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GALOIS ACTION ON HOMOLOGY OF GENERALIZED FERMAT CURVES

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Abstract

The fundamental group of Fermat and generalized Fermat curves is computed. These curves are Galois ramified covers of the projective line with abelian Galois groups H. We provide a unified study of the action of both cover Galois group H and the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the pro-ℓ homology of the curves in study. Also the relation to the pro-ℓ Burau representation is investigated.

1. Introduction

In [**15**] we have studied the actions of the braid group and of the absolute Galois group on a cyclic cover of the projective line. In this article, we have exploited the fact that in a ramified cover $X \to \mathbb{P}^1$ of the projective line we can remove the ramified points and in this way we obtain an open cover $X^0 \to X_s := \mathbb{P}^1 \backslash \{P_1, \ldots, P_s\}$ of the projective line minus the ramified points. Fix a point $x_0 \in X_s$ and an arbitrary but fixed preimage $x_0^{'} \in X^0$. By covering space theory the open curve X^0 can be described as a quotient of the universal covering space $\tilde{X}_s$ by the fundamental group of the open curve $\pi_1(X^0, x_0^{'}) < \pi_1(X_s, x_0) \cong F_{s-1}$, where F_{s-1} is the free group in $s-1$ generators. Also the group $\pi_1(X^0, x_0^{'})$ can be described as a subgroup of F_{s-1} by using the Screier lemma technique, see [**15**, sec. 3].

Y. Ihara in [**12**], [**11**] observed that for a fixed prime ℓ, if we pass to the pro-ℓ completions of the fundamental groups of the above curves, then the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ can be realized as a group of automorphisms of $\mathfrak{F}_{s-1}$ and it can also act on certain subgroups $\mathfrak{F}_{s-1}$, corresponding to topological covers of X_s, here by $\mathfrak{F}_{s-1}$ we denote the free pro-ℓ group in $s-1$ generators. The fundamental group of X_s admits the presentation

$$\pi_1(X_s, x_0) \cong \langle x_1, \ldots, x_s | x_1 x_2 \ldots x_s = 1 \rangle.$$

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In this way we can unify the study of both Braid group and the absolute Galois group on (co)homology spaces of cyclic curves. In [15, thm. 1] the fundamental group R_{ℓ^k} of the open cyclic cover Y_{ℓ^k} of the projective line minus s-points is computed to be equal to

$$R_{\ell^k} = \langle x_1^i x_j x_1^{-i+1}, 0 \le i \le \ell^k - 2, 2 \le j \le s-1, x_1^{\ell^k-1} x_j, 1 \le j \le s-1 \rangle. \tag{1}$$

Definition 1.1 Given a group F we will denote by F' the derived subgroup of F, that is $F' = [F, F]$.

In this article, we continue this study by focusing on the case of certain abelian coverings of the projective line. We begin by studying the classical Fermat curves Fer_n given as projective algebraic curves by the equation

$$\mathrm{Fer}_n := \{[x : y : z] \in \mathbb{P}^3 : x^n + y^n + z^n = 0\}.$$

The Fermat curve Fer_n forms a ramified Galois cover of the projective line ramified over three points, $\{0, 1, \infty\}$ with Galois group $H_0 = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. More precisely the fundamental group of $X_3 = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ is isomorphic to the free group F_2 in two generators a, b. Here we may fix a point x_0 in X_3 and take as a, b the homotopy classes of loops circling once clockwise around the points $0, 1$ of $\mathbb{P}^1$. There is also a third loop c circling around ∞, but the homotopy class of this loop can be expressed in terms of a, b in terms of the relation $abc = 1$. When we remove the preimages of the three ramified points $\{0, 1, \infty\}$ from Fer_n we obtain the open Fermat curve Fer_n^0, which is a topological covering of X_3. Let $R_{\mathrm{Fer}_n} = \pi_1(\mathrm{Fer}_n^0, x_0^{'})$ be the fundamental group of the open Fermat curve Fer_n^0 and $x_0^{'} \in \mathrm{Fer}_n^0$ is a fixed preimage of the point $x_0 \in X_3$. The group R_{Fer_n} is known to be isomorphic to $\langle a^n, b^n, [a, b] \rangle < F_2$, while

$$H_0 = F_2 / R_{\mathrm{Fer}_n} = F_2 / \langle a^n, b^n, [a, b] \rangle.$$

Using this quotient we can define the following generators for the group H_0, namely $\alpha = aR_{\mathrm{Fer}_n}$ and $\beta = bR_{\mathrm{Fer}_n}$. Let R'_{Fer_n} denote the commutator group of the fundamental group R_{Fer_n} of Fer_n. The group F_2 acts on F_2 by conjugation, that is for every two elements $x, y \in F_2$ we define $x^y = yxy^{-1}$. Notice also that the homology group $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ becomes an H_0-module by defining

$$x^\alpha = axa^{-1}, x^\beta = bxb^{-1} \text{ for } x \in R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}.$$

This action is well defined and independent of the selection of the representative of the class $\alpha, \beta \in H_0$.

Theorem 1.2 *The fundamental group R_{Fer_n} of the open Fermat curve Fer_n^0 is the subgroup of the free group $F_2 = \langle a, b \rangle$ on the generators*

$$A_1 = \left\{ (b^n)^{a^i} : 0 \le i \le n-1 \right\}, \qquad \#A_1 = n$$

$$A_2 = \left\{ [b^j, a]^{a^i} : 1 \le j \le n-1, 0 \le i \le n-2 \right\}, \qquad \#A_2 = (n-1)^2$$

$$A_3 = \{ a^n [a^{-1}, b^j] : 0 \le j \le n-1 \}, \qquad \#A_3 = n$$

The module $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ is generated as a $\mathbb{Z}[H_0]$-module by the elements a^n, b^n and $[a, b]$. An isomorphic image of the module $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ fits in the small exact sequence

$$0 \to \mathbb{Z}[\langle\alpha\rangle] \bigoplus \mathbb{Z}[\langle\beta\rangle] \to R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n} \to \mathbb{Z}[H_0]/I \to 0,$$

where I is the ideal of $\mathbb{Z}[H_0]$ generated by $\sum_{i=0}^{n-1} \alpha^i, \sum_{i=0}^{n-1} \beta^i$, or equivalently

$$\mathbb{Z}[H_0]/I = J_{\langle\alpha\rangle} \otimes J_{\langle\beta\rangle},$$

where $J_{\langle\alpha\rangle}$ (resp. $J_{\langle\alpha\rangle}$) denotes the coaugmentation module for the cyclic group $\langle\alpha\rangle$ (resp. $\langle\beta\rangle$), see Section 2.2. Finally if $\mathbb{F}$ is a field which contains the n-different nth roots of 1, then

$$H_1(\mathrm{Fer}_n, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F} = \bigoplus_{\substack{i,j=1\\ i+j\neq n}}^{n-1} \mathbb{F}\chi_{i,j}, \tag{2}$$

where $\chi_{i,j}$ is the character of H_0 such that $\chi_{i,j}(\alpha^\nu, \beta^\mu) = \zeta_n^{i\nu+j\mu}$ and ζ_n is a fixed primitive nth root of unity.

The proof of the above theorem can be found in Section 2.2 and in particular in Proposition 2.12.

The generalized Fermat curves play the role of Fermat curves in the more general setting of abelian coverings of $X_s = \mathbb{P}^1 \backslash \{P_1, \ldots, P_s\}$, $s > 3$. Their automorphism group was recently studied by R. Hidalgo, M. Leyton-Álvarez and the authors in [**10**].

A generalized Fermat curve of type $(k, s-1)$, where $k, s-1 \geq 2$ are integers, is a non-singular irreducible projective algebraic curve $C_{k,s-1}$ defined over a field $\Bbbk$ admitting a group of automorphisms $H_0 \cong (\mathbb{Z}/k\mathbb{Z})^{s-1}$ so that $C_{k,s-1}/H_0$ is the projective line with exactly s branch points, each one with ramification index k. Such a group H_0 is called a generalized Fermat group of type $(k, s-1)$. Let us consider a branched regular covering $\pi : C_{k,s-1} \to \mathbb{P}^1$, whose deck group is H_0. Let $R_{k,s-1}$ be the fundamental group of the open generalized Fermat curve $C^0_{k,s-1} = C_{k,s-1} \backslash \pi^{-1}(X_s)$ of type $(k, s-1)$.

By composing by a suitable Möbius transformation (that is, an element of $\mathrm{PSL}_2(\mathbb{C})$) at the left of π, we may assume that the branch values of π are given by the points $\infty, 0, 1, \lambda_1, \ldots, \lambda_{s-3}$, where $\lambda_i \in \mathbb{C}\backslash\{0, 1\}$ are pairwise different, that is we can take $X_s = \mathbb{P}^1 \backslash \{\infty, 0, 1, \lambda_1, \ldots, \lambda_{s-3}\}$, $s > 3$.

A generalized Fermat curve of type $(k, s-1)$ can be seen as a complete intersection in a projective space $\mathbb{P}^{s-1}$, defined by the following set of equations

$$C_{k,s-1} = C_k(\lambda_1, \ldots, \lambda_{s-3}) := \left\{ \begin{array}{rcl} x_0^k + x_1^k + x_2^k & = & 0 \\ \lambda_1 x_0^k + x_1^k + x_3^k & = & 0 \\ \vdots & \vdots & \vdots \\ \lambda_{s-3} x_0^k + x_1^k + x_{s-1}^k & = & 0 \end{array} \right\} \subset \mathbb{P}^{s-1}. \tag{3}$$

Observe that topologically the construction of generalized Fermat curves does not depend on the configuration of the ramification points. On the other hand, the Riemann surface/algebraic curve structure depends heavily on this configuration. For instance, the automorphism group depends on the configuration of these points, see [10].

The genus of $C_{k,s-1}$ can be computed using the Riemann–Hurwitz formula:

$$g_{(k,s-1)} = 1 + \frac{k^{s-2}}{2}((s-2)(k-1)-2). \tag{4}$$

It is known [6] that generalized Fermat curves, have the orbifold uniformization $\mathbb{H}/\Gamma$ in terms of the Fuchsian group

$$\Gamma_k = \langle x_1, x_2, \ldots, x_s \mid x_1^k = \cdots = x_s^k = x_1 x_2 \cdots x_s = 1 \rangle. \tag{5}$$

The surface group is given, see [6], [18] as $F_{s-1} \cdot \langle x_1^k, \ldots, x_{s-1}^k, (x_1 \cdots x_{s-1})^k \rangle$. We will compute the genus of the generalized Fermat curves by two more different methods in equation 19 and in Section 5.1.2.

Recall that $R_{k,s-1}$ denotes the fundamental group of the open generalized Fermat curve $C^0_{k,s-1}$ and set $\mathfrak{F}_{s-1,k} = \mathfrak{F}_{s-1}/\mathfrak{R}_k$, where $\mathfrak{R}_k$ is the smallest closed subgroup containing all elements $x_i^{\ell^k}, 0 \leq i \leq s$.

