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A generating set for the canonical ideal of HKG-curves

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Abstract

The canonical ideal for Harbater Katz Gabber covers satisfying the conditions of Petri's theorem is studied and an explicit non-singular model of the above curves is given.

Keywords: Canonical ideal, Petri's theorem, Harbater-Katz-Gabber curves, Weierstrass semigroup

1 Introduction

The study of the canonical embedding and the determination of the canonical ideal is a classical subject in algebraic geometry, see [1, III.3], [20], [15, p. 20], [21] for a modern account. On the other hand Harbater–Katz–Gabber curves (HKG-curves for short) grew out mainly due to work of Harbater [8] and of Katz and Gabber [11]. They are important because of the Harbater–Katz–Gabber compactification theorem of Galois actions on complete local rings and they proved to be an important tool in the study of local actions and the deformation theory of curves with automorphisms, see [2,5,6,10,14,16,17,19].

In [13] we have studied the relationship between the canonical ideal of a given curve and the action of the automorphism group on the space of holomorphic differentials. It is expected that a lot of information of the deformation of the action is hidden in the canonical ideal, see also [4,9].

In this article we will work over an algebraically closed field k of characteristic $p > 0$. Our aim is to calculate the canonical ideal of an HKG-curve X/k. In order to do so we use a recent result by Charalampous et al. [4] (integrated here as proposition 17) which roughly states that in order to show that a set of quadratic differentials generates the canonical ideal, it suffices to show that the "initial terms" of the differentials generate a large enough subspace of the degree 2 part of the polynomial ring of symmetric differentials. Additionally we employ the breakdown process of an HKG-curve into Artin-Schreier extensions as described in [10] and [14], while also expanding our understanding of the generating elements (Sect. 2). We will assume that the Galois group of the HKG-cover $X \to \mathbb{P}^1$ is a p-group. In this process we will use the symmetric Weierstrass semigroup H at the unique ramification point together with the explicit bases of polydifferentials based on the semigroup given in [10, proposition 42].

We define a set of possible generators of the canonical ideal (i.e. $\mathbf{A}+\mathbf{A}$) and then define an equivalence relation (Def. 4) appropriately which throws away the non-generators, a result in the spirit of the first isomorphism theorem (Sect. 3). There is a bijection (check Eq. 12)

$$\psi : H_2 \longrightarrow \mathbf{A}+\mathbf{A}/\sim .$$

where H_2 can be identified with a basis of the space of holomorphic differentials. This allows us to associate elements of a basis with sums of elements of $\mathbf{A}$ and we use these sums instead, since they are easier to manipulate. This bijection also allows us to work interchangeably between the space $\mathbf{A}+\mathbf{A}$ and the space of 2-differentials. Then in Sect. (4) we interpret the equations of the intermediate Artin-Schreier extensions as equations of quadratic differentials defining a set of relations K_0 and $K_{\bar{\nu},i}$, which we prove are part of the canonical ideal, see proposition 7 and 11. Of these two K_0 is the "trivial" part, imposed by the definition of the canonical map while $K_{\bar{\nu},i}$ is slightly less trivial and is derived from the tower of Artin-Schreier equations giving an HKG-curve. Notice that in order to be able to generate the canonical ideal by quadratic polynomials we have to assume that all intermediate extensions satisfy the assumptions of Petri's theorem, see Lemma 9.

In Sect. (5) we prove that the aforementioned sets generate the canonical ideal, using the bijection of the previous paragraph, by induction on the number of intermediate extensions of the function field.

In the last Sect. (6.1) we give several examples illustrating our construction. These examples are used to demonstrate the fact that, despite the possibly complicated definition of the generating sets (along with the proof), computations can be done efficiently in specific situations.

2 Preliminaries

Throughout this article, we work over an algebraically closed field k. Suppose X is a complete non-singular non-hyperelliptic curve of genus ≥ 3 over k. The canonical ideal I of X is described by the next theorem which is given following Saint-Donat's formulation in [20]:

Theorem 1 (Max Noether-Enriques-Petri) *Under the above assumptions the following hold:*

(1) *The canonical map*

$$\phi : \mathrm{Sym}(H^0(X,\Omega_X)) \to \bigoplus_{n\geq 0} H^0\big(X,\Omega_X^n\big)$$

*is surjective (*Sym *stands for the symmetric algebra).*

(2) *The kernel of ϕ, I, is generated by elements of degree 2 and 3.*

(3) *I is generated by elements of degree 2 except in the following cases;*

  (a) *X is trigonal*

  (b) *X is a plane quintic ($g=6$).*

For the remainder of this article, we assume that X is a Harbater– Katz–Gabber cover (HKG-cover for short). In other words, X is a Galois cover $X_{\mathrm{HKG}} \to \mathbb{P}^1$, such that there is a totally (and wildly) ramified point and at most one other ramified point that is tamely ramified. In this article we are mainly interested in p-groups so our HKG-covers have a unique ramified point P, which is totally and wildly ramified. We are also going to assume that our curves are non-trigonal, so that the third condition of Petri's theorem (Thm. 1) is satisfied. In Lemma 9 the reasons for this demand become apparent.

It is known [10,14] that an HKG curve is defined by a series of extensions $F_{i+1} = F_i(\bar{f}_i)$, where the irreducible polynomials of $\bar{f}_i$ are of the form

$$X^{p^{n_i}} + a^{(i)}_{n_i-1}X^{p^{n_i-1}} + \cdots + a^{(i)}_0 X - D_i, \tag{1}$$

where all the coefficients $a^{(i)}_{n_i-j} \in k, j = 1, \ldots, n_1$ and $D_i \in F_i$ has pole divisor $p^{n_i}\bar{m}_i P$.

The Weierstrass semigroup H is generated by the elements $\{|G_0|, \bar{m}_1, \ldots, \bar{m}_\xi\}$ where $\bar{m}_i = p^{n_{i+1}+\cdots+n_\xi} b_i$. Notice that the ramification groups are given by $|G_{b_{i+1}}| = p^{n_{i+1}+\cdots+n_\xi}$ and they form the following filtration sequence

$$\begin{aligned} G_0(P) = G_1(P) = \cdots = G_{b_1}(P) \gneqq G_{b_1+1}(P) = \cdots \\ \cdots = G_{b_2}(P) \gneqq \cdots \gneqq G_{b_\mu}(P) \gneqq \{1\}. \end{aligned}$$

We know that $(b_i, p) = 1$ and $|G_0| = p^{n_1+\cdots+n_\xi}$, see [10,14].

$$\begin{tikzcd}
F = F_{\xi+1} = F_\xi(\bar{f}_\xi) \arrow[d, dash, "p^{n_\xi}"] \\
F_\xi = F_{\xi-1}(\bar{f}_{\xi-1}) \arrow[d, dash, "p^{n_{\xi-1}}"] \\
F_{\xi-1} \arrow[d, dotted, dash] \\
F_2 = F_1(\bar{f}_1) \arrow[d, dash, "p^{n_1}"] \\
F_1 = k(\bar{f}_0) \arrow[u, dash, bend left=60, "|G_0|", to path={-- ([xshift=-3cm]\tikztostart.north) |- (\tikztotarget)}] \arrow[d, equal] \\
F_0 = F^{G_1(P)}
\end{tikzcd}$$

The above subset of the Weierstrass semigroup might not be the minimal set of generators, since this depends on whether $G_1(P)$ equals $G_2(P)$, see [10, Thm. 13]. We will denote by

$$H_s = \{h : h \in H, h \leq s(2g-2)\} \tag{2}$$

the part of the Weierstrass semigroup bounded by $s(2g-2)$. We will also denote by $\mathbf{A}$ the set

$$\mathbf{A} = \left\{(i_0, \ldots, i_\xi) \in \mathbb{N}^{\xi+1} : i_0|G_0| + \sum_{\nu=1}^{\xi} i_\nu \bar{m}_\nu \leq 2g-2\right\}. \tag{3}$$

