

COMPUTING POLYNOMIALS OF THE RAMANUJAN t_n CLASS INVARIANTS.

ELISAVET KONSTANTINOOU AND ARISTIDES KONTOGEORGIS

ABSTRACT. We compute the minimal polynomials of the Ramanujan values t_n , where $n \equiv 11 \pmod{24}$, using Shimura reciprocity law. These polynomials can be used for defining the Hilbert class field of the imaginary quadratic field $\mathbb{Q}(\sqrt{-n})$, and have much smaller coefficients than the Hilbert polynomials.

INTRODUCTION

Ramanujan on his third notebook, pages 392 and 393 in the pagination of [10, vol. 2] defined the values

$$(1) \quad t_n := \sqrt{3}q_n^{1/18} \frac{f(q_n^{1/3})f(q_n^3)}{f^2(q_n)}$$

where

$$q_n = \exp(-\pi\sqrt{n}).$$

The function f is equal to:

$$(2) \quad f(-q) := \prod_{n=1}^{\infty} (1 - q^n) = q^{-1/24} \eta(\tau)$$

where $q = \exp(2\pi i\tau)$, $\tau \in \mathbb{H}$ and $\eta(\tau)$ denotes the Dedekind eta-function. Without any further explanation on how he found them, Ramanujan gave the following table of polynomials $p_n(t)$ based on t_n for five values of n :

n	$p_n(t)$
11	$t - 1$
35	$t^2 + t - 1$
59	$t^3 + 2t - 1$
83	$t^3 + 2t^2 + 2t - 1$
107	$t^3 - 2t^2 + 4t - 1$

In [2] Bruce C. Berndt and Heng Huat Chan proved that these polynomials indeed have roots the Ramanujan values t_n . Unfortunately, their method could not be applied for higher values of n and they asked for an efficient way of computing the polynomials p_n for every n . Moreover, the authors proved that if the class number of $K_n := \mathbb{Q}(\sqrt{-n})$ is odd and $n \in \mathbb{N}$ is squarefree so that $n \equiv 11 \pmod{24}$ then t_n is a real unit generating the Hilbert class field.

It is known that the Hilbert class field can also be constructed by considering the irreducible polynomial of the algebraic integer $j(\theta)$ where $\theta = -1/2 + i\sqrt{n}/2$. The minimal polynomial of $j(\theta)$ is called the Hilbert polynomial. It is interesting to point out that the coefficients of the polynomials p_n have remarkably smaller size compared to the coefficients of the Hilbert polynomials. Therefore, finding an efficient and simple method for their construction is highly desirable and has a direct impact to applications where the explicit construction of class fields is

needed. Problems such as primality testing/proving [1], the generation of elliptic curve parameters [7] and the representability of primes by quadratic forms [4] can be considerably improved if the polynomials p_n could be constructed with an efficient and easily implemented way.

An explicit construction of the Hilbert class field has been done by N. Yui and D. Zagier in [12] using the Weber functions. Yui and Zagier use a clever construction of a function on quadratic forms $ax^2 + bxy + cy^2$ that does not depend on the equivalence class of quadratic forms. The construction of a similar function in the case of p_n polynomials seems very complicated and it is clear that a different approach must be followed. Our construction came from the enforcement of Shimura reciprocity law on the values t_n . Shimura reciprocity law has been proven to be a very powerful tool for attacking similar problems [3], [5], [6] and can provide methods for systematically determining the instances when a given function yields a class invariant and for computing the minimum polynomial of a class invariant.

The contribution of this paper is twofold. Firstly, we prove that the values t_n constitute class invariants for all values of $n \equiv 11 \pmod{24}$. Expanding the theorem in [2], we show that t_n generates the Hilbert class field not only in the case that the class number of K_n is odd but also when it is even. Secondly, we provide an efficient method for constructing the irreducible polynomials p_n from the Ramanujan values t_n and thus answer to the demand placed in [2] for a direct and easily applicable construction method. Moreover, we have implemented our method in *gp-pari* [9] and we present all polynomials p_n for all integers $107 < n \leq 1000$, where $n \equiv 11 \pmod{24}$ (table 1).

The rest of the paper is organized as follows. In the first section we fix the notation and we give the Ramanujan t_n values in terms of the Dedekind eta function. Next, we define six modular functions of level 72 $R, R_1, R_2, R_3, R_4, R_5$ and we compute the action of the generators of the group $SL_2(\mathbb{Z})$ on them. In section 2 we prove that t_n is indeed a class invariant, for all values $n \equiv 11 \pmod{24}$, and in final section we employ Shimura reciprocity law in order to compute the conjugates of t_n under the action of the class group and compute the minimal polynomial of t_n .

1. NOTATION

Let $SL_2(\mathbb{Z})$ be the group of matrices with integer entries and of determinant one. It is known [11, cor. 1.6] that the group $SL_2(\mathbb{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL_2(\mathbb{Z})$ induces an action on the upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\},$$

by sending $z \mapsto \frac{az+b}{cz+d}$.

Let η denote the Dedekind function:

$$(3) \quad \eta(\tau) = \exp(2\pi i\tau/24) \prod_{n=1}^{\infty} (1 - q^n), \text{ where } \tau \in \mathbb{H} \text{ and } q = \exp(2\pi i\tau).$$

Then η -function is transformed by S and T as follows [11, prop. 8.3]

$$(4) \quad \eta(\tau + 1) = e^{2\pi i/24} \eta(\tau) \text{ and } \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau).$$

From equation (4) we can compute the action of every element g of $SL_2(\mathbb{Z})$ on the η -function, since g can be written as a word in S, T .

We denote by H_n the Hilbert field of $K_n := \mathbb{Q}(\sqrt{-n})$, *i.e.*, the maximal Abelian unramified extension of K_n . The extension H_n/K_n is Galois with Galois group equal to the class group of fractional ideals modulo principal fractional ideals. For imaginary quadratic fields the class group can be represented as the space of binary quadratic forms $ax^2 + bxy + cy^2$ modulo an equivalence relation [4, th. 5.30]. We will denote by $[a, b, c]$ the quadratic form $ax^2 + bxy + cy^2$ and we will call two quadratic forms $[a_i, b_i, c_i]$ for $i = 1, 2$ equivalent if the corresponding roots $\tau_i \in \mathbb{H}$ are in the same orbit of $SL_2(\mathbb{Z})$ acting on \mathbb{H} . Using the identification of equivalence classes of quadratic forms with the ideal class group we can define the structure of an abelian group on the set of equivalence classes of quadratic forms.