Theorem 1.3 *The group $R_{k,s-1}$ is a free group generated by the union of the sets*

$$A_{s-1} = \left\{ (x_{s-1}^k)^{x_1^{i_1} \cdots x_{s-2}^{i_{s-2}}} \right\}$$

$$B_\nu = \left\{ [x^{\mathbf{i}}_{\nu+1,s-1}, x_\nu]^{x^{\mathbf{i}}_{1,\nu}} \right\} \quad 1 \leq i_\nu \leq k-2$$

$$B'_\nu = \left\{ \left(x_\nu^k [x_\nu^{-1}, x^{\mathbf{i}}_{\nu+1,s-1}] \right)^{x^{\mathbf{i}}_{1,\nu-1}} \right\}, \tag{6}$$

where $x^{\mathbf{i}}_{\ell_1,\ell_2} = x_{\ell_1}^{i_{\ell_1}} x_{\ell_1+1}^{i_{\ell_1+1}} \cdots x_{\ell_2}^{i_{\ell_2}}$, $\mathbf{i} = (i_1, \ldots, i_{s-1})$, *$0 \leq i_j \leq k-1$, $1 \leq j \leq s-1$. The group $R_{k,s-1}/R'_{k,s-1}$ is also generated (not necessary freely) by the union of the sets*

$$\tilde{A}_{s-1} = \{ (x_{s-1}^k)^{x_{1,s-2}} \}$$

$$\tilde{A}_\nu = \{ (x_\nu^k)^{x_{1,\nu-1} \cdot x_{\nu+1,s-1}} \}, \text{ for } 1 \leq \nu \leq s-2$$

$$\tilde{A}'_\nu = \{ [x_j, x_\nu]^{x_{1,\nu-1} \cdot x_\nu^{i_\nu} \cdot x_{\nu+1,s-1}} \}, \text{ for } 1 \leq \nu \leq s-2, \tag{7}$$

Proof. This theorem is proved using the Schreier lemma in Section 2.3. The transition from the first set of generators of equation (6) to the second set of non-free generators of equation (7) is done in Proposition 2.17. □

In our pro-ℓ setting we are interested in generalized Fermat curves of type $(\ell^k, s-1)$, so we will restrict ourselves to the study of curves $C_{\ell^k,s-1}$. Set $\mathscr{I} := (\mathbb{Z}\cap[0,\ell^k))^{s-1}$. Fix a primitive ℓ^k-root of unity ζ_{ℓ^k} and for each $\mathbf{i}=(i_1,\ldots,i_{s-1})\in\mathscr{I}$ define the characters $\chi_{\mathbf{i}}(\cdot)$ on the abelian group

$$H_0=\{\mathbf{x}=(\bar{x}_1^{\nu_1},\ldots,\bar{x}_{s-1}^{\nu_{s-1}}):\bar{x}_\mu^{\nu_\mu}\in\mathbb{Z}/\ell^k\mathbb{Z}\}\cong(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}$$

by

$$\chi_{\mathbf{i}}(\mathbf{x})=\zeta_{\ell^k}^{\sum_{\mu=1}^{s-1}\nu_\mu i_\mu}.$$

THEOREM 1.4 *The pro-ℓ homology of the closed curve is given by*

$$H_1(C_{\ell^k,s-1},\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}_\ell=\frac{\mathfrak{F}'_{s-1,k}}{\mathfrak{F}''_{s-1,k}}. \tag{8}$$

Let $\mathbb{F}$ be a field containing $\mathbb{Z}_\ell$ and the ℓ^k-roots of unity. We have the following decomposition:

$$H_1(C_{\ell^k,s-1},\mathbb{F})=\bigoplus_{\mathbf{i}\in\mathscr{I}}\mathbb{F}\cdot C(\mathbf{i})\chi_{\mathbf{i}},$$

where

$$C(\mathbf{i})=\begin{cases}s-z(\mathbf{i})-2 & \textit{if } \mathbf{i}\neq(0,\ldots,0)\\ s-z(\mathbf{i}) & \textit{if } \mathbf{i}=(0,\ldots,0)\end{cases}$$

and $z(\mathbf{i})$ is defined in equation (36). Moreover

$$\mathrm{rank}_{\mathbb{Z}_\ell}H_1(C_{\ell^k,s-1},\mathbb{Z}_\ell)=(s-1)\left(\ell^k\right)^{s-1}+2-s\left(\ell^k\right)^{s-2}.$$

Let $\Bbbk\subset\mathbb{C}$ and let $\bar{\Bbbk}$ be the algebraic closure of $\Bbbk$. The generalized Fermat curves behave in general in the same way if $\Bbbk$ is a field of characteristic $p\neq\ell$, but in this article we need to use fundamental groups and the theory of covering spaces so it is easier to assume that $\Bbbk\subset\mathbb{C}$, instead of working with algebraic fundamental groups in the sense of Grothendieck [**7**], [**8**]. *In general there will be no difference if we work over a field of characteristic zero.*

Fix the number of ramified points s. If $\bar{\Bbbk}(C_{\ell^k,s-1})$ is the function field of the generalized Fermat curve then

$$\frac{\mathfrak{F}'_{s-1,k}}{\mathfrak{F}''_{s-1,k}}=\mathrm{Gal}\left(\bar{\Bbbk}(C_{\ell^k,s-1})^{\mathrm{unrab}}/\bar{\Bbbk}(C_{\ell^k,s-1})\right),$$

where $\bar{\Bbbk}(C_{\ell^k,s-1})^{\mathrm{unrab}}$ is the maximal abelian unramified extension of the function field $\bar{\Bbbk}(C_{\ell^k,s-1})$.

Proof. The group theoretic interpretation of homology given in equation (8) is proved in Section 5.1. The analysis into characters is proved in Section 4 and in particular in Proposition 4.8. □

We would like to construct a 'curve' C_s which is a Galois cover of the projective line ramified over the set of s-points with Galois group $\mathrm{Gal}(C_s/\mathbb{P}^1) = \mathbb{Z}_\ell^{s-1}$. This 'curve' can only be defined as the limit case of the generalized Fermat curves $C_{\ell^k,s-1}$.

We can avoid the definition of such a 'curve' by working in the language of (infinite) Galois extensions of function fields as shown on the diagram on the right.

In this way instead of considering a simple generalized Fermat cover we consider all of them, together.

$$
\begin{tikzcd}
C_s \arrow[dd, "\mathbb{Z}_\ell^{s-1}"'] \arrow[dr, "(\ell^k\mathbb{Z}_\ell)^{s-1}"] & \\
& C_{\ell^k,s-1} \arrow[dl, "(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}"] \\
\mathbb{P}^1 &
\end{tikzcd}
\qquad
\begin{tikzcd}[arrows=dash]
M_s \arrow[dd, "\mathbb{Z}_\ell^{s-1}"'] \arrow[dr, "(\ell^k\mathbb{Z}_\ell)^{s-1}"] & \\
& \bar{\Bbbk}(C_{\ell^k,s-1}) \arrow[dl, "(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}"] \\
\bar{\Bbbk}(t) &
\end{tikzcd}
$$

The pro-ℓ limit

$$\mathbb{T} := \varprojlim_{\ell^k} T(\mathrm{Jac}(C_{\ell^k,s-1})) = \varprojlim_{\ell^k} \frac{\mathfrak{F}'_{s-1,k}}{\mathfrak{F}''_{s-1,k}}$$

corresponds to the $\mathbb{Z}_\ell$ homology of this 'curve' C_s and all the knowledge of the Galois module structure of all Tate modules of the curves $C_{\ell^k,s-1}$ is equivalent to the knowledge of the Galois module structure of $\mathbb{T}$.

The situation is similar to the pro-ℓ Burau representation, defined in [**15**]. We also in Section 5.1.5 how we can pass from the $\mathbb{Z}_\ell^{s-1}$-covers corresponding to generalized Fermat curves, to the $\mathbb{Z}_\ell$-case corresponding to the pro-ℓ Burau representation, using the ideas of [**16**].

Section 2 is devoted to the application of Schreier lemma to Fermat curves 2.1 and generalized Fermat curves 2.3 and the computation of homology by passing to the abelianization of the fundamental group. Section 3 is an introduction to Ihara's ideas on the study of the absolute Galois groups as a profinite braid [**11**], [**12**] following [**14**]. In Section 4.1, we compute the Alexander module for the generalized Fermat curves, while Section 5 is devoted to the $\mathbb{Z}_\ell^{s-1}$ cover of the projective line, seen as a limit of $C_{\ell^k,s-1}$ curves and the relation to the Tate modules of them. Finally we consider the passage to the Burau representation by comparing the corresponding Crowell sequences, in terms of the viewpoint developed in [**16**].

1.1. *Geometric interpretation*

We consider a Galois covering $\pi : \bar{Y} \to \mathbb{P}^1$ of the projective line ramified above the points in $S \subset \mathbb{P}^1_{\bar{\mathbb{Q}}}$, and the corresponding covering of compact Riemann surfaces. We also assume that the genus g of $\bar{Y}$ is ≥ 2. The curve $Y_0 = \bar{Y} \backslash \pi^{-1}(S)$ is a topological covering of $X_s = \mathbb{P}^1_{\mathbb{C}} \backslash S$, which can be described in terms of covering theory and corresponds to a subgroup R_0 of $\pi_1(\mathbb{P}^1_{\mathbb{C}} \backslash S) \cong \pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \backslash S) \to \pi_1^{\text{pro}-\ell}(\mathbb{P}^1_{\mathbb{C}})$. Denote by $\bar{R}_0$ the closure of R_0 in $\pi_1^{\text{pro}-\ell}(\mathbb{P}^1_{\mathbb{C}})$.

We have seen in [15] and we will see in Section 2.1 how this group R_0 can be computed by using the Schreier lemma. For an application of this method to cyclic covers of the projective line we refer to [15]. In order to pass from the open curve to the corresponding closed Riemann surface we consider the quotient by the group Γ, which is the closure in the subgroup of $\mathfrak{F}_{s-1}$ generated by the stabilizers of ramification points, that is

$$\Gamma = \langle x_1^{e_1}, \dots, x_s^{e_s} \rangle, \tag{9}$$

where $e_1, \dots, e_s$ are the ramification indices of the ramification points of $\pi : \bar{Y} \to \mathbb{P}^1$. In this article for some elements $g_1, \dots, g_t$ in a certain group we will denote by $\langle g_1, \dots, g_t \rangle$ the closed subgroup generated by the elements $\{g_1, \dots, g_t\}$.

Notice that if $e_1 = e_2 = \cdots = e_s$ then Γ is the closure of the group Γ_k defined in equation (5). Later, for $e_1 = \cdots = e_s = \ell^k$ we will denote this group by $\mathfrak{R}_k$. The group $R = R_0/R_0 \cap \Gamma$ corresponds to the closed curve $\bar{Y}$ as a quotient of the hyperbolic plane. This geometric situation can be expressed in terms of the short exact sequence of groups where the map ψ is the natural onto map defined by sending $a\Gamma \mapsto a\bar{R}_0 \cdot \Gamma$.

$$1 \to R = \frac{\bar{R}_0}{\Gamma \cap \bar{R}_0} \cong \frac{\bar{R}_0 \cdot \Gamma}{\Gamma} \to \frac{\mathfrak{F}_{s-1}}{\Gamma} \xrightarrow{\psi} \frac{\mathfrak{F}_{s-1}}{\bar{R}_0 \cdot \Gamma} \to 1. \tag{10}$$

In this article, we focus on the study of Fermat and generalized Fermat curves. Namely, in Sections 2.1 and 2.3 we compute the fundamental group of the corresponding curves. We also treat the classical Fermat curves $s = 3$ since this computation is elementary, while for the generalized Fermat curves $s \geq 3$ more advanced tools are needed, namely the usage of Alexander modules and the Crowell exact sequence.

2. Generalized Fermat curves

2.1. *Fermat curves*

These curves are ramified curves over $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ with deck group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. We have $\pi_1(\mathbb{P}^1 \backslash \{0, 1, \infty\}, x_0) \cong F_2 \cong \langle a, b \rangle$. This curve is a generalized Fermat curve $C_{n,s-1}$ for $s = 3$.

DEFINITION 2.1 The commutator $[a, b]$ of two elements a, b in a group is defined as $[a, b] = aba^{-1}b^{-1}$.

LEMMA 2.2 *For any two elements x, y of a group and any positive integer j we have*

(i) $[x^j, y] = [x, y]^{x^{j-1}} \cdot [x, y]^{x^{j-2}} \cdots [x, y]^x \cdot [x, y]$
(ii) $[x, y^j] = [x, y] \cdot [x, y]^y \cdot [x, y]^{y^2} \cdots [x, y]^{y^{j-1}}$.

Proof. See [5, 0.1 p.1]. □

We will employ the Schreier lemma for describing the fundamental group of the Fermat curve of level n, as explained in [15, sec. 3]. More precisely a (right) Schreier Transversal of a subgroup H of a free group $F_{s-1} = \langle x_1, \dots, x_{s-1} \rangle$ with basis $X = \{x_1, \dots, x_{s-1}\}$ is a set $T = \{t_1 = 1, \dots, t_n\}$ of reduced words such that each right coset of H in F_{s-1} contains a unique word of T called the representative of this class and all initial segments of these words also lie in T. For every $g \in F_{s-1}$ we will denote by $\bar{g}$ the element of T with the property $Hg = H\bar{g}$. Schreier's lemma, see [**15**, lemma 6] asserts that H is freely generated by the elements $\gamma(t,x) := tx\overline{tx}^{-1}$, $t \in T$, $x \in X$ and $tx \notin T$, $\gamma(t,x) \neq 1$.

