**Remark 24.** Observe that Artin-Schreier extension given by eq. (4) introduces more ramification primes. Indeed, every zero of the quantity \( \sum_{0 \leq \mu \leq i_1} c_\mu(\sigma)f_{\mu} \) is a ramified prime in the extension \( F_{i_1+1}/F_{i_1} \). Also the valuation of \( Y_{i_1+1} \) at the prime \( P \) in question is \( v_P(Y_{i_1+1}) = v_P(f_{i_1+1}) = v_P(\sum_{0 \leq \mu \leq i_1} c_\mu(\sigma)f_{\mu}) \).

**Remark 25.** It seems natural to ask the following question: Let \( L/K \) be a Galois extension with Galois group a \( p \)-group. Let \( P \) be a wild, totally ramified place of \( L \). If a pole number
in \( H(P) \) is divisible by \( p \), is it true that the corresponding element \( f \in L \) is \( \text{Gal}(L/K) \)-invariant? The answer to this question is negative as one sees by the following example: Consider the totally ramified elementary abelian extension of the rational function field given by the two Artin-Schreier extensions:

\[
y_1^p - y_1 = f_1(x), \quad y_2^p - y_2 = f_2(x),
\]

where \( f_i(x) \in k[x] \) for \( i = 1, 2 \). Consider the \( \mathbb{Z}/p\mathbb{Z} \)-extensions of \( K_1 := k(x,y_1), K_2 := k(x,y_2) \) of the rational function field. In those extensions only the infinite place \( \infty \) of \( k(x) \) is ramified. Moreover it is known [39, Korollar 1] that the Weierstrass semigroups at the places of \( K_i \) that are above \( \infty \) are \( p\mathbb{Z}_+ + m_i \mathbb{Z}_+ \) respectively where \( m_i = \deg f_i \). We have the following tower of fields and corresponding semigroups:

\[
\begin{array}{c|c|c}
L(x, y_1, y_2) & H(P) & \mathbb{Z}_+ \\
\hline
k(x, y_1) & \mathbb{Z}_+ + m_1 \mathbb{Z}_+ & \mathbb{Z}_+ \\
\hline
k(x) & \mathbb{Z}_+ + m_2 \mathbb{Z}_+ & \\
\end{array}
\]

Therefore, the Weierstrass semigroup \( H(P) \) of the place \( P \) of \( k(x, y_1, y_2) \) that is above \( \infty \), has \( p^2 \mathbb{Z}_+ + pm_1 \mathbb{Z}_+ + pm_2 \mathbb{Z}_+ \) as a subsemigroup. We have \( \text{div}_\infty(y_1) = pm_1 \) but \( y_2 \) is not in \( k(x, y_1) \), i.e., it is not \( \text{Gal}(k(x, y_1, y_2)/k(x,y_1)) \)-invariant. Suppose that \( m_1 < m_2 \). Recently, [49] and [1] showed that the jumps of the ramification filtration occur at \( m_1 \) and \( m_1 + p(m_2 - m_1) \).

The situation changes for the case of cyclic extensions of the rational function field as will see in the next section.

### 2.3. Cyclic totally ramified Galois extensions of function fields.

We will need the following:

**Proposition 26.** Let \( L_n \) be a cyclic extension of the rational function field \( L_0 = k(x) \) of degree \( p^n \). Assume also that every ramified place in the extension \( L_n/L_0 \) is ramified completely. Then

1. there is a unique tower of intermediate fields

\[
k(x) = L_0 \subset L_1 \subset \cdots \subset L_{m-1} \subset L_m
\]

such that \( [L_j : L_{j-1}] = p \) and this extension is given in terms of an Artin-Schreier extension

\[
L_j = L_{j-1}(y_j) : \quad y_j^p - y_j = B_j,
\]

where \( B_j \in L_{j-1} \) has the following property (standard form):

\[
\text{div}B_j = A - \sum \lambda_i \alpha_i P_i - 1,
\]

with \( A \) prime to the pole divisor of \( B_j \) and \( \lambda_i \) is either zero or a positive integer prime to the characteristic \( p \). Actually the integers \( \lambda_i \) are equal to the valuations \(-\lambda_i = v_{P_i}(B_j) = v_{P_i}(y_j)\). In the above notation we are using the notation of [27], namely in \( P_i \), the index \( j \) indicates that this prime is a prime of the field \( L_j \), while the index \( i \) shows that lies above the \( i \)-th prime of \( L_0 \) that ramifies. Moreover

2. The elements \( y_1^{\alpha_1}, y_2^{\alpha_2}, \ldots, y_n^{\alpha_n} \) form a basis of \( L_m \) over \( L_0 \), where \( 0 \leq \alpha_i < p \).
(3) For every ramified place \( P \) in \( L_m / L_0 \) and integer \( a \in \mathbb{N} \) we have the decomposition:

\[
L(aP) = \bigoplus_{0 \leq a_1, \ldots, a_m < p} (L(aP) \cap y_1^{a_1}y_2^{a_2} \cdots y_m^{a_m}k(x)),
\]

where \( y_1^{a_1}y_2^{a_2} \cdots y_m^{a_m}k(x) \) denotes the 1-dimensional \( k(x) \)-vector space generated by \( y_1^{a_1}y_2^{a_2} \cdots y_m^{a_m} \).

**Proof.** Part (1) is theorem 2 in [27], part (2) is lemma 3 in the same article and part (3) follows by observing that the divisor \( a^{-1} \) on page 311 can be replaced by any \( \text{Gal}(L_m/L_0) \)-invariant divisor. \( \square \)

**Definition 27.** We will denote by \( G(i) \) the subgroup of \( G_1(P) \) of order \( p^{m-i} \).

**Proposition 28.** Fix a place \( P_{i_0,0} \) of the rational function field \( L_0 \) that ramifies totally in the extension \( L_m/L_0 \). Then the set of places \( P_{i,j} \) of \( L_j \) that are above \( P_{i_0,0} \) will be denoted by \( P_j \) and \( \lambda_j = \lambda_{i_0,j}, 1 \leq j \leq m \).

Let \( H_{L_j}(P_j) \) denote the Weierstrass semigroup at the place \( P_j \) of the function field \( L_j \). We have:

\[
H_{L_j}(P_j) \subseteq p^j \mathbb{Z}_+ + p^{j-1}\lambda_1\mathbb{Z}_+ + \cdots + \lambda_j\mathbb{Z}_+.
\]

In particular

\[
H_{L_m}(P_m) \subseteq p^m \mathbb{Z}_+ + p^{m-1}\lambda_1\mathbb{Z}_+ + \cdots + \lambda_m\mathbb{Z}_+.
\]

If moreover \( P_{i_0,0} \) is the only place that ramifies in the extension \( L_m/L_0 \) then the above inclusions are equalities.

**Proof.** Assume that \( \mu < j \) and let \( y_\mu \) the generator of \( L_\mu \) over \( L_{\mu-1} \). Observe that on the function field \( L_j \) we have \( \nu_{P_j}(y_\mu) = -p^{j-\mu}\lambda_\mu \). According to prop. 26 part (3) every element in \( \cup_{\nu \geq 1} L(\nu P_m) \) comes from an element of the form \( y_1^{a_1}y_2^{a_2} \cdots y_m^{a_m}f(x) \) and these valuations at \( P_m \) form the semigroup \( p^m \mathbb{Z}_+ + p^{m-1}\lambda_1\mathbb{Z}_+ + \cdots + \lambda_m\mathbb{Z}_+ \). Therefore \( H_{L_m}(P_m) \subseteq p^m \mathbb{Z}_+ + p^{m-1}\lambda_1\mathbb{Z}_+ + \cdots + \lambda_m\mathbb{Z}_+ \).

