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AUTOMORPHISMS OF CURVES AND WEIERSTRASS SEMIGROUPS FOR HARBATER–KATZ–GABBER COVERS

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Abstract. We study p-group Galois covers $X \to \mathbb{P}^1$ with only one fully ramified point in characteristic $p > 0$. These covers are important because of the Harbater–Katz–Gabber compactification theorem of Galois actions on complete local rings. The sequence of ramification jumps is related to the Weierstrass semigroup of the global cover at the stabilized point. We determine explicitly the jumps of the ramification filtrations in terms of pole numbers. We give applications for curves with zero p-rank: we focus on curves that admit a big action. Moreover, we initiate the study of the Galois module structure of polydifferentials.

1. Introduction

Let X be a projective nonsingular algebraic curve of genus $g_X \geq 2$ defined over an algebraically closed field k of characteristic p. For technical reasons we will exclude the characteristics $p = 2, 3$ from our study. We will denote by F the function field of the curve X, and by G a subgroup of the automorphism group $\mathrm{Aut}(X)$.

In the literature there is a lot of interest concerning the properties of the automorphism group [19, 43, 44, 54], its size related to several topological invariants of the curve X [9, 16, 20, 28, 31, 36], the deformation theory of the couple (X, G) [2], and lifting problems [7, 8, 32–34].

An important tool in understanding the automorphism group is the localization of the action by considering the inertia group $G(P) = \{\sigma \in G : \sigma(P) = P\}$, acting on the local ring $\mathcal{O}_P$ at a k-rational point P. It is well known (see section 2.1) that the group $G(P)$ admits the following ramification filtration:

$$G(P) = G_{-1}(P) \supseteq G_0(P) \supseteq G_1(P) \supseteq G_2(P) \supseteq \cdots \supseteq \{\mathrm{id}\}.$$

The determination of the ramification filtration, and its *jumps*, i.e., the indices such that $G_i(P) \supsetneqq G_{i+1}(P)$, is a deep problem. For instance if $G_1(P)$ is abelian, then the Hasse–Arf theorem [41, Theorem, p. 76] puts very strong divisibility relations

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among the jumps. These jumps appear very often in a variety of cases in the literature in the local and global function fields, especially when one considers arithmetic problems in algebraic function fields; the most immediate application is the computation of the degree of the different of a Galois extension of function fields using Hilbert's celebrated formula [45, Chap. III.4].

The knowledge of the jumps is crucial for expressing obstructions to the lifting problem (see [7,8]) and is also related to the Artin representation [41, Chap. VI]. For local applications and their relation to the famous Hasse–Arf theorem, see [41, Chap. IV]; for an unexpected application to normal basis generators, the computation of normal bases is another active research area concerning local and finite fields; see the work in [6,49] and the references therein.

In contrast to their significance, we know only a very few things about jumps. More precisely, as far as we know, they have been computed explicitly in function fields only for some specific cases; see [23, Examples 1–4] for a nonexhaustive list, and for the more general cyclic p^n case see [52, Lemma 1]. For the abelian case $\mathbb{Z}/p^i\mathbb{Z}\times\mathbb{Z}/p^i\mathbb{Z}$, they have been computed only for $i = 1, 2$; for $i = 1$ see [1, section 3], [55, Theorem 3.11], while for $i = 2$ see [24].

Another direction toward understanding the automorphism group G is to consider the representation theory of G, acting on several naturally defined vector spaces. A natural choice for vector spaces acted on by the automorphism group involves the spaces

(i) $H^0(X, \Omega_X^{\otimes m})$, $m \in \mathbb{N}$, of holomorphic polydifferentials of X, and
(ii) Riemann–Roch spaces $L(D)$ for some G-invariant divisor D.

Concerning the first case, the determination of the Galois module structure is an interesting problem which has been solved in the following cases: for unramified Galois covers [48,50]; for the semisimple part, with respect to the Cartier operator, of $H^0(X, \Omega_X)$ for p-covers [26,30]; for cyclic and certain elementary abelian covers, [4,53] and [40], respectively. The main tool that was used in some of the above references, and in our work in this article as well, is the construction of an appropriate basis for the holomorphic polydifferentials. This also has rich connections with other subjects in the literature: the computation of n-Weierstrass points [54, Theorem 14.2.48], [5,12,13,47]; the computation of the rank of the Hasse–Witt matrix [27]; the classification of curves with a Hasse–Witt matrix of a certain rank [39]; and the study of the Artin–Schreier (sub)extensions of the rational function field [51].

Concerning the second case, when $D = P$, we can define an action of $G(P)$ on the spaces $L(nP)$ for $n \in \mathbb{N}$. One can ask if there is any relationship between the localized action of $G(P)$ on $\mathcal{O}_P$ and the natural linear representation on the spaces $\mathrm{GL}(L(nP))$. A way to answer this question is by considering the flag of vector spaces $L(nP)$ for $n \in \mathbb{N}$. The possible jumps of the dimension sequence of this natural flag lead to the notion of pole numbers and Weierstrass semigroups; see Definition 4.

More precisely, the Weierstrass semigroup $H(P) \subset \mathbb{N}$ is a numerical semigroup consisting of all $n \in \mathbb{N}$ such that there is a function f in the function field of the curve X with pole divisor $(f)_\infty = nP$. We will say that the numerical semigroup $H(P)$ has generators $d_1, \dots, d_r$ if

$$H(P) = \mathbb{Z}_+ d_1 + \cdots + \mathbb{Z}_+ d_r,$$

where $\mathbb{Z}_+ := \{d \in \mathbb{Z} : d \geq 1\}$. Each semigroup has a natural partial ordering: for two elements a and b in the semigroup we say that a is smaller than b if $b = a + c$ for another element c in the semigroup. The set of minimal elements with respect to this ordering is called a minimal set of generators for the semigroup; see [11].

An extreme example in the theory of numerical semigroups involves the symmetric ones. If we limit ourselves to the Weierstrass semigroups, then symmetric means that the maximum gap equals the largest possible value: $2g_X - 1$. Equivalently (see also [46, eq. (1.1)]), this symmetry is expressed by the following rule:

$$x \in H(P) \qquad \text{if and only if } 2g_X - 1 - x \notin H(P).$$

Symmetric numerical semigroups are closely connected to the geometry of the curve; see [35, section 7.2]. Moreover, every such semigroup is the Weierstrass semigroup of a Gorenstein curve [46]. For an introduction to numerical semigroups, and the importance of the symmetric condition, we refer to [11, 38].

It is known (see Proposition 7) that the gaps of the ramification filtration of $G(P)$ are related to the semigroup $H(P)$ since if $G_i(P) > G_{i+1}(P)$, for $i \geq 1$, then $i = m_r - m_\nu$, for some pole number m_ν, when m_r is the smallest pole number at P not divisible by the characteristic p.

One of our motivations for this study was to find the set of pole numbers which correspond to jumps of the ramification filtration.

In this note we have set the following aims:

(I) For “Harbater–Katz–Gabber covers”, or simply HKG-covers, i.e., Galois covers of the projective line with a unique wildly and at most one tamely ramified point, we will characterize exactly the lower ramification jumps in terms of pole numbers at their unique wildly ramified point and give a complete description for its symmetric Weierstrass semigroup. We remark that we have not made any assumption for $G_1(P)$ to be an abelian group.

(II) We will initiate the study of the Galois module structure of spaces of polydifferentials for HKG-covers and give a basis for their m-holomorphic polydifferentials. We will prove that HKG-covers arise in a natural way as Galois covers of curves with zero p-rank and apply these results to curves equipped with a “big action”, showing also that the module of their holomorphic polydifferentials is an indecomposable $k\,[G_1(P)]$-module.

Remark 1. We focus on the jumps of the ramification filtration. The ramification filtration might jump at -1, and in this case $G_{-1}(P)/G_0(P)$ is a nontrivial group isomorphic to the Galois group of the corresponding residue field extension. Moreover, we might have a jump at 0 if and only if there is tame ramification since $G_0(P)/G_1(P)$ equals the tame ramification degree. The crucial information regarding all of the other higher order ramification jumps lies on the p-part of G; this is the reason why we assume that our field is algebraically closed and we restrict ourselves to the p-part of the ramification filtration, i.e., to $G_1(P)$. Thus $G_{-1}(P) = G_0(P) = G_1(P)$.

Although it seems possible to extend all of our results over perfect, instead of algebraically closed, base fields, there are certain places that have to be treated with some extra attention. In the following proofs we have used explicitly the fact that k is an algebraically closed field: in the proof of Proposition 27, and in the proofs of Theorem 29 and Corollary 32, respectively. In the latter cases we use [45, Prop. III.7.10], which requires certain polynomials to have all of their roots in k.

It is clear that the group $G_1(P)$ acts on the vector spaces $L(m_iP)$ for each $i \in \mathbb{N}$, defining representations

$$\rho_i : G_1(P) \to \mathrm{GL}\big(L(m_iP)\big). \tag{1}$$

The second author proved that all but a finite number of these representations are faithful.

Proposition 2 ([23, Lemmata 2.1 and 2.2]). *If $g_X \geq 2$ and $p \neq 2,3$, then there is at least one pole number $m_r \leq 2g_X - 1$ not divisible by the characteristic p. Then there is a faithful representation*

$$\rho : G_1(P) \to \mathrm{GL}\big(L(m_rP)\big),$$

where m_r is the smallest pole number not divisible by the characteristic.

Remark 3. Observe that for a general decomposition group with tame ramification, the above defined representation might not be faithful.

Proposition 2 is the starting point for defining a new filtration of $G_1(P)$, which we will call the ramification filtration. More precisely, the ith group is just the kernel of the linear representation ρ_i defined in equation (1). We refer to section 2.4, for a more detailed definition. The set of jumps of the representation filtration are easier to understand since their definition is based on representations of the general linear group. Our study of the jumps of the ramification filtration, with the aid of the representation filtration and the Weierstrass semigroup theory, gives a complete description of them.