For each $h \in H_1$ there is a fixed element $\bar{f}_h$ with unique pole at P of order h. These elements are the field generators, such that $F_{i+1} = F_i(\bar{f}_i)$. The sets H_1 and $\mathbf{A}$ have the same cardinality and moreover the map

$$H_s \ni h \longmapsto f_h df_0^{\otimes s}, \tag{4}$$

gives rise to a basis of $H^0(X, \Omega^s)$, see [10, proposition 42]. We will also denote $f_h df_0^{\otimes s}$ by ω_h and since each element of $\mathbf{A}$ corresponds to an element $L \in H_1$ we will define $\omega_L := \omega_h$. This implies that the cardinality of H_s is given by

$$\#H_s = \begin{cases} g & \text{if } s = 1 \\ (2s-1)(g-1) & \text{if } s > 1. \end{cases}$$

We will denote by $\mathbb{T}^2$ the monomials of $\mathrm{Sym}H^0(X, \Omega_X)$ of degree two (i.e. of the form $\omega_L\omega_K$. For a graded ring S we will use $(S)_2$ to denote elements of degree 2.

The information of the successive extensions is encoded in the coefficients $a_j^{(i)}$ of the additive left part of Eq. (1) *and* in the elements $D_i \in F_i$. Equation (1) vanishes at $\bar{f}_i$, yielding the equality

$$\bar{f}_i^{p^{n_i}} + a_{n_i-1}^{(i)} \bar{f}_i^{p^{n_i-1}} + \cdots + a_0^{(i)} \bar{f}_i = D_i$$

where, taking valuations on both sides, yields that the valuation of D_i is $-p^{n_i}\bar{m}_i$. Notice that the minus sign comes from the fact that $\bar{f}_i$ has a *pole* at P and since it is of order $\bar{m}_i$, one has $v_P(D_i) = v_P(\bar{f}_i^{p^{n_i}}) = -p^{n_i}\bar{m}_i$. Since D_i belongs to $F_i = F^{G_1(P)}(\bar{f}_1, \ldots, \bar{f}_{i-1})$ and $F^{G_1(P)} = k(\bar{f}_0)$ (see [10, remark 21]), one can express D_i as

$$D_i(\bar{f}_0, \ldots, \bar{f}_{i-1}) = \sum_{(\ell_0,\ldots,\ell_{i-1}) \in \mathbb{N}^i} \alpha_{\ell_0,\ldots,\ell_{i-1}}^{(i)} \bar{f}_0^{\ell_0} \cdots \bar{f}_{i-1}^{\ell_{i-1}}. \tag{5}$$

where $\alpha_{\ell_0,\ldots,\ell_{i-1}}^{(i)} \in k$ are some coefficients, not to be confused with the coefficients in Eq. (1). We will need the following:

Lemma 2 *Assume that* $(\ell_0, \ldots, \ell_{i-1}), (w_0, \ldots, w_{i-1}) \in \mathbb{N}^i$ *such that*

$$1 \le \ell_\lambda, w_\lambda < p^{n_\lambda} \text{ for all } 1 \le \lambda \le i-1 \tag{6}$$

and

$$\ell_0|G_0| + \ell_1\bar{m}_1 + \cdots + \ell_{i-1}\bar{m}_{i-1} = w_0|G_0| + w_1\bar{m}_1 + \cdots + w_{i-1}\bar{m}_{i-1}. \tag{7}$$

Then $(\ell_0, \ldots, \ell_{i-1}) = (w_0, \ldots, w_{i-1})$.

Proof Assume that $(\ell_0, \ldots, \ell_{i-1}) \neq (w_0, \ldots, w_{i-1})$. We have by assumption, after cancelling $p^{n_i+\cdots+n_\xi}$ from both sides,

$$\begin{aligned} &\ell_0 p^{n_1+\cdots+n_{i-1}} + \sum_{\nu=1}^{i-2} \ell_\nu p^{n_{\nu+1}+\cdots+n_{i-1}} b_\nu + \ell_{i-1}b_{i-1} \\ &\quad = w_0 p^{n_1+\cdots+n_{i-1}} + \sum_{\nu=1}^{i-2} w_\nu p^{n_{\nu+1}+\cdots+n_{i-1}} b_\nu + w_{i-1}b_{i-1}. \end{aligned} \tag{8}$$

By the coprimality of b_{i-1} and p we get that $p^{n_{i-1}}$ divides $w_{i-1} - \ell_{i-1}$. Suppose that the last difference is not zero and assume without loss of generality that it is positive i.e.

$$w_{i-1} - \ell_{i-1} = \lambda p^{n_{i-1}},\ \lambda > 0.$$

Then w_{i-1} is strictly greater than $p^{n_{i-1}}$ which contradicts the inequality (6) so we must have $w_{i-1} = \ell_{i-1}$. Cancelling the corresponding terms on either side of Eq. 7 allows us to perform the same procedure yielding $w_{i-2} = \ell_{i-2}$. Proceeding with induction we get $w_1 = \ell_1$ which means that also w_0 equals ℓ_0, a contradiction since the elements were assumed different. □

The following lemma allows us to manipulate the elements D_i:

Lemma 3 *Let $F = F_{\xi+1}$ be the top field, with generators $\bar{f}_i$, $i = 0, \ldots, \xi$ and associated irreducible polynomials A_i as in Eq. (1):*

$$A_i(X) = X^{p^{n_i}} + a^{(k)}_{n_i-1} X^{p^{n_i-1}} + \cdots + a^{(k)}_0 X - D_i,$$

where D_i is given in Eq. (5),

$$D_i(\bar{f}_0, \ldots, \bar{f}_{i-1}) = \sum_{(\ell_0,\ldots,\ell_{i-1})\in\mathbb{N}^i} a^{(i)}_{\ell_0,\ldots,\ell_{i-1}} \bar{f}_0^{\ell_0} \ldots \bar{f}_{i-1}^{\ell_{i-1}}.$$

Then one of the monomials $\bar{f}_0^{\ell_0} \ldots \bar{f}_{i-1}^{\ell_{i-1}}$ has also pole divisor $p^{n_i}\bar{m}_i P$ and this holds for all $i = 1, \ldots, \xi$.

Proof Recall that $D_i \in F_i, \bar{f}_i \in F_{i+1} - F_i$ and the pole divisor of D_i is $p^{n_i}\bar{m}_i P$. Suppose on the contrary (for D_i) that, none of the monomial summands of D_i has pole divisor of the desired order, $p^{n_i}\bar{m}_i P$. In other words,

$$\ell_0|G_0| + \ell_1\bar{m}_1 + \cdots + \ell_{i-1}\bar{m}_{i-1} \neq p^{n_i}\bar{m}_i$$

for all $\ell_0, \ldots, \ell_{i-1}$ appearing as exponents. We can assume that ℓ_λ, w_λ satisfy the inequality of Eq. (6) for all exponents of all monomial summands of D_i since, otherwise, we can substitute the corresponding element $\bar{f}_\lambda^{\ell_\lambda}$ with terms of smaller exponents because of its irreducible polynomial, see also Eq. (1).