We don’t know if the elements \( y_\mu \) have other poles so they may not be in \( L(\kappa P_m) \) for some integer \( \kappa \). But if there is only one ramified point then the pole divisors of \( y_\mu \) is supported only at \( P_m \) and \( \lambda_\mu p^{m-i} \) is in the semigroup. \( \square \)

Using the description of \( L_m \) we are able to see the following

**Proposition 29.** If \( \text{Gal}(L_m/L_0) \) is cyclic of order \( p^m \) and \( H(P) \) is the Weierstrass semigroup at a totally ramified point \( P \) of \( L_m \), and \( \text{div}_\infty(f) = aP \) for an integer \( a \) such that \( p^a \mid a \) then \( f \) is in \( L_{m-\kappa} \), i.e. \( f \) is fixed by the unique subgroup of \( \text{Gal}(L_m/L_0) \) of order \( p^\kappa \).

**Proof.** Using the valuation of elements of the form \( y_1^{a_1}y_2^{a_2} \cdots y_m^{a_m} \) we see that if \( p^\kappa \mid a \) then \( f \) is a \( L_0 \) linear combination of elements \( y_1^{a_1}y_2^{a_2} \cdots y_m^{a_m} \) with \( \alpha_m = \cdots = \alpha_{m-(\kappa-1)} = 0 \) that are in \( L_{m-\kappa} \). \( \square \)

In eq. (5) we introduced the sequence of numbers \( \lambda_1 < \lambda_2 < \cdots \) corresponding to the valuations of the generating elements \( y_\mu \) of the field \( L_m \). Is there any relation to the sequence \( m_1 < m_2 < \cdots \) of pole numbers? What is the relation with the jumps of the representation filtration?

Assume that the jumps of the representation filtration appear at the integers \( j_1, j_2, \ldots, j_n \), i.e. \( \ker \rho_i > \ker \rho_{i+1} \) for all \( i = 1, \ldots, n \). We have the following:

\[
G_1(P) = \ker \rho_1 = \ker \rho_2 = \cdots = \ker \rho_{j_1} > \ker \rho_{j_1+1} = \cdots = \ker \rho_r = \{1\}.
\]

Notice that the kernel of \( \rho_r \) is trivial since \( \rho_r \) is a faithful representation by proposition 3. Observe that the number of jumps in both the representation and the ramification filtrations is the same, since in both filtrations the elementary abelian quotients at indices where a
jump occur, are just cyclic groups of order $p$, therefore $m$, the number of jumps in the ramification filtration, equals $n$, the number of jumps in the representation filtration. The jumps at the representation filtration occur at $j_1, j_2, \ldots, j_n$, and $\ker p_{j_n} = G(\nu)$.

Let $j_\nu$ be an index such that $G(\nu) = \ker p_{j_\nu} \supsetneq \ker p_{j_\nu + 1} = G(\nu + 1)$. Since we are in the cyclic group case we have $\ker p_{j_\nu} / \ker p_{j_\nu + 1} \cong \mathbb{Z}/p\mathbb{Z}$. Let $\sigma$ be a generator of the later quotient. We can write the action on $f_{j_\nu + 1}$ as

$$\sigma f_{j_\nu + 1} = f_{j_\nu + 1} + \sum_{\mu \leq j_\nu} c_\mu(\sigma)f_\mu.$$ 

Since all $f_\mu, \mu \leq j_\nu$ are $\sigma$-invariant the element

$$(6) \quad Y_{j_\nu + 1} = \frac{f_{j_\nu + 1}}{\sum_{\mu \leq j_\nu} c_\mu(\sigma)f_\mu}$$

is acted on by $\sigma$ as $\sigma Y_{j_\nu + 1} = Y_{j_\nu + 1} + 1$, and it is an Artin-Schreier generator of the extension $F_{j_\nu + 1}/F_{j_\nu}$.

**Definition 30.** Consider a tower $F = F_n \supseteq F_{n-1} \supseteq \cdots \supseteq F_1 \supseteq F_0$ of fields. On this tower of fields we consider the sequence of jumps $F_n \supseteq F_{j_\nu} \supseteq F_{j_\nu - 1} \supseteq \cdots \supseteq F_1 = \cdots = F_0$. In the notation of proposition 26 the fields $F_{j_\nu + 1}$ are just the fields $L_{\nu}$. Fix a place $P$ of $F$ and let $Q_\nu$ be the restriction of $P$ on the field $F_\nu$. Notice that the places $P_\nu$ of $L_{\nu}$ defined in proposition 26 are equal to the places $Q_{j_\nu + 1}$.

An element $f_j \in F_j$ can be seen as an element in all fields $F_{j'}$ with $j' \geq j$. For $j \geq i$ we will denote by $0 > -m_{j,i}$ the valuation of $f_j$ as an element in $F_j$ at the place $Q_j$, where the function $f_j$ corresponds to the pole number $m_{i,j}$ of $H(P)$. We are interested in the elements $f_{j_\nu + 1}$, i.e. for the indices where a jump occurs in the representation filtration. Notice that since $f_{j_\nu + 1} \notin \langle f_i : i \leq j_\nu \rangle$ the valuation $v_{Q_{j_\nu + 1}}(f_{j_\nu + 1})$ is prime to $p$ according to proposition 29. Therefore the number $m_{j_\nu + 1,j_\nu + 1}$ is prime to $p$ and we will denote it by $\mu_{j_\nu+1}$.

We compute using eq. (6)

$$v_{Q_{j_\nu + 1}}(Y_{j_\nu + 1}) = -m_{j_\nu + 1,j_\nu + 1} - v_{Q_{j_\nu + 1}} \left( \sum_{\mu \leq j_\nu} c_\mu(\sigma)f_\mu \right)$$

$$(7) \quad = -\mu_{j_\nu + 1} + \max\{m_{j_{\nu},\mu} : c_\mu(\sigma) \neq 0\}.$$ 

Notice that $\lambda_{i,j}$ the valuations of generators $y_j$, of the fields given in the field tower of proposition 26 part (1), are prime to the characteristic from the standard form hypothesis. Notice also that there are $\nu$ generators $Y_{j_\nu + 1}$. We will abuse the notation and denote with $Y_\nu, 1 \leq \nu \leq n$ the generators of the field extension $F_{j_\nu + 1}/F_{j_\nu}$. Now, since $Y_\nu$ and $y_\nu$ are both Artin-Schreier generators their valuation, which is prime to the characteristic from the discussion above, equals the conductor of the extension and must be equal, thus the elements $Y_\nu$ and $y_\nu$ have the same valuation at $Q_{j_\nu + 1} = P_\nu$. The absolute value of the valuation computed in eq. (7) equals $\lambda_\nu$, where $\lambda_\nu$ are the numbers introduced in proposition 28.

**Corollary 31.** We keep the notation from definition 30. The valuations $\lambda_\nu$ of the elements $y_\nu$ at $P_\nu$ satisfy $\lambda_\nu \leq \mu_\nu$. Equality holds only if there is only one place $P$ ramified in extension $F_{n}/F_0$, i.e., we are in the case of a Katz-Gabber cover with cyclic group of order $p^n$.

**Proof.** The inequality is a direct consequence of eq. (7). If $\lambda_\nu = \mu_\nu$ then all coefficients $c_\mu(\sigma) \mu > 0$ in eq. (7) are zero, i.e. only the coefficient $c_0(\sigma)$ of the constant function $f_0$ can be non-zero. Then $Y_\nu = f_\nu$ and there is only one place ramified in the extension $F_{\nu+1}/F_\nu$.  

$\blacksquare$
Proposition 32. Let \( P \) be as in definition 30. The jumps of the ramification filtration at \( P \) are exactly at the the integers \( \lambda_\nu \), where \( \lambda_\nu \) are the valuations of the elements \( y_\nu \) at \( P_\nu \).