In general the ramification filtration can be introduced and studied in terms of general local rings; see [41]. In the case of spectra $\mathcal{O}$ of local rings of the form $k[[t]]$ acted on by a group G_0, where k is a perfect field of characteristic $p > 0$, we can pass from the local case to the global one with the HKG-covers; see Definition 12. These covers can be seen as a minimal compactification of a local action, and there is a lot of interest in them; for instance they appear in deformation and lifting problems [3, 7, 8].

By considering the HKG compactification of an action on the local ring $k[[t]]$, we have the advantage of being able to attach global invariants, like genus, p-rank, differentials, etc., to the local case. Also finite subgroups of the automorphism group $\mathrm{Aut}k[[t]]$, a subject difficult to understand, but crucial for studying deformation theory of curves with automorphisms [2], become subgroups of $\mathrm{GL}(V)$ for a finite dimensional vector space V.

We would like to point out that in the case of Riemann surfaces such a relation among the group $G(P)$ and the Weierstrass semigroup at P is known. Morisson and Pinkham [29] studied this connection in characteristic 0 for *Galois Weierstrass points*: a point P on a compact Riemann surface Y is called Galois Weierstrass if for a meromorphic function f on Y such that $(f)_\infty = dP$, where d is the least pole number in the Weierstrass semigroup at P, the function $f : Y \to \mathbb{P}^1(\mathbb{C})$ gives rise to a Galois cover. This article can be seen as a natural generalization of some results in that article in positive characteristic. Notice that in the latter case, the first nonzero element in $H(P)$ is not enough to grasp the group structure. We have to go up to the first pole number in $H(P)$ that is not divisible by p to do so. And of course the stabilizer $G(P)$ and its p-part $G_1(P)$ do not have to be cyclic groups anymore.

Our motivation for studying actions on HKG-covers was the deformation theory of curves with automorphisms. Bertin and Mézard in [2] proved that a local global principle that can be used to show that the "difficult part" of the study of the deformation functor of curves with automorphisms resides in the local deformation functors.

This is too vast an object of study to describe here; the reader is advised to look at [2] for more information. Local actions can be compactified to HKG-covers, and at least the dimension of the tangent space of the deformation functor is reflected by the space of 2-holomorphic differentials $H^0(X, \Omega_X^{\otimes 2})$ of the corresponding HKG-cover. Indeed, in [22] the second author related the dimension of the space of coinvariants of global sections of 2-polydifferentials $\dim H^0(X, \Omega_X^{\otimes 2})_G$ to the dimension of the tangent space of the deformation functor of curves with automorphisms. This computation is a complicated task and deserves further study.

The structure of the article is as follows: In section 2 we review some basic notions for the ramification filtration (see section 2.1) and the Weierstrass semigroup at a fixed point of our curve X (see section 2.2). After that, we see how these two notions are related in section 2.3 and focus on the HKG-covers, where the Galois group is not necessarily an abelian group. We finally define the representation filtration and give all the necessary background in order to state our two main results in section 2.4. Section 3 provides some information concerning the Weierstrass semigroup at a totally ramified point for a general Galois cover. Section 4 is the heart of this note, providing the proofs for the computation of the ramification jumps (in upper and lower numbering). Section 5 is devoted to applications of our main results: we focus mainly on curves with big actions; see section 5.1. These curves, like any other HKG-cover, are curves with zero p-rank; see section 5.2. Finally, in section 5.3 we interpret the Hasse–Arf theorem in terms of our results. In section 6 we provide a basis for holomorphic polydifferentials. This will characterize all of the Weierstrass semigroups that we have previously computed as symmetric; on the other hand, this will also be the starting point for studying the Galois module structure.

2. Definitions and main results

2.1. **Ramification filtration.** Let $\mathcal{O}_P$ be the completed local ring at the k-rational point P, and let m_P be its maximal ideal. The subgroup $G_i(P) \subset G(P)$ is defined as the subgroup of $\sigma \in G(P)$ which acts trivially on $\mathcal{O}_P/m_P^{i+1}$. The groups $G_i(P)$ form a filtration:

$$G(P) = G_{-1}(P) \supseteq G_0(P) \supseteq G_1(P) \supseteq G_2(P) \supseteq \cdots \supseteq \{\mathrm{id}\}.$$

It is known that $G_1(P)$ is the p-part of $G(P)$ and that $G_0(P)/G_1(P)$ is a cyclic group of order prime to p, while for $i \geq 1$ the quotients G_i/G_{i+1} are elementary abelian groups. The quotient $G(P)/G_0(P)$ is isomorphic to the Galois group $\mathrm{Gal}\left(\frac{\mathcal{O}_P}{P} / \frac{\mathcal{O}_P^G}{\mathcal{O}_P^G \cap P}\right)$. The latter group is trivial if k is algebraically closed. By Remark 1 we will restrict ourselves to the study of the jumps of the p-part $G_1(P)$.

Let us fix the notation for the jumps:

$$G_0(P) = G_1(P) = G_{b_1} > G_{b_2} > \cdots > G_{b_\mu} > \{\mathrm{id}\}. \tag{2}$$

This means that $G_{b_\nu} \supsetneq G_{b_\nu+1} = G_{b_{\nu+1}}$ for every $1 \leq \nu \leq \mu$ and that there are μ jumps.

The theory of ramification filtrations can be considered more generally for complete discrete valuation rings; see [41, Chap. IV]. We will see in section 2.5 that such local actions on rings $k[[t]]$ can always come from actions on curves.

2.2. **Weierstrass semigroups.** Consider the flag of vector spaces

$$k = L(0) = L(P) = \cdots = L((i-1)P) < L(iP) \leq \cdots \leq L((2g_X - 1)P),$$

where

$$L(iP) := \{f \in F : \operatorname{div}(f) + iP \geq 0\} \cup \{0\}.$$

We will write $\ell(D) = \dim_k L(D)$ for a divisor D.

Definition 4. An integer i will be called a pole number if there is a function $f \in F^*$ such that $(f)_\infty = iP$ or equivalently $\ell\big((i-1)P\big) + 1 = \ell\big(iP\big)$. If i is not a pole number, we call it a gap. The set of pole numbers at P form a numerical semigroup $H(P)$ which is called the Weierstrass semigroup at P.

Note that 0 is always a pole number; thus from now on when we write $H(P)$ we always assume that $\{0\} \in H(P)$ for every Weierstrass semigroup. It is known that there are exactly g_X pole numbers that are smaller than or equal to $2g_X - 1$ and that every integer $i \geq 2g_X$ is in the Weierstrass semigroup; see [45, I.6.7].

2.3. **Action on Riemann–Roch spaces.**

Definition 5. Let m_r be the smallest pole number at P not divisible by p. Denote by

$$0 = m_0 < \cdots < m_{r-1} < m_r$$

all of the pole numbers at P in increasing sequence which are $\leq m_r$. From now on, $f_i \in F$, for $0 \leq i \leq r$, will denote a selection of a function such that $(f_i)_\infty = m_i P$.

Remark 6. Observe that a function which has a unique pole at P of order m_i is not unique. If f_i, f_i' are two functions such that $(f_i)_\infty = (f_i')_\infty = m_i P$, then by examining the Laurent expansion of f_i, f_i', there is constant $C \in k^*$ such that

$$f_i' = Cf_i + g,$$

where g is a function in $L(m_{i-1}P)$.

Concerning the jumps of the ramification filtration, we have the following characterization.

Proposition 7 ([23, Prop. 2.3])**.** *Let X be a curve acted on by the group G. For every fixed point P on X we consider the corresponding faithful representation defined in Proposition* 2*:*

$$\rho : G_1(P) \to \mathrm{GL}_{\ell(m_r P)}(k).$$

If $G_i(P) > G_{i+1}(P)$ for $i \geq 1$, then $i = m_r - m_\nu$, for some pole number m_ν.

Since we characterize exactly the jumps and the structure of the Weierstrass semigroups at the unique ramified point of an HKG-cover, we also characterize exactly the set of pole numbers m_ν for $\nu < r$ such that $m_r - m_\nu$ is a jump.

2.4. **Representation filtration.** Recall that

$$0 = m_0 < \cdots < m_{r-1} < m_r$$

are all of the pole numbers at P in increasing sequence up to m_r.

Definition 8. For each $0 \leq i \leq r$ we consider the representations

$$\rho_i : G_1(P) \to \mathrm{GL}(L(m_i P)).$$

We form the decreasing sequence of groups

$$G_1(P) = \ker\rho_0 \supseteq \ker\rho_1 \supseteq \ker\rho_2 \supseteq \cdots \supseteq \ker\rho_r = \{1\}. \tag{3}$$

We will call this sequence of groups the *representation filtration.*

Remark 9. The ith ramification group is the kernel of the map

$$\phi_i : G_1(P) \to \mathrm{Aut}\left(\mathcal{O}_p/m_P^{i+1}\right),$$

while the ith representation group is the kernel of the map

$$\rho_i : G_1(P) \to \mathrm{GL}\big(L(m_i P)\big),$$

where $L(m_i P)$ can also be seen as a quotient of $L(m_r P) = \Big(L(m_i P) \oplus W\Big)/W$, where W is the vector space complement of $L(m_i P)$ in $L(m_r P)$.

Note that the spaces $L(m_i P)$ are fixed by the action of $G_1(P)$. The filtration of equation (3) leads to a successive sequence of elementary abelian p-group extensions of the field $F^{G_1(P)}$:

$$F^{G_1(P)} = F^{\ker\rho_0} \subseteq F^{\ker\rho_1} \subseteq F^{\ker\rho_2} \subseteq \cdots \subseteq F^{\ker\rho_r} = F. \tag{4}$$

We call an index i a jump of the representation filtration if and only if $\ker\rho_i \gneq \ker\rho_{i+1}$. Let us also fix the notation for the representation jumps:

$$G_1(P) = \ker\rho_0 = \cdots = \ker\rho_{c_1} > \cdots > \ker\rho_{c_{n-1}} > \ker\rho_{c_n} > \{\mathrm{id}\}.$$

In other words, the above sequence jumps at n integers. These integers will be called jumps of the representation filtration,

$$c_1 < c_2 < \cdots < c_{n-1} < c_n = r - 1. \tag{5}$$

The last equality $c_n = r-1$ comes from the faithful representation of Proposition 2 since $\ker\rho_r = \{1\}$, coupled with Lemma 24, which will be proved later. Notice that $c_i \in \{1, \ldots, r\}$ for all $1 \leq i \leq n$.