Let $\ell_0 := (1, 1, \frac{1-d}{4})$ ($d = -n \equiv 1 \pmod{4}$, $n \in \mathbb{N}$) be the zero element in this group. This element corresponds to the root

$$\tau_{\ell_0} = -\frac{1}{2} + i\frac{\sqrt{n}}{2}.$$

Set

$$q_n = \exp(-\pi\sqrt{n}) = -\exp(2\pi i\tau_{\ell_0}).$$

Then

$$f(q_n) = f(-\exp(2\pi i\tau_{\ell_0})) = \exp(2\pi i\tau_{\ell_0})^{-1/24}\eta(\tau_{\ell_0}),$$

$$f(q_n^3) = \exp(2\pi i\tau_{\ell_0})^{-3/24}\eta(3\tau_{\ell_0}),$$

$$f(q_n^{1/3}) = (-1)^{1/18} \exp(2\pi i\tau_{\ell_0})^{-\frac{1}{3 \cdot 24}} \eta\left(\frac{\tau_{\ell_0}}{3} + \frac{2}{3}\right)$$

Taking equation (1) and all the above equations into consideration we arrive easily at the following:

Lemma 1.1. *The Ramanujan value t_n is given by*

$$t_n = \sqrt{3}R_2(\tau_{\ell_0}),$$

where

$$R_2(\tau) = \frac{\eta(3\tau)\eta(\frac{1}{3}\tau + \frac{2}{3})}{\eta^2(\tau)}$$

Now let $N \in \mathbb{N}$ and $\Gamma(N)$ be the group

$$\Gamma(N) := \left\{ \gamma \in SL_2(\mathbb{Z}), \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

The field of modular functions of level N consists of the meromorphic functions g of the upper half plane \mathbb{H} that are invariant under the group $\Gamma(N)$, *i.e.* $g(\gamma\tau) = g(\tau)$ for every $\tau \in \mathbb{H}$ and $\gamma \in \Gamma(N)$. Every modular function is periodic with period N and thus it admits a Fourier expansion of the form

$$g(q) = \sum_{\nu=-i}^{\infty} a_{\nu}q^{\nu},$$

where $q = \exp(2\pi i\tau/N)$. We will limit ourselves to modular functions where all coefficients of the Fourier expansions are elements of the field $\mathbb{Q}(\zeta_N)$. The Galois group $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ is isomorphic to the group $(\frac{\mathbb{Z}}{N\mathbb{Z}})^*$ by defining $\sigma_d(\zeta_N) = \zeta_N^d$ for every $(d, N) = 1$.

The action of the group $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ can be extended to the field of modular functions of level N with coefficients in $\mathbb{Q}(\zeta_N)$, as follows:

$$(5) \quad g(q)^{\sigma_d} = \sum_{\nu=-i}^{\infty} \sigma_d(a_{\nu})q^{\nu}, \quad \sigma_d \in \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}).$$

Moreover, the action of an element $A \in GL_2(\mathbb{Z})$ on modular functions $g(q)$ can be expressed as

$$(6) \quad g^A = (g^B)^{\sigma_{\det(A)}}$$

where $A = B \cdot \begin{pmatrix} 1 & 0 \\ 0 & \det(A) \end{pmatrix}$ and $B \in SL_2(\mathbb{Z})$.

Lemma 1.2. *The following functions are modular functions of level 72.*

$$\begin{aligned} R(\tau) &= \frac{\eta(3\tau)\eta(\tau/3)}{\eta^2(\tau)} \\ R_1(\tau) &= \frac{\eta(3\tau)\eta(\tau/3 + 1/3)}{\eta^2(\tau)} \\ R_2(\tau) &= \frac{\eta(3\tau)\eta(\tau/3 + 2/3)}{\eta^2(\tau)} \\ R_3(\tau) &= \frac{\eta(\tau/3)\eta(\tau/3 + 2/3)}{\eta^2(\tau)} \\ R_4(\tau) &= \frac{\eta(\tau/3)\eta(\tau/3 + 1/3)}{\eta^2(\tau)} \\ R_5(\tau) &= \frac{\eta(\tau/3 + 2/3)\eta(\tau/3 + 1/3)}{\eta^2(\tau)}. \end{aligned}$$

Moreover, the element $\sigma_d : \zeta_{72}^d \mapsto \zeta_{72}^d$ for $(d, n) = 1$ acts on them as follows:

$$(7) \quad \begin{aligned} \sigma_d(R) &= R, \\ \sigma_d(R_1) &= \begin{cases} \zeta_{72}^{d-1} R_1 & \text{if } d \equiv 1 \pmod{3} \\ \zeta_{72}^{d-2} R_2 & \text{if } d \equiv 2 \pmod{3} \end{cases} \\ \sigma_d(R_2) &= \begin{cases} \zeta_{72}^{2d-2} R_2 & \text{if } d \equiv 1 \pmod{3} \\ \zeta_{72}^{2d-1} R_1 & \text{if } d \equiv 2 \pmod{3} \end{cases} \\ \sigma_d(R_3) &= \begin{cases} \zeta_{72}^{d-1} R_3 & \text{if } d \equiv 1 \pmod{3} \\ \zeta_{72}^{d-2} R_2 & \text{if } d \equiv 2 \pmod{3} \end{cases} \\ \sigma_d(R_4) &= \begin{cases} \zeta_{72}^{2d-2} R_4 & \text{if } d \equiv 1 \pmod{3} \\ \zeta_{72}^{2d-1} R_3 & \text{if } d \equiv 2 \pmod{3} \end{cases} \\ \sigma_d(R_5) &= \zeta_{72}^{3d-3} R_5. \end{aligned}$$