Proof. Observe that by [20, proof, Prop. 2.3], the integer \( \lambda_\nu \) equals the lower jump of the representation filtration \( \ker \rho_j / \ker \rho_{j+1} \cong \mathbb{Z}/p\mathbb{Z} \) (keep in mind that in that paper the enumeration is decreasing). Notice that \( \ker \rho_j = G(\nu) \) and \( \ker \rho_{j+1} = G(\nu + 1) \). The ramification filtration of \( G(\nu) \) is given by \( G(\nu)_\mu = G(\mu)(P) \cap G(\nu) \). Observe that the groups \( G(\nu) \) are elements of the ramification filtration therefore there is no need to consider the upper ramification filtration, in order to relate the ramification filtrations of \( G_1(P) \) and \( G_1(P)/G(\nu) \) [37, corollary after prop 3. IV.1], i.e.,

\[
(\frac{G_1(P)/G(\nu)}{G(\nu)})_j = G_j(P)/G(\nu).
\]

In the same way we have

\[
\frac{G(\nu)}{G(\nu + 1)}_j = \frac{G(\nu)}{G(\nu + 1)} \cdot \frac{G_j(P) \cap G(\nu)}{G(\nu + 1)}.
\]

This implies that \( \lambda_\nu \) are the jumps of the ramification filtration. \( \square \)

Example 33. We consider now the case of a cyclic group extension \( L/L_0 \) of order \( p^n \) with only one full ramified place such that only \( L_0 \) is rational.

We have the following tower of \( \mathbb{Z}/p\mathbb{Z} \)-cyclic extensions:

\[
\begin{array}{ccc}
L & P^* & p^n\mathbb{Z}_+ + p^{n-1}\lambda_1\mathbb{Z}_+ + p^{n-2}\lambda_2\mathbb{Z}_+ + \cdots + \lambda_n\mathbb{Z}_+
\\
\downarrow & \downarrow & \downarrow
\\
P_2 & P_1 & p^2\mathbb{Z}_+ + p\lambda_1\mathbb{Z}_+ + \lambda_2\mathbb{Z}_+
\\
\downarrow & \downarrow & \downarrow
\\
P_1 & P_0 & p\mathbb{Z}_+ + \lambda_1\mathbb{Z}_+
\\
\downarrow & \downarrow & \downarrow
\\
P_0 = L^{G_1(P)} & \mathbb{Z}_+
\end{array}
\]

The place \( P \) is fully ramified in extension \( L/L_0 \) and we will denote by \( P_i \) the corresponding place of \( L^{G_i(P)} \). The field \( L^{G_i(P)} \) is assumed to be rational and has Weierstrass semigroup \( \mathbb{Z}_+ \). The field \( L^{G(1)} \) is an Artin-Schreier extension of \( L^{G_1(P)} \) and the Weierstrass semigroup at \( P_1 \) has a part \( p\mathbb{Z}_+ \) coming as the semigroup of \( \mathbb{Z}/p\mathbb{Z} = G_1(P)/G(1) \)-invariant elements plus a new non invariant element that is not divisible by \( p \), the \( \lambda_1\mathbb{Z}_+ \) which is not the trivial extension. The Weierstrass semigroup at \( P_2 \) is \( p^2\mathbb{Z}_+ + p\lambda_1\mathbb{Z}_+ \).

The next step is to consider the Weierstrass semigroup of \( P_2 \). It has a part \( p^2\mathbb{Z}_+ + p\lambda_1\mathbb{Z}_+ \) coming from the \( G(1)/G(2) \cong \mathbb{Z}/p\mathbb{Z} \)-invariant elements and also some extra elements that should contain a semigroup of the form \( \lambda_2\mathbb{Z}_+ \), where \( (\lambda_2, p) = 1 \). But eq. (5) gives us that \( H(P_2) \subseteq p^2\mathbb{Z}_+ + p\lambda_1\mathbb{Z}_+ + \lambda_2\mathbb{Z}_+ \) and since there is a unique ramified place the above inclusion is actually an equality.

This way we can go all the way up to \( L \) and find that the Weierstrass semigroup is given by

\[
H(P) = p^n\mathbb{Z}_+ + p^{n-1}\lambda_1\mathbb{Z}_+ + p^{n-2}\lambda_2\mathbb{Z}_+ + \cdots + \lambda_n\mathbb{Z}_+.
\]

2.4. Holomorphic differentials in the cyclic totally ramified case. In this section we are going to use known bases for the space of holomorphic differentials in order to express the gaps of the Weierstrass semigroups in terms of the Boseck invariants of the curves. In [17] we have defined Boseck invariants to be the values \( \Gamma_k(m) \) in definition 34 below and we expressed the Galois module structure of holomorphic \( m \)-polydifferentials in terms of them. Here we omit the \( m \) from the notation since we are interested only for the \( m = 1 \)
case. The motivation for their name was the work of Boseck [4] and in what follows we will call them the Boseck invariants of the curve.

We assume that \( F \) is a cyclic, totally ramified extension of order \( p^n \) of the rational function field.

Let \( k \) be an integer with \( p \)-adic expansion
\[
(8) \quad k = a_1^{(k)} + a_2^{(k)} p + \cdots + a_n^{(k)} p^{n-1}.
\]
The set \( w_k = 1^{(k)} y_1^{(k)} y_2^{(k)} \cdots y_n^{(k)} \), \( 0 \leq k \leq p^n - 1 \) is an \( k(x) \)-basis \( E \)-basis of \( F \) [47]. The valuations of the basis elements \( w_k \) are given by
\[
(9) \quad v_{\Gamma_k} (w_k) = - \sum_{j=1}^{n} a_j^{(k)} \lambda_{ij} p^{n-j},
\]
where the \( \lambda_{ij} \) are given in proposition 26, part (1).

We denote the ramified places of \( k(x) \), with \( (x - \alpha_i), \ 1 \leq i \leq s \), since in a rational function field every ramified place corresponds to an irreducible polynomial, which is linear since the the field \( k \) is algebraically closed. We set
\[
g_k(x) = \prod_{i=1}^{s} (x - \alpha_i)^{\nu_{ik}(1)}.
\]

**Definition 34.** For \( k = 0, 1, \ldots, p^n - 1 \), we define
\[
\Gamma_k := \sum_{i=1}^{s} \nu_{ik},
\]
where
\[
(10) \quad \nu_{ik} = \left[ \frac{\delta_i - \sum_{j=1}^{n} a_j^{(k)} \lambda_{ij} p^{n-j}}{p^n} \right],
\]
where \( \delta_i \) denotes the different exponent \( \delta(P_{i,n}/(x - \alpha_i)) \) and the \( 1 \leq j \leq n \) runs over the intermediate fields in the tower of proposition 26, part 1. Finally
\[
(11) \quad \rho_i^{(k)} = \left( \delta_i - \sum_{j=1}^{n} a_j^{(k)} \lambda_{ij} p^{n-j} \right) - \nu_{ik} \cdot p^n,
\]
is the remainder of the division of the quantity \( \delta_i + v_{\Gamma_k} (w_k) \) by \( p^n \).

**Proposition 35.** Let \( X \) be a cyclic extension of degree \( p^n \) of the rational function field. The set
\[
\left\{ \omega_{k}^{(\alpha_i)} = (x - \alpha_i)^{\nu_i^{(k)}} g_k(x)^{-1} w_k dx : 0 \leq \nu_i^{(k)} \leq \Gamma_k - 2, 0 \leq k \leq p^n - 2 \right\}
\]
forms a basis for the set of holomorphic differentials for a cyclic extension of the rational function field of order \( p^n \).

**Proof.** We take the basis of [17, Lemma 10], set \( m = 1 \) and modify it in order to evaluate holomorphic differentials in the ramified primes of the extension. The same construction is given by Garcia in [7, Theorem 2, Claim] where the elementary abelian, totally ramified case is studied. The proof is identical to the one given there.

**Remark 36.** Observe from eq. (10) that if \( \Gamma_k = 0 \), then \( \nu_{ik} = 0 \) for all \( i = 1, \ldots, r \) and that could happen only when \( k = p^n - 1 \) since \( \nu_{ik} \geq 0 \). Thus \( \Gamma_k \)'s attain the maximum value when \( k = 0 \), i.e \( \Gamma_0 \geq \Gamma_k \), for all \( 0 \leq k \leq p^n - 1 \) and for \( 0 \leq k \leq p^n - 2 \), we have that \( \rho_1^{(k)} \leq p^n - 2 \).