A Schreier transversal T for the subgroup $R_{\mathrm{Fer}_n} \subset F_2$ such that $F_2/R_{\mathrm{Fer}_n} \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is given by $a^i b^j$, $0 \leq i,j \leq n-1$. The fundamental group of the Fermat curve is isomorphic to R_{Fer_n}.

LEMMA 2.3 *The group R_{Fer_n} is characteristic, that is every automorphism $\sigma \in \mathrm{Aut}F_2$ keeps R_{Fer_n} invariant.*

Proof. The group $R_{\mathrm{Fer}_n} \subset F_2 = \langle a,b \rangle$, can be generated by the elements $a^n, b^n, [a,b]$. The automorphism group of the free group F_n, and in particular of F_2, is generated by Nielsen transformations n_i and n_{ij} [**4**, th. 1.5 p. 125] which are defined as follows: The automorphism n_i sends a free generator $x_i \mapsto x_i^{-1}$ and leaves all other generators unchanged while the automorphism n_{ij} sends $x_i \mapsto x_i x_j$ and leaves all other generators unchanged. It is evident from the relations of R_{Fer_n}, see also lemma 2.4, that $n_i(R_{\mathrm{Fer}_n}) = R_{\mathrm{Fer}_n}$ and $n_{ij}(R_{\mathrm{Fer}_n}) = R_{\mathrm{Fer}_n}$. □

We also compute:

$$\overline{a^i b^j b} = \begin{cases} a^i b^{j+1} & \text{if } j < n-1 \\ a^i & \text{if } j = n-1 \end{cases}$$

and

$$\overline{a^i b^j a} = \begin{cases} a^{i+1} b^j & \text{if } i < n-1 \\ b^j & \text{if } i = n-1 \end{cases}$$

Thus

$$\begin{aligned} a^i b^j b \left(\overline{a^i b^j b}\right)^{-1} &= \begin{cases} a^i b^j b b^{-j-1} a^{-i} = 1 & \text{if } j < n-1 \\ a^i b^n a^{-i} & \text{if } j = n-1 \end{cases} \\ a^i b^j a \left(\overline{a^i b^j a}\right)^{-1} &= \begin{cases} a^i b^j a b^{-j} a^{-i-1} & \text{if } i < n-1, j \neq 0 \\ 1 & \text{if } i < n-1, j = 0, \\ a^{n-1} b^j a b^{-j} & \text{if } i = n-1 \end{cases} \end{aligned}$$

Consider the generators $\alpha = aR_{\mathrm{Fer}_n}, \beta = bR_{\mathrm{Fer}_n}$ of the group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Observe that there is a well-defined action of α (resp. β) on $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ given by conjugation, that is

$$x^\alpha = x^a = axa^{-1} \qquad x^\beta = x^b = bxb^{-1}$$

for all $x \in R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$. Notice that this is indeed an action which implies that

$$(x^\alpha)^\beta = x^{\alpha\beta} = x^{\beta\alpha} = (x^\beta)^\alpha$$

that is the actions of α and β commute.

LEMMA 2.4 *The generators of the free group R_{Fer_n} as union of the tree following sets:*

$$\begin{aligned}
A_1 &= \left\{(b^n)^{a^i} : 0 \le i \le n-1\right\}, && \#A_1 = n\\
A_2 &= \left\{[b^j, a]^{a^i} : 1 \le j \le n-1, 0 \le i \le n-2\right\} && \#A_2 = (n-1)^2\\
A_3 &= \{a^n[a^{-1}, b^j] : 0 \le j \le n-1\} && \#A_3 = n
\end{aligned}$$

Proof. This is a direct consequence of the Schreier lemma. Notice also that the above given sets together give rise to n^2+1 generators as predicted by Schreier index formula. Indeed, we compute $\#A_1 + \#A_2 + \#A_3 = n + (n-1)^2 + n = n^2 + 1$. □

LEMMA 2.5 *Fix $0 \le i \le n-2$. We will prove that the $\mathbb{Z}$-module generated by the elements*

$$\Sigma_1(i) := \{[b^j, a]^{\alpha^i}, \quad 1 \le j \le n-1\}$$

is the same as the $\mathbb{Z}$-module generated by the elements

$$\Sigma_2(i) := \{[b, a]^{\alpha^i \beta^j}, \quad 1 \le j \le n-2\}.$$

Proof. We will use additive notation here. By lemma 2.2(i) for $1 \le j \le n-1$ and $0 \le i \le n-2$ we have

$$[b^j, a]^{\alpha^i} = [b, a]^{(\beta^{j-1} + \beta^{j-2} + \cdots + \beta + 1)\alpha^i}. \tag{11}$$

Similar to equation (11)

$$[a^j, b]^{\beta^i} = [a, b]^{(\alpha^{j-1} + \alpha^{j-2} + \cdots + \alpha + 1)\beta^i}. \tag{12}$$

This proves that the elements of the set $\Sigma_1(i)$ are transformed to the elements of the set $\Sigma_2(i)$ in terms of an invertible block matrix where each block is the invertible $(n-1)\times(n-1)$ matrix with entries in $\mathbb{Z}$:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix}$$

Therefore $\Sigma_1(i)$ and $\Sigma_2(i)$ generate the same $\mathbb{Z}$-module. □

Notice also that

$$(a^n)^{\beta^j} = b^j a^{n-1} b^{-j} a^{-n+1} \cdot a^{n-1} b^j a b^{-j} = [b^j, a^{n-1}] + \underbrace{a^{n-1} b^j a b^{-j}}_{\in A_3}$$
$$= [b^j, a]^{\alpha^{n-2}+\alpha^{n-3}+\cdots+\alpha+1} + a^{n-1} b^j a b^{-j}. \tag{13}$$

Set $\Sigma_1 = \cup_i \Sigma_1(i)$. The above computation shows that $(a^n)^{\beta^j}$ can be written as a $\mathbb{Z}$-linear combination of elements of Σ_1 (which generate A_2) and A_3. Moreover

$$a^n[a^{-1}, b^j] = (a^n)^{\beta^j} - [b,a]^{\left(\sum_{k=0}^{j-1} \beta^k\right)\left(\sum_{\lambda=0}^{n-2} \alpha^\lambda\right)}.$$

We have shown that

LEMMA 2.6 *The free $\mathbb{Z}$-module $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ can be generated by the n^2+1 elements*

$$(a^n)^{\beta^i}, (b^n)^{\alpha^i},\ 0 \le i \le n-1 \text{ and } [a,b]^{\alpha^i \cdot \beta^j}, 0 \le i,j \le n-2.$$

2.2. *Structure as a $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$-module*

We can now consider the homology group as the rank n^2+1 free $\mathbb{Z}$-module $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$. By Lemma 2.3 R_{Fer_n} is a characteristic subgroup, so the group $H_0 = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} = \langle\alpha\rangle \times \langle\beta\rangle$ acts on $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ by conjugation making $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ a H_0-module.

For a finite group G the coaugmenation ideal J_G is defined as the quotient $J_G = \mathbb{Z}[G]/\langle \sum_{g\in G} g\rangle$.

LEMMA 2.7 *Set $H_0 = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. The module $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ is generated as a $\mathbb{Z}[H_0]$-module by the elements a^n, b^n, $[a,b]$. An isomorphic image of the module $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ fits in the short exact sequence*

$$0 \to \mathbb{Z}[\langle\alpha\rangle] \bigoplus \mathbb{Z}[\langle\beta\rangle] \to R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n} \to \mathbb{Z}[H_0]/I \to 0,$$

where I is the ideal of $\mathbb{Z}[H_0]$ generated by $\sum_{i=0}^{n-1} \alpha^i, \sum_{i=0}^{n-1} \beta^i$, or equivalently

$$\mathbb{Z}[H_0]/I \cong J_{\langle\alpha\rangle} \otimes J_{\langle\beta\rangle}.$$

Proof. By the $\mathbb{Z}$-basis given in Lemma 2.6 it is evident that a^n, b^n, $[a,b]$ indeed generate $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$. The elements a^n, b^n are acted by the groups $\langle\alpha\rangle$, $\langle\beta\rangle$ and form a $\mathbb{Z}[H_0]$-submodule of $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ isomorphic to $\mathbb{Z}[\langle\alpha\rangle] \bigoplus \mathbb{Z}[\langle\beta\rangle]$. Indeed, since the action of α on a is trivial we can identify the elements in $\mathbb{Z}[H_0]a^n$ to the set of elements of $\mathbb{Z}[\langle\beta\rangle]$ and $\mathbb{Z}[H_0]b^n$ can be similarly identified to $\mathbb{Z}[\langle\alpha\rangle]$. Notice also that $\mathbb{Z}[H_0]a^n \cap \mathbb{Z}[H_0]b^n = \{0\}$.

Observe now that the elements $[a,b]^{\alpha^i \beta^j}$ are subject to the condition given in equation (11) which implies that for all i

$$[a,b]^{\alpha^i(1+\beta+\beta^2+\cdots+\beta^{n-1})} = [a,b^n]^{\alpha^i} = a^{i+1} b^n a^{-i-1} - a^i b^n a^{-i} \in \mathbb{Z}[\langle\alpha\rangle] \bigoplus \mathbb{Z}[\langle\beta\rangle]. \tag{14}$$

In the above formula, we have used the additive structure of $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$. Equation (14) shows that the operator $1+\beta+\cdots+\beta^{n-1}$ in the quotient of $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ by $\mathbb{Z}[\langle\alpha\rangle] \bigoplus \mathbb{Z}[\langle\beta\rangle]$ is zero. Similarly, by equation (12) we obtain that $\sum_{i=0}^{n-1} \alpha^i = 0$ in the quotient. Therefore, in the $\mathbb{Z}[H_0]$-module generated by $[a,b]$ we have that $1+\beta+\cdots+\beta^{n-1}$ and $1+\alpha+\cdots+\alpha^{n-1}$ both annihilate $[a,b]$. We compute that

$$\mathbb{Z}[H_0]/\langle 1+\beta+\beta^2+\cdots+\beta^{n-1}\rangle = \bigoplus_{i=0}^{n-1} \alpha^i J_{\langle\beta\rangle}.$$

The result follows. □

REMARK 2.8 The theorem of Maschke implies that after a scalar extension to $\mathbb{Q}$ we have

$$R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[\langle\alpha\rangle] \bigoplus \mathbb{Q}[\langle\beta\rangle] \bigoplus \mathbb{Q}[H_0]/I,$$

We will now prove that in $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ there are exactly $3n$ elements which are fixed by an element of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

First note, the $2n$ elements $(a^n)^{\beta^i}$ (respectively $(b^n)^{\alpha^i}$) are fixed by $\langle\alpha\rangle$ (respectively $\langle\beta\rangle$).

The other n-elements are the elements $((ab)^n)^{\alpha^i}$ which are fixed by $\langle ab\rangle$.