Remark 10. Every element $\sigma \in \ker\rho_i$ fixes by definition all f_ν such that $(f_\nu)_\infty = m_\nu P$ for $\nu \leq i$. A nonnegative integer i is a representation jump whenever the function f_{i+1} is not $\ker\rho_i$ invariant.

We will prove in Proposition 20 that if c_i is a representation jump, then m_{c_i+1} is a minimal generator of $H(P)$. At every jump of the sequence of the groups $\ker\rho_{c_i}$, the corresponding sequence of fields will also jump and moreover,

$$F^{\ker\rho_{c_{i+1}}} = F^{\ker\rho_{c_i}}(f_{c_i+1}). \tag{6}$$

Definition 11. In order to simplify notation, we set $F_i = F^{\ker\rho_{c_i}}$, $\bar{m}_i = m_{c_i+1}$, and $\bar{f}_i = f_{c_i+1}$. Denote also $p^{h_i} = |\ker\rho_{c_{i+1}}|$ for all $1 \leq i \leq n-1$, and $p^{h_0} = |G_1(P)|$.

Thus equation (6) can be written as

$$F_{i+1} = F_i(\bar{f}_i).$$

We will prove in Lemma 24 that in every extension we add an extra function $f_{c_i+1} = \bar{f}_i$. Define $Q_i = F_i \cap P$ for $1 \leq i \leq n+1$ to be the *unique* ramification point of the tower defined in equation (4). At the level of the Weierstrass semigroups, the field generator $\bar{f}_i$ adds a new generator $\bar{m}_i$ in the image of the semigroup $H(Q_i)$ on $H(Q_{i+1})$. In section 3 in Lemma 16, we will see how the Weierstrass semigroups at the ramified points of a Galois extension of fields are related. Using this relation, the semigroup of F_2 at Q_2 is

$$H(Q_2) = \left| \frac{\ker\rho_{c_1}}{\ker\rho_{c_2}} \right| \mathbb{Z}_+ + \lambda_1 \mathbb{Z}_+ = p^{h_0 - h_1}\mathbb{Z}_+ + \lambda_1 \mathbb{Z}_+,$$

with $(\lambda_1, p) = 1$.

Notice that $\lambda_1 = 1$ if and only if $F^{\ker\rho_{c_2}}$ is rational. We proceed in this way to obtain

$$H(Q_{i+1}) = p^{h_{i-1} - h_i} H(Q_i) + \lambda_i \mathbb{Z}_+ \qquad \text{for all } 1 \leq i \leq n,$$

where $(\lambda_i, p) = 1$. We will see in Proposition 20 that the elements

$$p^{h_1}\lambda_1 < p^{h_2}\lambda_2 < \cdots < p^{h_{n-1}}\lambda_{n-1} < \lambda_n = \frac{m_{c_n+1}}{|\ker\rho_{c_{n+1}}|} = m_r$$

are inside the set of generators of the Weierstrass semigroup at P. If we add the element p^{h_0}, then Proposition 26 will give

$$\langle p^{h_0}, p^{h_1}\lambda_1, \ldots, p^{h_{n-1}}\lambda_{n-1}, \lambda_n \rangle_{\mathbb{Z}_+} = H(P).$$

We have the following picture of fields, groups, places and semigroups:

$$\begin{array}{cccc}
F & \{1\} & P & H(P) \\
\Big| & \Big| & \Big| & \Big| \\
F_{i+1} = F^{\ker\rho_{c_{i+1}}} & \ker\rho_{c_{i+1}} & Q_{i+1} & H(Q_{i+1}) = \langle p^{h_{i-1}-h_i} H(Q_i), \lambda_i \rangle_{\mathbb{Z}_+} \\
\Big| & \Big| & \Big| & \Big| \\
F_i = F^{\ker\rho_{c_i}} & \ker\rho_{c_i} & Q_i & H(Q_i) \\
\Big| & \Big| & \Big| & \Big| \\
F_0 = F^{G_1(P)} & G_1(P) & Q_1 & \mathbb{Z}_+
\end{array}$$

2.5. **HKG-covers.**

Definition 12. An HKG-cover is a Galois cover $X_{\mathrm{HGK}} \to \mathbb{P}^1$ such that there are at most two k-rational points $p_1, p_2 \in \mathbb{P}^1$ such that p_1 is tamely ramified and p_2 is fully ramified. All other geometric points of $\mathbb{P}^1$ remain unramified. In this article we are interested in p-groups, so for us HKG-covers have a unique ramified point.

So far we have started with a subgroup of $\mathrm{Aut}(X)$ that is the isotropy group $G(P)$ of a fixed point P of X. On the other hand, in section 5 we will see in Theorem 40 that X_{HKG} has zero p-rank and thus that every p-subgroup G of $\mathrm{Aut}(X_{\mathrm{HKG}})$ can be realized as the stabilizer of a unique point P (see [19, paragraph 11.13]); thus $G = G(P)$.

The HKG compactification theorem [18], [21, Th. 1.4.1] for the case of p-groups asserts that there is an HKG-cover $X_{\mathrm{HKG}} \to \mathbb{P}^1$ ramified only at one point P, with Galois group $G = \mathrm{Gal}(X_{\mathrm{HKG}}/\mathbb{P}^1) = G_0$ such that $G_0(P) = G_0$, and the action of G_0 on the completed local ring $\hat{\mathcal{O}}_{X_{\mathrm{HKG}},P}$ coincides with the original action of G_0 on $\mathcal{O}$.

For the case of HKG-covers, we will show in Corollary 32 that the subset of minimal generators $\bar{m}_1, \ldots, \bar{m}_n$ of the Weierstrass semigroup described in Proposition 20 is the whole set of minimal generators unless $G_1(P) = G_2(P)$. In the latter case we will prove in Proposition 34 that we also have to add $|G_1(P)|$ in order to obtain the full set of minimal generators of the semigroup. In Proposition 27 we will describe the action of the Galois group on the generators of the tower of the fixed fields by the kernels; this will be a fundamental step for the computation of the jumps that will be given in Theorem 29. We will also prove in Corollary 30 that the representation and ramification filtrations coincide. By these two results the jumps of the ramification filtration are completely determined. A basis of holomorphic polydifferentials will be given in Proposition 42; this will help us to derive some useful information for their Galois module structure in Proposition 44. Finally, this basis of holomorphic polydifferentials also proves, in Corollary 43, that the Weierstrass semigroup at the ramified point is symmetric.

2.6. **Main results.** Now we are ready to state our two main theorems: concerning the structure of $H(P)$, the Weierstrass semigroup at P, we have the following.

Theorem 13.

(1) *For every jump of the representation filtration c_i, $1 \leq i \leq n$, there exists a generator of $H(P)$ of the form $\bar{m}_i = m_{c_i+1} = p^{h_i}\lambda_i$, where $(\lambda_i, p) = 1$.*

(2) *The first ramification jump affects the structure of $H(P)$ in the following way:*
   - (a) *If $G_1(P) > G_2(P)$, then the extension $F/F^{G_2(P)}$ is also HKG, and the Weierstrass semigroup $H(P)$ is minimally generated by $\bar{m}_i$, with $1 \leq i \leq n$. Moreover, $|G_2(P)| = \bar{m}_1 = m_1$.*
   - (b) *If $G_1(P) = G_2(P)$, then we need $\bar{m}_i$, $1 \leq i \leq n$ together with $p^{h_0} = |G_1(P)|$ in order to generate $H(P)$. In this case $|G_1(P)| \neq \bar{m}_i$ for all $1 \leq i \leq n$.*

   In both cases the semigroup $H(P)$ is symmetric.

Proof. Part (1) will be proved in Proposition 20; part 2a will be proved in Corollary 32, while part (2b) will be proved in Proposition 34 and Lemma 25. Finally, the assertion about the symmetric Weierstrass semigroup will follow from Corollary 43. □

The relationship between the representation and the ramification filtrations is given in terms of the following.

Theorem 14. *Assume that $X \to X/G_1(P) = \mathbb{P}^1$ is an HKG-cover. Then the following are true:*

(1) *The jumps of the ramification filtration are the integers λ_i for $1 \leq i \leq n$, i.e., $\lambda_i = b_i$ for every such i, while the number of ramification and representation jumps coincide, i.e., $\mu = n$.*

(2) $G_{b_i} = \ker \rho_{c_i}$ *for all $2 \leq i \leq n$.*

Proof. Part (1) will be proved in Theorem 29, while (2) will be proved in Corollary 30. □

Remark 15. In view of Remark 1 let us assume that we have an HKG-cover which is also tamely ramified. Since $G_1(P) \lhd G_0(P)$, we have the following picture of curves, function fields, and ramified places:

$$\begin{tikzcd}
X \arrow[d] & F \arrow[d, dash] & P \arrow[d, dash] & \\
\mathbb{P}^1 \cong X/G_1(P) \arrow[d, "C_n"] & F^{G_1(P)} \arrow[d, dash] & Q \arrow[d, dash] & Q' \arrow[d, dash] \\
\mathbb{P}^1 \cong X/G_0(P) & F^{G_0(P)} & q & q'
\end{tikzcd}$$

Keep in mind that $G_0(P)$ is the semidirect product of the cyclic $G_0(P)/G_1(P)$ and $G_1(P)$. Now the lower ramification jumps are at 0, while the rest of them are given by Theorem 14.

3. Totally ramified Galois covers

We begin our study by relating the Weierstrass semigroups at totally ramified points of Galois covers over algebraically closed fields in positive characteristic. We remark that the results obtained in this section are not limited to p-groups. Consider a Galois cover $\pi : X \to Y = X/G$ of algebraic curves, and let P be a fully ramified k-rational point of X. How are the Weierstrass semigroup sequences of P and $\pi(P)$ related?