We will need the following definition. For more information concerning the theory of linear series the reader is referred to [42].
Definition 37. Let $\mathcal{D}$ be a linear series of degree $d$ and dimension $r$, i.e., a linear subspace of $\mathbb{P}(L(E))$. The linear series can be seen as a set of linear equivalent divisors. We can form a decreasing sequence

$$\mathcal{D}_i(P) = \{D \in \mathcal{D} : D \geq iP\}.$$ 

The sequence of $(\mathcal{D}, P)$-orders is a sequence of integers:

$$j^\mathcal{D}_0(P) < \cdots < j^\mathcal{D}_r(P),$$

such that $\mathcal{D}_j(P) \supseteq \mathcal{D}_{j+1}(P)$ for all $j$ in the above sequence.

For all but finite points of $X$ the $(\mathcal{D}, P)$-orders do not depend on $P$. The exceptional points are called $(\mathcal{D}, P)$-Weierstrass points.

Let $E$ be the generic $(\mathcal{D}, P)$-order sequence. We will call it the order sequence of the linear series.

A curve will be called classical with respect to the linear series $\mathcal{D}$ if and only if $E_\mathcal{D} = \{0, \ldots, r\}$.

We have the following:

**Proposition 38.** Let $X$ be a cyclic extension of degree $p^n$ of the rational function field. The gap sequence at the ramified primes $P_i$ that lies over the place $(x - \alpha_i)$, with $0 \leq i \leq s$ of $k(x)$ is given by

$$\mathcal{G}(P_i) = \left\{\nu(k) \cdot p^n + \rho_i^{(k)} + 1 \mid 0 \leq k \leq p^n - 2, \ 0 \leq \nu(k) \leq \Gamma_k - 2\right\},$$

where $\Gamma_k$ is the Boseck invariant associated to the extension $F/k(x)$.

**Proof.** This computation is based on the valuation of the elements in the basis of holomorphic differentials given in proposition 35. Let $\rho_i^{(k)}$ be as in Eq. (11). We denote by $j^K_i(P_i) = j^K$ the $(K, P_i)$-orders and $K$ the canonical linear series. By computing the valuations at $P_i$ of the basis elements given in proposition 35, we are able to compute that the $(K, P_i)$ orders are given by

$$\left\{\nu(k) \cdot p^n + \rho_i^{(k)} \mid 0 \leq k \leq p^n - 2, \ 0 \leq \nu(k) \leq \Gamma_k - 2\right\}. \tag{12}$$

In the above computation it is essential to notice that for different values of $k, \nu(k)$, $0 \leq k \leq p^n - 2$, $0 \leq \nu(k) \leq \Gamma_k - 2$ the values $\nu(k) \cdot p^n + \rho_i^{(k)}$ are different, therefore the valuation of a linear combination of the differentials in proposition 35, is just the minimal of the valuation of each summand. We know that knowledge of the $(K, P_i)$'s orders, is equivalent to the knowledge of the $(0, P_i)$'s gaps, i.e. the gap sequence at $P_i$ will be given by $j^K_i(P_i) + 1$.

Notice that every element has a different valuation with respect to the place $P_i$. Notice also that the dimension of the holomorphic differentials is $g$ the same number as the number of the gaps. Also the $n = 1$ case was studied by Boseck in [4, Satz. 17] \hfill \Box

**Remark 39.** Boseck in his seminal paper [4, Satz 18], where the $n = 1$ case is studied, states that as $k$ takes all the values $0 \leq k \leq p - 2$ the remainder of the Boseck’s basis construction $\rho_i^{(k)}$ takes all the values $0 \leq \rho_i^{(k)} \leq p - 2$ and thus all the numbers $1, \ldots, p - 1$ are gaps. This is not entirely correct as we will show in example 42. The problem appears if there is exactly one ramified place in the Galois extension.

**Lemma 40.** If all $\Gamma_k \geq 2$ then all numbers $1, \ldots, p^n - 1$ are gaps. If there are Boseck invariants $\Gamma_k = 1$, then the set of gaps smaller than $p^n$ is exactly the set $\{\rho_i^{(k)} + 1 \mid k : 0 \leq k \leq p^n - 2, \Gamma_k \geq 2\}$.
Proof. Recall that in eq. (11) the elements $\rho^i_k$ were defined to be the remainder of the division of $\delta_i = \sum_{j=1}^n (a_j^{(k)} \lambda_{i,j} p^{n-1})$ by $p^x$. As $k$ runs in $0 \leq k \leq p^n - 2$ the $\rho^i_k$ run in $0, \ldots, p^n - 2$. Indeed, let us define the function
\[ \Psi : \{0, \ldots, p^n - 2\} \to \{0, \ldots, p^n - 2\}, \]
\[ k \mapsto \rho^i_k. \]
Since the expressions in eq. (12) are all different the function $\Psi$ is onto.

But the $\Gamma_k$ that are equal to 1 have to be excluded since they give not rise to a holomorphic differentials in proposition 35, see [17, Eq. (21)] and example 42.

\[ \square \]

Remark 41. Notice that elements $\Gamma_k = 1$ can appear only for primes $p \geq \lambda_{i,j}$ and only if there is only one ramified place. But in our representation theoretic viewpoint the interesting case is the one of small primes.

Example 42. We consider the now the case of an Artin-Schreier extension of the function field $k(x)$, of the form $y^p - y = 1/x^m$. In this extension only the place $(x-0)$ is ramified with different exponent $\delta_1 = (m+1)(p-1)$. The Boseck invariants in this case are
\[ \Gamma_k = \left( \frac{(m+1)(p-1) - km}{p} \right) \text{ for } k = 0, \ldots, p-2. \]
The Weierstrass semigroup is known [39] to be $m\mathbb{Z}_+ + p\mathbb{Z}_+$. Let us now try to find the small gaps by using lemma 40. If $p < m$ then all numbers $1, \ldots, p-1$ are gaps. If $p > m$ then $m$ is a pole number smaller than $p$. Indeed, $\Gamma_{p-2} = 1$ is the remainder of the division of $(m+1)(p-1) - (p-2)m$ by $p$ is $\rho^{(p-2)} = m-1$. But then $\rho^{(p-2)} + 1 = m$ is not a gap.

Definition 43. In the literature, see Oliveira [31], semigroups $H$ of genus $g$ such the $g$-th gap is $2g - 1$ are called symmetric.

A semigroup $H$ is called symmetric, because the symmetry is expressed in the semigroup in the following way
\[ (13) \quad n \in H \text{ if and only if } 2g - 1 - n \notin H. \]

Lemma 44. Let $X$ be a cyclic $p$-group extension of the rational function field with only one totally ramified point $P_0$. The biggest gap is equal to $(\Gamma_0 - 2)p^n + \rho^{(0)} + 1 = \delta - 2p^n + 1$ and the Weierstrass semigroup at that point is symmetric.

Proof. Since there is only one ramified place $\nu_{o,k} = \Gamma_k$. From remark 36 and proposition 38 we see that the biggest gap is equal to $(\Gamma_0 - 2)p^n + \rho^{(0)} + 1 = \delta - 2p^n + 1$, where $\delta$ is the different exponent at the unique ramified place and is equal to the degree of the different. For the last equality keep in mind that $\Gamma_0 = \left\lfloor \frac{\delta}{p^n} \right\rfloor$ and $\delta = p^n \Gamma_0 + \rho^{(0)}$.

A direct computation with Riemann–Hurwitz formula, shows that
\[ 2g - 1 = -2p^n + 1 + \delta, \]
hence the semigroup is symmetric.

\[ \square \]

Does lemma 44 holds for the general case when we have more than one totally ramified primes? The answer is no, in general, but the following lemma gives a necessary and sufficient condition for this.

Lemma 45. Assume that $X$ is as in lemma 44 but now there are $s \geq 2$ totally ramified places $P_i$, with $1 \leq i \leq s$. Let $\delta_i$ denote the different exponent at the prime $i$. The semigroup at some ramified place $P_i$ is symmetric if and only if
\[ (14) \quad \sum_{i' \neq i} \delta_{i'} = p^n \sum_{i' \neq i} \left\lfloor \frac{\delta_{i'}}{p^n} \right\rfloor. \]
Moreover, let $\lambda_{i,j}$ be the valuations of the generators of the field extensions given in Proposition 26 part (1). Then eq. (14) holds if and only if $\lambda_{i'} \equiv -1 \mod p^n$ for all $i' \neq i$.