LEMMA 2.9 *We can write the elements* $((ab)^n)^{\alpha^i}$ *as follows:*

$$(ab)^n = [b,a]^{\alpha^{n-1}\left(\sum_{\nu=0}^{n-2}\beta^\nu\right)+\cdots+\alpha^2(\beta+1)+\alpha} a^n b^n.$$

Proof. We begin by computing the $n=2$ case:

$$(ab)^2 = abab = ab\overbrace{ab^{-1}a^{-1}}^{[b,a]}abb = a[b,a]a^{-1}aabb = [b,a]^\alpha a^2b^2.$$

Similarly, for the $n=3$ case we have

$$\begin{aligned}(ab)^3 &= (ab)^2(ab) = [b,a]^\alpha a^2b^2ab = [b,a]^\alpha a^2b\overbrace{bab^{-1}a^{-1}}^{[b,a]}abb\\ &= [b,a]^\alpha\cdot[b,a]^{\alpha^2\beta}a^2bab^2 = [b,a]^{\alpha+\alpha^2\beta}a^2\overbrace{bab^{-1}a^{-1}}^{[b,a]}abb^2\\ &= [b,a]^{\alpha+\alpha^2\beta}[b,a]^{\alpha^2}a^3b^3 = [b,a]^{\alpha^2(\beta+1)+\alpha}a^3b^3.\end{aligned}$$

Now assume that for k we have

$$(ab)^k = [b,a]^{\alpha^{k-1}\left(\sum_{\nu=0}^{k-2}\beta^\nu\right)+\cdots+\alpha^2(\beta+1)+\alpha} a^k b^k$$

Set $E(k)=\alpha^{k-1}\left(\sum_{\nu=0}^{k-2}\beta^{\nu}\right)+\cdots+\alpha^2(\beta+1)+\alpha$. We will now consider

$$\begin{aligned}(ab)^{k+1}&=[b,a]^{E(k)}a^kb^kab\\&=[b,a]^{E(k)}a^k[b^k,a]ab^kb=[b,a]^{E(k)}[b^k,a]^{\alpha^k}a^{k+1}b^{k+1}\\&=[b,a]^{E(k)}[b,a]^{\alpha^k\left(\beta^{k-1}+\beta^{k-2}+\cdots+\beta+1\right)}a^{k+1}b^{k+1}\\&=[b,a]^{\alpha^k\left(\sum_{\nu=0}^{k-1}\beta^{\nu}\right)+\alpha^{k-1}\left(\sum_{\nu=0}^{k-2}\beta^{\nu}\right)+\cdots+\alpha^2(\beta+1)+\alpha}a^{k+1}b^{k+1},\end{aligned}$$

as desired. □

The above lemma gives us

$$((ab)^n)^{\alpha^i}=[b,a]^{\alpha^{n-1+i}\left(\sum_{\nu=0}^{n-2}\beta^{\nu}\right)+\cdots+\alpha^{2+i}(\beta+1)+\alpha^{1+i}}a^n\,(b^n)^{\alpha^i}.$$

We can see that the transformation matrix from elements $[b^j,a]^{\alpha^i}$ to elements of the form $[b,a]^{\beta^j\alpha^i}$ is invertible. This allows us to prove that the elements in the sets A_2 and A_3 can be written as linear combinations of elements of the form $[b,a]^{\alpha^i\beta^j}$ and $(a^n)^{\beta^j}$ for $1\le j\le n-1$, $0\le i\le n-2$. It is clear that the elements $(a^n)^{\beta^i},(b^n)^{\alpha^i},((ab)^n)^{\alpha^i}$ as given in the table below are fixed by the cyclic group mentioned in the third column. The elements γ_i are the n-elements $(b^n)^{\alpha^i}$ fixed by β, the n-elements $(a^n)^{\beta^i}$ fixed by α and the n invariant elements $((ab)^n)^{\alpha^i}$ in the module generated by commutators. In the following table we enumerate the fixed elements γ_i:

Invariant element γ_i	Index	Fixed by
$(a^n)^{\beta^i}$	$1\le i\le n$	$\langle\alpha\rangle$
$(b^n)^{\alpha^i}$	$n+1\le i\le 2n$	$\langle\beta\rangle$
$((ab)^n)^{\alpha^i}$	$2n+1\le i\le 3n$	$\langle\alpha\beta\rangle$

So far we have computed the open Fermat curve admitting a presentation

$$R_{\mathrm{Fer}_n}=\langle a_1,b_1,\ldots,a_g,b_g,\gamma_1,\ldots,\gamma_{3n}|\gamma_1\gamma_2\cdots\gamma_{3n}\cdot[a_1,b_1][a_2,b_2]\cdots[a_g,b_g]=1\rangle,$$

where g is the genus of the closed Fermat curve which equals to $(n-1)(n-2)/2$. Every ramification point of the Fermat curve is surrounded by a path γ_i and there are $3n$ such paths, see Figure 1. We can verify that our computation is correct so far, by computing the genus of the closed Fermat curve. We add the $3n$ missing points and we observe that the rank of R_{Fer_n} equals $2g+3n-1$ so the Schreier index formula implies:

$$2g+3n-1=n^2+1\Rightarrow g=\frac{n^2+2-3n}{2}=\frac{(n-1)(n-2)}{2}.$$

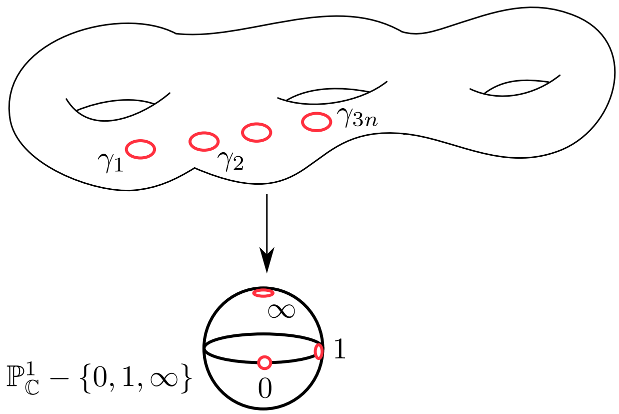


Figure 1. Open Fermat curve as cover of the projective line

We have that

$$H_1(X_F, \mathbb{Z}) = \frac{R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}}{\langle \gamma_1, \dots, \gamma_{3n} \rangle}.$$

Definition 2.10 We will denote by ζ_n a fixed primitive nth root of unity and by $\chi_{i,j}$ the character such that $\chi_{i,j}(\alpha^\nu \beta^\mu) = \zeta_n^{i\nu + j\mu}$.

Definition 2.11 Let Γ be free $\mathbb{Z}$-module generated by $\langle \gamma_1, \dots, \gamma_{3n} \rangle$.

Proposition 2.12 *A basis for the $\mathbb{Z}$-module $H_1(X_F, \mathbb{Z})$ consists of the set:*

$$\{[b,a]^{\alpha^i \beta^j} \mod \Gamma : 0 \le i \le n-2, 0 \le j \le n-3\}.$$

Let $\mathbb{F}$ be a field that contains n different nth roots of 1. Then

$$H_1(X_F, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F} = \bigoplus_{\substack{i,j=1 \\ i+j \neq n}}^{n-1} \mathbb{F}\chi_{i,j}.$$

Proof. The first assertion follows by considering the action modulo the elements which are invariant by an element of H_0. Indeed, in order to compute the quotient we change the basis of $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ by replacing each one of the elements $[b,a]^{\alpha^{n-1+i}\beta^{n-2}}$ by $((ab)^n)^{\alpha^i}$ for all $0 \le i \le n-1$, which belongs to the group $\langle \gamma_1, \dots, \gamma_{3n} \rangle$ and is considered to be zero.

For the second assertion let us write

$$(J_{\langle \alpha \rangle} \otimes \mathbb{F}) \bigotimes (J_{\langle \beta \rangle} \otimes \mathbb{F}) = \left(\bigoplus_{i=1}^{n-1} \mathbb{F}\chi_{i,0} \right) \bigotimes \left(\bigoplus_{j=1}^{n-1} \mathbb{F}\chi_{0,j} \right) = \bigoplus_{i,j=1}^{n-1} \mathbb{F}\chi_{i,j}$$

We are looking for the elements which are stabilized by $\alpha\beta$, that is $\chi_{i,j}(\alpha\beta) = \zeta^{i+j} = 1$. This is the module $\oplus_{i=1}^{n-1}\mathbb{F}\chi_{i,n-i}$, which has n-elements. The desired result follows. □

Observe that the above computation agrees with $\dim_{\mathbb{F}} H_1(X_F, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F} = (n-1)(n-2)$.

Braid group action
We will now consider the action of the Braid group B_3 on $H_1(X_F, \mathbb{Z})$ of the closed Fermat surface. By the faithful Artin representation we observe that the braid group in three strings is generated by the elements σ_1, σ_2, where

$$\sigma_1(a) = aba^{-1} \qquad \sigma_2(a) = a \qquad \sigma_1(b) = a \qquad \sigma_2(b) = a^{-1}b^{-1}.$$

Notice that the above two automorphism in the abelianized free group with two generators acts like the matrices

$$\bar{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \bar{\sigma}_2 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

in $\mathrm{GL}(2, \mathbb{Z})$, reflecting the fact that $B_3/Z(B_3) \cong \mathrm{PSL}(2, \mathbb{Z})$. Therefore, working in $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ we have

$$\begin{aligned} \sigma_1[a,b] &= [aba^{-1}, a] \ = [b,a]^{\alpha} = -[a,b]^{\alpha} \\ \sigma_2[a,b] &= [a, a^{-1}b^{-1}] = [b^{-1}, a^{-1}]. \end{aligned}$$

and more generally

$$\sigma_1([b,a]^{\alpha^i\beta^j}) = -[b,a]^{\alpha^{j+1}\beta^i} \qquad \sigma_2([b,a]^{\alpha^i\beta^j}) = -[b^{-1}, a^{-1}]^{\alpha^{i-j}\beta^{-j}}$$

Indeed, we compute

$$\begin{aligned} \sigma_1([b,a]^{\alpha^i\beta^j}) &= \sigma_1\left(a^ib^j[b,a]b^{-j}a^{-i}\right) = ab^ia^{-1}a^j[a,b]^{\alpha}a^{-j+1}b^{-i}a^{-1} \\ &= -[b,a]^{\alpha^{j+1}\beta^i} \end{aligned}$$

and

$$\begin{aligned} \sigma_2([b,a]^{\alpha^i\beta^j}) &= \sigma_2\left(a^ib^j[b,a]b^{-j}a^{-i}\right) = -a^i(a^{-1}b^{-1})^j[b^{-1}, a^{-1}](a^{-1}b^{-1})^{-j}a^{-i} \\ &= -[b^{-1}, a^{-1}]^{\alpha^i(\alpha\beta)^{-j}} = -[b^{-1}, a^{-1}]^{\alpha^{i-j}\beta^{-j}}. \end{aligned}$$

In the above equations we have used that H_0 is abelian and its action on $R_{\mathrm{Fer}_n}/R'_{\mathrm{Fer}_n}$ is well defined. We also compute

$$\sigma_1((b^n)^{\alpha^i}) = (a^n)^{\beta^i} \qquad \sigma_1((a^n)^{\beta^j}) = (b^n)^{\alpha^{j+1}}$$
$$\sigma_2((b^n)^{\alpha^i}) = ((ba)^n)^{-\alpha^i} \qquad \sigma_2((a^n)^{\beta^j}) = (a^n)^{(\beta\alpha)^{-j}}.$$

2.3. *The generalized Fermat curve*

Application of the Schreier lemma

Consider the open curve $X_s = \mathbb{P}^1 \backslash \{0, 1, \infty, \lambda_1, \ldots, \lambda_{s-3}\}$ and let x_0 be a fixed base point in X_s. For the fundamental group we have

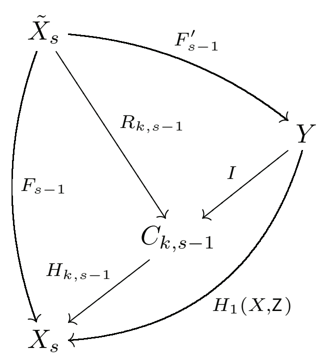
$$\pi_1(X_s, x_o) \cong F_{s-1} = \langle x_1, \ldots, x_{s-1} \rangle.$$

Let $\tilde{X}_s$ denote the universal covering space and $Y = \tilde{X}_s / F'_{s-1}$ be the cover of X_s corresponding to the group F'_{s-1}, that is

$$\mathrm{Gal}(Y/X_s) \cong F_{s-1}/F'_{s-1} \cong H_1(X_s, \mathbb{Z}) \cong \mathbb{Z}^{s-1}.$$

Let $H_{k,s-1} \cong (\mathbb{Z}/k\mathbb{Z})^{s-1}$ be the abelian group fitting in the short exact sequence

$$0 \to I \to H_1(X_s, \mathbb{Z}) \to H_{k,s-1} \to 0. \tag{15}$$



If we denote, in additive notation, $H_1(X_s, \mathbb{Z}) = \bigoplus_{\nu=0}^{s-1} x_\nu \mathbb{Z}$, then $I = \bigoplus_{\nu=0}^{s-1} k x_\nu \mathbb{Z}$.