Proof. The fact that the above equations are indeed modular of level 72 is a direct computation using the transformations of the η -functions under the generators T, S of $SL_2(\mathbb{Z})$ given in (4). The action of σ_d given in (7) is computed by considering the Fourier expansions of the η -factors of the functions R_i . For instance let us compute the action of the element σ_d on R_2 . We begin by computing its action on $\eta(\tau/3 + 2/3)$:

$$(8) \quad \begin{aligned} \eta(\tau/3 + 2/3) &= \exp\left(\frac{2\pi i}{24}(\tau/3 + 2/3)\right) \sum_{\nu=0}^{\infty} a_n \exp\left(\frac{2\pi i\nu}{3}\tau + \frac{2\pi i\nu}{3}\right) = \\ &= \exp\left(\frac{2\pi i}{24}(\tau/3)\right) \zeta_{72}^2 \sum_{\nu=0}^{\infty} \zeta_3^{2\nu} a_n \exp\left(\frac{2\pi i\nu}{3}\tau\right), \end{aligned}$$

where $\zeta_3 = \zeta_{72}^{24}$ is a primitive third root of unity. The desired formulas of equations (7) follow from the definition of the action of σ_d on ζ_{72} and arguing as above. For example (notice that from (3) the Fourier expansion for $\eta(\tau)$ has rational coefficients so it is invariant under the action of σ_d),

$$\sigma_d(R_2) = \sigma_d\left(\frac{\eta(3\tau)\eta(\tau/3 + 2/3)}{\eta^2(\tau)}\right) = \frac{\eta(3\tau)}{\eta^2(\tau)} \sigma_d(\eta(\tau/3 + 2/3)).$$

By (8) we have that

$$\sigma_d(\eta(\tau/3 + 2/3)) = \exp\left(\frac{2\pi i}{24}(\tau/3)\right) \zeta_{72}^{2d} \sum_{\nu=0}^{\infty} \zeta_3^{2\nu d} a_n \exp\left(\frac{2\pi i \nu}{3}\tau\right).$$

If $d \equiv 1 \pmod{3}$ then $\zeta_3^{2\nu d} = \zeta_3^{2\nu}$, thus

$$\sigma_d(R_2) = \zeta_{72}^{2d-2} R_2.$$

If $d \equiv 2 \pmod{3}$ then $\zeta_3^{2\nu d} = \zeta_3^\nu$, and

$$\sigma_d(\eta(\tau/3 + 2/3)) = \eta(\tau/3 + 1/3) \zeta_{72}^{2d-1} \Rightarrow \sigma_d(R_2) = \zeta_{72}^{2d-1} R_1.$$

□

Later we will use some computer algebra programs in order to prove that t_n is indeed a class invariant and find the minimal polynomials of t_n . For this reason it is convenient to have the actions of the elements S, T, σ_d in matrix form. Using (4) we give the following matrix action of the elements S, T, σ_d on the functions R, R_1, \dots, R_5 :

$$(9) \quad \begin{pmatrix} R(\tau+1) \\ R_1(\tau+1) \\ R_2(\tau+1) \\ R_3(\tau+1) \\ R_4(\tau+1) \\ R_5(\tau+1) \end{pmatrix} = A_T \begin{pmatrix} R(\tau) \\ R_1(\tau) \\ R_2(\tau) \\ R_3(\tau) \\ R_4(\tau) \\ R_5(\tau) \end{pmatrix}, \quad \begin{pmatrix} R(\frac{-1}{\tau}) \\ R_1(\frac{-1}{\tau}) \\ R_2(\frac{-1}{\tau}) \\ R_3(\frac{-1}{\tau}) \\ R_4(\frac{-1}{\tau}) \\ R_5(\frac{-1}{\tau}) \end{pmatrix} = A_S \begin{pmatrix} R(\tau) \\ R_1(\tau) \\ R_2(\tau) \\ R_3(\tau) \\ R_4(\tau) \\ R_5(\tau) \end{pmatrix},$$

$$\begin{pmatrix} \sigma_d R \\ \sigma_d R_1 \\ \sigma_d R_2 \\ \sigma_d R_3 \\ \sigma_d R_4 \\ \sigma_d R_5 \end{pmatrix} = A_{\sigma_d} \begin{pmatrix} R \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{pmatrix},$$

where

$$(10) \quad A_T := \begin{pmatrix} 0 & \zeta_{72}^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_{72}^3 & 0 & 0 & 0 \\ \zeta_{72}^6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\zeta_{72}^3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\zeta_{72}^6} \\ 0 & 0 & 0 & \frac{1}{\zeta_{72}^3} & 0 & 0 \end{pmatrix},$$

$$A_S := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\zeta_{72}^3(-\zeta_{72}^{30} + \zeta_{72}^6)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\zeta_{72}^3}{-\zeta_{72}^{30} + \zeta_{72}^6} & 0 \\ 0 & -\zeta_{72}^{33} + \zeta_{72}^9 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-\zeta_{72}^{30} + \zeta_{72}^6}{\zeta_{72}^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

and

$$A_{\sigma_d} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{72}^{d-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_{72}^{2d-d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{72}^{2d-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_{72}^{d-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{72}^{3d-3} \end{pmatrix} \quad \text{if } d \equiv 1 \pmod{3},$$

$$A_{\sigma_d} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_{72}^{d-2} & 0 & 0 & 0 \\ 0 & \zeta_{72}^{2d-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_{72}^{2d-1} & 0 \\ 0 & 0 & 0 & \zeta_{72}^{d-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{72}^{3d-3} \end{pmatrix} \text{ if } d \equiv 2 \pmod{3},$$

2. SHIMURA RECIPROCITY LAW AND t_n CLASS INVARIANTS

Let $K_n = \mathbb{Q}(\sqrt{-n})$ be an imaginary quadratic number field, \mathcal{O} the ring of integers of K_n and $\theta = \frac{1}{2} - i\frac{\sqrt{n}}{2}$. It is known that $j(\theta)$ is an algebraic integer that generates the Hilbert class field of K_n , and moreover that the conjugates $j(\theta)$ under the action of the class group are given by

$$j(\theta)^{[a, -b, c]} = j(\tau_{[a, b, c]}),$$

where $\tau_{[a, b, c]}$ is the unique root of $ax^2 + bx + c$ with positive imaginary part, *i.e.* $\tau_{[a, b, c]} = \frac{-b + i\sqrt{-D}}{2a}$.