Proof. Fix an $i$. Then from remark 36 and proposition 38, we see that the biggest gap at $P_i$ is $(\Gamma_0 - 2)p^n + \rho_i^{(0)} + 1$ and using one more time the Riemann–Hurwitz formula, we see that this gap equals to $2g - 1$ is equivalent to the condition

$$\sum_{i'=1}^s \delta_{i'} = 2p^n + 1.$$  

Now, $\Gamma_0 = \sum_{i'=1}^s \left\lfloor \frac{\delta_{i'}}{p^n} \right\rfloor$, the left hand of eq. (15) equals to

$$p^n \sum_{i' \neq i} \left\lfloor \frac{\delta_{i'}}{p^n} \right\rfloor + \delta_i = 2p^n + 1.$$  

Thus in order eq. (15) to be valid we should have

$$\sum_{i' \neq i} \delta_{i'} = p^n \sum_{i' \neq i} \left\lfloor \frac{\delta_{i'}}{p^n} \right\rfloor.$$  

The right hand of eq. 16 equals to $\sum_{i' \neq i} (\delta_{i'} - \rho_i^{(0)})$. Since $\rho_i^{(0)}$’s are by definition non negatives, eq. 16 holds if and only if

$$\rho_i^{(0)} = 0 \iff \delta_i \equiv 0 \mod p^n,$$  

for every $i' \neq i$.  

For $n = 1$, $\delta_{i'} = (\lambda_{i',j} + 1)(p - 1)$ and the above condition is equivalent to $\lambda_{i',j} \equiv -1 \mod p$, for $i' \neq i$.

For $n > 1$, $\delta_{i'} = (p-1)\sum_{j=1}^n (\lambda_{i',j} + 1)p^{n-j}$ (see [47, p.110]), and the above condition is equivalent to $\sum_{j=1}^{n-1} (\lambda_{i',j} + 1)p^{n-j} \equiv \sum_{j=1}^n (\lambda_{i',j} + 1)p^{n-j} \mod p^n$, for $i' \neq i$, or $\lambda_{i',j} \equiv -1 \mod p^n$, for $i' \neq i$.

□

Corollary 46. Let $X$ be as in lemmata 44,45, i.e. assume that there exists a $P_{\alpha_0}$ that is totally ramified and that its Weierstrass semigroup is symmetric. Then the Weierstrass sequence up to $2g$, at the place $P_{\alpha_0}$, that lies over the place $(x - \alpha_{\alpha_0})$, is given by

$$H(P_{\alpha_0}) = \{2g - 1 - a|a \in \mathcal{G}(P_{\alpha_0})\},$$  

where $\mathcal{G}(P_{\alpha_0})$ is the gap sequence at $P_{\alpha_0}$ from proposition 38.

Proof. This is a direct consequence of lemmata 44, 45 and eq. (13).  

□

Remark 47. In the theory of numerical semigroups the following construction is frequently used in order to describe the semigroup [29]: Let $d(P)$ be the least positive element of $H(P)$. All elements $\mu \in \{1, \ldots, d(P) - 1\}$ are gaps and for every $\mu$ we denote by $b_\mu(P)$ the minimal element of $H(P)$ such that $b_\mu(P) \equiv \mu \mod d(P)$. This means that $b_\mu(P) = \nu_\mu(P)d(P) + \mu$, and $\nu_\mu(P) = \left\lfloor \frac{b_\mu(P)}{d(P)} \right\rfloor$ equals the number of gaps that are congruent to $\mu$ modulo $d(P)$.

Assume that the smallest pole number is $p^n$ and that there is only one ramified place in the field $F_0/F_0$. Then the integers $\nu_\mu(P)$ in this description of the semigroup are equal to the Boseck invariants $\nu_\mu(P) = \Gamma_{\mu-1} - 1$ since both integers count the number of gaps that are equal to $\rho_\mu^{(k)} + 1 \mod p^n$ by proposition 38. Notice also that a theorem due to Lewittes [29, th. 1.3], [24, th. 5] in characteristic zero, has an interpretation for the trivial group (the prime to $p$-part of a $p$-group acting on our curve):

$$g = \sum_{\mu=1}^{p^n-1} \nu_\mu = \sum_{\mu=1}^{p^n-1} (\Gamma_{\mu-1} - 1),$$  

where
since \( g \) is the trace of the trivial representation on holomorphic differentials and \( \Gamma_{\mu} - 1 \) counts the number of gaps that are equivalent to \( \mu \) modulo \( p^n \). This is equivalent to the formula proved in [17, rem. 7] for \( m = 1 \).

**Proposition 48.** We use the notation of theorem 19. Consider an Artin-Schreier cyclic extension of the rational function field \( F_0 \), i.e., the Galois group is isomorphic to \( \mathbb{Z}/p\mathbb{Z} \). Then for every \( i \) there is an integer \( k(i) \) with \( 0 \leq i, k(i) \leq p - 1 \), such that \( \left\lfloor \frac{mi}{p} \right\rfloor = \Gamma_{k(i)} - 1 \) for some Boseck invariant.

**Proof.** According to theorem 19 the number of gaps equivalent to \( \mu \) modulo \( p \) equals \( \left\lfloor \frac{mi}{p} \right\rfloor \). By remark 47 this number equals to \( \Gamma_k - 1 \) for the \( k \) such that \( p^{k} + 1 = mi \) modp. □

3. SEMIGROUPS

Let \( H \neq \mathbb{Z}_+ \) be a semigroup of natural numbers and suppose that there is a natural number \( n \) such that for all \( s \geq n \) we have \( s \in H \). We select the minimal such number \( n \), i.e. \( n - 1 \notin H \). Observe that when that semigroup is the Weierstrass semigroup of a curve, then the genus of the curve and the genus of the semigroup (i.e. the number of gaps of the semigroup) coincide, and we should have that \( g + 1 \leq n \leq 2g \). Let \( d_1 \) be an element in \( H \) and let \( d_2 \) be the minimal element in \( H \setminus d_1 \mathbb{Z}_+ \) such that \((d_1, d_2) = 1\). Write \( d_2 = d_1 \left\lfloor \frac{d_2}{d_1} \right\rfloor + u \), \( 0 < u < d_1 \).

**Lemma 49.** The numbers \( d_1, d_2 \) generate a subsemigroup \( d_1 \mathbb{Z}_+ + d_2 \mathbb{Z}_+ \subset H \), such that every number \( s > (d_1 - 1)(d_2 - 1) \) is in \( d_1 \mathbb{Z}_+ + d_2 \mathbb{Z}_+ \subset H \). In particular

\[
(17) \quad n \leq (d_1 - 1)(d_2 - 1).
\]

If the numbers \( d_1, d_2 \) generate the semigroup \( H \) then

\[
(18) \quad n = (d_1 - 1)(d_2 - 1).
\]

**Proof.** Write \( d_2 = d_1 \left\lfloor \frac{d_2}{d_1} \right\rfloor + u \), \((u, d_1) = 1\), \( 0 < u < d_1 \). In what follows we will describe the integers which can be written as linear combination with \( \mathbb{Z}_+ \)-coefficients of \( d_1, d_2 \).

By considering elements of the form \( \nu d_1 + d_2 \) we obtain all elements of the form \( \left( \nu + \left\lfloor \frac{d_2}{d_1} \right\rfloor \right) d_1 + u \), \( \nu \geq 0 \). Let \( I_\lambda \) denote the interval \([\lambda d_1, (\lambda + 1)d_1)\). For \( \lambda \geq \left\lfloor \frac{d_2}{d_1} \right\rfloor \) every interval \( I_\lambda \) contains an integer that is equivalent to \( u \) modulo \( d_1 \).