Lemma 16. *Let $F(X)$, $F(Y) = F(X)^G$ denote the function fields of the curves X and Y, respectively. The morphisms*

$$N_G : F(X) \to F(Y)$$

and

$$\pi^* : F(Y) \to F(X),$$

sending $f \in F(X)$ to $N_G(f) = \prod_{\sigma \in G} \sigma f$ and $g \in F(Y)$ to $\pi^* g \in F(X)$, respectively, induce injections

$$N_G : H(P) \to H(Q)$$

and

$$\pi^* : H(Q) \xrightarrow{\times |G|} H(P),$$

where $Q := \pi(P)$.

Proof. For every element $f \in F(X)$ such that $(f)_\infty = mP$, the element $N_G(f)$ is a G-invariant element, so it is in $F(Y)$. Moreover, the pole order of $N_G(f)$ seen as a function on $F(X)$ is $|G| \cdot m$. But since P is fully ramified, the valuation of $N_G(f)$ expressed in terms of the local uniformizer at $\pi(P)$ is just $-m$.

On the other hand, an element $g \in F(Y)$ seen as an element of $F(X)$ by considering the pullback $\pi^*(g)$ has for the same reason a valuation at P multiplied by the order of G. □

Remark 17. The condition of full ramification is necessary in the above lemma. Indeed, if a point $Q \in Y$ has more than one element in $\pi^{-1}(Q)$, then the pullback of g is supported on $\pi^{-1}(Q)$ and gives no information for the Weierstrass semigroup at any of the points $P \in \pi^{-1}(Q)$.

Corollary 18. *The order $|G| \in H(P)$ if and only if $g_{X/G} = 0$.*

Another immediate consequence of Lemma 16 is the following.

Corollary 19. *If an element f is such that $(f)_\infty = aP$ is invariant under the action of a subgroup $H < G$, then $|H|$ divides a.*

Proof. Since f is invariant, it is the pullback of a function $g \in F(X/H)$. The result now follows from Lemma 16. □

4. Enumerating jumps

Recall that an index i is a jump of the representation filtration if and only if $\ker \rho_i \gneq \ker \rho_{i+1}$, and that we have the following sequence for the representation jumps:

$$G_1(P) = \ker \rho_0 = \cdots = \ker \rho_{c_1} > \cdots > \ker \rho_{c_{n-1}} > \ker \rho_{c_n} = \ker \rho_{r-1} > \{\mathrm{id}\}.$$

Proposition 20. *If* $\ker\rho_{c_i} \supsetneq \ker\rho_{c_i+1}$*, i.e., when c_i is a representation jump, then $\bar{m}_i = m_{c_i+1}$ is a minimal generator of $H(P)$.*

Proof. Fix elements $f_i \in L(m_r P)$ with pole numbers m_i. Suppose that $\ker \rho_{c_i} \supsetneq \ker \rho_{c_i+1}$; then the element f_{c_i+1} is not fixed by $\ker \rho_{c_i}$. Observe also that every function in $L(mP)$, with $m < m_{c_i+1}$, is by definition fixed by $\ker \rho_{c_i+1}$.

If m_{c_i+1} is in the semigroup $\langle m_1, \ldots, m_{c_i} \rangle_{\mathbb{Z}_+}$ generated by all $m_1, \ldots, m_{c_i}$ then

$$m_{c_i+1} = \sum_{j \le c_i} \nu_j m_j, \qquad \text{where } \nu_j \in \mathbb{Z}_+, \tag{7}$$

and there is a constant $C \in k^*$ such that

$$f_{c_i+1} = C \cdot \prod_{j \le c_i} f_j^{\nu_j} + \Lambda_{c_i+1}, \tag{8}$$

where Λ_{c_i+1} is a sum of terms such that the degree of their polar part is smaller than m_{c_i+1}. But this is impossible since every element $\sigma \in \ker\rho_{c_i}$ fixes the right-hand side of the last equation; therefore, $\ker \rho_{c_{i+1}} = \ker \rho_{c_i}$, which is a contradiction. The reader should notice that, in general, the expression given in equation (7) is not unique. This fact does not affect the proof of the proposition. □

Remark 21. The fields F, F_i, $i = 1, \ldots, n$ given in equation (4) and in Definition 8 are generated by the elements $\bar{f}_i$ that we introduced in each step, i.e.,

$$F_{i+1} = F_i(\bar{f}_i) = F^{G_1(P)}(\bar{f}_1, \ldots, \bar{f}_i).$$

Moreover, $F^{G_1(P)} = k(f_{i_0})$ for some index i_0, and $F = k(f_{i_0}, \bar{f}_1, \ldots, \bar{f}_n = f_r)$. We form the fields F_i by successive extensions of the rational function field $F^{G_1(P)}$. At every jump c_i of the representation filtration we add an extra element $\bar{f}_i$ to the field F_i.

4.1. **Examples.** The converse of Proposition 20 is wrong. We will give examples of curves where m_{i+1}, for some index $i \in \mathbb{N} \cup \{0\}$, is a generator of the Weierstrass semigroup, but i is not a representation jump; i.e., $\ker \rho_i = \ker \rho_{i+1}$. In the first example provided below we can take $i = 0$.

Example 22. Consider the Artin–Schreier extension of the rational function field given by the equation

$$y^p - y = f(x),$$

where $f(x)$ is a polynomial that has a unique pole at ∞ and $\deg f(x) = m_r$, $(p, m_r) = 1$. Suppose that $m_r > p$. It is well known that the Weierstrass semigroup at P, the point above ∞, is given by $\langle p, m_r \rangle_{\mathbb{Z}_+}$ [44, p. 618]. Notice that $|G| = |G_1(P)| = |\ker \rho_0| = p$, with $m_1 = p$ being a generator of the Weierstrass semigroup *but* $\ker \rho_0 = \ker \rho_1$ since $|\ker \rho_0|$ divides m_1, so f_1 is a $\ker \rho_0$-invariant element and 0 is not a representation jump. Notice that here $m_r = -v_P(y) = -v_\infty(f(x))$ is the unique ramification jump of $G_1(P)$.

Next we will give an example, namely, the Giulietti–Korchmáros curve (see [15]), where m_{i+1} is a Weierstrass generator at P with $i \neq 0$ such that $\ker \rho_i = \ker \rho_{i+1}$.

Example 23 (The GK curve)**.** Let $\xi = p^\alpha$ for a positive integer α and $q = \xi^3$. Let

$$h(X) = \sum_{\kappa=0}^{\xi} (-1)^{\kappa+1} X^{\kappa(\xi-1)}.$$

In the three dimensional projective space over $\bar{\mathbb{F}}_{q^2}$, the curve X_{GK} that results as the complete intersection of the surface with affine equation

$$Z^{\xi^2-\xi+1} = Y h(X),$$

and the Hermitian cone with affine equation

$$X^\xi + X = Y^{\xi+1}$$

is called the GK curve [15]. It has a unique infinite point P, and it is maximal over $\bar{\mathbb{F}}_{q^2}$ [15, Theorem 1], i.e., the number of its $\mathbb{F}_{q^2}$-rational points attains the Hasse–Weil upper bound $q^2 + 1 + 2g_{X_{\mathrm{GK}}} q$. This example provides us with one of the few known families of curves that are maximal. Note that in [14] a generalization of the above curve is given, the so-called generalized GK curve. The Weierstrass semigroup at P is generated by $\langle m_1, m_2, m_3 \rangle_{\mathbb{Z}_+}$, with $m_1 = \xi^3 - \xi^2 + \xi$, $m_2 = \xi^3$ and $m_3 = \xi^3 + 1$ [15, Proposition 1]. Notice that $m_2 = \xi^3 = |G_1(P)|$ (see [10,15]) and that $F^{G_1(P)} = k(f_2)$. We compute the representation filtration, and the picture is the following:

$$G_1(P) = \ker \rho_0 \supsetneq \ker \rho_1 = \ker \rho_2 \supsetneq \{\mathrm{id}\}.$$

That is, m_2 is a generator, *but* 1 is not a representation jump (notice also that $|\ker \rho_2| = \xi$). Here $F^{\ker \rho_2} = k(f_1, f_2) = F^{G_1(P)}(f_1)$; see [10]. Moreover, there are two ramification jumps for this case [10, Proposition 4.2]: $m_r = -v_P(f_3)$ and $\frac{m_1}{|\ker \rho_2|}$.

4.2. **Structure of the Weierstrass semigroups, Galois action, and computation of ramification jumps.** Recall that $F_i = F^{\ker \rho_{c_i}}$, and $Q_i := F_i \cap P$ for $1 \leq i \leq n+1$ is the restriction of the place P to the intermediate field F_i. Keep in mind that r counts the number of elements in the Weierstrass semigroup up to the first pole number that is not divisible by p, while n counts the number of representation jumps.

Lemma 24. *For all $1 \leq i \leq n$, the semigroup $H(Q_{i+1})$ is generated by elements of the semigroup $H(Q_i)$ multiplied by $p^{h_{i-1}-h_i} = [\ker \rho_{c_i} : \ker \rho_{c_{i+1}}]$ and an extra prime to p minimal generator:*

$$-v_{Q_{i+1}}(\bar{f}_i) = \frac{m_{c_i+1}}{|\ker \rho_{c_{i+1}}|} = \bar{m}_i p^{-h_i},$$

where c_i is a representation jump and $\bar{m}_i$ the minimal extra generator for $H(Q_{i+1})$ compared to $H(Q_i)$, by Proposition 20.