There is an efficient algorithm for computing a set of non-equivalent quadratic forms and the minimal polynomial $f_D(x)$ of $j(\theta)$ can easily be computed from the floating point approximation of the values $j(\tau_{[a, b, c]})$:

$$f_D(x) = \prod_{[a, b, c] \in Cl(\mathcal{O})} (x - j(\tau_{[a, b, c]})).$$

For instance, the polynomial for the quadratic extension of discriminant -107 is:

$$f_{-107}(x) = x^3 + 129783279616 \cdot 10^3 x^2 - 6764523159552 \cdot 10^6 x + 337618789203968 \cdot 10^9.$$

The disadvantage of using the above polynomials for the construction of the Hilbert class field is the very large size of their coefficients compared to the coefficients of the polynomials p_n . Notice that the polynomial p_n for $n = 107$ is equal to $x^3 - 2x^2 + 4x - 1$. In the literature there are alternative explicit constructions of the Hilbert class fields based on the Weber functions [12], or by other modular functions [3], [5], [6]. In [2] the authors proved that also the values t_n can generate the Hilbert class field providing the following theorem.

Theorem 2.1. *If $n \equiv 11 \pmod{24}$ and the class group of K_n is odd then t_n generates the Hilbert field.*

Proof. [2, th. 4.1]. □

The Shimura reciprocity law can be applied in order to compute the minimal polynomial of t_n and give an alternative proof of theorem 2.1 by removing the odd class number requirement.

In order to prove that t_n is a class invariant also for the case that the class group of K_n is even, we will use the following construction:

Theorem 2.2. *Let $\mathcal{O} = \mathbb{Z}[\theta]$ be the ring of algebraic integers of the imaginary quadratic field K , and assume that $x^2 + Bx + C$ is the minimal polynomial of θ . Let $N > 1$ be a natural number, x_1, \dots, x_r be generators of the abelian group $(\mathcal{O}/N\mathcal{O})^*$ and $\alpha_i + \beta_i\theta \in \mathcal{O}$ be a representative of the class of the generator x_i . For each representative we consider the matrix:*

$$A_i := \begin{pmatrix} \alpha_i - B\beta_i & -C\beta_i \\ \beta_i & \alpha_i \end{pmatrix}.$$

If f is a modular function of level N and if for all matrices A_i it holds that

$$(11) \quad f(\theta) = f^{A_i}(\theta), \text{ and } \mathbb{Q}(j) \subset \mathbb{Q}(f)$$

then $f(\theta)$ is a class invariant.

Proof. [5, Cor. 4] □

We know by lemma 1.1 that the Ramanujan invariant t_n can be constructed by evaluating the modular function $\sqrt{3}R_2$ of level 72 at τ_{ℓ_0} . Thus, we begin by constructing the generators of $(\mathcal{O}/72\mathcal{O})^*$ as Theorem 2.2 dictates. Since $n \equiv 11 \pmod{24}$ we can take

$$\theta = \frac{1}{2} + \frac{1}{2}\sqrt{-n}$$

as a generator of the ring of algebraic integers of $K = \mathbb{Q}(\sqrt{-n})$. The minimal polynomial of θ is $x^2 - x + \frac{n+1}{4}$ and thus B and C in Theorem 2.2 are equal to -1 and $\frac{n+1}{4}$ respectively. The form of the minimal polynomial implies that the prime $p = 2$ stays inert in the extension K/\mathbb{Q} while the prime $p = 3$ splits.

In order to prove that $t_n := \sqrt{3}R_2(\theta)$ is indeed a class invariant we have to prove that

$$(12) \quad (\sqrt{3}R_2)^{A_i} = \sqrt{3}R_2, \text{ for all matrices } A_i \text{ and for } n \equiv 11 \pmod{24}.$$

We observe that the structure of the group $(\frac{\mathcal{O}}{72\mathcal{O}})^*$ depends only on the value of $n \pmod{72}$, and there are exactly three equivalence classes $n \pmod{72}$ so that $n \equiv 11 \pmod{24}$, namely $n = 11, 35, 59 \pmod{72}$. Thus, we have reduced (12) to be checked for finite number of n .

Using Chinese remainder theorem we can express the group $(\frac{\mathcal{O}}{72\mathcal{O}})^*$ as a direct product

$$\left(\frac{\mathcal{O}}{72\mathcal{O}}\right)^* \cong \left(\frac{\mathcal{O}}{9\mathcal{O}}\right)^* \times \left(\frac{\mathcal{O}}{8\mathcal{O}}\right)^*.$$

We will study the structure of the above two summands separately.

We compute that

$$\left(\frac{\mathcal{O}}{9\mathcal{O}}\right)^* \cong \frac{\mathbb{Z}}{6\mathbb{Z}} \times \frac{\mathbb{Z}}{6\mathbb{Z}}.$$

A selection of generators for this group is cumbersome to do by hand. We have used brute force method, (we have checked all elements one by one if there are invertible and then we have computed their orders) using *magma* [8] algebra system in order to compute that for $C = \frac{n+1}{4} \in \{3, 9, 15\}$ a set of generators is given by $7\theta + 4, 5$. Moreover the group

$$\left(\frac{\mathcal{O}}{8\mathcal{O}}\right)^* \cong \frac{\mathbb{Z}}{12\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}},$$

and we have computed again using *magma* the following selection of generators

n	$\frac{n+1}{4}$	Generators
11	3	$\theta, 7, 4\theta + 7$
35	9	$5\theta + 6, 7, 4\theta + 7$
59	15	$\theta, 7, 4\theta + 7$

From the above generators and from the Chinese remainder theorem we can construct generators for the group $(\mathcal{O}/72\mathcal{O})^*$ and map them to the matrices A_i defined in theorem 2.2. We have totally 5 generators for the group $(\frac{\mathcal{O}}{72\mathcal{O}})^*$: the first two are generators of the group $(\mathcal{O}/9\mathcal{O})^*$ and the last three are generators of the group $(\mathcal{O}/8\mathcal{O})^*$. In order to compute the term f^{A_i} in (11) we have to consider any lift of A_i in $GL_2(\mathbb{Z})$ and write it as a product $w_i(S, T)\text{diag}(1, \det(A_i))$, where $w_i(S, T)$ is a word in S, T .