REMARK 2.13 The short exact sequence is in some sense a generalization of the winding number exact sequence given in definition 9 of [15].

We will now employ the Schreier lemma [4, chap. 2 sec. 8], [19, sec. 2.3 th. 2.7] in order to compute the free subgroup $R_{k,s-1} \subset F_{s-1}$, where $R_{k,s-1}$ is the subgroup of F_{s-1} corresponding to the curve $C_{k,s-1}$ and is isomorphic to the fundamental group of $C_{k,s-1}$. We will introduce some new notation first:

DEFINITION 2.14 For any $\mathbf{i} := (i_1, \ldots, i_{s-1}) \in \mathbb{Z}^{s-1}$, define

$$\mathbf{x}^{\mathbf{i}} := x_1^{i_1} x_2^{i_2} \cdots x_{s-1}^{i_{s-1}}.$$

We also set

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \mathbf{e}_{s-1} = (0, \ldots, 0, 1).$$

LEMMA 2.15 *A Schreier Transversal for $R_{k,s-1} < F_{s-1}$ is given by*

$$T = \{\mathbf{x}^{\mathbf{i}} : \mathbf{i} = (i_1, \ldots, i_j, \ldots, i_{s-1}) \in \mathbb{Z}^{s-1} \textit{ and } 0 \leq i_j \leq k-1\},$$

Proof. Notice, that the set T contains $\#H_{k,s-1}$ elements which are different modulo $R_{k,s-1}$. To see this we can use the fact that

$$H_{k,s-1} \cong \frac{F_{s-1}}{R_{k,s-1}} = \frac{H_1(X_s, \mathbb{Z})}{I},$$

and the special form of I. The condition concerning the initial segments is trivially satisfied by the special form of the elements in T. $\square$

For given $1 \leq \nu \leq s-1$ we have:

$$\overline{\mathbf{x}^{\mathbf{i}} \cdot x_\nu} = \begin{cases} \mathbf{x}^{\mathbf{i}+\mathbf{e}_\nu} & i_\nu < k-1 \\ \mathbf{x}^{\mathbf{i}-i_\nu \mathbf{e}_\nu} & i_\nu = k-1 \end{cases}$$

Case I For $1 \leq \nu \leq s-1$:

$$\mathbf{x}^{\mathbf{i}} \cdot x_\nu \cdot \left(\overline{\mathbf{x}^{\mathbf{i}} \cdot x_\nu}\right)^{-1} = \begin{cases} \mathbf{x}^{\mathbf{i}} \cdot x_\nu \cdot \mathbf{x}^{-\mathbf{i}-\mathbf{e}_\nu} & \text{if } i_\nu < k-1 \\ \mathbf{x}^{\mathbf{i}} \cdot x_\nu \cdot \mathbf{x}^{-\mathbf{i}+(k-1)\mathbf{e}_\nu} & \text{if } i_\nu = k-1 \end{cases}$$

Notice that in the second case

$$\mathbf{x}^{-\mathbf{i}+(k-1)\mathbf{e}_\nu} = x_1^{-i_1} x_2^{-i_2} \cdots x_{\nu-1}^{-i_{\nu-1}} x_{\nu+1}^{-i_{\nu+1}} \cdots x_{s-1}^{-i_{s-1}}.$$

Case II For $\nu = s-1$:

$$\mathbf{x}^{i} \cdot x_{s-1} \cdot \left(\overline{\mathbf{x}^{\mathbf{i}} x_{s-1}}\right)^{-1} = \begin{cases} 1 & i_{s-1} < k-1 \\ \mathbf{x}^{\mathbf{i}} \cdot x_{s-1}^{k} \cdot (\mathbf{x}^{\mathbf{i}})^{-1} & i_{s-1} = k-1 \end{cases}$$

$$x^{\mathbf{i}}_{\ell_1,\ell_2} = \begin{cases} x_{\ell_1}^{i_{\ell_1}} x_{\ell_1+1}^{i_{\ell_1+1}} \cdots x_{\ell_2}^{i_{\ell_2}} & \text{if } \ell_1 \leq \ell_2 \\ 1 & \text{if } \ell_1 > \ell_2 \end{cases}$$

The generators of the free group $R_{k,s-1}$ are falling in the following categories:

$$
\begin{aligned}
A_{s-1} &= \left\{ (x_{s-1}^k)^{x_1^{i_1}\cdots x_{s-2}^{i_{s-2}}} \right\} \\
B_\nu &= \left\{ x^{\mathbf{i}}_{1,\nu-1} \cdot x_\nu^{i_\nu} x^{\mathbf{i}}_{\nu+1,s-1} \cdot x_\nu \cdot \left(x^{\mathbf{i}}_{\nu+1,s-1}\right)^{-1} \cdot x_\nu^{-i_\nu-1} \cdot \left(x^{\mathbf{i}}_{1,\nu-1}\right)^{-1} \right\} \\
&= \left\{ [x^{\mathbf{i}}_{\nu+1,s-1}, x_\nu]^{x^{\mathbf{i}}_{1,\nu-1}\cdot x_\nu^{i_\nu}} \right\} \\
&= \left\{ [x^{\mathbf{i}}_{\nu+1,s-1}, x_\nu]^{x^{\mathbf{i}}_{1,\nu}} \right\} \quad 1 \le i_\nu \le k-2 \\
B'_\nu &= \left\{ x^{\mathbf{i}}_{1,\nu-1} \cdot x_\nu^{k-1} \cdot x^{\mathbf{i}}_{\nu+1,s-1} \cdot x_\nu \cdot \left(x^{\mathbf{i}}_{\nu+1,s-1}\right)^{-1} \cdot \left(x^{\mathbf{i}}_{1,\nu-1}\right)^{-1} \right\} \\
&= \left\{ \left(x_\nu^k [x_\nu^{-1}, x^{\mathbf{i}}_{\nu+1,s-1}]\right)^{x^{\mathbf{i}}_{1,\nu-1}} \right\}.
\end{aligned} \tag{16}
$$

Notice that A_{s-1} corresponds to Case II, while the sets B_ν, B'_ν for $1 \le \nu \le s-2$ correspond to Case I, for $i_\nu < k-1$ and $i_\nu = k-1$ subcases, respectively. We now count the sizes of the above sets.

$$
\begin{aligned}
\#A_{s-1} &= k^{s-2} \\
\#B_\nu &= (k-1)\cdot k^{\nu-1} \cdot (k^{s-1-\nu} - 1), \text{ for } 1 \le \nu \le s-2 \\
\#B'_\nu &= k^{\nu-1} \cdot k^{s-1-\nu} = k^{s-2}, \text{ for } 1 \le \nu \le s-2
\end{aligned}
$$

which gives in total

$$
\#A_{s-1} + \sum_{\nu=1}^{s-2} \#B_\nu + \sum_{\nu=1}^{s-2} \#B'_\nu = (s-2)\cdot k^{s-1} + 1. \tag{17}
$$

Elements stabilized

Let us not consider the action of $H_{k,s-1} = (F_{s-1}/R_{k,s-1})$ on $(R_{k,s-1}/R'_{k,s-1})$. Let us now denote

$$
\frac{F_{s-1}}{R_{k,s-1}} = \langle \xi_1, \ldots, \xi_{s-1} \rangle \cong \left(\frac{\mathbb{Z}}{k\mathbb{Z}} \right)^{s-1},
$$

where $\xi_j = x_j R_{k,s-1}$. Let us write $\xi^{\mathbf{i}}_{\ell_1,\ell_2} = x^{\mathbf{i}}_{\ell_1,\ell_2} R_{k,s-1}$. Observe first that the group generated by ξ_{s-1} stabilizes $(x_{s-1}^k)^{\xi^{\mathbf{i}}_{1,s-2}}$, since

$$
\begin{aligned}
\left((x_{s-1}^k)^{\xi^{\mathbf{i}}_{1,s-2}} \right)^{\xi_{s-1}} &= (x_{s-1}^k)^{\xi_{s-1}\cdot \xi^{\mathbf{i}}_{1,s-2}} \\
&= (x_{s-1} x_{s-1}^k x_{s-1}^{-1})^{\xi^{\mathbf{i}}_{1,s-2}} = (x_{s-1}^k)^{\xi^{\mathbf{i}}_{1,s-2}}.
\end{aligned}
$$

In this way we see that all k^{s-2} elements of A_{s-1} have non-trivial stabilizer. Now we observe that

$$
\begin{aligned}
&x^{\mathbf{i}}_{1,\nu-1} \cdot x_\nu^{k-1} \cdot x^{\mathbf{i}}_{\nu+1,s-1} \cdot x_\nu \cdot \left(x^{\mathbf{i}}_{\nu+1,s-1}\right)^{-1} \cdot \left(x^{\mathbf{i}}_{1,\nu-1}\right)^{-1} = \\
&\qquad = [x_\nu^{k-1}, x^{\mathbf{i}}_{\nu+1,s-1}]^{\xi^{\mathbf{i}}_{1,\nu-1}} \cdot (x_\nu^k)^{\xi^{\mathbf{i}}_{1,\nu-1}\cdot \xi^{\mathbf{i}}_{\nu+1,s-1}}.
\end{aligned}
$$

Observe that for each ν, $1 \le \nu \le s-1$, $\langle \xi_\nu \rangle$ stabilizes the k^{s-2} elements of B'_ν of the form $(x_\nu^k)^{\xi^{\mathbf{i}}_{1,\nu-1}\cdot \xi^{\mathbf{i}}_{\nu+1,s-1}}$ and the element $\langle \xi_1 \cdots \xi_{s-1} \rangle$ stabilizes all elements $((x_1 \cdots x_{s-1})^k)^{\xi^{\mathbf{i}}_{1,s-2}}$, of which there are k^{s-2}.

Invariant element γ_i	Cardinal	Fixed by
$(x_{s-1}^k)^{\xi^{\mathbf{i}}_{1,s-2}}$	k^{s-2}	$\langle \xi_{s-1} \rangle$
$(x_\nu^k)^{\xi^{\mathbf{i}}_{1,\nu-1}\cdot \xi^{\mathbf{i}}_{\nu+1,s-1}}$	$(s-2)k^{s-2}$	$\langle \xi_\nu \rangle$, where $1 \le \nu \le s-2$
$((x_1 \cdots x_{s-1})^k)^{\xi^{\mathbf{i}}_{1,s-2}}$	k^{s-2}	$\langle \xi_1 \cdots \xi_{s-1} \rangle$

In total we have sk^{s-2} fixed elements γ_i.