Proof. From Lemma 16 in every step of the representation tower we have

$$\left| \frac{\ker \rho_{c_i}}{\ker \rho_{c_{i+1}}} \right| H(Q_i) = p^{h_{i-1}-h_i} H(Q_i) \subset H(Q_{i+1}).$$

We will apply Proposition 20 to the extension $F_{i+1}/F^{G_1(P)}$. The group $\ker \rho_{c_{i+1}}$ is a normal subgroup of $G_1(P)$ as a kernel of a homomorphism. Recall that $F_{i+1} = F^{\ker \rho_{c_{i+1}}}$, and notice now that the field extension $F_{i+1}/F^{G_1(P)}$ is also HKG and their representation filtration is obtained from the quotients of the representation filtration of $F/F^{G_1(P)}$ by the group $\ker \rho_{c_{i+1}}$. Therefore, according to Proposition 20 and from basic properties arising from the definition, see Remarks 10 and 21, $H(Q_{i+1})$ will have an extra generator compared to $H(Q_i)$, which is coming from the generator of the extension F_{i+1}/F_i, which is $\bar{f}_i$. Using Lemma 16, we have

$$-v_{Q_{i+1}}(\bar{f}_i) = \frac{\bar{m}_i}{|\ker \rho_{c_{i+1}}|}. \tag{9}$$

We will now prove that $\bar{f}_i$ has prime to p pole order. We know by Proposition 2 that there is a prime to p pole number m minimally chosen in $H(Q_{i+1})$ together with an element g such that $(g)_\infty = mQ_{i+1}$, and the action ρ_m of $\mathrm{Gal}(F_{i+1}/F_i)$ on $L(mQ_{i+1})$ is faithful. This proves that g generates F_{i+1} over F_i. Indeed, if this was not the case, then $\{\mathrm{id}\} \neq \mathrm{Gal}\,(F_{i+1}/F_i(g)) \subseteq \ker \rho_m = \{\mathrm{id}\}$. It is clear that $\frac{\bar{m}_i}{|\ker \rho_{c_{i+1}}|} \leq m$ since $\frac{\bar{m}_i}{|\ker \rho_{c_{i+1}}|}$ is the smallest element in $H(Q_{i+1})$ not in $p^{h_{i-1}-h_i} H(Q_i)$; note also that if $\frac{\bar{m}_i}{|\ker \rho_{c_{i+1}}|} > m$, then g would be, by construction, $\ker \rho_{c_i}$-invariant.

Every element in the semigroup $H(Q_{i+1})$ should be the pole number of a polynomial in $k[\bar{f}_0, \ldots, \bar{f}_{i-1}, g]$. Thus $\bar{f}_i = P_1(g)$ for an appropriate $P_1 \in k[\bar{f}_0, \ldots, \bar{f}_{i-1}]$. On the other hand, since $\bar{f}_i$ is by construction another generator of the field extension F_{i+1}/F_i, we have similarly $g = P_2(\bar{f}_i)$ for an appropriate $P_2 \in k[\bar{f}_0, \ldots, \bar{f}_{i-1}]$. Composing P_1 and P_2, it is easy to see that $P_1 \circ P_2 = \mathrm{id}$. But this is possible only if P_1 is linear on the g variable, i.e.,

$$\bar{f}_i = \alpha g + \beta \qquad \text{for some } \alpha, \beta \in k[\bar{f}_0, \ldots, \bar{f}_{i-1}]. \tag{10}$$

Recall that all of the pole numbers in $H(Q_{i+1})$ that arise as the polar part of the functions $\bar{f}_0, \ldots, \bar{f}_{i-1}$ are coming from the push forward of the $H(Q_i)$ multiplied by

$p^{h_{i-1}-h_i}$ via the map π^* of Lemma 16. Notice now that there are only two possible cases:

(1) If $\alpha \notin k^*$ or $-v_{Q_{i+1}}(\beta) > -v_{Q_{i+1}}(g) = m$, then the two summands on the right-hand side of equation (10) must have equal valuations. If not, we contradict our hypothesis $\frac{\bar{m}_i}{|\ker \rho_{c_{i+1}}|} \leq m$. With this in mind, we get that m is a multiple of $p^{h_{i-1}-h_i}$, which again contradicts our hypothesis.

(2) If $\alpha \in k^*$ and $-v_{Q_{i+1}}(\beta) < m$, then $\frac{\bar{m}_i}{|\ker \rho_{c_{i+1}}|} = m$; compare this also to Remark 6.

With another simple argument we will now show that $\bar{m}_i$ is the *only* extra generator of $H(Q_{i+1})$ compared to $H(Q_i)$. Suppose not, and let $h \in k[\bar{f}_0, \ldots, \bar{f}_{i-1}, \bar{f}_i]$ be a rational function such that $(h)_\infty = nQ_{i+1}$ with $\frac{\bar{m}_i}{|\ker \rho_{c_{i+1}}|} < n$ a minimal generator of $H(Q_{i+1})$. Again we will view h as a polynomial in $\bar{f}_i$. Note that the degree of h with respect to this variable is less than $p^{h_{i-1}-h_i}$ since $\bar{f}_i$ generates the extension. Write

$$h = \sum_{\nu=0}^{p^{h_{i-1}-h_i}-1} \alpha_\nu \bar{f}_i^\nu, \qquad \text{with } \alpha_\nu \in k[\bar{f}_0, \ldots, \bar{f}_{i-1}].$$

All of the summands have different valuations. Indeed, if this is not the case, then there are indices $s \lneq j$ such that $v_{Q_{i+1}}(\alpha_s \bar{f}_i^s) = v_{Q_{i+1}}(\alpha_j \bar{f}_i^j)$, or

$$p^{h_{i-1}-h_i} \cdot \delta = \frac{\bar{m}_i}{|\ker \rho_{c_{i+1}}|}(j-s), \qquad \text{for some positive integer } \delta.$$

This is impossible since $(j-s) < p^{h_{i-1}-h_i}$ and $\frac{\bar{m}_i}{|\ker \rho_{c_{i+1}}|}$ is prime to p. In this way we manage to write $-v_{Q_{i+1}}(h)$ as an $\mathbb{N}$-linear combination of smaller minimal generators of the Weierstrass semigroup. This implies that $-v_{Q_{i+1}}(h)$ itself cannot be a minimal generator. $\square$

According to Proposition 20, since $\{c_1, \ldots, c_n\}$ are the jumps of the representation filtration, the elements $\{\bar{m}_1, \ldots, \bar{m}_n = m_r\}$ are generators of the Weierstrass semigroup $H(P)$. But it is not true that every generator of $H(P)$ occurs this way, as we have already seen in the examples of this section and as the following lemma indicates.

Lemma 25. *Let M be a minimal generator of the Weierstrass semigroup at P such that $M \neq \bar{m}_\nu$ for all $1 \leq \nu \leq n$. Then any function $f_M \in F$ with $(f_M)_\infty = M$ is $G_1(P)$-invariant. The number of representation jumps is either equal to the number of minimal generators of the Weierstrass semigroup or it is equal to the number of minimal generators of the Weierstrass semigroup minus 1 and $|G_1(P)| = M$.*

Proof. If there is such a generator M_i of $H(Q_i)$, then this generator is a multiple of a generator of $H(Q_{i-1})$ by Lemma 24. This means that any function $f_{M_i} \in F_i$ which has pole number M_i at Q_i, is an element invariant under the Galois group of the extension F_i/F_{i-1}. Using this argument inductively, we arrive at the conclusion that the function f_M is $G_1(P)$-invariant, and thus, by Corollary 19, $|G_1(P)|$ divides M and thus $M = |G_1(P)|$. Finally, if such an f_M exists, it is unique since $F^{G_1(P)}$ is rational by our hypothesis. This completes the proof. $\square$

We sum up all the information concerning the Weierstrass semigroups of the field tower arising from the representation filtration in the following.

Proposition 26. *The Weierstrass semigroups of the fields F_i at $Q_i = P \cap F_i$ for every $1 \le i \le n$ and $\ker \rho_{c_1} = G_1(P)$ are given by*

$$H(Q_{i+1}) = \langle \bar{m}_j p^{-h_i}, |G_1(P)|p^{-h_i} \rangle = \left\langle \frac{m_{c_j+1}}{|\ker \rho_{c_{i+1}}|}, \left| \frac{G_1(P)}{\ker \rho_{c_{i+1}}} \right| \right\rangle_{\mathbb{Z}_+},$$

where j runs through the indices $1 \le j \le i$. For the Weierstrass semigroup at P we get

$$H(P) = \langle \bar{m}_j, |G_1(P)| \rangle_{\mathbb{Z}_+}, \quad \textit{where } 1 \le j \le n, \quad \textit{while } H(Q_1) = \mathbb{Z}_+.$$

Proposition 27. *Assume that $\sigma \in \ker\rho_{c_i} - \ker\rho_{c_{i+1}}$. Then*

$$\begin{aligned} \sigma(f_\nu) &= f_\nu \qquad \textit{for all } \nu \le c_i \\ \sigma(f_{c_i+1}) = \sigma(\bar{f}_i) &= \bar{f}_i + c(\sigma), \qquad \textit{where } c(\sigma) \in k^*. \end{aligned} \tag{11}$$

Proof. In general $\sigma(\bar{f}_i) = \alpha \cdot \bar{f}_i + c(\sigma)$, where $c(\sigma) \in k[f_1, \dots f_{c_i}]$, and $\alpha \in k^*$. Since σ has order a power of p, we see that $\alpha = 1$. But if $c(\sigma)$ is not constant, then it has a root $Q \neq Q_i$ since the field k is assumed to be algebraically closed. We will prove that Q is then a ramified point, and this will lead to a contradiction since only one place can ramify, and this is Q_i.

Consider the ring $A := \mathcal{O}(X - Q_i)$, where $\mathcal{O}$ denotes the structure sheaf of a nonsingular projective model of our curve X that corresponds to the function field F_i. The ring A is by definition

$$A = \bigcup_{\nu=0}^{\infty} L(\nu Q_i) = k[f_1, \dots, f_{c_i}],$$

where the elements $f_1, \dots, f_{c_i}$ are subject to several relations coming from the function field of the curve. Observe that when ν becomes greater than or equal to $\bar{m}_{i-1}p^{-h_{i-1}}$ (i.e., is greater than all the generators of the Weierstrass semigroup at Q_i) the algebra generated by $f_1, \dots, f_{c_i}$ as elements of the vector space $L(\nu Q_i)$ is the ring A. Keep in mind that the vector space $L(\nu Q_i)$ is inside the function field of the curve, so there is a well defined notion of multiplication on elements of $L(\nu Q_i)$. Every place $Q \neq Q_i$ of the function field F_i corresponds to a unique maximal ideal of the ring A.