By the strict triangle inequality there will be at least two different monomials $\bar{f}_0^{\ell_0} \ldots \bar{f}_{i-1}^{\ell_{i-1}}$, $\bar{f}_0^{w_0} \ldots \bar{f}_{i-1}^{w_{i-1}}$ in the sum of D_i sharing the same valuation and the contradiction follows from Lemma 2. □

3 Preparation for the main theorem

Define the Minkowski sum (recall the definition of $\mathbf{A}$ given in Eq. (3))

$$\mathbf{A} + \mathbf{A} = \{L + K : L, K \in \mathbf{A}\},$$

where $L + K = (i_0 + j_0, \ldots, i_\xi + j_\xi)$ for $L = (i_0, \ldots, i_\xi), K = (j_0, \ldots, j_\xi)$. There is a natural map

$$\mathbb{N}^{\xi+1} \ni (i_0, i_1, \ldots, i_\xi) = \bar{h} \longmapsto ||\bar{h}|| = i_0|G_0| + \sum_{\nu=1}^{\xi} i_\nu \bar{m}_\nu \in \mathbb{N}, \tag{9}$$

which restricts to the map

$$\begin{aligned} \mathbf{A} + \mathbf{A} &\xrightarrow{||\cdot||} H_2 \\ L + K &\longmapsto \left(L + K\right)\begin{pmatrix} |G_0| \\ \bar{m}_1 \\ \vdots \\ \bar{m}_\xi \end{pmatrix} = (i_0 + j_0)|G_0| + \sum_{\nu=1}^{\xi}(i_\nu + j_\nu)\bar{m}_\nu. \end{aligned} \tag{10}$$

The map given in Eq. (10) is not one to one. In order to bypass this we introduce a suitable equivalence relation $\sim$ on $\mathbf{A} + \mathbf{A}$ so that there is a bijection

$$\psi : (\mathbf{A} + \mathbf{A})/\sim \quad \longrightarrow H_2' := \mathrm{Im}\psi \subset H_2.$$

Definition 4 Define the equivalence relation $\sim$ on $\mathbf{A}+\mathbf{A}$, by the rule

$$(L+K)\sim(L'+K') \text{ if and only if } ||L+K|| = ||L'+K'||.$$

The function ψ together with Eq. (4) allows us to express a quadratic differential ω_h corresponding to an element $h \in H_2'$ as an element in $\mathbf{A}+\mathbf{A}$ by selecting a representative $L+K \in \mathbf{A}+\mathbf{A}$ of the class of $\psi(f)$. That is for every element $h \in H_2'$ we can write

$$\psi([L_h+K_h]) = h \text{ for certain elements } L_h, K_h \in \mathbf{A}. \tag{11}$$

It is clear by our definitions that the following equality holds.

$$\left|\frac{\mathbf{A}+\mathbf{A}}{\sim}\right| = |H_2'| \leq |H_2| = 3g-3. \tag{12}$$

as we mentioned in the introduction, the reasons for the definition of the equivalence relation will be clear later but the curious reader may check proposition 17.

We will need the following:

Lemma 5 *The equivalence class of the element* $L+K = (i_0+j_0,\ldots,i_\xi+j_\xi) \in \mathbf{A}+\mathbf{A}$ *corresponds under the assignment*

$$A+B \in \mathbf{A}+\mathbf{A} \mapsto \omega_A\omega_B$$

to the following set of degree 2 *monomials*

$$\Gamma_{L+K} := \left\{\begin{array}{l} \omega_A\omega_B \in \mathrm{Sym}H^0(X,\Omega_X): \textit{ for } A=(a_0,\ldots,a_\xi), B=(b_0,\ldots,b_\xi) \\ \textit{such that:} \\ ((a_0+b_0)-(i_0+j_0))|G_0| + \displaystyle\sum_{\nu=1}^{\xi-1}\big(a_\nu+b_\nu-(i_\nu+j_\nu)\big)\bar{m}_\nu = \lambda\bar{m}_\xi p^{n_\xi} \\ \textit{and } a_\xi+b_\xi-(i_\xi+j_\xi) = -\lambda p^{n_\xi} \textit{ for some } \lambda\in\mathbb{Z} \end{array}\right\},$$

Proof The equivalence class of $L+K$ is a subset of $\mathbf{A}+\mathbf{A}$ which corresponds to holomorphic differentials as described below: Notice first that two equivalent elements $L+K$, $L'+K'$ satisfy

$$(i_0+j_0-(i_0'+j_0'))|G_0| + \sum_{\nu=1}^{\xi}(i_\nu+j_\nu-(i_\nu'+j_\nu))\bar{m}_\nu = 0$$

which, combined with the facts that $(\bar{m}_\xi, p)=1$ and $\bar{m}_i = p^{n_{i+1}+\cdots+n_\xi}b_i$ yields that there is an integer λ such that

$$\big(i_0+j_0-(i_0'-j_0')\big)\frac{|G_0|}{p^{n_\xi}} + \sum_{\nu=1}^{\xi-1}(i_\nu+j_\nu-(i_\nu'+k_\nu'))\frac{\bar{m}_\nu}{p^{n_\xi}} = \lambda\bar{m}_\xi \tag{13}$$

$$\text{and } i_\xi'+j_\xi'-(i_\xi+j_\xi) = \lambda p^{n_\xi}. \tag{14}$$

$\square$

Remark 6 By Petri's theorem the canonical map ϕ (check Eq. (1)) maps a degree 2 polynomial in the symmetric algebra of $H^0(X,\Omega_X)$ to $f_h df_0^{\otimes 2} \in H^0(X,\Omega_X^{\otimes 2})$, that is

$$\phi\left(\sum_\nu a_\nu\omega_{L_\nu}\omega_{K_\nu}\right) = f_h df_0^{\otimes 2}, \quad a_\nu \in k. \tag{15}$$

It is not correct that a holomorphic 2-differential $f_h df_0^{\otimes 2}$ is the image of a single element $\omega_L\omega_K$. Indeed, for the genus 9 Artin-Schreier curve

$$y^7 - y = x^4$$

a basis for the set of holomorphic differentials corresponds to the set

$$\mathbf{A} = \{[0,0], [0,1], [0,2], [0,3], [0,4], [1,0], [1,1], [1,2], [2,0]\}$$
$$\omega_{0,0} = x^0y^0dx, \quad \omega_{0,1} = x^0y^1dx, \quad \omega_{0,2} = x^0y^2dx, \quad \omega_{0,3} = x^0y^3dx, \quad \omega_{0,4} = x^0y^4dx,$$
$$\omega_{1,0} = x^1y^0dx, \quad \omega_{1,1} = x^1y^1dx, \quad \omega_{1,2} = x^1y^2dx, \quad \omega_{2,0} = x^2y^0dx$$

while the holomorphic 2-differential $x^4ydx^{\otimes 2}$ cannot be expressed as a single monomial of the above differentials, but as the following linear combination

$$\omega_{0,4}^2 - \omega_{0,2}^2 = y(y^7 - y)dx^{\otimes 2} = x^4ydx^{\otimes 2}.$$

If the 2-differential $f_0^{i_0}\cdots f_\xi^{i_\xi} df_0^{\otimes 2}$ is the image of a single monomial $\omega_K\omega_L$ with $K + L = (i_0, \ldots, i_\xi)$, then it is clear that the element $h = |G_0|i_0 + \sum_{\nu=1}^{\xi} \bar{m}_\nu i_\nu$ in H_2 is the image of $L + K \in \mathbf{A} + \mathbf{A}$.

4 The generating sets of the canonical ideal

For any element $K = (i_0, \ldots, i_\xi) \in \mathbb{N}^{\xi+1}$ we will denote by f_K the element $f_0^{i_0}\cdots f_\xi^{i_\xi}$.

Proposition 7 *Consider the sets of quadratic holomorphic differentials:*

$$K_0 := \{\omega_L\omega_K - \omega_{L'}\omega_{K'} \in \mathrm{Sym}H^0(X, \Omega_X) : L + K = L' + K', L, K, L', K' \in \mathbf{A}\}.$$

Then K_0 is contained in the canonical ideal.