The following lemma gives us the decomposition of a matrix in $SL_2(\mathbb{Z}/p^r\mathbb{Z})$ as a word in the generators of the group $SL_2(\mathbb{Z}/p^r\mathbb{Z})$. Therefore, we must consider the matrices A_i modulo 8 or 9 and define $A_{i,8} \in GL_2(\mathbb{Z}/8\mathbb{Z})$ and $A_{i,9} \in GL_2(\mathbb{Z}/9\mathbb{Z})$,

so that $A_i \equiv A_{i,8} \pmod{8}$ and $A_i \equiv A_{i,9} \pmod{9}$. Notice that $A_{i,8} \equiv \text{Id} \pmod{8}$ for $i = 1, 2$, *i.e.* the first two generators and $A_{i,9} \equiv \text{Id} \pmod{9}$ for $i = 3, 4, 5$, *i.e.* the last three generators.

Lemma 2.3. *Let p^r be a prime power and let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}/p^r\mathbb{Z})$ so that either a or c is invertible modulo p^r . Let $\bar{S}_{p^r} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\bar{T}_{p^r} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be two generators of the group $SL_2(\mathbb{Z}/p^r\mathbb{Z})$. Set $y = (1+a)c^{-1} \pmod{p^r}$ if $(c, p) = 1$, otherwise set $z = (1+c)a^{-1} \pmod{p^r}$. Then*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{cases} \bar{T}_{p^r}^y \bar{S}_{p^r} \bar{T}_{p^r}^c \bar{S}_{p^r} \bar{T}_{p^r}^{dy-b} \pmod{p^r} & \text{if } (c, p) = 1 \\ \bar{S}_{p^r} \bar{T}_{p^r}^{-z} \bar{S}_{p^r} \bar{T}_{p^r}^{-a} \bar{S}_{p^r} \bar{T}_{p^r}^{bz-d} \pmod{p^r} & \text{if } (a, p) = 1. \end{cases}$$

Proof. [5, lemma 6]. □

The generators $\bar{S}_{p^r}, \bar{T}_{p^r}$ modulo $p^r = 8, 9$ are then lifted to elements $S_8, S_9, T_8, T_9 \in SL_2(\mathbb{Z})$ so that

$$\begin{aligned} S_8 &\equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \pmod{8}, \text{ and } S_8 \equiv \text{Id} \pmod{9} \\ S_9 &\equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \pmod{9}, \text{ and } S_9 \equiv \text{Id} \pmod{8} \\ T_8 &\equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \pmod{8}, \text{ and } T_8 \equiv \text{Id} \pmod{9} \\ T_9 &\equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \pmod{9}, \text{ and } T_9 \equiv \text{Id} \pmod{8}. \end{aligned}$$

Using Chinese remainder theorem we compute that

$$(13) \quad \begin{aligned} S_8 &= T^{-1} S T^{-10} S T^{-1} S T^{-162}, & T_8 &= T^9, \\ S_9 &= T^{-1} S T^{-65} S T^{-1} S T^{1096} & \text{and } T_9 &= T^{-8} \end{aligned}$$

We observe that the elements S_8, T_8 commute with S_9, T_9 modulo 72.

The matrices A_{i,p^r} , $p^r = 8$ or 9 can be decomposed as products

$$A_{i,p^r} = B_{i,p^r} \begin{pmatrix} 1 & 0 \\ 0 & \det(A_{i,p^r}) \end{pmatrix},$$

where the matrices B_{i,p^r} have determinant 1 $\pmod{p^r}$ and can be expressed, using lemma 2.3, as words $w_{p^r}(S, T)$ in the generators S, T . The matrices of the form A_i act on the field of modular functions of level 72 with coefficients in $\mathbb{Q}(\zeta_{72})$ as the product of $w_8(S, T) \cdot w_9(S, T) \cdot \text{diag}(1, d_i)$, where d_i is the determinant of A_i , *i.e.* the unique integer so that $d_i \equiv \det(A_{i,9}) \pmod{9}$ and $d_i \equiv \det(A_{i,8}) \pmod{8}$.

For instance the generator $7\theta+4$ of $(\mathcal{O}/9\mathcal{O})^*$ for $C = 3$ corresponds to the matrix $A := \begin{pmatrix} 11 & -21 \\ 7 & 4 \end{pmatrix} \equiv \begin{pmatrix} 2 & 6 \\ 7 & 4 \end{pmatrix} \pmod{9}$ (and to the identity matrix modulo 8). This is a matrix of determinant 2 $\pmod{9}$ and it is decomposed as

$$\begin{pmatrix} 2 & 6 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 7 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

where $B := \begin{pmatrix} 2 & 3 \\ 7 & 2 \end{pmatrix}$ is a matrix of determinant 1 $\pmod{9}$. Using 2.3 we find that

$$B = \begin{pmatrix} 2 & 3 \\ 7 & 2 \end{pmatrix} = \bar{T}_9^3 \bar{S}_9 \bar{T}_9^7 \bar{S}_9 \bar{T}_9^3.$$

A lift of the elements \bar{S}_9, \bar{T}_9 in $SL_2(\mathbb{Z})$ is given by equation (13). This means that we replace each \bar{S}_9 of the above formula to $S_9 = T^{-1} S T^{-65} S T^{-1} S T^{1096}$ and each