Notice also that the automorphism group acts on A. We will prove that the ideal Q is left invariant under the action of σ. Let Q be a root of $c(\sigma)$, and denote by Q the corresponding ideal of A. It is finitely generated, so $Q = \langle g_j \rangle$, where g_j are polynomial expressions in f_i, where $1 \le i \le c_i$. We will prove that

$$\sigma(g_j) \in Q \qquad \text{for all } j.$$

Indeed, write

$$g_j = \sum_{\nu_1, \dots, \nu_{c_i}} a_{\nu_1, \dots, \nu_{c_i}} f_1^{\nu_1} \cdots f_{c_i}^{\nu_{c_i}}.$$

Then

$$\begin{aligned}\sigma(g_j) &= \sum_{\nu_1,\ldots,\nu_{c_i}} a_{\nu_1,\ldots,\nu_{c_i}} f_1^{\nu_1}\cdots(f_{c_i}+c(\sigma))^{\nu_i}\\ &= \sum_{\nu_1,\ldots,\nu_{c_i}} a_{\nu_1,\ldots,\nu_{c_i}} f_1^{\nu_1}\cdots f_{c_i}^{\nu_{c_i}}\\ &\quad+\sum_{\nu_1,\ldots,\nu_{c_i}} a_{\nu_1,\ldots,\nu_{c_i}} \sum_{\mu=1}^{\nu_i} f_1^{\nu_1}\cdots f_{c_i-1}^{\nu_{c_i-1}}\binom{\nu_{c_i}}{\mu} c(\sigma)^{\mu} f_{c_i}^{\nu_{c_i}-\mu}.\end{aligned}$$

But Q is a root of $c(\sigma)$, and this is equivalent to $c(\sigma)\in Q$, so the second summand of the last equation is an element in Q.

We would like also to point out how we can construct the curve $X - Q_i$. If ν is big enough, then the projective map Φ corresponding to the linear series $|\nu Q_i|$ is an embedding; see, for example, [17, Theorem 4.3.15]. The image $\Phi(X)$ is then a nonsingular curve; removing the point $\Phi(Q_i)$, we obtain the affine nonsingular curve with coordinate ring A. Notice that, by construction, X is the projective closure of that curve, with Q_i being the point at infinity, while the function fields for both curves are just F_i. □

In what follows we will use the following.

Lemma 28. *Let $f\in F$ such that $p\nmid v_P(f)$. If $\sigma\in G_i\setminus G_{i+1}$, then $\sigma(f)=f+f'$, with $f'\neq 0$ and $i=-v_P(f)+v_P(f')$.*

Proof. This is [19, Lemma 11.83]. □

Theorem 29. *Let P be the totally ramified place of the HKG-cover. Recall that $Q_i = P\cap F_i$, with $1\le i\le n+1$. Let μ be the number of ramification jumps of equation* (2)*, and let n be the number of representation jumps; see equation* (5)*.*

(i) *The groups $\ker\rho_{c_i}/\ker\rho_{c_{i+1}}$, for each $1\le i\le n$, have exactly one lower ramification jump which is equal to $-v_{Q_{i+1}}(\bar f_i)$.*

(ii) *The jumps mentioned in* (i) *are equal to the ramification jumps of the groups $G_{b_i}/G_{b_{i+1}}$, for $1\le i\le\mu$, thus $\mu=n$, and they exhaust all of the ramification jumps of $G_1(P)$.*

Proof. We first prove (i). Lemma 24 implies $\gcd(v_{Q_{i+1}}(\bar f_i),p)=1$. Using Proposition 27 and Lemma 28, we obtain that the jump for $\ker\rho_{c_i}/\ker\rho_{c_{i+1}}=\mathrm{Gal}(F_{i+1}/F_i)$ is indeed $-v_{Q_{i+1}}(\bar f_i)$ since $\sigma(\bar f_i)=\bar f_i+c(\sigma)$, where $c(\sigma)$ is constant and has valuation 0. This jump is also unique by Lemma 28. Moreover, the extension F_{i+1}/F_i is elementary abelian since $c: G_1(P)\to k$ gives rise to an isomorphism from $\mathrm{Gal}(F_{i+1}/F_i)$ to a p-subgroup of the additive group of k. Compare also to [45, Prop. III.7.10].

In order to prove (ii) we are going to apply (i) step by step. In the first step we consider the group $\ker\rho_{c_n}$, which is elementary abelian with a unique jump at m_r. Since this group is a subgroup of $G_1(P)$ and this is the maximum jump, we can have (see Proposition 7), we obtain $m_r=b_\mu$.

For the next step we consider the lower ramification jumps of the filtration of the group $\ker\rho_{c_{n-1}}$. From the previous step, we see that the quotient $\ker\rho_{c_{n-1}}/\ker\rho_{c_n}$ has a unique lower jump at $-v_{Q_n}(\bar f_{n-1})$. It is well known that the first jumps in the lower and upper numbering coincide since the Herbrand function ϕ, as it is defined in [41, IV.3, p. 73], is the identity for values smaller than the first lower jump. Thus

$-v_{Q_n}(\bar{f}_{n-1})$ is also the first jump in the upper numbering for $\ker \rho_{c_{n-1}}/\ker \rho_{c_n}$. Using the well-known property of the upper ramification filtration—that for all normal subgroups H of G and u an upper jump we have $(G/H)^u = G^u H/H$ [41, IV, Prop. 14]—we derive that $-v_{Q_n}(\bar{f}_{n-1})$ equals also the first upper, and thus the first lower jump of $\ker \rho_{c_{n-1}}$. Note that since $\ker \rho_{c_n}$ is a normal subgroup of $\ker \rho_{c_{n-1}}$, the latter group inherits the lower ramification jump of the first step. That is, m_r is also a lower jump for $\ker \rho_{c_{n-1}}$, the greatest one, since by equation (9) we have

$$-v_{Q_n}(\bar{f}_{n-1}) = \frac{m_{c_{n-1}+1}}{|\ker \rho_{c_n}|} = \frac{\bar{m}_{n-1}}{|\ker \rho_{c_n}|} < \frac{\bar{m}_n}{|\ker \rho_{c_{n+1}}|} = -v_{Q_{n+1}}(\bar{f}_n).$$

Notice that $\bar{m}_n = m_{c_n+1} = m_r$ and $|\ker \rho_{c_{n+1}}| = 1$.

We continue like this, using the fact that every ramification jump of a subgroup of $G_1(P)$ is a ramification jump of $G_1(P)$ as well [41, Proposition IV.2, p. 62], and get that all of the positive integers $-v_{Q_{i+1}}(\bar{f}_i)$ are indeed jumps of $G_1(P)$.

Are there more jumps of the ramification filtration? By construction $\ker \rho_{c_1} = G_1(P)$ and $\ker \rho_{c_1}$ has at least n lower ramification jumps, since n is the number of representation jumps and by (i) every representation jump gives rise to a lower ramification jump. If the number of the ramification jumps is strictly greater than n, then some of the Galois groups $\ker \rho_{c_i}/\ker \rho_{c_{i+1}}$ should have more than one lower ramification jump, which is impossible from the computations done above. □

Corollary 30. *The following groups are equal:*

$$G_{b_i} = \ker \rho_{c_i} \qquad \text{for all } 1 \leq i \leq \mu = n.$$

Proof. We will prove first that $\ker \rho_{r-1} \subset G_{b_\mu}$. But $b_\mu = m_r$, thus for an element $\sigma \in \ker \rho_{r-1}$ we have $\sigma(f_r) = f_r + c(\sigma)$, with $c(\sigma) \in k^*$, so

$$v_P(\sigma(t) - t) = m_r + 1 = b_\mu + 1 \Rightarrow \sigma \in G_{b_\mu}.$$

Now we will prove that $\ker \rho_{r-1} \supset G_{b_\mu}$.

Notice that every element in G_{b_μ} satisfies $v_P(\sigma(t)-t) = b_\mu + 1 = m_r + 1$. Let c_{i^*} be maximal such that $G_{b_\mu} \subset \ker \rho_{c_{i^*}}$. Then by construction there is an element $\sigma' \in G_{b_\mu}$ that does not belong to $\ker \rho_{c_{i^*+1}}$; that is (using Proposition 27),

$$\sigma'(f_j) = f_j \qquad \text{for all } j \leq c_{i^*}$$

and

$$\sigma'(f_{c_{i^*}+1}) = f_{c_{i^*}+1} + \sigma'(c) \qquad \text{for some } \sigma'(c) \in k^*.$$

For a Galois group G of a local field extension L/K consider the function i_G defined by $i_G(\sigma) = v_L(\sigma(t) - t)$; see [41, Chap. IV, p. 62]. We consider this function for the Galois extension $\frac{\ker \rho_{c_{i^*}}}{\ker \rho_{c_{i^*+1}}}$,

$$i_{\frac{\ker \rho_{c_{i^*}}}{\ker \rho_{c_{i^*+1}}}}\left(\sigma' \ker \rho_{c_{i^*+1}}\right) = -v_{Q_{i^*+1}}(f_{c_{i^*}+1}) + 1,$$

using Lemma 28. On the other hand, this value should be equal to b_μ. Notice that since G_{b_μ} is elementary abelian with a unique jump, the lower and upper ramification filtrations coincide. So $m_r = b_\mu = -v_{Q_{i^*+1}}(f_{c_{i^*}+1})$. Thus $i^* = n$ and $c_{i^*} = c_n = r-1$. This proves that $\ker \rho_{r-1} = G_{b_\mu}$, i.e., the last groups in both filtrations coincide.