Proof For the canonical map $\phi : \mathrm{Sym}(H^0(X, \Omega_X)) \to \bigoplus_{n\geq 0} H^0(X, \Omega_X^n)$ one has;

$$\phi(\omega_K\omega_L - \omega_{K'}\omega_{L'}) = f_{K+L}df_0^{\otimes 2} - f_{K'+L'}df_0^{\otimes 2} = 0.$$

□

Remark 8 Since K_0 is included in the canonical ideal we have that

$$\omega_{K_h}\omega_{L_h} = \omega_{K_{h'}}\omega_{L_{h'}}$$

modulo the canonical ideal for any selection of $K_h + L_h, K_{h'} + L_{h'}$ representing $h, h' \in \mathbf{A} + \mathbf{A}$ such that $K_h + L_h = K_{h'} + L_{h'}$. Therefore, we will denote 2-differentials by $\omega_h^{\otimes 2}$.

Using this notation we can rewrite the summands of D_i in Eq. (5) as 2-differentials as explained below:

Lemma 9 *The elements $D_i \in F_i$ have degree less than $4g - 4$, yielding that $D_i \cdot df_0^{\otimes 2}$ are 2-holomorphic differentials in F. In particular every monomial summand $\bar{f}_0^{\ell_0}\cdots \bar{f}_{i-1}^{\ell_{i-1}}$ that appears in the expression of D_i given in Eq. (5) can be given as an element*

$$(0, \ldots, 0) + (\ell_0, \ldots, \ell_{i-1}, 0, \ldots, 0) \in \mathbf{A} + \mathbf{A}.$$

and the element D_i can be written as a 2-differential as

$$D_i \cdot df_0^{\otimes 2} = \sum_{\substack{\bar{\lambda}=(\ell_0,\ldots,\ell_{i-1},0,\ldots,0)\in\mathbf{A}+\mathbf{A} \\ ||\bar{\lambda}||\leq p^{n_i}\bar{m}_i}} a_{\bar{\lambda}}^{(i)}\omega_{\bar{\lambda}}^{\otimes 2}, \tag{16}$$

Proof By Eq. (1) we have that the absolute value of the valuation of D_i in F_{i+1} is $p^{n_i} b_i$. We will first show that $p^{n_i} b_i \leq 4g_{F_{i+1}} - 4$.

According to the Riemann-Hurwitz formula the genera of F_{i+1} and F_i are related by

$$2(g_{F_{i+1}} - 1) = p^{n_i} 2(g_{F_i} - 1) + (b_i + 1)(p^{n_i} - 1). \tag{17}$$

Therefore

$$\begin{aligned} 4(g_{F_{i+1}} - 1) - p^{n_i} b_i &= 2p^{n_i} 2(g_{F_i} - 1) + p^{n_i} b_i - 2b_i + 2p^{n_i} - 2 \\ &= 2p^{n_i} 2(g_{F_i} - 1) + (p^{n_i} - 2)b_i + 2(p^{n_i} - 1). \end{aligned} \tag{18}$$

If $g_{F_i} \geq 1$ then we have the desired inequality. Suppose that $g_{F_i} = 0$. This can only happen for $i = 1$ since $p^{n_i} > 1$ and $b_i > 1$. Therefore we need to show that

$$b_1 p^{n_1} - 2p^{n_1} - 2b_1 - 2 \geq 0$$

and we are working over the rational function field. The assumption on our curve being non-hyperelliptic implies that $p^{n_i} > 2$ as well as $b_i > 2$ and the last inequality becomes

$$b_i \geq \frac{2p^{n_i} + 2}{p^{n_i} - 2} \tag{19}$$

which is satisfied for $p^n > 7$. Also the remaining cases, i.e. $p^{n_i} = 5, 7$ require b_i to be ≥ 4 which is also true since $b_i = 2$ is exluded by non-hyperellipticity and $b_i = 3$ by non-trigonality.

Now the rest can be proved by induction as follows; We showed that

$$p^{n_i} b_i \leq 4g_{F_{i+1}} - 4 \tag{20}$$

When we move from F_{i+1} to F_{i+2} the absolute value of the valuation of D_i becomes $p^{n_{i+1}+n_i} b_i$ and we need to show that

$$p^{n_{i+1}+n_i} b_i \leq 4g_{F_{i+2}} - 4$$

By 20 it suffices to show that $p^{n_{i+1}}(4g_{F_{i+1}} - 4) \leq 4g_{F_{i+2}} - 4$ which by the Riemann-Hurwitz formula (stated above) is equivalent to $(b_{i+1} + 1)(p^{n_{i+1}} - 1)$ being non-negative, which holds. □

Remark 10 If we assume that F_i is neither trigonal nor hyperelliptic then the same holds for all fields F_k for $k \geq i$, see [18, Appendix].

The set K_0 does not contain all elements of the canonical ideal. For instance, it does not contain the information of the defining equation of the Artin-Schreier extension and also the canonical ideal is not expected to be binomial.

Before the definition of the other generating sets of the canonical ideal, let us provide some insight into the process used to construct the elements of these sets.

Equation (1) is satisfied by the element $\bar{f}_i$, i.e,

$$\bar{f}_i^{p^{n_i}} + a_{n_i-1}^{(i)} \bar{f}_i^{p^{n_i-1}} + \cdots + a_0^{(i)} \bar{f}_i - D_i = 0.$$

This equation can be multiplied by elements of the form $\bar{f}_0^{\nu_0} \cdots \bar{f}_\xi^{\nu_\xi}$ for any $\nu_0, \ldots, \nu_\xi$, giving rise to

$$\bar{f}_0^{\nu_0} \cdots \bar{f}_\xi^{\nu_\xi} \left(\bar{f}_i^{p^{n_i}} + a_{n_i-1}^{(i)} \bar{f}_i^{p^{n_i-1}} + \cdots + a_0^{(i)} \bar{f}_i - D_i \right) = 0,$$

which equals

$$\bar{f_0}^{\nu_0}\cdots\bar{f_i}^{\nu_i+p^{n_i}}\cdots\bar{f_\xi}^{\nu_\xi}+\ldots+a_0^{(i)}\bar{f_0}^{\nu_0}\cdots\bar{f_i}^{\nu_i+1}\cdots\bar{f_\xi}^{\nu_\xi}-\bar{f_0}^{\nu_0}\cdots\bar{f_i}^{\nu_i}\cdots\bar{f_\xi}^{\nu_\xi}D_i=0.$$

If the exponents $(\nu_0,\ldots,\nu_\xi)$ are selected so that each summand in the last equation is an element in $\mathbf{A}+\mathbf{A}$, then the equation gives rise to an element in the canonical ideal.

Proposition 11 *Set*

$$\bar{\nu}:=(\nu_0,\ldots,\nu_\xi)\in\mathbb{N}^{\xi+1}$$
$$\bar{\gamma}_{\bar{\nu},i,\nu}:=(\nu_0,\ldots,\nu_i+p^{n_i-\nu},\nu_{i+1},\ldots,\nu_\xi),\ 0\leq\nu\leq n_i$$

such that $||\bar{\gamma}_{\bar{\nu},i,0}||\leq 4g-4$. *Also set*

$$\Lambda_i=\{\bar{\lambda}=(\ell_0,\ldots,\ell_{i-1})\in\mathbb{N}^i:0\leq\ell_\nu<p^{n_\nu}\ \textit{for}\ 1\leq\nu\leq i\}$$
$$\bar{\beta}_{\bar{\nu},i,\bar{\lambda}}:=(\ell_0,\ldots,\ell_{i-1},0,\ldots,0)+\bar{\nu}\in\mathbf{A}+\mathbf{A}.$$

Define

$$K_{\bar{\nu},i}:=\left\{\omega^{\otimes 2}_{\bar{\gamma}_{\bar{\nu},i,0}}+\sum_{\nu=1}^{n_i}a_\nu^{(i)}\omega^{\otimes 2}_{\bar{\gamma}_{\bar{\nu},i,\nu}}-\sum_{\substack{\bar{\lambda}\in\Lambda_i\\ ||\bar{\lambda}||\leq p^{n_i}\bar{m}_i}}a_{\bar{\lambda}}^{(i)}\omega^{\otimes 2}_{\bar{\beta}_{\bar{\nu},i,\bar{\lambda}}}\right\}\tag{21}$$

Then $K_{\bar{\nu},i}$ *is contained in the canonical ideal for* $1\leq i\leq\xi$.