By considering elements of the form \( \nu d_1 + 2d_2 \) we obtain all elements of the form \( \left( \nu + 2 \left\lfloor \frac{d_2}{d_1} \right\rfloor \right) d_1 + 2u \) etc.

In order to obtain an element with arbitrary residue modulo \( d_1 \) we have to consider all elements of the form \( \nu d_1 + \mu d_2 \), where \( \mu \) takes all values in \( 0, \ldots, d - 1 \). Fix such an \( \mu \).

We consider the combination

\[
(19) \quad \left( \nu + \mu \left\lfloor \frac{d_2}{d_1} \right\rfloor \right) d_1 + \mu u = \left( \nu + \mu \left\lfloor \frac{d_2}{d_1} \right\rfloor + \mu \frac{u}{d_1} \right) d_1 + \left( \mu u - \frac{\mu u}{d_1} d_1 \right).
\]

This proves that for every \( \lambda \geq \left( \frac{\mu}{d_1} \left\lfloor \frac{d_2}{d_1} \right\rfloor + \frac{\mu u}{d_1} \right) \) every interval \( I_\lambda \) contains an integer that is equivalent to \( \mu u \) modulo \( d_1 \).

The greatest value for \( \mu \) is \( d_1 - 1 \) so for

\[
\lambda \geq (d_1 - 1) \left\lfloor \frac{d_2}{d_1} \right\rfloor + \left\lfloor \frac{(d_1 - 1)u}{d_1} \right\rfloor = (d_1 - 1) \left\lfloor \frac{d_2}{d_1} \right\rfloor + u - 1.
\]

For this value of \( \mu \) the coefficient in front of \( d_1 \) in eq. (19) is \( (d_1 - 1) \left\lfloor \frac{d_2}{d_1} \right\rfloor + u - 1 \), and this means that every natural number \( \geq d_1(d_1 - 1) \left\lfloor \frac{d_2}{d_1} \right\rfloor + d_1(u - 1) \) is in \( H \). But now we
replace $d_1 \left\lfloor \frac{d_2}{d_1} \right\rfloor$ by $d_2 - u$ and we verify that:

\[
d_1(d_1 - 1) \left\lceil \frac{d_2}{d_1} \right\rceil + d_1(u - 1) = (d_1 - 1)(d_2 - 1) + u - 1.
\]

We now observe that in the interval $I_{d_1 - 2}$ there is exactly one gap that is equivalent to $d_1 - u \mod d_1$. The value of this gap equals $(d_1 - 1)(d_2 - 1) + u - 1 - u = (d_1 - 1)(d_2 - 1) - 1$.

Therefore, all numbers greater than this gap are in $d_1\mathbb{Z}_+ + d_2\mathbb{Z}_+$. This means that

\[
s \geq (d_1 - 1)(d_2 - 1) \Rightarrow s \in d_1\mathbb{Z}_+ + d_2\mathbb{Z}_2.
\]

Notice that we have proved that $(d_1 - 1)(d_2 - 1) - 1 \notin d_1\mathbb{Z}_+ + d_2\mathbb{Z}_+$. If $H = d_1\mathbb{Z}_+ + d_2\mathbb{Z}_+$ then $(d_1 - 1)(d_2 - 1) - 1$ is a gap and $n = (d_1 - 1)(d_2 - 1)$.

Let $m_i$ be the sequence that enumerates the Weierstrass semigroup, $m_0 = 0$ is always a pole number. Observe that if $2g - 1 \notin H(P)$ then $n = 2g$. Indeed, we know that if the function field $F$ is not hyperelliptic then $m_i \geq 2i + 1$ for $i = 1, \ldots, g - 2$ and $m_{g-1} \geq 2g - 2$ [42, lemma 1.25]. This means that we have two cases for $m_{g-1}$, namely either $m_{g-1} = 2g - 2$ or $m_{g-1} = 2g - 1$.

Let $K$ be the canonical linear series. Observe that at a generic point $P$ of a $K$–classical curve $X$, we have $m_i(P) = g + i$, for $i \geq 1$. In that case the gaps $\mathcal{G}(P)$ and the generic order sequence $\mathcal{E}(P)$ are classical and they are equal to

\[
\mathcal{G}(P) = \{1, \ldots, g\} \text{ and } \mathcal{E}(P) = \{0, \ldots, g - 1\}.
\]

In what follows assume that $m_{g-1} = 2g - 2$ at a point $P$ of the curve. Notice that this condition implies that the maximum gap at $P$ equals to $2g - 1$. Therefore this leads to the study where the curve must satisfy at least one of the following conditions:

1. the curve is not $K$–classical,
2. $P$ is a Weierstrass point.

In fact there is more to say. A more carefull analysis indicates that if $m_{g-1} = 2g - 2$, then the second condition, i.e. $P$ should be a Weierstrass point, is always satisfied. Indeed, we distinguish the following cases:

**Case 1.** The curve $X$ is not $K$–classical and $P$ is ordinary, meaning that $2g - 1 \in \mathcal{G}(P) \neq \{1, \ldots, g\}$ and

\[
\mathcal{G}(P) = \{\epsilon_0^K + 1, \ldots, \epsilon_{g-1}^K + 1\} = \{j_i(P)_K + 1\} 0 \leq i \leq g - 1\}.
\]

Thus, we must have that $\epsilon_{g-1}^K = 2g - 2 = j_{g-1}^K(P)$; This case cannot occur, see [42, Lemma 2.31 p.30.], [8].

**Case 2.** The point $P$ is a Weierstrass point with respect to $K$, and $X$ is a $K$–classical curve. That is $2g - 1 \in \mathcal{G}(P) \neq \{1, \ldots, g\}$.

\[
\mathcal{G}(P) = \{j_i^K(P) + 1\} 0 \leq i \leq g - 1\} \neq \{\epsilon_i^K + 1\} 0 \leq i \leq g - 1\},
\]

where $j_{g-1}^K(P) = 2g - 2$ but $\epsilon_{g-1}^K = g - 1 \neq j_{g-1}^K(P)$.

**Case 3.** The point $P$ is a Weierstrass point with respect to $K$ and the curve $X$ is not $K$–classical. That is $g - 1 < \epsilon_{g-1}^K \leq j_{g-1}^K(P)$.

We have proved the following:

**Proposition 50.** If a curve has symmetric Weierstrass semigroup at a point $P$, then this point is a Weierstrass point. Moreover if the Weierstrass semigroup at this point is generated by two elements, $d_1, d_2$, i.e. when we have equality in lemma 49, eq. (18), the genus of the curve is given by

\[
g = \frac{(d_1 - 1)(d_2 - 1)}{2}.
\]

If the Weierstrass semigroup at this point cannot be generated by two elements, then

\[
g < \frac{(d_1 - 1)(d_2 - 1)}{2}.
\]
Example 51. Consider the case of Artin–Schreier curve defined in example 42 or the most general case
\[ y^q - y = f, \]
where \( f \) is a polynomial of degree \( m \) which is prime to \( p \), and \( q = p^k \) have genus \( g = \frac{(q-1)(m-1)}{2} \). Therefore, using the theory of Boseck invariants we see that the biggest gap is
\[ \delta - 2q + 1 = (m + 1)(q - 1) - 2q + 1 = (m - 1)(q - 1) - 1 = 2q - 1, \]
and \( n = 2q \).

To the same conclusion we arrive using lemma 49. Indeed, it is known [39] that the Weierstrass semigroup is generated by \( m, q \). Thus \( n = (q - 1)(m - 1) = 2q \).

Observe that certain function fields like the Hermitian function fields and their quotients \( y^q + y = x^m, m | q + 1, q = p^k \), satisfy \( 2n = g \). Also Matignon-Lehr curves given by [23, 4.1], are given as Artin-Schreier extensions of the rational function field and satisfy \( 2n = g \). Moreover if \( k = \mathbb{F}_{q^2} \), where \( \mathbb{F}_{q^2} \) is a finite field with \( q^2 \) elements, then Hermitian and their quotients \( y^q + y = x^m, m | q + 1 \), are certain maximal curves and can be viewed as the Artin-Schreier curves \( y^q - y = f(x) \), where \( f(x) \in \mathbb{F}_q[x] \) and \( \gcd(d, f, p) = 1, \) see [11, Theorem 5.4].