We now consider the HKG extension of the rational function field given by

$$F^{G_{b_\mu}} = k(X/\ker \rho_{c_n}) = F^{G_1(P)}(\bar{f}_1, \ldots, \bar{f}_{n-1}).$$

This extension has ramification filtration

$$\frac{G_1(P)}{G_{b_\mu}} \geq \cdots \geq \frac{G_i}{G_{b_\mu}} \geq \cdots \geq \frac{G_{b_\mu - 1}}{G_{b_\mu}} > \{1\}.$$

Indeed, we know by [41, Corollary, p. 64] that the ramification filtration of the quotient group G/H when $H = G_j$ is a subgroup of the ramification filtration is given by $(G/H)_i = G_i/H$ for $i \leq j$ and $(G/H)_i = \{1\}$ for $i \geq j$. The representation filtration of $\frac{G_1(P)}{G_{b_\mu}}$ is formed by the quotients of the representation filtration of $\ker \rho_{c_1}$ by $\ker \rho_{r-1}$. Using the previous argument, we see that the last groups in both filtrations are equal and we proceed inductively using Theorem 29. □

We will now focus on the case where the first jump equals 1.

Corollary 31. *The condition $G_1(P) > G_2(P)$ is equivalent to F_2 being rational.*

Proof. Let $[G_1(P) : \ker \rho_{c_2}] =: q$. The group $G_1(P)/\ker \rho_{c_2}$ is elementary abelian of order q with a unique jump, say, at υ. The Riemann–Hurwitz theorem implies that

$$2g_{F_2} - 2 = -2q + (\upsilon + 1)(q - 1)$$

and that $\upsilon = 1$ if and only if $g_{F_2} = 0$. □

Corollary 32. *Suppose that $G_1(P) > G_2(P)$. Let i_0 be the index such that $-v_P(f_{i_0}) = m_{i_0} = |G_1(P)|$ and $k(f_{i_0}) = F^{G_1(P)}$ as given in Lemma* 21*. Concerning the structure of the Weierstrass semigroups $H(Q_{i+1})$ given in Proposition* 26*, we have*

$$H(Q_{i+1}) = \left\langle \bar{m}_j p^{-h_j} : 1 \leq j \leq i \right\rangle_{\mathbb{Z}_+},$$

while

$$H(P) = \langle \bar{m}_j : 1 \leq j \leq n \rangle_{\mathbb{Z}_+}.$$

More precisely, $|G_2(P)| = m_1$, i.e., the order of the second lower ramification group equals the first pole number and

$$m_r = m_{r-1} + 1.$$

Proof. The element f_{i_0} is not needed for the generation of $F_j = F^{\ker \rho_{c_j}}$ for every $j > 1$, that is, $\left\langle \bar{m}_j p^{-h_i} \right\rangle_{\mathbb{Z}_+} \ni \left| \frac{G_1(P)}{\ker \rho_{c_{i+1}}} \right|$. Indeed, from Corollary 31 we have $F^{G_1(P)}(\bar{f}_1) = F_2$ being rational. The element f_{i_0} is a rational function of $\bar{f}_1$. Moreover, in this case, we can normalize the Artin–Schreier generator $\bar{f}_1$ for the elementary abelian extension with a unique ramification jump, and apply [45, Proposition III.7.10] such that

$$f_{i_0} = \bar{f}_1^q - \bar{f}_1,$$

where q equals $[G_1(P) : \ker \rho_{c_2}]$.

Corollary 18 implies that $|G_1(P)|$ can result as a pole number as a multiple of $|\ker \rho_{c_2}|$, which is a pole number since $F^{\ker \rho_{c_2}} = F_2 = F_1(\bar{f}_1)$ is rational. Moreover, from Corollary 30 we have $|G_2(P)| = |\ker \rho_{c_2}|$, while $|\ker \rho_{c_2}| = \bar{m}_1$ and thus

$$\left| \frac{G_1(P)}{\ker \rho_{c_{i+1}}} \right| \in \left\langle \frac{\bar{m}_1}{|\ker \rho_{c_{i+1}}|} \right\rangle_{\mathbb{Z}_+} \qquad \text{for every } 1 \leq i \leq n.$$

Notice that in this case $\bar{m}_1 = m_1$, and that the first nonzero pole number is always a minimal generator.

Finally, the last assertion about m_r comes directly from Proposition 7. □

At this point, we would like to discuss the case in which $|G_1(P)|$ is a generator of the semigroup. It turns out that this happens if and only if 1 is *not* a ramification jump, i.e., $G_1(P) = G_2(P)$. We have seen that the minimal generators of the semigroup $H(P)$ are of two types:

(1) they are induced by jumps of the representation filtration, and
(2) $|G_1(P)|$.

We will need the following.

Lemma 33. *Assume that S is a numerical semigroup and that E is the semigroup such that $E = p^\ell S + N\mathbb{Z}_+$, for some $\ell \in \mathbb{N}$, where $(N, p) = 1$. Suppose further that the semigroups S, E have the same cardinality of minimal generators. Then N is a generator of the semigroup S.*

Proof. This is [42, Proposition A.0.15] in the Ph.D. thesis of Smith. Notice that there the result is proved only for $p^\ell = p$, but the same proof can be used for the more general case of higher values of ℓ. □

Proposition 34. *The number $|G_1(P)|$ is a minimal generator of the Weierstrass semigroup at P if and only if $G_1(P) = G_2(P)$.*

Proof. If $G_1(P) > G_2(P)$, $F^{G_2(P)}$ is rational, $|G_2(P)|$ equals the first pole number from Corollary 32, and since $|G_2(P)|$ divides $|G_1(P)|$, $|G_1(P)|$ cannot be a minimal generator.

For the other direction assume that $|G_1(P)|$ is not a minimal generator; then we will prove that $G_1(P) > G_2(P)$. By our hypothesis there is a semigroup $H(Q_i)$ where $|G_1(P)|/|\ker \rho_{c_i}|$ is not a generator for some $c_i < r$. Let ν^* be the first index such that $|G_1(P)|/|\ker \rho_{c_i}|$ is a generator for $i \leq \nu^*$ and that $|G_1(P)|/|\ker \rho_{c_{\nu^*+1}}|$ is not a generator for $H(Q_{\nu^*+1})$. We have the following generating sets for the semigroups:

$$H(Q_{\nu^*}) = \left\langle \left| \frac{G_1(P)}{\ker \rho_{c_{\nu^*}}} \right|, \frac{\bar{m}_j}{|\ker \rho_{c_{\nu^*}}|} : 1 \leq j < \nu^* \right\rangle_{\mathbb{Z}_+},$$

$$H(Q_{\nu^*+1}) = \left\langle \frac{\bar{m}_j}{|\ker \rho_{c_{\nu^*+1}}|} : 1 \leq j \leq \nu^* \right\rangle_{\mathbb{Z}_+};$$

i.e., both semigroups have the same number of generators. According to Lemma 24, the semigroup $H(Q_{\nu^*+1})$ is generated by elements of the semigroup $H(Q_{\nu^*})$ multiplied by $[\ker \rho_{c_{\nu^*}} : \ker \rho_{c_{\nu^*+1}}]$ and an extra prime to p generator $\frac{\bar{m}_{\nu^*}}{|\ker \rho_{c_{\nu^*+1}}|}$, i.e.,

$$H(Q_{\nu^*+1}) = [\ker \rho_{c_{\nu^*}} : \ker \rho_{c_{\nu^*+1}}] \cdot H(Q_{\nu^*}) + \mathbb{Z}_+ \frac{\bar{m}_{\nu^*}}{|\ker \rho_{c_{\nu^*+1}}|}.$$

We will now complete the proof by applying Lemma 33. The prime to p generator $N = \frac{\bar{m}_{\nu^*}}{|\ker \rho_{c_{\nu^*+1}}|}$ should be a generator of $H(Q_{\nu^*})$, but it cannot be any of the $\frac{\bar{m}_j}{|\ker \rho_{c_{\nu^*}}|} : 1 \leq j < \nu^*$ since it is the greatest of these, so the only remaining case is $N = \left| \frac{G_1(P)}{\ker \rho_{c_{\nu^*}}} \right|$. But since N is prime to p, we have $|G_1(P)| = |\ker \rho_{c_{\nu^*}}|$, $N = 1$ and thus $\nu^* = 1$ and $H(Q_1) = H(Q_2) = \mathbb{Z}_+$, but this contradicts the nonrationality of the field $F^{G_2(P)}$. □

Remark 35. For HKG-covers the field $F^{G_2(P)}$ is always rational; see [19, Theorem 11.78(iii)].

Remark 36 (Upper ramification jumps). The reader should notice that by computing the jumps of the lower ramification filtration, we gain information on the jumps of the *upper* ramification filtration through the Herbrand's formula; see [41, section IV]. As an application of this we get that, for p-groups, upper and lower ramification jumps are connected by the following formula:

$$b_i = \sum_{j=1}^{i} (u_j - u_{j-1}) p^{h_0 - h_{j-1}} \qquad \text{for every } 1 \le i \le n,$$

where $u_1, \ldots, u_n$ are the upper jumps of $G_1(P)$ and here $b_0 = u_0 = 0$.

5. Applications

5.1. **Big actions.** A case where the order of $G_1(P)$ is not a generator of $H(P)$, due to Proposition 34, is when we focus on big actions as this notion is defined in the work of Lehr and Matignon [25] and studied further by Rocher and Matignon [28, 36].

Definition 37. A curve X together with a subgroup G of the automorphism group of X is called a big action if G is a p-group and

$$\frac{|G|}{g_X} > \frac{2p}{p-1}.$$

All big actions have the following property.

Proposition 38 ([25, Prop. 8.5])**.** *Assume that (X, G) is a big action. There is a unique point P of X such that $G_1(P) = G$, the group $G_2(P)$ is not trivial and strictly contained in $G_1(P)$, and the quotient $X/G_2(P) \cong \mathbb{P}^1$. Moreover, the group G is an extension of groups*

$$0 \to G_2(P) \to G = G_1(P) \xrightarrow{\pi} (\mathbb{Z}/p\mathbb{Z})^v \to 0.$$

The first jump for their ramification filtration is equal to 1, while the other jumps are given by Theorem 29. Moreover, we are now able able to compute explicitly the Weierstrass semigroup at the ramified point.