Notice here that $\bar{\nu}$ is fixed while $\bar{\lambda}$ is running.

Proof Again consider $\phi:\mathrm{Sym}(H^0(X,\Omega_X))\to\bigoplus_{n\geq 0}H^0(X,\Omega_X^n)$. Then

$$\begin{aligned}
&\phi\left(\omega^{\otimes 2}_{\bar{\gamma}_{\bar{\nu},i,0}}+\sum_{\nu=1}^{i}a_\nu^{(i)}\omega^{\otimes 2}_{\bar{\gamma}_{\bar{\nu},i,\nu}}-\sum_{\substack{\bar{\lambda}\in\Lambda_i\\ ||\bar{\lambda}||\leq p^{n_i}\bar{m}_i}}a_{\bar{\lambda}}^{(i)}\omega^{\otimes 2}_{\bar{\beta}_{\bar{\nu},i,\bar{\lambda}}}\right)\\
&=\left(f_{(\nu_0,\ldots,\nu_i+p^{n_i},\ldots,\nu_\xi)}+\sum_{\nu=1}^{i}a_\nu^{(i)}f_{(\nu_0,\ldots,\nu_i+p^{n_i-\nu},\ldots,\nu_\xi)}-\right.\\
&\left.-\sum_{\substack{\bar{\lambda}\in\Lambda_i\\ ||\bar{\lambda}||\leq p^{n_i}\bar{m}_i}}a_{\bar{\lambda}}^{(i)}f_{(\ell_0+\nu_0,\ldots,\ell_{i-1}+\nu_{i-1},\nu_i,\ldots,\nu_\xi)}\right)df_0^{\otimes 2}\\
&=f_{(\nu_0,\ldots,\nu_\xi)}\left(\bar{f_i}^{p^{n_i}}+\sum_{\nu=1}^{i}a_\nu^{(i)}\bar{f_i}^{p^{n_i-\nu}}-\sum_{\substack{\bar{\lambda}\in\Lambda_i\\ ||\bar{\lambda}||\leq p^{n_i}\bar{m}_i}}a_{\bar{\lambda}}^{(i)}f_{(\ell_0\cdots+\ell_{i-1},0,\ldots,0)}\right)df_0^{\otimes 2},
\end{aligned}$$

which equals 0 due to the relation satisfied by the irreducible polynomial of $\bar{f_i}$. □

5 The main theorem

We define a term order which compares products of differentials as follows: Let $\omega_{I_1}\omega_{I_2}\cdots\omega_{I_d},\omega_{I'_1}\omega_{I'_2}\cdots\omega_{I'_{d'}}$ be two such products and consider the $(k+1)-$tuples $I_1+\cdots+I_d=(\nu_0,\ldots,\nu_\xi)$, $I'_1+\cdots+I'_{d'}=(\nu'_0,\ldots,\nu'_\xi)$.

Define

$$\omega_{I_1}\omega_{I_2}\cdots\omega_{I_d} \prec \omega_{I'_1}\omega_{I'_2}\cdots\omega_{I'_{d'}} \Leftrightarrow (\nu_0,\ldots,\nu_\xi) <_{\mathrm{colex}} (\nu'_0,\ldots,\nu'_\xi)$$

that is

- $\nu_\xi < \nu'_\xi$ or
- $\nu_\xi = \nu'_\xi$ and $\nu_{\xi-1} < \nu'_{\xi-1}$ or
- $\vdots$
- $\nu_i = \nu'_i$ for all $i = k,\ldots,1$ and $\nu_0 < \nu'_0$.

We are going to work with the initial terms of the sets defined in the last two propositions where, by "initial term" we mean a maximal term with respect to the colexicographical order. We denote initial terms with $\mathrm{in}_\prec(\cdot)$.

Lemma 12 *For the element $K_{\bar{\nu},i}$ of proposition 11 we have that*

$$\mathrm{in}_\prec(K_{\bar{\nu},i}) = \omega_{\bar{\gamma}_{\bar{\nu},i,0}}.$$

and also, in the polynomial $K_{\bar{\nu},i}$ there is another summand which is smaller colexicographically than $\omega_{\bar{\gamma}_{\bar{\nu},i,0}}$ but has the same $||\cdot||$-value.

Proof Indeed, in Eq. (21) there are two elements of maximal value in terms of $||\cdot||$. Namely $\omega_{\bar{\gamma}_{\bar{\nu},i,0}}$ and $a^{(i)}_{\bar{\lambda}}\omega^{\otimes 2}_{\bar{\beta}_{\bar{\nu},i,\bar{\lambda}}}$, for the $\bar{\lambda} = (\ell_0,\ldots,\ell_{i-1},0,\ldots,0) \in \mathbf{A}+\mathbf{A}$ corresponding to the monomial $\bar{f}_0^{\ell_0}\cdots\bar{f}_{i-1}^{\ell_{i-1}}$ of minimum valuation which exists due to lemma 3. Of these two elements, $\omega_{\bar{\gamma}_{\bar{\nu},i,0}}$ is bigger since it corresponds to the element $(\nu_0,\ldots,\nu_i+p^{n_i},\ldots,\nu_\xi)$, while the other corresponds to the smaller element $(\nu_0+l_0,\ldots,\nu_{i-1}+l_{i-1},\nu_i,\ldots,\nu_\xi)$, with respect to the colexicographical order. □

We are now ready to state our main result. Recall that we have assumed throughout this article that X is a Harbater–Katz–Gabber cover which is non-elliptic of genus ≥ 3 over k. We also have assumed that X is non-trigonal so that the canonical ideal is generated by elements of degree 2 (see also Theorem 1).

Theorem 13 *The canonical ideal is generated by K_0 and by $K_{\bar{\nu},i}$, for $1 \leq i \leq \xi$ and for the $\bar{\nu} \in \mathbb{N}^{\xi+1}$ satisfying the inequality $||\bar{\gamma}_{\bar{\nu},i,0}|| \leq 4g-4$.*

Remark 14 In the above theorem the condition $||\bar{\gamma}_{\bar{\nu},i,0}|| \leq 4g-4$ implies the condition $||\bar{\gamma}_{\bar{\nu},i,\nu}|| \leq 4g-4$ for $0 \leq \nu \leq n_i$. We will prove in Lemma 15 that it also implies the condition $||\bar{\beta}_{\bar{\nu},i,\bar{\lambda}}|| \leq 4g-4$. This means that the condition $||\bar{\gamma}_{\bar{\nu},i,\nu}|| \leq 4g-4$ for $0 \leq \nu \leq n_i$ guarantees that, in $K_{\bar{\nu},i}$, not only the first term (i.e. $\omega^{\otimes 2}_{\bar{\gamma},i,0}$), but also all the others correspond to 2-differentials.

Lemma 15 *The condition $||\bar{\gamma}_{\bar{\nu},i,0}|| \leq 4g-4$, or in other words,*

$$\nu_0|G_0| + \sum_{\nu=1}^{\xi}\nu_\nu\bar{m}_\nu + p^{n_i}\bar{m}_i \leq 4g-4 \tag{22}$$

implies that $\bar{\beta}_{\bar{\nu},i,\bar{\lambda}}$ lies in $\mathbf{A}+\mathbf{A}$, that is, it is also a 2-differential, for all $\bar{\lambda}$ associated with the monomials of D_i.