\bar{T}_9 to T^{-9} . This gives us the desired lift of B to an element in $SL_2(\mathbb{Z})$. Using this lift, and the transformation matrices A_S, A_T given in (10) we compute that the action of A on the modular functions R_i is given in terms of the following matrix:

$$E := \begin{pmatrix} 0 & 0 & 0 & \frac{-2\zeta_{72}^{18}}{3} + \frac{\zeta_{72}^6}{3} & 0 & 0 \\ 0 & 0 & \zeta_{72}^{15} - \zeta_{72}^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\zeta_{72}^{15}}{3} + \frac{\zeta_{72}^3}{3} \\ 0 & 0 & 0 & 0 & -\zeta_{72}^9 & 0 \\ -2\zeta_{72}^{21} + \zeta_{72}^9 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{72}^{18} + \zeta_{72}^6 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let V be the $\mathbb{Q}(\zeta_{72})$ -vector space of modular functions generated by the elements $R, R_1, R_2, R_3, R_4, R_5$. The vector space V can be identified to the vector space $\mathbb{Q}(\zeta_{72})^6$, in terms of the map

$$V \rightarrow \mathbb{Q}(\zeta_{72})^6$$

$$a_0R + a_1R_1 + \cdots + a_5R_5 \mapsto (a_0, a_1, \dots, a_5),$$

$a_i \in \text{Hom}(V, \mathbb{Q}(\zeta_{72}))$. The space $V^* = \mathbb{Q}(\zeta_{72})^6$ is the dual space of V and we have to see how the action of the elements T, S, σ_d act on V^* . The elements A_T, A_S defined in (10) act on $\mathbb{Q}(\zeta_{72})^6$ in terms of the transpose matrices A_T^t, A_S^t , while the action of A_{σ_d} on R, R_1, \dots, R_5 given on (10) acts on $\mathbb{Q}(\zeta_{72})^6$ in terms of the contragredient action, *i.e.* by considering the transpose of the matrix A_{σ_d} . By chinese remainder theorem we compute that the element $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ acts on $\mathbb{Q}(\zeta_{72})$ as the automorphism $\sigma_{65} : \zeta_{72} \mapsto \zeta_{72}^{65}$. Indeed, 65 is an integer $65 \equiv 2 \pmod{9}$ and $65 \equiv 1 \pmod{8}$.

Since $d = 65 \equiv 2 \pmod{3}$ we compute that the vector $(0, 0, 1, 0, 0, 0)^t$ corresponding to the element R_2 is mapped to $A_{\sigma_{-65}} E^{\sigma_{-65}} (0, 0, 1, 0, 0, 0)^t = (0, 0, -1, 0, 0, 0)$, where by $E^{\sigma_{-65}}$ we denote the matrix where all elements are acted on by σ_{-d} . Notice that $\sqrt{3} = \zeta_{72}^6 - \zeta_{72}^{30}$. Indeed, the value $i\sqrt{3}$ can be expressed as a difference of two primitive 3-roots of unity ζ_3, ζ_3^2 since $i = \zeta_{72}^{18}$ and $\zeta_3 = \zeta_{72}^{24}$. Moreover, $\sigma_{-65}(\sqrt{3}) = \zeta_{72}^{-6 \cdot 65} - \zeta_{72}^{-30 \cdot 65} = -\sqrt{3}$ and thus $\sqrt{3}R_2$ is left invariant. Following the same procedure it can be proven that $\sqrt{3}R_2$ stays invariant for all matrices A_i .

Theorem 2.4. *The Ramanujan value t_n is a class invariant for $n \equiv 11 \pmod{24}$.*

Proof. The condition $\mathbb{Q}(j) \subset \mathbb{Q}(f)$ of theorem 2.2 is known [2, proof of th. 4.1]. By machine computation¹, it turns out that (12) holds for all matrices A_i and thus t_n is a class invariant for all $n \equiv 11 \pmod{24}$. \square

3. COMPUTING THE POLYNOMIALS p_n

In this section we provide a method for the construction of the minimal polynomial of t_n . Following the article of A. Gee [5, eq. 17] we give the following definition:

Definition 3.1. *Let $N \in \mathbb{N}$ and $[a, b, c]$ be a representative of the equivalence class of an element in the class group. Let p be a prime number and p^r be the maximum power of p that divides N . Assume that the discriminant $D = b^2 - 4ac \equiv 1 \pmod{4}$.*

¹The magma program used for this computation is available in the following web location <http://eloris.samos.aegean.gr/papers.html>

We define the matrix

$$A_{[a,b,c],p^r} = \begin{cases} \begin{pmatrix} a & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} & \text{if } p \nmid a \\ \begin{pmatrix} \frac{-b-1}{2} & -c \\ 1 & 0 \end{pmatrix} & \text{if } p \mid a \text{ and } p \nmid c \\ \begin{pmatrix} \frac{-b-1}{2} - a & \frac{1-b}{2} - c \\ 1 & -1 \end{pmatrix} & \text{if } p \mid a \text{ and } p \mid c. \end{cases}$$

Chinese remainder theorem implies that

$$GL_2(\mathbb{Z}/N\mathbb{Z}) \cong \prod_{p|N} GL_2(\mathbb{Z}/p^r\mathbb{Z}).$$

We define $A_{[a,b,c]}$ as the unique element in $GL_2(\mathbb{Z}/N\mathbb{Z})$ that it is mapped to $A_{[a,b,c],p^r}$ modulo p^r . This matrix $A_{[a,b,c]}$ can be written uniquely as a product

$$(14) \quad A_{[a,b,c]} = B_{[a,b,c]} \begin{pmatrix} 1 & 0 \\ 0 & d_{[a,b,c]} \end{pmatrix},$$

where $d_{[a,b,c]} = \det A_{[a,b,c]}$ and $B_{[a,b,c]}$ is a matrix with determinant 1.