Corollary 39. *If (X, G) is a big action, then $|G_1(P)|$ is not a minimal generator of $H(P)$. Moreover,*

$$H(P) = \langle \bar{m}_j : 1 \le j \le n \rangle_{\mathbb{Z}_+}, \qquad |G_2(P)| = m_1,$$

and

$$m_r = m_{r-1} + 1;$$

i.e., the structure of $H(P)$ is given by Corollary 32.

5.2. **Curves with zero p-rank.** The p-rank of the Jacobian is an important invariant of an algebraic curve, which also controls the automorphism group of the curve; see [31]. The case of zero p-rank curves corresponds to curves X with a huge number of automorphisms [31, Theorem 1(iv)]. In this class of curves the most automorphisms occur exactly when $X/G_1(P)$ is rational. Otherwise $|G_1(P)|$ is less than or equal to the genus of the curve; see [19, Theorem 11.78(i)]. This is exactly the HKG p-case.

Theorem 40. *The following conditions are equivalent for a p-group $G \subseteq \mathrm{Aut}(X)$:*

(1) *The curve X has zero p-rank and $|G|$ is a pole number at the unique point $P \in X$ that G stabilizes.*
(2) *The cover $X \to X/G$ is an HKG-cover.*

Proof.
$1 \Rightarrow 2$ By [19, Lemma 11.129] every element of order p fixes exactly one point. This means that $G = G(P)$—i.e., G can be realized as the stabilizer of a point $P \in X$—and that for the cover $X \to X/G(P)$, P is the unique totally ramified point. By Corollary 18, $|G| = |G(P)|$ is a pole number at P if and only if $X/G(P)$ is a rational curve.

$2 \Rightarrow 1$ Use the Deuring–Shafarevich formula [30, equation (1.1)] and the definition of an HKG p-cover. □

5.3. **Hasse–Arf divisibility conditions.** The Hasse–Arf theorem for abelian groups gives certain divisibility conditions for the jumps of the ramification filtration. Using Theorem 14 restricted to the case of an abelian group $G_1(P)$, these divisibility conditions can be interpreted in terms of the Weierstrass semigroup at P.

Corollary 41 (Hasse–Arf theorem)**.** *Assume that an HKG-cover has abelian Galois p-group $G_1(P)$. Let $p^{r_i} = [G_{b_i} : G_{b_{i+1}}]$ for all $1 \leq i \leq n-1$. Then the generators of the Weierstrass semigroup that result from the jumps of the representation filtration satisfy*

$$\frac{\bar{m}_{i+1}}{|G_{b_{i+2}}|} \equiv \frac{\bar{m}_i}{|G_{b_{i+1}}|} \bmod p^{\sum_{j=1}^{i} r_j}$$

or

$$\left|\frac{G_{b_{i+1}}}{G_{b_{i+2}}}\right| \bar{m}_{i+1} \equiv \bar{m}_i \bmod |G_{b_1}| \, .$$

Proof. We will use an equivalent form of the Hasse–Arf theorem (see [37]): the condition for the upper jumps u_i to be integers can be directly translated to congruences for the lower ramification jumps. Namely, every two subsequent lower ramification jumps b_{i+1}, b_i must satisfy

$$b_{i+1} \equiv b_i \bmod p^{\sum_{j=1}^{i} r_j}, \quad \text{where } p^{r_i} := [G_{b_i} : G_{b_{i+1}}] \quad \text{for every } 1 \leq i \leq n-1.$$

Now replace b_i with $\frac{\bar{m}_i}{|G_{b_{i+1}}|}$ for every $1 \leq i \leq n$ in order to derive the desired result. □

6. Holomorphic polydifferentials

In what follows X is always an HKG-cover with a Galois group as a p-group. We can construct a basis for the m-holomorphic polydifferentials of X as follows.

Let f_{i_0} be the function generating the rational field $F^{G_1(P)} = k(f_{i_0})$. The function f_{i_0} can be selected so that it has a simple unique pole at infinity which is the restriction of the place P to $k(f_{i_0})$. Let $p^{h_0} = |G_1(P)|$. We observe first that

$$\operatorname{div}(df_{i_0}^{\otimes m}) = \left(-2mp^{h_0} + m \sum_{i=1}^{n} (b_i - b_{i-1})(p^{h_{i-1}} - 1) \right) P, \tag{12}$$

where

$$b_0 = -1, \; p^{h_0} = |G_1(P)|, \; p^{h_i} = |\ker \rho_{c_{i+1}}| = |G_{b_{i+1}}| \qquad \text{for } i \geq 1.$$

The right-hand side of equation (12) equals $m(2g_X-2)P$ by the Riemann–Hurwitz formula.

Proposition 42. *For every pole number μ we select a function f_μ such that $(f_\mu)_\infty = \mu P$. The set $\{f_\mu df_{i_0}^{\otimes m} : \deg \mathrm{div}(f_i) \leq m(2g_X-2)\}$ is a basis for the space of m-holomorphic (poly)differentials of X for every positive integer $m \geq 1$.*

Proof. All m-holomorphic differentials are of the form $gdf_{i_0}^{\otimes m}$. Therefore, the condition for being holomorphic is translated into the condition $g \in L(m(2g_X-2)P)$. This means that the linear independent elements $f_i df_{i_0}^{\otimes m}$ with $\deg \mathrm{div} f_i = m_i \leq m(2g_X-2)$ are holomorphic. In order to see that all of the holomorphic differentials are of this form, we will count them.

Case $m=1$. Notice that $\ell((2g_X-2)P) = g_X$. On the other hand, $\ell((2g_X-1)P) = g_X$ from the Weierstrass gap theorem [45, I.6.7]. This means that in the interval $[0, 2g_X-2]$ there are exactly g_X pole numbers; equivalently, $2g_X-1$ is a gap.

Case $m>1$. Similarly, observe, using the Riemann–Roch theorem, that the space of m-holomorphic differentials has dimension

$$\dim L(mW) = m(2g_X-2) + 1 - g_X = (2m-1)g_X - 2m + 1.$$

On the other hand, the number of f_i such that $\deg \mathrm{div}(f_i) \leq m(2g_X-2)$ can be computed as follows.

In the interval $[0, 2g_X-1]$ there are g_X such elements and every number greater than $2g_X$ is a pole number using again the Riemann–Roch theorem. So in the interval $(2g_X-1, m(2g_X-2)]$ there are $m(2g_X-2)-(2g_X-1) = 2mg_X-2m-2g_X+1$ elements. In total there are $2mg_X-2m-2g_X+1+g_X = (2m-1)g_X-2m+1$, and this coincides with the dimension of the space of m-holomorphic differentials. $\square$

Corollary 43. *The Weierstrass semigroup at P is symmetric, i.e., $2g_X-1$ is a gap.*

We have proved in Proposition 26 that the elements m_{c_i+1} for $1 \leq i \leq n$ together with the element p^{h_0} generate the Weierstrass semigroup. A numerical semigroup Σ that is not of the form $a\mathbb{Z}_+$ has a minimal element $\kappa(\Sigma)$ called *the conductor* such that all integers $n \geq \kappa(\Sigma)$ are in the semigroup.

Since the semigroup is symmetric, we see that $\kappa(H(P)) = 2g_X$. Recall that $2g_X-1$ is a gap in this case and that the Riemann–Roch theorem implies that all integers $\geq 2g_X$ are in $H(P)$.

We will now focus on the representation theory of HKG-covers.

Proposition 44. *Let $p^{h_0} = |G| = |G_1(P)|$. The module $\Omega_X^{\otimes m}$ is the direct sum of at most*

$$N := \left\lfloor \frac{m(2g-2)}{p^{h_0}} \right\rfloor = -2m + \left\lfloor m \frac{\sum_{i=1}^n (b_i - b_{i-1})(p^{h_{i-1}}-1)}{p^{h_0}} \right\rfloor$$

direct indecomposable summands.

Proof. We have a representation of the group $G_1(P)$ in terms of lower diagonal matrices in $\Omega_X^{\otimes m} \cong L(m(2g_X-2)P)$. For an element f in $L(m(2g_X-2)P)$ we have the function $v_P : L(m(2g_X-2)P) \to \mathbb{N}$ sending f to $-v_P(f)$ and $v_P(\sigma(f)-f) > v_P(f)$.

Assume that the space $L(m(2g_X-2)P)$ admits a decomposition

$$L(m(2g_X-2)P)=\bigoplus W_i$$

as a direct sum of G-modules W_i. We will prove that we can find a basis of elements $e_1,\dots e_{\dim W_i}$ of W_i that have different valuations. Indeed, start from any basis of W_i. If there are two basis elements a,b of W_i such that $v_P(a)=v_P(b)$, then these are, locally at P, of the form

$$a=a_1\frac{1}{t^v}+\text{higher order terms},\qquad b=b_1\frac{1}{t^v}+\text{ higher order terms}.$$

Therefore, there is an element λ such that $a-\lambda b\neq 0$ has a different valuation than a,b, $(\lambda=a_1/b_1)$. We replace the element b with the element $a-\lambda b$. Proceeding this way, we construct the desired basis elements with different valuations. Now,

$$\sigma(e_i)=e_i+b_i(\sigma),\qquad \text{with } b_i(\sigma)=0 \text{ or } |v_P(b_i(\sigma))|<|v_P(e_i)|,$$

and this proves that every direct summand W_i has an upper triangular representation matrix, so it contains at least one invariant element.

Therefore, the number of indecomposable summands is smaller than the number of $G_1(P)$-invariant elements. The space of invariant elements has a basis of elements of the form $f_{i_0}^j$ such that $-v_P(f_{i_0}^j)\leq m(2g_X-2)$, and the result follows. □

Corollary 45. *If $|G_1(P)|>m(2g_X-2)$, then the module $H^0(X,\Omega^{\otimes m})$ is indecomposable. In particular the space of holomorphic differentials $H^0(X,\Omega)$ is indecomposable for a curve X that admits a big action.*

Proof. If $|G_1(P)|>m(2g_X-2)$, then the only $G_1(P)$-invariant elements belonging to $L(2m(g_X-1))$ are the constants. Thus this space includes a unique copy of the one dimensional irreducible representation and therefore is indecomposable. The assertion for curves admitting big action comes directly now from their definition. □

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