Proof For $\bar{\lambda} \in \Lambda_i$ let

$$\bar{\beta}_{\bar{\nu},i,\bar{\lambda}} = (\nu_0 + \ell_0, \ldots, \nu_{i-1} + \ell_{i-1}, \nu_i, \ldots, \nu_\xi).$$

We need to show that

$$(\nu_0 + \ell_0)|G_0| + \sum_{\nu=1}^{\xi} \nu_\nu \bar{m}_\nu + \sum_{\nu=1}^{i-1} \ell_\nu \bar{m}_\nu \leq 4g - 4.$$

By (22) we need to show that

$$\ell_0|G_0| + \sum_{\nu=1}^{i-1} \ell_\nu \bar{m}_\nu \leq p^{n_i} \bar{m}_i.$$

Note that $\bar{\lambda}$ is the exponents of a monomial summand of D_i and, by the valuation's strict triangle inequality one has;

$$\begin{aligned} v(f_{\bar{\lambda}}) \geq v(D_i) \Leftrightarrow \\ -(\ell_0|G_0| + \sum_{\nu=1}^{i-1} \ell_\nu \bar{m}_\nu) \geq -p^{n_i} \bar{m}_i \end{aligned}$$

as expected, where $f_{\bar{\lambda}}$ is $\bar{f}_0^{\ell_0} \cdots \bar{f}_{i-1}^{\ell_{i-1}}$ □

Definition 16 Define J to be the set of elements in the canonical ideal consisting of the elements $K_0, K_{\bar{\nu},i}$ for $1 \leq i \leq \xi$ and for the appropriate $\bar{\nu} \in \mathbb{N}^{\xi+1}$ satisfying the inequality $||\bar{\gamma}_{\bar{\nu},i,0}|| \leq 4g - 4$.

In order to prove Theorem 13, we need to show that J is the canonical ideal. We will use the following proposition, for a proof see [4].

Proposition 17 *Let J be a set of homogeneous polynomials of degree 2 containing the elements G_0 and an extra set of generators G' and let I be the canonical ideal. Assume that the hypotheses imposed by Petri's theorem in order for the canonical ideal to be generated by polynomials of degree two are fulfilled. If $\dim_L (S/\langle \mathrm{in}_\prec J\rangle)_2 \leq 3(g-1)$, then $I = \langle J\rangle$, where $S = \mathrm{Sym}(H^0(X, \Omega_X)$ is the symmetric algebra of $H^0(X, \Omega_X)$.*

In order to apply proposition 17 we will show that

$$\left|\frac{\mathbf{A}+\mathbf{A}}{\sim}\right| = \dim\left(\frac{S}{\langle \mathrm{in}_\prec(J)\rangle}\right)_2, \tag{23}$$

where we already know, see Eq. (12), that the cardinality of the first quotient is $\leq |H_2| = 3g-3$. We identify a k-basis of $(S/\langle \mathrm{in}_\prec\langle J\rangle)_2$ with $\mathbb{T}^2 - \{\mathrm{in}_\prec(f) : f \in J\}$ and, in order to prove equality (23), we define the map

$$\begin{aligned} \Phi : \mathbb{T}^2 - \{\mathrm{in}_\prec(f) : f \in J\} &\longrightarrow \frac{\mathbf{A}+\mathbf{A}}{\sim} \\ \omega_L \omega_K &\longmapsto [L+K] \end{aligned} \tag{24}$$

Lemma 18 *If $(u_0, \ldots, u_\xi) \in \mathbf{A}+\mathbf{A}$ then every $(u'_0, \ldots, u'_\xi)$ with $0 \leq u'_\nu \leq u_\nu$ for $1 \leq \nu \leq \xi$ is also in $\mathbf{A}+\mathbf{A}$.*

Proof Since $\bar{u} = (u_0, \ldots, u_\xi) \in \mathbf{A} + \mathbf{A}$ there are $\bar{a} = (a_0, \ldots, a_\xi)$, $\bar{b} = (b_0, \ldots, b_\xi)$ with $\bar{u} = \bar{a} + \bar{b}$ and $\bar{a}, \bar{b} \in \mathbf{A}$, that is $||\bar{a}||, ||\bar{b}|| \leq 2g - 2$. But then every $\bar{a}'$ (resp. $\bar{b}'$) with $\bar{a}' = (a'_0, \ldots, a'_\xi)$ (resp. $\bar{b}' = (b'_0, \ldots, b'_\xi)$) such that $0 \leq a'_\nu \leq a_\nu$ (resp. $0 \leq b'_\nu \leq b_\nu$) for $0 \leq \nu \leq \xi$ satisfies $||\bar{a}'|| \leq ||\bar{a}|| \leq 2g - 2$ (resp. $||\bar{b}'|| \leq ||\bar{b}|| \leq 2g - 2$), that is $\bar{a}', \bar{b}' \in \mathbf{A}$. The result follows. □

We start by showing that Φ is one-to-one.

Lemma 19 *The map Φ is injective.*

Proof Consider the following elements of $\mathbf{A}$:

$$L = (i_0, i_1, \ldots, i_\ell, \ldots, i_\xi) \qquad K = (j_0, j_1, \ldots, j_\ell, \ldots, j_\xi)$$
$$L' = (i'_0, i'_1, \ldots, i'_\ell, \ldots, i'_\xi) \qquad K' = (j'_0, j'_1, \ldots, j'_\ell, \ldots, j'_\xi)$$

such that, $\omega_K \omega_L, \omega_{L'} \omega_{K'}$ are in $\mathbb{T}^2 - \{\mathrm{in}(f) : f \in J\}$. Assume that $\Phi(\omega_L \omega_K) = \Phi(w'_L w'_K)$, i.e. $L + K \sim L' + K'$. Suppose that $i_\xi + j_\xi = i'_\xi + j'_\xi$. Then we have the following equality:

$$(i_0 + j_0)|G_0| + \sum_{\ell=1}^{\xi} (i_\ell + j_\ell)\bar{m}_\ell = (i'_0 + j'_0)|G_0| + \sum_{\ell=1}^{\xi} (i'_\ell + j'_\ell)\bar{m}_\ell$$

from which we cancel the last terms and divide by p^{n_ξ} in order to have

$$(i_0 + j_0)p^{n_1 + \cdots + n_{\xi-1}} + \sum_{\ell=1}^{\xi-1} (i_\ell + j_\ell)\frac{\bar{m}_\ell}{p^{n_\xi}} = (i'_0 + j'_0)p^{n_1 + \cdots + n_{\xi-1}} + \sum_{\ell=1}^{\xi-1} (i'_\ell + j'_\ell)\frac{\bar{m}_\ell}{p^{n_\xi}}.$$

By repeating the above process we can assume that there is an $\ell \leq \xi$ such that $i'_\nu + j'_\nu = i_\nu + j_\nu$ for $\ell < \nu \leq \xi$ and $i'_\ell + j'_\ell \neq i_\ell + j_\ell$ and assume without loss of generality that $i'_\ell + j'_\ell > i_\ell + j_\ell$. Then by Lemma 5, we would have

$$i'_\ell + j'_\ell - (i_\ell + j_\ell) = \lambda p^{n_\ell} \tag{25}$$

for $\lambda > 0$. Using this we will show that $\omega_{L'} \omega_{K'}$ belongs to $\mathrm{in}_\prec(J)$. In order to do that, we need to build an element $K_{i,\bar{\nu}}$ which has $\omega_{L'} \omega_{K'}$ as its initial term. In other words we look for an element of the following form;

$$\omega^{\otimes 2}_{\bar{\gamma}_{\bar{\nu},i,0}} + \sum_{\nu=1}^{n_i} a^{(i)}_\nu \omega^{\otimes 2}_{\bar{\gamma}_{\bar{\nu},i,\nu}} - \sum_{\substack{\lambda \in \Lambda_i \\ ||\bar{\lambda}|| \leq p^{n_i} \bar{m}_i}} a^{(i)}_{\bar{h}} \omega^{\otimes 2}_{\bar{\beta}_{\bar{\nu},i,\bar{\lambda}}}, \tag{26}$$

where $\omega^{\otimes 2}_{\bar{\gamma}_{\bar{\nu},i,0}} = \omega^{\otimes 2}_{L'+K'}$ and everything else should be as defined in Proposition 11. This comes down to finding $\bar{\nu} = (\nu_0, \ldots, \nu_\xi) \in \mathbb{N}^{\xi+1}$ such that

$$(\nu_0, \ldots, \nu_\ell + p^{n_\ell}, \nu_{\ell+1} \ldots, \nu_\xi) = \left(\nu_0, \ldots, \nu_\ell + p^{n_\ell}, i'_{\ell+1} + j'_{\ell+1} \ldots, i'_\xi + j'_\xi\right) = L' + K'.$$

Indeed, recall that if we match our element with an initial term corresponding to $\bar{f}_\ell^{p^{n_\ell}}$ then all the other terms can be defined by the equation of the irreducible polynomial of $\bar{f}_\ell$.