Shimura reciprocity law gives us [5, lemma 20] the action of $[a, b, c]$ on $\sqrt{3}R_2(\theta)$ for $\theta = 1/2 - i\sqrt{n}/2$:

$$(\sqrt{3}R_2(\theta))^{[a,-b,c]} = (\zeta_{72}^{6d_{[a,b,c]}} - \zeta_{72}^{30d_{[a,b,c]}}) R_2 \left(\frac{\alpha_{[a,b,c]}\tau_{[a,b,c]} + \beta_{[a,b,c]}}{\gamma_{[a,b,c]}\tau_{[a,b,c]} + \delta_{[a,b,c]}} \right)^{\sigma_{d_{[a,b,c]}}},$$

where $\begin{pmatrix} \alpha_{[a,b,c]} & \beta_{[a,b,c]} \\ \gamma_{[a,b,c]} & \delta_{[a,b,c]} \end{pmatrix} = A_{[a,b,c]}$ and $\tau_{[a,b,c]}$ is the (complex) root of $az^2 + bz + c$ with positive imaginary part.

If we try to implement this method in order to compute the polynomial for t_n we face a problem. Even though we can compute a floating point approximation of the conjugate $R_2(A_{[a,b,c]})$, it is not possible to use this approximation in order to compute the action of $\sigma_{d_{[a,b,c]}}$ on it. There is however a simple approach that we can follow and solve this problem. We can express the matrix $A_{[a,b,c]}$ as a product of a matrix $B_{[a,b,c]}$ as in (14) and then compute the expansion of $B_{[a,b,c]}$ as a word of the matrices S, T .

We begin our computation by computing a full set of representatives of equivalence classes $[a, b, c]$. Since $72 = 2^3 \cdot 3^2$ we have to compute matrices

$$A_{[a,b,c],p^r} \in GL_2(\mathbb{Z}/p^r\mathbb{Z}) \text{ for } p^r = 8, 9.$$

Then we compute the determinant $d_{[a,b,c],p^r}$ of the matrix $A_{[a,b,c],p^r}$ and we find a decomposition

$$A_{[a,b,c],p^r} = B_{[a,b,c],p^r} \begin{pmatrix} 1 & 0 \\ 0 & d_{[a,b,c],p^r} \end{pmatrix}.$$

The matrices $B_{[a,b,c],p^r}$ are elements of $SL_2(\mathbb{Z}/p^r\mathbb{Z})$ and can be written as words of $\bar{S}_{p^r}, \bar{T}_{p^r}$ by using lemma 2.3.

So, if $B_{[a,b,c],8} = w(\bar{T}_8, \bar{S}_8)$ and $B_{[a,b,c],9} = w'(\bar{T}_9, \bar{S}_9)$ are the decompositions of $B_{[a,b,c],p^r}$ as words of $\bar{S}_{p^r}, \bar{T}_{p^r}$ we take the lift

$$B_{[a,b,c]} = w(T_8, S_8)w'(T_9, S_9) \in SL_2(\mathbb{Z}),$$

and the corresponding action on the functions R, R_1, \dots, R_5 is computed by using (9).

The determinants $d_{[a,b,c],8} \in \mathbb{Z}/8\mathbb{Z}$ and $d_{[a,b,c],9} \in \mathbb{Z}/9\mathbb{Z}$ can be lifted in an element $d_{[a,b,c]}$ of $\mathbb{Z}/72\mathbb{Z}$ (so that it reduces to $d_{[a,b,c],8} \pmod{8}$ and $d_{[a,b,c],9} \pmod{9}$ respectively) by using again the Chinese remainder theorem.

The desired polynomial p_n can then be computed:

$$p_n(t) = \prod_{[a,b,c]} \left(t - (\sqrt{3}R_2(\frac{-1+i\sqrt{n}}{2}))^{[a,-b,c]} \right).$$

We have used the *gp-pari*² program in order to perform this computation. The resulting polynomials p_n for $107 \leq n < 1000$ are given in table 1.

REFERENCES

1. A. O. L. Atkin and F. Morain, *Elliptic curves and primality proving*, Math. Comp. **61** (1993), no. 203, 29–68. MR **MR1199989** (**93m**:11136)
2. Bruce C. Berndt and Heng Huat Chan, *Ramanujan and the modular j -invariant*, Canad. Math. Bull. **42** (1999), no. 4, 427–440. MR **MR1727340** (**2002a**:11035)
3. Heng Huat Chan, Alice Gee, and Victor Tan, *Cubic singular moduli, Ramanujan’s class invariants λ_n and the explicit Shimura reciprocity law*, Pacific J. Math. **208** (2003), no. 1, 23–37. MR **MR1979370** (**2004h**:11038)
4. David A. Cox, *Primes of the form $x^2 + ny^2$* , A Wiley-Interscience Publication, John Wiley & Sons Inc., New York, 1989, Fermat, class field theory and complex multiplication. MR **MR1028322** (**90m**:11016)
5. Alice Gee, *Class invariants by Shimura’s reciprocity law*, J. Théor. Nombres Bordeaux **11** (1999), no. 1, 45–72, Les XXèmes Journées Arithmétiques (Limoges, 1997). MR **MR1730432** (**2000i**:11171)
6. Alice Gee and Peter Stevenhagen, *Generating class fields using Shimura reciprocity*, Algorithmic number theory (Portland, OR, 1998), Lecture Notes in Comput. Sci., vol. 1423, Springer, Berlin, 1998, pp. 441–453. MR **MR1726092** (**2000m**:11112)
7. E. Konstantinou, A. Kontogeorgis, Y. Stamatiou, and C. Zaroliagis, Generating Prime Order Elliptic Curves: Difficulties and Efficiency Considerations, in *International Conference on Information Security and Cryptology – ICISC 2004*, Lecture Notes in Computer Science Vol. 3506, pp. 261–278, Springer, 2005.
8. The Magma Computational Algebra System. Available from: <http://magma.maths.usyd.edu.au/magma/>
9. Pari/GP Number Theory System. Available from: <http://www.pari-gp-home.de/>
10. Srinivasa Ramanujan, *Notebooks. Vols. 1, 2*, Tata Institute of Fundamental Research, Bombay, 1957. MR **MR0099904** (20 #6340)
11. Joseph H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994. MR **MR1312368** (**96b**:11074)
12. Noriko Yui and Don Zagier, *On the singular values of Weber modular functions*, Math. Comp. **66** (1997), no. 220, 1645–1662. MR **MR1415803** (**99i**:11046)

DEPARTMENT OF INFORMATION AND COMMUNICATION SYSTEMS ENGINEERING, UNIVERSITY OF THE AEGEAN, 83200 KARLOVASSI, SAMOS, GREECE.