LEMMA 2.16 *The following equality holds.*

$$[x^{\mathbf{i}}_{\nu+1,s-1}, x_\nu]^{\xi^{\mathbf{i}}_{1,\nu}} = \left([x_{\nu+1}^{i_{\nu+1}}, x_\nu] + [x_{\nu+2}^{i_{\nu+2}}, x_\nu]^{\xi_{\nu+1}{}^{i_{\nu+1}}} + \cdots + [x_{s-1}^{i_{s-1}}, x_\nu]^{\xi^{\mathbf{i}}_{\nu+1,s-2}}\right)^{\xi^{\mathbf{i}}_{1,\nu}}.$$
$$= \left(\sum_{j=\nu+1}^{s-1} [x_j^{i_j}, x_\nu]^{\xi^{\mathbf{i}}_{\nu+1,j-1}}\right)^{\xi^{\mathbf{i}}_{1,\nu}}$$

Proof. The lemma will be proved by induction. Notice that it is enough to prove

$$[x^{\mathbf{i}}_{\nu+1,s-1}, x_\nu] = [x_{\nu+1}^{i_{\nu+1}}, x_\nu] + [x_{\nu+2}^{i_{\nu+2}}, x_\nu]^{\xi_{\nu+1}{}^{i_{\nu+1}}} + \cdots + [x_{s-1}^{i_{s-1}}, x_\nu]^{\xi^{\mathbf{i}}_{\nu+1,s-2}}$$
$$= \sum_{j=\nu+1}^{s-1} [x_j^{i_j}, x_\nu]^{\xi^{\mathbf{i}}_{\nu+1,j-1}}.$$

For $\nu+1 = s-1$ the desired equality is trivial. We will use the following commutator identity, which can be easily verified:

$$[xz, y] = [z, y]^x \cdot [x, y].$$

Assume that the equality holds for the next product $x^{\mathbf{i}}_{\nu,s-1}$ we compute

$$[x^{\mathbf{i}}_{\nu,s-1}, x_\nu] = [x_\nu^{i_\nu} \cdot x^{\mathbf{i}}_{\nu+1,s-1}, x_\nu] = [x^{\mathbf{i}}_{\nu+1,s-1}, x_\nu]^{\xi_\nu^{i_\nu}} \cdot [x_\nu^{i_\nu}, x_\nu].$$

Hence writing the above equality additively we obtain

$$[x^{\mathbf{i}}_{\nu,s-1}, x_\nu] = [x_\nu^{i_\nu}, x_\nu] + \left([x_{\nu+1}^{i_{\nu+1}}, x_\nu] + [x_{\nu+2}^{i_{\nu+2}}, x_\nu]^{\xi_{\nu+1}^{i_{\nu+1}}} + \cdots + [x_{s-1}^{i_{s-1}}, x_\nu]^{\xi^{\mathbf{i}}_{\nu+1,s-2}}\right)^{\xi_\nu^{i_\nu}}$$
$$= [x_\nu^{i_\nu}, x_\nu] + [x_{\nu+1}^{i_{\nu+1}}, x_\nu]^{\xi_\nu^{i_\nu}} + [x_{\nu+2}^{i_{\nu+2}}, x_\nu]^{\xi_\nu^{i_\nu}\xi_{\nu+1}^{i_{\nu+1}}} + \cdots + [x_{s-1}^{i_{s-1}}, x_\nu]^{\xi^{\mathbf{i}}_{\nu,s-2}}$$

□

Similar to the computation of the classical Fermat curves we change to a more suitable basis.

PROPOSITION 2.17 *Recall that* $\mathbf{i} = (i_1, \dots, i_{s-1})$, $0 \le i_1, \dots, i_{s-1} \le k-1$. *A generating set for* $R_{k,s-1}/R'_{k,s-1}$ *is given by*

$$\begin{aligned} \tilde{A}_{s-1} &= \{(x_{s-1}^k)^{\xi^{\mathbf{i}}_{1,s-2}}\} \\ \tilde{A}_\nu &= \{(x_\nu^k)^{\xi^{\mathbf{i}}_{1,\nu-1}\cdot\xi^{\mathbf{i}}_{\nu+1,s-1}}\}, \text{ for } 1 \le \nu \le s-2 \\ \tilde{A}'_\nu &= \{[x_j, x_\nu]^{\xi^{\mathbf{i}}_{1,s-1}}\}, \text{ for } 0 \le j < \nu \le s-1, 1 \le i_j, i_\nu \le k-2. \end{aligned}$$

Proof. Notice that equation (13) for $i \neq \nu$ implies that

$$\begin{aligned} \left(x_i^k\right)^{\xi_\nu^j} &= [x_\nu^j, x_i^{k-1}] + x_i^{k-1} x_\nu^j x_i x_\nu^{-j} \\ &= [x_\nu^j, x_i]^{\xi_i^{k-2}+\xi_i^{k-3}+\dots+\xi_i+1} + x_i^k [x_i^{-1}, x_\nu^j]. \end{aligned} \tag{18}$$

The result follows using the generating set given in equation (16), Lemma 2.16 and equation (18). ☐

REMARK 2.18 For the homology of the closed curve we have:

$$H_1(C_{k,s-1}, \mathbb{Z}) = \frac{R_{k,s-1}/R'_{k,s-1}}{\langle \gamma_1, \dots, \gamma_{sk^{s-1}} \rangle}.$$

Using equation (4) and the fact that rank $H_1(C_{k,s-1}, \mathbb{Z}) = 2g_{C_{k,s-1}}$ it is easy to verify that

$$(s-2)k^{s-1} + 1 - (s \cdot k^{s-2} - 1) = 2g_{C_{k,s-1}}. \tag{19}$$

In the above formula we have subtracted one from the number of invariant elements γ_i since $\gamma_1 \cdots \gamma_{sk^{s-2}} = 1$.

Describing the action in this case is not as straightforward as it was for the case of classical Fermat curve. We will use the theory of Alexander modules instead and postpone this computation to Section 4.1.

3. On the representation of Ihara

3.1. *Pro-ℓ braid groups*

Let ℓ be a prime number and let $\mathfrak{F}_s$ denote the pro-ℓ free group with s free generators. Let $S \subset \mathbb{P}^1_{\bar{\mathbb{Q}}}$ be a set consisted of s points, $s \ge 3$, on the projective line and suppose that $P \in \mathbb{Q}$ for all $P \in S - \{\infty\}$. In this way the absolute Galois group corresponds to 'pure braids'. Ihara in [11] introduced the monodromy representation

$$\mathrm{Ih}_S : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\mathfrak{F}_{s-1}).$$

Here the group $\mathfrak{F}_{s-1} \cong \pi_1^{\text{pro}-\ell}(\mathbb{P}^1_{\bar{\mathbb{Q}}} - S)$ is the pro-ℓ étale fundamental group and is known to admit a presentation

$$\mathfrak{F}_{s-1} = \overline{\langle x_1, \dots, x_s | x_1 x_2 \cdots x_s = 1 \rangle}, \tag{20}$$

where $\overset{\wedge}{\cdot}$ denotes the pro-ℓ completion of a finitely generated group. Given a set $\{x_i, i\in I\}$ in a topological group we will denote by $\langle x_i, i\in I\rangle$ the topological closure of the group generated by the group elements $x_i,\ i\in I$. In [11] Ihara studied the case $S=\{0,1,\infty\}$. This is an interesting case since by Belyi's theorem [2] the branched covers of $\mathbb{P}^1\backslash\{0,1,\infty\}$ are exactly the curves defined over $\bar{\mathbb{Q}}$. The case $s\geq 3$ is also interesting and was also considered by Ihara, see [13]. Using a Möbious transformation we can assume that the set S consists of the elements $0,1,\lambda_1,\cdots,\lambda_{s-3},\infty$.

The Ihara representation can be explained in terms of Galois theory as follows: Consider the maximal pro-ℓ extension $\mathcal{M}$ of $\mathbb{Q}(t)$ unramified outside the set S. The Galois group $\mathrm{Gal}(\mathcal{M}/\bar{\mathbb{Q}}(t))$ is known to be isomorphic to the pro-ℓ free group $\mathfrak{F}_{s-1}$ of rank $s-1$. A selection of generators $x_1,\ldots,x_{s-1}$ corresponds to an isomorphism $i:\mathfrak{F}_{s-1}\rightarrow \mathrm{Gal}(\mathcal{M}/\bar{\mathbb{Q}}(t))$, such that $i(x_\nu)$ $(1\leq\nu\leq s)$ generates the inertia group of some place ξ_ν of $\mathcal{M}$ extending the place P_i of $\bar{\mathbb{Q}}(t)$, corresponding to the ith element of the set S.

We have the following exact sequence:

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Gal}(\mathscr{M}/\bar{\mathbb{Q}}(t)) \arrow[r] \arrow[d, "i"', "\cong"] & \mathrm{Gal}(\mathscr{M}/\mathbb{Q}(t)) \arrow[r] & \mathrm{Gal}(\bar{\mathbb{Q}}(t)/\mathbb{Q}(t)) \arrow[r] \arrow[d, "\cong"] & 1 \\ & \mathfrak{F}_{s-1} & & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & \end{tikzcd} \tag{21}$$

Every element $\rho\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ gives rise to an element $\rho^*\in\mathrm{Gal}(\mathcal{M}/\mathbb{Q}(t))$, which is unique up to an element of $\mathrm{Gal}(\mathcal{M}/\bar{\mathbb{Q}}(t))$, and so we obtain an isomorphism $x\mapsto\rho^*x\rho^{-1}\in\tilde{P}(\mathfrak{F}_{s-1})/\mathrm{Int}(\mathfrak{F}_{s-1})$, where

$$\tilde{P}(\mathfrak{F}_{s-1}):=\left\{\phi\in\mathrm{Aut}(\mathfrak{F}_{s-1})|\phi(x_i)\sim x_i^{N(\phi)}(1\leq i\leq s)\text{for some } N(\phi)\in\mathbb{Z}_\ell^*\right\},$$

and $\sim$ denotes the conjugation equivalence.

Y. Ihara [11, p.52], proved that the action of $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the topological generators of $\mathfrak{F}_{s-1}$ is in $\tilde{P}(\mathfrak{F}_{s-1})$ that is

$$\sigma(x_i)=w_i(\sigma)x_i^{N(\sigma)}w_i(\sigma)^{-1},$$

where $N(\sigma)\in\mathbb{Z}_\ell^*$ and $w_i(\sigma)\in\mathfrak{F}_{s-1}$ is the element defining the conjugation. In this way the outer Galois representation

$$\Phi_S:\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\rightarrow\tilde{P}(\mathfrak{F}_{s-1})/\mathrm{Int}(\mathfrak{F}_{s-1})$$

is defined.

By selecting the representatives of elements $\tilde{P}(\mathfrak{F}_{s-1})$ we can define the Ihara representation

$$\mathrm{Ih}_S:\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\rightarrow P(\mathfrak{F}_{s-1})\subset\mathrm{Aut}(\mathfrak{F}_{s-1}),$$

where

$$P(\mathfrak{F}_{s-1})=\left\{\phi\in\mathrm{Aut}(\mathfrak{F}_{s-1})\;\middle|\;\begin{array}{l}\phi(x_i)\sim x_i^{N(\phi)},\ (1\leq i\leq s-2),\phi(x_{s-1})\approx x_{s-1}^{N(\phi)}\\ \phi(x_s)=x_s^{N(\phi)},\ \text{for some } N(\phi)\in\mathbb{Z}_\ell^\times\end{array}\right\}, \tag{22}$$

where $\approx$ denotes conjugacy by an element of the subgroup of $\mathfrak{F}_s$ generated by the commutator $\mathfrak{F}_s'$ and $x_1,\dots,x_{s-3}$. The composition $N\circ \mathrm{Ih}_S$ equals the cyclotomic character $\chi_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^*$. For more details on these constructions see [**11**, prop.3 p.55], [**14**, prop. 2.2].

3.2. *Magnus embedding*

We will explain now the Magnus embedding following [**14**]. This embedding is given by the map

$$\mathfrak{F}_{s-1} \to \mathbb{Z}_\ell[[u_1,u_2,\dots,u_{s-1}]]_{\mathrm{nc}} \tag{23}$$

of $\mathfrak{F}_{s-1}$ into the 'non-commutative' formal power series algebra ($x_i \mapsto 1+u_i$ for $1\le i\le s-1$). Let $\mathfrak{H}$ denote the abelianization of $\mathfrak{F}_{s-1}$, and H the abelianization of F_{s-1}

$$H := \mathrm{gr}_1(F_{s-1}) = H_1(F_{s-1},\mathbb{Z}) \qquad \mathfrak{H} =: \mathrm{gr}_1(\mathfrak{F}_{s-1}) = H_1(\mathfrak{F}_{s-1},\mathbb{Z}_\ell) = H\otimes_{\mathbb{Z}}\mathbb{Z}_\ell.$$

The term gr_1 above has its origin on the graded Lie algebra corresponding to a (pro-ℓ) free group, see [**11**, p. 58] and [**17**]. Following [**14**], [**20**] we consider the tensor algebras

$$T(H) = \bigoplus_{n\ge 0} H^{\otimes n}, \qquad T(\mathfrak{H}) = \bigoplus_{n\ge 0} \mathfrak{H}^{\otimes n},$$

where $\mathfrak{H}^0 = \mathbb{Z}_\ell$ and $\mathfrak{H}^{\otimes n} := \mathfrak{H}\otimes_{\mathbb{Z}_\ell}\cdots\otimes_{\mathbb{Z}_\ell}\mathfrak{H}$ (n-times) (resp. $H^0=\mathbb{Z}$, $H^{\otimes n} = H\otimes_{\mathbb{Z}}\cdots\otimes_{\mathbb{Z}} H$)). If $u_0,\dots,u_{s-1}$ is a $\mathbb{Z}_\ell$ basis of the free $\mathbb{Z}_\ell$-module $\mathfrak{H}$, then

$$T(\mathfrak{H}) = \mathbb{Z}_\ell\langle u_1,\dots,u_{s-1}\rangle,$$

is the non-commutative polynomial algebra $\mathbb{Z}_\ell[[u_1,u_2,\dots,u_{s-1}]]_{\mathrm{nc}}$ over $\mathbb{Z}_\ell$, appearing in the right-hand side of equation (23).