Here it is nice to point out the connection of maximal curves with Weierstrass semigroups: Assume that \( X \) is a maximal curve over \( \mathbb{F}_{q^2} \) of genus \( g \). Let \( X(\mathbb{F}_{q^2}) \) be the set of all \( \mathbb{F}_{q^2} \)-rational points of \( X \). Let also \( P \) be a point in \( X(\mathbb{F}_{q^2}) \) and let \( m_i = m_i(P) \) be a pole number at \( P \).

Then according to Lewittes, [26, th 1(b)], [5, p.46]
\[ \#X(\mathbb{F}_{q^2}) = N \leq q^2 m_1 + 1. \]

Combining the above with the Hasse-Weil bound for a maximal curve, we obtain the following bound \( (m_1, \text{any pole number}) \)
\[ \#X(\mathbb{F}_{q^2}) = q^2 + 1 + 2qg \leq q^2 m_1 + 1, \]
or
\[ g \leq \frac{q(m_1 - 1)}{2}. \]

If \( m_i = q \) the above is a result due to Ihara’s, [16] and the equality is obtained when \( X \) is the Hermitian curve. Notice that if \( P \) is \( \mathbb{F}_{q^2} \) rational point then \( q, q + 1 \) are always pole numbers, [5, proposition 1.5,(iv)].

4. HASSE-WITT MATRIX AND SEMIGROUPOS

In [41] K.O. Stöhr and P.Viana introduced a completely local construction of the Hasse-Witt matrix [15]. One of their results that will be useful to us is the following

Proposition 52. Let \( (a_{ij}) \) be the Hasse-Witt matrix and consider the product
\[ A_r := (a_{ij}) (a_{ij}^p) \cdots (a_{ij}^{p^{r-1}}) \]
Let \( P \) be any point in the curve in question. For each positive \( r \) the rank of the matrix \( A_r \)
is larger than or equal to the number of gaps at divisible by \( p^r \).

Proof. [41, cor. 2.7] \( \square \)

Notice that \( (a_{ij}) \) is dual to the Cartier operator and the matrix \( A_r \) corresponds to the application of \( r \)-times of the corresponding \( p \)-linear map. For more information concerning the Hasse-Witt matrix and the Cartier operator we refer to [3]. The rank of \( A_r \) equals the rank of the Cartier Operator.

Also notice that if the rank is zero, then there are no gaps divisible by \( p^r \) and every number divisible by \( p^r \) is a pole number. This can also be seen by different methods, see [32].
Remark 53. The $p$-rank of the Jacobian is the rank of the matrix $A_p$. Having $p$-rank zero does not give us any information about pole numbers since every number greater than $2g$ is a pole number. For this notice that for $p \geq 5$ we have $2g < p^2$.

Proposition 54. If a curve has Hasse-Witt matrix zero then every integer divisible by $p$ is a pole number. This implies that $G_1(P)$ is at most an extension of an elementary abelian group with a cyclic group of order $p$.

Proof. If there was a pole number $m$ such that $m < p$ then $G_1(P)$ is faithfully represented in $L(mP)$ and it should be elementary abelian. If $p < m$, then $m_1 = p$ and by lemma 9 we have that $\ker \rho_1$ divides $p$ therefore $\ker \rho_1$ is either trivial or isomorphic to a cyclic group of order $p$. The group $G_1(P)$ is given by a short exact sequence

$$1 \to \ker \rho_1 \to G_1(P) \to V \to 1$$

where $V$ is an elementary abelian group. □

Example 55. A classical example of a curve with nilpotent Cartier operator is given by the Hermitian curve

$$y^{p^r} + y = x^{p^r+1},$$

(which is isomorphic to the Fermat curve $x^{p^r+1} + y^{p^r+1} + 1 = 0$, with Cartier operator satisfying $C' = 0$ [32].

Corollary 56. If $X$ has Hasse-Witt matrix zero, and is a big action then $G_2(P)$ is cyclic of order $p$.

Proof. If the first pole number is not divisible by $p$, then we have a faithful two dimensional representation of $G_1(P)$ on $L(mP)$, so $G_1(P)$ is elementary abelian, therefore bounded by a linear bound on $g$ [30]. The first pole number is divisible by $p$ since the Hasse-Witt matrix is assumed to be zero. Also the first pole number is $m_1 = |G_2(P)|$ by lemma 18. Thus, $G_2(P)$ is a cyclic group of order $p$. □

Corollary 57. Let $X$ be a curve of genus $g \geq 2$. If the curve $X$ has nilpotent Cartier operator, i.e., $C' = 0$, and moreover $p^\ell \leq g$, then the curve is non-classical with respect to the canonical linear series. Moreover all curves with zero Cartier operator that are equipped with an automorphism group that has a wild ramified point and $g \neq p - 1$, are non-classical with respect to the canonical linear series with only one hyperelliptic exception, namely $y^2 = x^p - x$.

Proof. Recall that a curve is classical for the canonical divisor if and only if the gap sequence for a non-Weierstrass point is given by $\{1, 2, \ldots, g\}$. Proposition 52 implies that $p^\ell$ is a pole number. Since $p^\ell \leq g$ the curve can’t be classical.

Assume now that the curve $X$ has an automorphism group $G$ such that there is a wild ramified point in the cover $X \to X/G$, and has zero Cartier operator. Therefore $\ell = 1$, and according to [34] the existence of wild ramification forces $p - 1 \leq g$ or the curve is the hyperelliptic curve $y^2 = x^p - x$. For the case $p - 1 < g$ the result follows. We don’t know what happens for the $g = p - 1$ case (for small primes, $p < 5$, these curves are classical because they satisfy the criterion in eq. (21) below).

The hyperelliptic curve $y^2 = x^p - x$ is a curve of genus $g = (p - 1)/2$ and it is also an Artin-Schreier extension of the rational function field. In [43] it is proved that this curve has zero Cartier operator. In fact this is the superspecial hyperelliptic curve with the biggest possible genus (see [2, Theorem 1.2]). It is well known that all the hyperelliptic curves of arbitrary characteristic are $K$-classical, [36, Satz 8]. □

Notice that all superspecial hyperelliptic curves $X$ are $K$-classical and they satisfy the following equation

$$p > \deg K = 2g - 2, \implies X \text{ is classical with respect to } K,$$
(see for instance [22, Theorem 15] and [48, prop. 14.2.64, p. 561]), since from [2, Theorem 1.2] their genera are upper bounded by $\frac{p-1}{2}$. Moreover when $\text{char} k = 0$ or when eq. (21) is valid then $X$ is K-classical.

The remarkable fact is that neither of the wild ramified coverings $X \longrightarrow X/G$, with $X$ being classical or not with respect to the canonical linear series, satisfy equation 21 for $g \geq 3$ (the reader should exclude the hyperelliptic exceptional case considered in corollary 57). This statement follows from the simple facts that for these curves $p \leq g+1$, and $2g-2 \geq g+1$ for every $g \geq 3$. Keep also in mind that there do not exist non K-classical curves for $g \leq 3$, with only one exception for $p = g = 3$, see [19].

**Remark 58.** Corollary 57, restricted to the the world of maximal curves is similar to the construction [5, proposition 1.7.]. Indeed, from theorem 3.3 in [11], every maximal and minimal curve over $F_{q^2}$, $q = p^f$ have nilpotent Cartier operator with $C^\ell = 0$. The small difference in the lower bound that is given there, $p^f - 1 \leq g$ is explained because if $X$ is classical and $g = p^f - 1$ then $n_1 = p^f$ and from [5, proposition 1.5 and the remark just before this] this curve should be the Hermitian, that is a contradiction since the Hermitian curve has genus $p^f(p^f-1)/2$. Observe also that the genus of the nilpotent curves is bounded by

$$g \leq \frac{p^f(p^f-1)}{2},$$

where $\ell$ is the rank of nilpotency [32, th. 4.1].