E-mail address: ekonstantinou@aegean.gr

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF THE AEGEAN, 83200 KARLOVASSI, SAMOS, GREECE, <http://eloris.samos.aegean.gr>

E-mail address: kontogar@aegean.gr

²The pari program used for this computation is available at <http://eloris.samos.aegean.gr/papers.html>.

TABLE 1. Polynomials p_n for $107 \leq n < 1000$.

n	$p_n(t)$
107	$x^3 - 2x^2 + 4x - 1$
131	$x^5 + x^4 - x^3 - 3x^2 + 5x - 1$
155	$x^4 + 2x^3 + 5x^2 + 4x - 1$
179	$x^5 - 2x^4 + 5x^3 - x^2 + 6x - 1$
203	$x^4 - 3x^3 + 7x - 1$
227	$x^5 - 5x^4 + 9x^3 - 9x^2 + 9x - 1$
251	$x^7 + 5x^6 + 6x^5 - 2x^4 - 4x^3 + 2x^2 + 9x - 1$
275	$x^4 - x^3 + 6x^2 - 11x + 1$
299	$x^8 + x^7 - x^6 - 12x^5 + 16x^4 - 12x^3 + 15x^2 - 13x + 1$
323	$x^4 - x^3 + 4x^2 + 13x - 1$
347	$x^5 + 7x^4 + 21x^3 + 27x^2 + 13x - 1$
371	$x^8 + 9x^6 - 10x^5 + 14x^4 + 8x^3 - 23x^2 + 18x - 1$
395	$x^8 - x^7 + 5x^6 + 16x^5 + 28x^4 + 24x^3 + 27x^2 + 17x - 1$
419	$x^9 - 6x^8 + 12x^7 - 7x^6 + 12x^5 - 8x^4 + 31x^3 + 10x^2 + 20x - 1$
443	$x^5 - 4x^4 - 3x^3 + 17x^2 + 22x - 1$
467	$x^7 + 6x^6 + 7x^5 - 3x^4 + 3x^3 - 23x^2 + 26x - 1$
491	$x^9 + x^8 + 16x^7 + 2x^6 + 37x^5 - 31x^4 + 44x^3 - 40x^2 + 29x - 1$
515	$x^6 + 8x^5 + 32x^4 + 60x^3 + 68x^2 + 28x - 1$
539	$x^8 - 6x^7 + 28x^6 - 56x^5 + 77x^4 - 56x^3 + 28x^2 - 34x + 1$
563	$x^9 + 4x^8 + 6x^7 - 11x^6 + 44x^5 - 76x^4 + 91x^3 - 64x^2 + 38x - 1$
587	$x^7 + x^6 + 16x^5 - 12x^4 + 20x^3 + 24x^2 + 39x - 1$
611	$x^{10} - 8x^9 + 35x^8 - 62x^7 - x^6 + 116x^5 - 65x^4 - 100x^3 + 125x^2 - 46x + 1$
635	$x^{10} - 11x^9 + 50x^8 - 121x^7 + 201x^6 - 192x^5 + 87x^4 + 51x^3 - 98x^2 + 49x - 1$
659	$x^{11} - 7x^{10} + 7x^9 + 27x^8 + 19x^7 - 43x^6 - 5x^5 + 91x^4 + 157x^3 + 97x^2 + 49x - 1$
683	$x^5 + 6x^4 - 5x^3 - 41x^2 + 56x - 1$
707	$x^6 + 4x^5 + 30x^4 + 72x^3 + 108x^2 + 58x - 1$
731	$x^{12} + 7x^{11} + 25x^{10} + 12x^9 + 41x^8 + 9x^7 +$ $+92x^6 + 73x^5 - 133x^4 + 216x^3 - 153x^2 + 67x - 1$
755	$x^{12} - 2x^{11} + 18x^{10} + 50x^9 + 82x^8 + 182x^7 + 360x^6 + 522x^5 +$ $+598x^4 + 486x^3 + 262x^2 + 66x - 1$
779	$x^{10} + 8x^9 + 24x^8 - 8x^7 - 11x^6 + 26x^5 + 81x^4 + 220x^3 + 98x^2 + 74x - 1$
803	$x^{10} + 3x^9 + 26x^8 + 11x^7 - 65x^6 + 16x^5 + 7x^4 - 83x^3 + 150x^2 - 83x + 1$
827	$x^7 - 7x^6 + 38x^5 - 54x^4 + 112x^3 - 146x^2 + 89x - 1$
851	$x^{10} - 7x^9 - x^8 + 86x^7 + 69x^6 - 201x^5 - 219x^4 + 94x^3 + 103x^2 - 95x + 1$
875	$x^{10} - 10x^9 + 25x^8 + 10x^7 + 15x^6 + 94x^5 - 35x^4 - 120x^3 + 85x^2 + 100x - 1$
899	$x^{14} + 16x^{13} + 97x^{12} + 308x^{11} + 666x^{10} + 1086x^9 +$ $+1490x^8 + 1766x^7 + 1800x^6 + 1556x^5 + 998x^4 + 698x^3 + 229x^2 + 106x - 1$
923	$x^{10} - x^9 + 30x^8 - 81x^7 - 29x^6 + 56x^5 + 211x^4 - 27x^3 - 110x^2 - 115x + 1$
947	$x^5 + 5x^4 + 7x^3 - 103x^2 + 125x - 1$
971	$x^{15} - x^{14} + 21x^{13} + 133x^{12} + 264x^{11} + 310x^{10} + 216x^9 +$ $+62x^8 - 100x^7 - 300x^6 + 152x^5 + 338x^4 + 79x^3 - 285x^2 + 135x - 1$
995	$x^8 + 12x^7 + 59x^6 + 78x^5 + 12x^4 + 66x^3 + 289x^2 + 140x - 1$