We will denote by $\widehat{T}(\mathfrak{H})$ the completion of $T(\mathfrak{H})$ with respect to the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the two sided ideal generated by $u_1,\dots,u_{s-1}$ and ℓ. This algebra is the algebra of non-commutative formal power series over $\mathbb{Z}_\ell$ with variables $u_1,\dots,u_{s-1}$:

$$\widehat{T}(\mathfrak{H}) = \prod_{n\ge 0}\mathfrak{H}^{\otimes n} = \mathbb{Z}_\ell\langle\langle u_1,\dots,u_{s-1}\rangle\rangle.$$

Let $\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]$ be the complete group algebra of $\mathfrak{F}_{s-1}$ over $\mathbb{Z}_\ell$, and let

$$\varepsilon_{\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]} : \mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]] \to \mathbb{Z}_\ell$$

be the augmentation homomorphism. Denote by $I_{\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]} := \ker \varepsilon_{\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]}$ the augmentation ideal. The correspondence $x_i \mapsto 1+u_i$ for $1\le i\le s-1$ induces an isomorphism of topological $\mathbb{Z}_\ell$-algebras, the *pro-ℓ Magnus isomorphism.*

$$\Theta : \mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]] \xrightarrow{\cong} \widehat{T}(\mathfrak{H}).$$

EXAMPLE 3.1 The map Θ sends $\mathbb{Z}_\ell[\mathbb{Z}] = \mathbb{Z}_\ell[t,t^{-1}]$ to $\mathbb{Z}_\ell[[u]]$ by mapping $\Theta(t) = 1+u$ and $\Theta(t^{-1}) = (1+u)^{-1} = \sum_{i=0}^{\infty}(-1)^i u^i$. The image $\Theta(\mathbb{Z}_\ell[t,t^{-1}])$ is not onto $\hat{T}(\mathfrak{H})$, but $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]] = \mathbb{Z}_\ell[[\mathfrak{F}_1]]$ is mapped isomorphically to $\hat{T}(\mathfrak{H})$ by Θ.

For an multi-index $I=(i_1,\dots,i_{s-1})$ we set $u_I = u_{i_1}\cdots u_{i_{s-1}}$. The coefficient of u_I in $\Theta(\alpha)$ is called the Magnus coefficient of α and it is denoted by $\mu(I,\alpha)$, that is

$$\Theta(\alpha) = \varepsilon_{\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]}(\alpha) + \sum_{|I|\geq 1} \mu(I,\alpha) u_I.$$

For certain properties of the Magnus embedding and a fascinating application to ℓ-adic Milnor invariants we refer to [20, chap. 8], [14, sec. 3.2].

3.3. *Milnor invariants*

Consider the group $\mathfrak{H} := \mathfrak{F}_{s-1}^{\mathrm{ab}} = \mathfrak{F}_{s-1}/[\mathfrak{F}_{s-1},\mathfrak{F}_{s-1}]$. For $f\in\mathfrak{F}_{s-1}$ denote by $[f]$ its image in $\mathfrak{H}$. We will write $\mathfrak{H}$ as an additive $\mathbb{Z}_\ell$-module, which is generated by $[u_1],\dots,[u_{s-1}]$. Notice that the following relation holds:

$$[u_1]+\cdots+[u_{s-1}]+[u_s]=0.$$

Every automorphism $\phi\in\mathrm{Aut}(\mathfrak{F}_{s-1})$ gives rise to a linear automorphism of the free $\mathbb{Z}_\ell$-module $\mathfrak{H}$ and we will denote it by $[\phi]\in\mathrm{GL}(\mathfrak{H})$.

LEMMA 3.2 *The elements $w_i(\sigma)\in\mathfrak{F}_{s-1}$ can be selected uniquely so that*

(i) $\mathrm{Ih}_S(\sigma)(x_i) = w_i(\sigma) x_i^{\chi_\ell(\sigma)} w_i(\sigma)^{-1}$, *where χ_ℓ is the ℓ-cyclotomic character.*
(ii) *In the expression* $[w_i(\sigma)] = c_1^{(i)}[u_1]+\cdots+c_{s-1}^{(i)}[u_{s-1}]$, $c_j^{(i)}\in\mathbb{Z}_\ell$, *we have* $c_i^{(i)}=0$.

Proof. See [14, lemma 3.2]. □

For a multi-index $I=(i_1,\dots,i_n)$, $1\leq i_1,\dots,i_n\leq s-1$ the ℓ-adic Milnor number for $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is defined as the ℓ-adic Magnus coefficient of $w_i(\sigma)$, for $I'=(i_1,\dots,i_{n-1})$, that is

$$\mu(\sigma,I) := \mu(I', w_{i_n}(\sigma)),$$

see [14, equation 3.2]. It is clear that the selection of $w_i(\sigma)$ describes completely the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\mathfrak{F}_{s-1}$.

The commutative Magnus ring

In this article, we will consider actions of $\mathrm{Aut}(F_{s-1})$ or $\mathrm{Aut}(\mathfrak{F}_{s-1})$ on certain $\mathbb{Z}$-modules ($\mathbb{Z}_\ell$-modules) M defined as quotients of subgroups of the (pro-ℓ) free group. For example on F_{s-1}^{ab} or on $\mathfrak{F}_{s-1}^{\mathrm{ab}}$. We would like for M to be an abelian group (we also choose to write M additively) and we will entirely focus on the case $M=R/R'$, where $R<\mathfrak{F}_{s-1}$ (or $R<F_{s-1}$).

The group F_{s-1} (resp. $\mathfrak{F}_{s-1}$) acts on itself by conjugation. This action can be translated as an $T(H)$ (resp. $\hat{T}(\mathfrak{H})$) module structure on M, by setting

$$\alpha w \alpha^{-1} = \Theta(\alpha)\cdot w,$$

for $w\in F_{s-1}$ (resp. $w\in\mathfrak{F}_{s-1}$).

LEMMA 3.3 *If $M = R/R'$ and $[R, \mathfrak{F}'_{s-1}] \subset R'$ (resp. $[R, F'_{s-1}] \subset R'$) then the induced conjugation action on M satisfies*

$$ab \cdot m = ba \cdot m, \text{for all } a, b \in \hat{T}(\mathfrak{H}) \text{(resp. } T(H)\text{) and } m \in M. \tag{24}$$

Notice that the inclusion $\mathfrak{F}'_{s-1} \subset R$ (resp. $F'_{s-1} \subset R$) implies the desired condition for the action to commute.

Proof. For $a, b \in \mathfrak{F}_{s-1}$ and $r \in R$ we compute

$$abrb^{-1}a^{-1} = ba[a^{-1}, b^{-1}]r[a^{-1}, b^{-1}]^{-1}a^{-1}b^{-1}.$$

So a sufficient condition for equation (24) to hold is $[R, \mathfrak{F}'_{s-1}] \subset R'$ (resp. $[R, F'_{s-1}] \subset R'$). This condition is satisfied if $\mathfrak{F}'_{s-1} \subset R$ (resp. $F_{s-1} \subset R$) then equation (24) holds. □

Therefore, if the assumption of Lemma 3.3 holds, instead of considering the action of the non-commutative ring $\hat{T}(\mathfrak{H})$ (resp. $T(H)$) it makes sense to consider the action of the corresponding abelianized ring.

DEFINITION 3.4 Consider the commutative $\mathbb{Z}_\ell$-algebra of formal power series

$$\begin{aligned} \mathscr{A} &= \mathbb{Z}_\ell[[u_i : 1 \le i \le s]] / \langle (1+u_1)(1+u_2)\cdots(1+u_s) - 1 \rangle \\ &\cong \mathbb{Z}_\ell[[u_i : 1 \le i \le s-1]]. \end{aligned} \tag{25}$$

The algebra $\mathscr{A}$ is the symmetric algebra of $\mathfrak{H}$ over $\mathbb{Z}_\ell$, and there is a natural quotient map $\hat{T}(\mathfrak{H}) \to \mathrm{Sym}(\mathfrak{H}) = \mathscr{A}$.

REMARK 3.5 As we noticed already the action of $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ can be described in terms of the cocycles $w_1(\sigma), \ldots, w_{s-1}(\sigma)$. But then we can find elements

$$\varpi_1(\sigma) = \Theta(w_1(\sigma)), \ldots, \varpi_{s-1}(\sigma) = \Theta(w_{s-1}(\sigma)) \in \mathscr{A}$$

such that

$$\sigma(x_i) = \varpi_i(\sigma) \cdot x_i^{\chi_\ell(\sigma)}. \tag{26}$$

Therefore, in order to understand the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $M = \mathfrak{F}_{s-1}/\mathfrak{F}'_{s-1}$ it makes sense to consider the $\mathscr{A}$-module structure of M.

4. Alexander modules

4.1. *Definition and Crowell exact sequence*

We will use the notation of Section 1.1 for the groups $\bar{R}_0, R = \bar{R}_0/\Gamma \cap \bar{R}_0 \cong \bar{R}_0 \cdot \Gamma/\Gamma, \Gamma$. Consider the short exact sequence in equation (10). The group $G = \mathfrak{F}_{s-1}/\Gamma$ admits the presentation:

$$G = \overline{\langle x_1, \dots, x_s | x_1^{e_1} = \cdots = x_s^{e_s} = x_1 \cdots x_s = 1 \rangle}. \tag{27}$$

On the other hand since we assumed that $\mathfrak{F}'_{s-1} \subset \bar{R}_0$, (see Lemma 3.3) the group $\mathfrak{F}_{s-1}/\bar{R}_0 \cdot \Gamma$ is isomorphic to a quotient of the abelian group $\mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_{s-1}\mathbb{Z}$.

Recall that $\psi : \mathscr{F}_{s-1}/\Gamma \to \mathscr{F}_{s-1}/\bar{R}_0 \cdot \Gamma$. Set

$$\mathscr{A}^{\bar{R}_0,\Gamma} = \mathbb{Z}_\ell[[\mathscr{F}_{s-1}/\bar{R}_0 \cdot \Gamma]],$$

and define the map $\varepsilon_{\mathscr{A}^{\bar{R}_0,\Gamma}} : \mathbb{Z}_\ell[[\mathscr{F}_{s-1}/\bar{R}_0 \cdot \Gamma]] \to \mathbb{Z}_\ell$ to be the augmentation map corresponding functorially to the map $\mathscr{F}_{s-1}/\bar{R}_0 \cdot \Gamma \to \{1_{\mathscr{F}_{s-1}/\bar{R}_0 \cdot \Gamma}\}$, see [20, 8.3 p.99].

Consider also $\mathscr{A}_\psi^{\bar{R}_0,\Gamma}$ to be the *Alexander module*, a free $\mathbb{Z}_\ell$-module

$$\mathscr{A}_\psi^{\bar{R}_0,\Gamma} = \left(\bigoplus_{g \in \mathfrak{F}_{s-1}/\Gamma} \mathscr{A}^{\bar{R}_0,\Gamma} dg \right) / \left\langle d(g_1 g_2) - dg_1 - \psi(g_1) dg_2 : g_1, g_2 \in \mathfrak{F}_{s-1}/\Gamma \right\rangle_{\mathscr{A}^{\bar{R}_0,\Gamma}},$$

where the denominator in the above quotient denotes the $\mathscr{A}^{\bar{R}_0,\Gamma}$-module generated by the relations inside $\langle \dots \rangle_{\mathscr{A}^{\bar{R}_0,\Gamma}}$.