**Remark 59.** Combining the results from proposition [5, Proposition 1.10], lemma 49 here, and [21, theorem 2.5], we get the following:

Consider a maximal curve over $F_{q^2}$, with genus $g \geq 2$ and the set $\Sigma$ of $F_{q^2}$-rational points such that the Weierstrass semigroup up to $q+1$, and hence all the Weierstrass semigroup, is generated by two integers. Then the Weierstrass semigroup at all points of $\Sigma$ is symmetric, i.e. their max gap is always at $2g-1$.

Since $q, q+1$ must always be in the Weierstrass semigroup at such a point, this condition to the numbers of generators, corresponds to the minimum number of generators that a maximal curve can have. Thus we can rephrase: The maximal curves over $F_{q^2}$ with minimal set of generators for their Weierstrass semigroups at a $F_{q^2}$-rational point, have symmetric Weierstrass semigroups at this point.

The next proposition shows that the condition on the number of generators of the Weierstrass semigroup in a $F_{q^2}$-rational point is not necessary so that the point has symmetric Weierstrass semigroup.

**Proposition 60.** Let $X_{GK}$ be the maximal curve over $F_{q^2}$ defined in Giulietti–Korchmáros, [13]. Then the Weierstrass semigroup at the rational point $X_{\infty}$ is symmetric. This is an example of curve where the equality in proposition 50 fails.

**Proof.** We will use the notation from [13]. Let $n = p^h$, $p$ a prime, $h \geq 1$ and $q = n^3$. From [13, equation 10, p. 236], we can write

$$2g_{GK} - 1 = \sum_{i=1}^{3} \left( \frac{d_i - 1}{d_i} \right) \alpha_i$$

$$= -\alpha_1 + (n^2 - n)\alpha_2 + (n-1)\alpha_3,$$

where $\alpha_1 = n^3 - n^2 + n, \alpha_2 = n^3, \alpha_3 = n^3 + 1$, are the generators of $H(X_{\infty})$, [13, proposition 5], $d_0 = 0, d_1 = \alpha_1, d_2 = \text{g.c.d} (\alpha_1, \alpha_2), \text{ and } d_3 = \text{g.c.d} (\alpha_1, \alpha_2, \alpha_3)$. Suppose now that $2g_{GK} - 1$ is a pole number. From [13, lemma 5] there are uniquely determined no negative integers $j_i$, with $i = 1, \ldots, 3$ and $j_1, j_2 \leq n^2 - n, j_3 \leq n - 1$ such that

$$2g_{GK} - 1 = \alpha_1 j_1 + \alpha_2 j_2 + \alpha_3 j_3.$$
From eq. 23 this is equivalent to
\[(j_1 + 1) \alpha_1 + j_2 \alpha_2 + j_3 \alpha_3 = (n^2 - n) \alpha_2 + (n - 1) \alpha_3\]
which shows that \(j_2 = n^2 - n, j_3 = n - 1, j_1 = -1\), a contradiction!

From [13, theorem 2], we have that \(2g_X = (n^3 + 1)(n^2 - 2) + 2\). With an immediate calculation we show that \(2g_X < (\alpha_1 - 1)(\alpha_2 - 1)\). Indeed, notice that \((\alpha_1, \alpha_2) = n\) so we can not apply lemma 49. However we can show that \(2g_{GK} < (\alpha_1 - 1)(\alpha_3 - 1)\) since the right hand of the inequality is greater than \((\alpha_1 - 1)(\alpha_2 - 1)\). Also \(2g_{GK} \leq (\alpha_2 - 1)(\alpha_3 - 1)\) since otherwise \(X_{GK}\) is Hermitian, a contradiction.

**Remark 61.** Let \(X\) be the Hermitian, or the Garcia–Stichtenoth curve, [9], or the Giulietti–Korchmáros maximal curve over \(\mathbb{F}_{q^2}\). In fact these are three of the five known families of maximal curves over \(\mathbb{F}_{q^2}\) (in the sense that every known maximal curve arise as an \(\mathbb{F}_{q^2}\)-cover of these curves). The other two families are the Deligne–Lusztig curves that are \(\mathbb{F}_{q^2}\) maximal curves with \(q\) being a certain power of three and two respectively (see [13, introduction]). Then \(X\) has symmetric Weierstrass semigroups in a \(\mathbb{F}_{q^2}\) rational point \(P\). The Hermitian and the Garcia–Stichtenoth curves are examples where we obtain the equality in proposition 50.

Indeed, Hermitian and the Garcia–Stichtenoth curves have symmetric Weierstrass semigroups in \(\mathbb{F}_{q^2}\) rational points because there are Artin–Schreier curves, thus the Weierstrass semigroup in a \(\mathbb{F}_{q^2}\) rational point is generated by 2 elements. From remark 59, the Weierstrass semigroup in any rational point is symmetric. For the Hermitian case, we have that the generators of \(H(P)\) are \(d_1 = q, d_2 = q + 1\) and thus \(g_{H} = \frac{q(q-1)}{2}\). For the Garcia–Stichtenoth curve \(y^2 - y = x^{q^2 - q + 1}, d_1 = \ell^2 - \ell + 1, d_2 = \ell^2\) and \(g_{G} = \frac{(d_2 - 1)(d_2 - 1)}{2}\) [9, theorem 1]. The assertion for the Giulietti–Korchmáros maximal curve comes from proposition 60.

**Remark 62.** Let \(X\) be the Hermitian, or the Garcia–Stichtenoth curve, or the Giulietti–Korchmáros maximal curve over \(\mathbb{F}_{q^2}\). Denote by \(P \in X\) the \(\mathbb{F}_{q^2}\)-rational place where \(H(P)\) is symmetric and with \(F\) its function field. Take \(G\) to be a \(p\) subgroup of the automorphism group of \(X\). Take the Galois cover \(\pi : X \rightarrow X/G\) and suppose that there is only one place ramified in that cover. We will prove that the Weierstrass semigroup at \(P\) is symmetric if and only if the Weierstrass semigroup at the point \(\pi(P)\) is symmetric.

Since every \(p\)-group is solvable we can decompose the cover \(\pi\) to a sequence of Artin–Schreier covers. So it is enough to consider the case of covers of the form \(F = F_0(y)\) where
\[
y^{p^h} - y = f, \text{ where } f \in F_0,
\]
and \(f\) has a unique pole at the rational place \(\pi(P)\) of \(F_0\), with \(v_{F_0}(f) = -m < 0\). Then \(F/F_0\) is totally ramified at the place \(P\), with \(P|\pi(P)\). Notice that the conditions of Lewittes theorem, 19 here, are satisfied. Now we can say the following: Lewittes showed that if \(g_{X/G} = 0\) then \(H(P)\) is symmetric; if \(g_{X/G} > 0\) and \(H(\pi(P))\) is symmetric then \(H(P)\) is symmetric, see [25, p.36 after corollary].

Now we will prove that if \(H(P)\) is symmetric then \(H(\pi(P))\) is symmetric. According to Lewittes, theorem 19 here, the max gap in \(P\) which equals to \(2g_X - 1\) is given by
\[
2g_X - 1 = (m + 1)(p^h - 1) + p^h h_{g_{X/G}} = (m + 1)(p^h - 1) + p^h h_{g_{X/G}} - p^h + 1
\]
\[
\deg \text{Diff} + p^h (h_{g_{X/G}} - 1) + 1.
\]
Combining eq. (25) and Riemann–Hurwitz formula we should have that \(h_{g_{X/G}} = 2g_{X/G} - 1\), and that means that the Weierstrass semigroup at \(\pi(P)\) is also symmetric.

Notice that \(\pi(P)\) is also an \(\mathbb{F}_{q^2}\) rational point from the transitivity of the relative degrees. Finally notice that all the curves considered by Stichtenoth–Garcia–Xing in [10,
Section 3, Theorem 3.2] (see also [9, theorem 2, \( m = 1 \) case]), have symmetric Weierstrass semigroups at \( \pi(P) \).

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