Define $\bar{\nu}$ as follows:

$$\nu_s = \begin{cases} i'_s + j'_s & \text{for } s \neq \ell \\ i'_\ell + j'_\ell - p^{n_\ell} & \text{for } s = l \end{cases}$$

The element $(\nu_0, \ldots, \nu_\xi)$ lies in $\mathbf{A} + \mathbf{A}$. Indeed, since $L' + K'$ is in $\mathbf{A} + \mathbf{A}$, according to Lemma 18 we only need to show that $0 \leq \nu_\nu$ for all $0 \leq \nu \leq \xi$. The only thing that needs to be checked is whether ν_ℓ is nonnegative. Equivalently, whether $i'_\ell + j'_\ell \geq p^{n_\ell}$. Now recall that $i'_\ell + j'_\ell = \lambda p^{n_\ell} + (i_\ell + j_\ell)$ and hence $\nu_\ell = i'_\ell + j'_\ell - p^{n_\ell} = i_\ell + j_\ell + (\lambda - 1)p^{n_\ell}$ by Eq. (25). Since $\lambda \geq 1$ we get

$$\lambda p^{n_\ell} + (i_\ell + j_\ell) \geq p^{n_\ell}$$

as expected.

This proves that $\omega_{L'+K'}$ is the initial term of $K_{\bar{\nu},\ell}$ for $\bar{\nu} = (\nu_0, \ldots, \nu_\xi)$, check also Lemma 12, giving us a contradiction so the map Φ is injective. ◻

Lemma 20 *The map Φ is surjective.*

Proof Take an equivalence class $[L+K]$ in $(\mathbf{A}+\mathbf{A})/\sim$. Recall the definition of the set Γ_{L+K} given in Lemma 5. Consider the minimal element of Γ_{L+K} , i.e. $\min\Gamma_{L+K} := \omega_A\omega_B \in \mathbb{T}^2$. There is such a minimal element since Γ_{L+K} is nonempty (for example $\omega_L\omega_K \in \Gamma_{L+K}$) and since our order is a total order. We still need to show that $\omega_A\omega_B$ is not in $\mathrm{in}_\prec(J)$.

Firstly suppose that $\omega_A\omega_B \in \mathrm{in}_\prec(K_0)$. Then there is $\omega_I\omega_J$ such that $\omega_I\omega_J \prec \omega_A\omega_B$ and $A+B = I+J$. By the last equality, $||A+B|| = ||I+J||$ so $A+B \sim I+J$. But this means that $\omega_I\omega_J$ is also in Γ_{L+K} and is colexicographically smaller than $\omega_A\omega_B$, a contradiction.

Suppose now that $\omega_A\omega_B \in \mathrm{in}_\prec(K_{\bar{\nu},i})$ for some $\bar{\nu}, i$. Then according to lemma 12 there is a second element in the polynomial $K_{\bar{\nu},i}$ which has the same value when $||\cdot||$ is applied, but is smaller in $\prec$ (a contradiction since, having the same $||\cdot||-$value means that they are equivalent i.e. they both lie in Γ_{L+K}). ◻

6 Examples

We provide here some explicit examples of our method for calculating the canonical ideal of HKG curves.

6.1 Artin–Schreier curves

Here we write down the generating sets of the canonical ideal corresponding to Artin-Schreier curves of the form

$$X : y^{p^n} - y = x^m, \qquad (m,p) = 1, \tag{27}$$

where the values of m, p are given in the following table. Notice that these curves form an example of an HKG-cover extension for the $k = 1$ case.

m	Petri's theorem requirement
$m > 5$	$p^n > 3$
$m = 4, 5$	$p^n \geq 5$

In this case the genus g of the curve is $g > 6$ and also the curve is not hyperelliptic nor trigonal.

Indeed the above given curves have Weierstrass semigroup

$$H := m\mathbb{Z}_+ + p^n\mathbb{Z}_+ \tag{28}$$

at the unique ramified point P. Let G be the p^n order Artin-Schreier cover group generated by the automorphism $\tau : y \mapsto y+1, x \mapsto x$. Assume that there is a degree two covering $X \to \mathbb{P}^1$. This is a Galois covering with Galois group generated by the hyperelliptic involution $j : X \to X$. The hyperelliptic involution cannot be in the p^n order Galois group

G of the Artin-Schreier extension, since p is odd. On the other hand it is well known that the hyperelliptic involution is in the center of the automorphism group of X, [3]. Since $\tau(j(P)) = j\tau(P) = P$ we have $j(P) = P$, otherwise the Galois cover $X \to X/G = \mathbb{P}^1$ has two ramified points, a contradiction. But then 2 should be a pole number of the semigroup H, contradicting Eq. (28).

In order to prove that X is also not trigonal, we can employ the fact that with the assumptions given in the table above we can indeed find a quadratic basis of the canonical ideal. Alternatively we can argue as follows: In characteristic zero we know that at a non ramified point P in the degree 3 cover $X \to \mathbb{P}^1$ of a trigonal curve the first few elements in the Weierstrass semigroup at P are $3n, 3n+2, 3n+3$ or $3n, 3n+1, 3n+3, 3n+4$ or $2n+2$ or $2n+1, 2n+3$ for $(g-1)/n \leq n \leq g/2$, see [12, thm p.172]. On the other hand for a Weierstrass point of the trigonal curve which is not ramified in the degree 3 cover, the Weierstrass semigroup at P is of the form

$$a, a+1, a+2, \ldots, a+(s-g), s+2, s+3, \ldots$$

for some $g \leq a \leq \lfloor (s+1)/2 \rfloor + 1$ and $g-1 < s \leq 2g-2$, [12, Lemma 2.5]. The Lefschetz principle implies that this is the structure of Weierstrass semigroups for a big enough prime p. On the other hand, the ramified point P in the Artin-Schreier cover is a Weierstrass point, see [7, Th. 1]. The semigroup structure at P given in Eq. (28) is not compatible with any of the Weierstrass semigroups of trigonal curves, therefore the curve X is not trigonal at least for big enough p. Unfortunately the bound for the prime p comes from Lefschetz principle and can not be determined.

Recall that H_i denotes the bounded parts of the Weierstrass semigroup (Eq. 2). For the case at hand we have that

$$|H_1| = g = (m-1)(p^n - 1)/2$$
$$|H_2| = 3(g-1).$$

Also $\mathbf{A} = \{L := (i_0, i_1) : i_0 p^n + i_1 m \leq 2(g-1)\}$ and

$$\mathbf{A} + \mathbf{A} = \{L + K = (i_0 + j_0, i_1 + j_1) \mid L := (i_0, i_1) \in \mathbf{A},\ K := (j_0, j_1) \in \mathbf{A}\}.$$

The equivalence class of $L + K \in \mathbf{A} + \mathbf{A}$, as described in Lemma 5, corresponds to the following set of degree 2 monomials

$$\Gamma_{L+K} = \{\omega_A \omega_B \in \mathrm{Sym} H^0(X, \Omega_X) : A + B - (L+K) = (\lambda m, -\lambda p^n) \text{ for some } \lambda \in \mathbb{Z}\}.$$

According to proposition 7 K_0 is defined by

$$K_0 := \{\omega_L \omega_K - \omega_{L'} \omega_{K'} \in \mathrm{Sym} H^0(X, \Omega_X) : L + K = L' + K', L, K, L', K' \in \mathbf{A}\}.$$

The sets $K_{\bar{v},i}$ containing the information of the Artin-Schreier extension now adopt the following, much simpler form:

$$K_{(v_0,v_1),1} = \left\{ \omega^{\otimes 2}_{(v_0, v_1 + p^n)} - \omega^{\otimes 2}_{(v_0, v_1+1)} - \omega^{\otimes 2}_{(v_0+m, v_1)} \right\}$$

for the $\bar{v} := (v_0, v_1)$ satisfying $||(v_0, v_1 + p^n)|| \leq 4g - 4$, equivalently,

$$v_0 p^n + v_1 m + p^n m \leq 4g - 4.$$

Notice that if p, n and m are given specific values, the last inequality can be solved explicitly and the generating sets can be written down.

Example 21 Recall that $\omega_{ij} = x^i y^j dx$. Consider the Artin-Schreier curve $y^7 - y = x^4$ of genus 9. The canonical ideal is generated by the set K_0 given by

$$\begin{aligned}&\{-\omega_{0,4}\omega_{1,0}+\omega_{0,3}\omega_{1,1}, -\omega_{1,0}\omega_{1,1}+\omega_{0,1}\omega_{2,0}, \omega_{0,4}\omega_{1,0}-\omega_{0,3}\omega_{1,1}, \omega_{1,0}\omega_{1,1}-\omega_{0,1}\omega_{2,0}, -\omega_{0,2}^2+\omega_{0,1}\omega_{0,3}, \omega_{0,2}^2-\omega_{0,1}\omega_{0,3},\\
&-\omega_{0,1}\omega_{1,1}+\omega_{0,0}\omega_{1,2}, \omega_{0,1}\omega_{1,1}-\omega_{0,0}\omega_{1,2}, -\omega_{0,2}\omega_{1,1}+\omega_{0,1}\omega_{1,2}, \omega_{0,2}\omega_{1,1}-\omega_{0,1}\omega_{1,2}, -\omega_{1,1}\omega_{1,2}+\omega_{0,3}\omega_{2,0}, -\omega_{0,2}^2+\omega_{0,0}\omega_{0,4},\\
&\omega_{0,2}^2-\omega_{0,0}\omega_{0,4}, \omega_{1,1}\omega_{1,2}-\omega_{0,3}\omega_{2,0}, -\omega_{0,1}^2+\omega_{0,0}\omega_{0,2}, \omega_{0,1}^2-\omega_{0,0}\omega_{0,2}, -\omega_{0,3}\omega_{1,1}+\omega_{0,2}\omega_{1,2}, -\omega_{1,1}^2+\omega_{0,2}\omega_{2,0},\\
&\omega_{0,3}\omega_{1,1}-\omega_{0,2}\omega_{1,2}, -\omega_{1,0}\omega_{1,2}+\omega_{0,2}\omega_{2,0}, \omega_{1,0}\omega_{1,2}-\omega_{0,2}\omega_{2,0}, \omega_{1,1}^2-\omega_{0,2}\omega_{2,0}, -\omega_{0,2}\omega_{1,0}+\omega_{0,0}\omega_{1,2}, \omega_{0,2}\omega_{1,0}-\omega_{0,0}\omega_{1,2},\\
&\omega_{0,3}\omega_{1,0}-\omega_{0,2}\omega_{1,1}, -\omega_{0,3}\omega_{1,0}+\omega_{0,1}\omega_{1,2}, \omega_{0,3}\omega_{1,0}-\omega_{0,1}\omega_{1,2}, -\omega_{0,2}\omega_{1,0}+\omega_{0,1}\omega_{1,1}, -\omega_{1,0}^2+\omega_{0,0}\omega_{2,0}, \omega_{0,2}\omega_{1,0}-\omega_{0,1}\omega_{1,1},\\
&\omega_{1,0}^2-\omega_{0,0}\omega_{2,0}, -\omega_{1,1}^2+\omega_{1,0}\omega_{1,2}, \omega_{1,1}^2-\omega_{1,0}\omega_{1,2}, -\omega_{0,1}\omega_{0,2}+\omega_{0,0}\omega_{0,3}, -\omega_{0,4}\omega_{1,1}+\omega_{0,3}\omega_{1,2}, \omega_{0,4}\omega_{1,1}-\omega_{0,3}\omega_{1,2},\\
&-\omega_{0,2}\omega_{0,3}+\omega_{0,1}\omega_{0,4}, \omega_{0,2}\omega_{0,3}-\omega_{0,1}\omega_{0,4}, -\omega_{0,3}^2+\omega_{0,2}\omega_{0,4}, \omega_{0,3}^2-\omega_{0,2}\omega_{0,4}, -\omega_{0,1}\omega_{0,3}+\omega_{0,0}\omega_{0,4}, \omega_{0,1}\omega_{0,3}-\omega_{0,0}\omega_{0,4},\\
&-\omega_{0,4}\omega_{1,0}+\omega_{0,2}\omega_{1,2}, \omega_{0,4}\omega_{1,0}-\omega_{0,2}\omega_{1,2}, -\omega_{0,1}\omega_{1,0}+\omega_{0,0}\omega_{1,1}, \omega_{0,1}\omega_{1,0}-\omega_{0,0}\omega_{1,1}, \omega_{1,2}^2-\omega_{0,4}\omega_{2,0},\\
&-\omega_{1,2}^2+\omega_{0,4}\omega_{2,0}, \omega_{0,1}\omega_{0,2}-\omega_{0,0}\omega_{0,3}, -\omega_{0,3}\omega_{1,0}+\omega_{0,2}\omega_{1,1}\}\end{aligned}$$

and one trinomial

$$-\omega_{0,0}\omega_{0,1}+\omega_{0,3}\omega_{0,4}-\omega_{2,0}^2$$

6.2 HKG-covers with *p*-cyclic group

This is a case where all the intermediate subextensions F_i/F_{i-1} are of degree p and the corresponding irreducible polynomials are

$$X^p + a^{(i)}X - D_i$$

In this case the generating sets of the canonical ideal are

$$K_0 := \{\omega_L\omega_K - \omega_{L'}\omega_{K'} \in \mathrm{Sym}H^0(X,\Omega_X) : L+K = L'+K', L, K, L', K' \in \mathbf{A}\}$$

$$K_{\bar{\nu},i} := \left\{ \omega^{\otimes 2}_{(\nu_0,\ldots,\nu_i+p,\ldots,\nu_\xi)} + a^{(i)}\omega^{\otimes 2}_{(\nu_0,\ldots,\nu_i,\ldots,\nu_\xi)} - \sum_{\substack{\bar{\lambda}\in\mathbf{A}+\mathbf{A}\\ ||\bar{\lambda}||\le p\bar{m}_i}} a^{(i)}_{\bar{h}}\omega^{\otimes 2}_{\bar{\beta}_{\bar{\nu},i,\bar{\lambda}}} \right\} \tag{29}$$

such that $||\bar{\gamma}_{\bar{\nu},i,0}|| \le 4g-4$ where $\bar{\beta}_{\bar{\nu},i,\bar{\lambda}} = (l_0,\ldots,l_{i-1},0,\ldots,0)+\bar{\nu}$ as defined before.

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