Define also the map $\theta_1 : R^{\mathrm{ab}} \to \mathscr{A}_\psi^{\bar{R}_0,\Gamma}$ given by

$$R^{\mathrm{ab}} \ni n \mapsto dn \tag{28}$$

and the map $\theta_2 : \mathscr{A}_\psi^{\bar{R}_0,\Gamma} \to \mathscr{A}^{\bar{R}_0,\Gamma}$ to be the homomorphism induced by

$$dg \mapsto \psi(g) - 1 \text{for } g \in G.$$

We will use the Crowell Exact sequence [20, sec. 9.2, sec. 9.4],

$$0 \to R^{\mathrm{ab}} = R/R' \xrightarrow{\theta_1} \mathscr{A}_\psi^{\bar{R}_0,\Gamma} \xrightarrow{\theta_2} \mathscr{A}^{\bar{R}_0,\Gamma} \xrightarrow{\varepsilon_{\mathscr{A}^{\bar{R}_0,\Gamma}}} \mathbb{Z}_\ell \to 0. \tag{29}$$

For a description of the Alexander module in terms of differentials in non-commutative algebras we refer to [16]. Notice that when the group $\mathfrak{F}_{s-1}/\bar{R}_0 \cdot \Gamma$ is finite then we will write $\mathbb{Z}_\ell[\mathfrak{F}_{s-1}/\bar{R}_0 \cdot \Gamma]$ instead of $\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}/\bar{R}_0 \cdot \Gamma]]$. In this case $\varepsilon_{\mathscr{A}^{\bar{R}_0,\Gamma}}$ is the augmentation map sending finite sums $\sum_{g \in \mathscr{F}_{s-1}/\bar{R}_0 \cdot \Gamma} a_g g$ to $\sum_{g \in \mathscr{F}_{s-1}/\bar{R}_0 \cdot \Gamma} a_g \in \mathbb{Z}_\ell$.

PROPOSITION 4.1 *The module $\mathscr{A}_{\psi}^{\bar{R}_0,\Gamma}$ admits the following free resolution as an $\mathscr{A}^{\bar{R}_0,\Gamma}$-module:*

$$\left(\mathscr{A}^{\bar{R}_0,\Gamma}\right)^{s+1} \xrightarrow{Q} \left(\mathscr{A}^{\bar{R}_0,\Gamma}\right)^{s} \longrightarrow \mathscr{A}_{\psi}^{\bar{R}_0,\Gamma} \longrightarrow 0, \tag{30}$$

*where s is the number of generators of G, given in equation (27) and $s+1$ is the number of relations. Let $\beta_1,\dots,\beta_{s+1} \in \mathscr{A}^{\bar{R}_0,\Gamma}$. The map Q is expressed in form of Fox derivatives [**3**, sec. 3.1], [**20**, chap. 8], as follows*

$$\begin{pmatrix}\beta_1\\ \vdots \\ \beta_{s+1}\end{pmatrix} \mapsto \begin{pmatrix} \psi\pi\left(\frac{\partial x_1^{e_1}}{\partial x_1}\right) & \psi\pi\left(\frac{\partial x_2^{e_2}}{\partial x_1}\right) & \cdots \psi\pi\left(\frac{\partial x_s^{e_s}}{\partial x_1}\right) & \psi\pi\left(\frac{\partial x_1\cdots x_s}{\partial x_1}\right) \\ \psi\pi\left(\frac{\partial x_1^{e_1}}{\partial x_2}\right) & \psi\pi\left(\frac{\partial x_2^{e_2}}{\partial x_2}\right) & \cdots \psi\pi\left(\frac{\partial x_s^{e_s}}{\partial x_2}\right) & \psi\pi\left(\frac{\partial x_1\cdots x_s}{\partial x_2}\right) \\ \vdots & \vdots & & \vdots \\ \psi\pi\left(\frac{\partial x_1^{e_1}}{\partial x_s}\right) & \psi\pi\left(\frac{\partial x_2^{e_2}}{\partial x_s}\right) & \cdots \psi\pi\left(\frac{\partial x_s^{e_s}}{\partial x_s}\right) & \psi\pi\left(\frac{\partial x_1\cdots x_s}{\partial x_s}\right) \end{pmatrix} \begin{pmatrix}\beta_1\\ \vdots \\ \beta_{s+1}\end{pmatrix},$$

where π is the natural epimorphism $\mathfrak{F}_s \to G$ defined by the presentation given in equation (27).

Proof. See [**20**, cor. 9.6]. □

If in equation (29) $\bar{R}_0 = \mathfrak{F}'_{s-1}$ and $\Gamma = \{1\}$, then $\mathscr{A}^{\mathfrak{F}'_{s-1},\{1\}} = \mathbb{Z}_\ell[[u_1,\dots,u_{s-1}]] = \mathscr{A}$, as defined in equation (25).

To summarize, for $H_0 = \mathfrak{F}_{s-1}/\bar{R}_0 \cdot \Gamma$, the Alexander module $\mathscr{A}_{\psi}^{R,\Gamma}$ can be computed as a cokernel of the function Q:

$$\mathscr{A}_{\psi}^{\bar{R}_0,\Gamma} = \text{coker } Q, \qquad \left(\mathscr{A}^{\bar{R}_0,\Gamma}\right)^{s+1} = \mathbb{Z}_\ell[[H_0]]^{s+1} \xrightarrow{Q} \mathbb{Z}_\ell[[H_0]]^{s} = \left(\mathscr{A}^{\bar{R}_0,\Gamma}\right)^{s}. \tag{31}$$

The exponents in the above formula reflect the fact that the group G is generated by $(s+1)$ relations over s free variables.

PROPOSITION 4.2 *If $\Gamma = \{1\}$ in equation (10) the Crowell exact sequence gives the Blanchfield-Lyndon exact sequence:*

$$0 \longrightarrow R^{\mathrm{ab}} \longrightarrow \left(\mathscr{A}^{\bar{R}_0,\{1\}}\right)^{s-1} \xrightarrow{d_1} \mathscr{A}^{\bar{R}_0,\{1\}} \xrightarrow{\varepsilon} \mathbb{Z}_\ell \longrightarrow 0. \tag{32}$$

Proof. See [**20**, p.118] for the discrete case and the pro-ℓ case follows similarly. □

Alexander modules for generalized Fermat curves

It is clear that the group $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ is generated as an $\mathscr{A}$-module by the elements $[x_i,x_j]$ for $1 \le i < j \le s-1$.

In what follows $\mathfrak{F}'_{s-1} = \bar{R}_0$ in the context of equation (10). The structure of $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ as an $\mathscr{A}$-module is expressed in terms of the Crowell exact sequence, see Section 4.1, related to the short exact sequence:

$$1 \to \mathfrak{F}'_{s-1} \to \mathfrak{F}_{s-1} \xrightarrow{\psi} \mathfrak{F}^{\mathrm{ab}}_{s-1} \to 1,$$
$$0 \to (\mathfrak{F}'_{s-1})^{\mathrm{ab}} = \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} \to \mathscr{A}_\psi \to \mathbb{Z}_\ell[[u_1,\ldots,u_{s-1}]] \to \mathbb{Z}_\ell \to 0,$$

where $\mathscr{A}_\psi = \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\{1\}}$ is the Alexander module and

$$\mathscr{A} = \mathscr{A}^{\mathfrak{F}'_{s-1},\{1\}} = \mathbb{Z}_\ell[[u_1,\ldots,u_{s-1}]].$$

EXAMPLE 4.3 Assume that in equation (10) the group $H_0 = (\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}$ and consider the open generalized Fermat curve with fundamental group $\bar{R}_0 = \mathfrak{F}'_{s-1}$. Let $\mathfrak{R}_k = \Gamma$ be the smallest closed normal subgroup of $\mathfrak{F}_{s-1}$ generated by $x_1^{\ell^k},\ldots,x_{s-1}^{\ell^k}$. The group $G = \mathfrak{F}_{s-1,k} = \mathfrak{F}_{s-1}/\mathfrak{R}_k$ admits the presentation:

$$\mathfrak{F}_{s-1,k} = \left\langle x_1,\ldots,x_s | x_1^{\ell^k} = \cdots = x_s^{\ell^k} = x_1\cdots x_s = 1\right\rangle.$$

Denote the images of the elements x_i in H_0 by $\bar{x}_i$. It is clear that $\mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ is a free $\mathbb{Z}_\ell$-module of rank

$$\mathrm{rank}_{\mathbb{Z}_\ell}(\mathrm{coker}\ Q) = s(\ell^k)^{(s-1)} - \mathrm{rank}_{\mathbb{Z}_\ell}(Q).$$

Observe that $\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} \cong \mathbb{Z}_\ell[H_0]$ is a free $\mathbb{Z}_\ell$-module of rank $(\ell^k)^{s-1}$. By induction we can prove

$$\frac{\partial x_i^{\ell^k}}{\partial x_j} = \delta_{ij}(1 + x_i + x_i^2 + \cdots + x_i^{\ell^k-1}) \text{ for } 1 \le j \le s$$

$$\frac{\partial x_1x_2\cdots x_s}{\partial x_j} = x_1\cdots x_{j-1} \tag{33}$$

Set $\Sigma_i = 1 + \bar{x}_i + \cdots + \bar{x}_i^{\ell^k-1}$. The map Q in equation (31) is given by the matrix on the left of the following equation [20, cor. 9.6]

$$\begin{pmatrix} \Sigma_1 & 0 & \cdots & 0 & 1 \\ 0 & \Sigma_2 & \ddots & \vdots & \bar{x}_1 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \Sigma_s & \bar{x}_1\bar{x}_2\cdots\bar{x}_{s-1} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{s+1} \end{pmatrix} = \begin{pmatrix} \Sigma_1\beta_1 + \beta_{s+1} \\ \Sigma_2\beta_2 + \bar{x}_1\beta_{s+1} \\ \vdots \\ \Sigma_s\beta_s + \bar{x}_1\cdots\bar{x}_{s-1}\beta_{s+1} \end{pmatrix} \tag{34}$$

where $\beta_i \in \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ for $1 \le i \le s$. Observe that

$$\Sigma_i\bar{x}_i^\nu = \Sigma_i \text{ for all } 0 \le \nu \le \ell^k - 1.$$

LEMMA 4.4 *For $1 \leq i \leq s-1$ the following equation holds*

$$\Sigma_i \cdot \mathbb{Z}_\ell[H_0] = \Sigma_i \cdot \mathbb{Z}_\ell \left[\bigoplus_{\substack{\nu=1 \\ \nu \neq i}}^{s-1} \mathbb{Z}/\ell^k\mathbb{Z} \right].$$

On the other hand the module $\Sigma_s \mathbb{Z}_\ell[H_0]$ contains all elements invariant under the action of the product $\bar{x}_1 \cdots \bar{x}_{s-1}$ and is a free $\mathbb{Z}_\ell$-submodule of $\mathbb{Z}_\ell[H_0]$.

Proof. Write

$$\mathbb{Z}_\ell[H_0] = \mathbb{Z}_\ell \left[\bigoplus_{\nu=1}^{s-1} \mathbb{Z}/\ell^k\mathbb{Z} \right] = \bigotimes_{\nu=1}^{s-1} \mathbb{Z}_\ell \left[\mathbb{Z}/\ell^k\mathbb{Z} \right].$$

Therefore the multiplication by Σ_i gives rise to the tensor product

$$\left(\bigotimes_{\nu=1}^{i-1} \mathbb{Z}_\ell \left[\mathbb{Z}/\ell^k\mathbb{Z} \right] \right) \bigotimes \left(\Sigma_i \mathbb{Z}_\ell \left[\mathbb{Z}/\ell^k\mathbb{Z} \right] \right) \bigotimes \left(\bigotimes_{\nu=i+1}^{s-1} \mathbb{Z}_\ell \left[\mathbb{Z}/\ell^k\mathbb{Z} \right] \right) =$$

$$\left(\bigotimes_{\nu=1}^{i-1} \mathbb{Z}_\ell \left[\mathbb{Z}/\ell^k\mathbb{Z} \right] \right) \bigotimes \left(\Sigma_i \mathbb{Z}_\ell \right) \bigotimes \left(\bigotimes_{\nu=i+1}^{s-1} \mathbb{Z}_\ell \left[\mathbb{Z}/\ell^k\mathbb{Z} \right] \right)$$

and the desired result follows.

For the case of $\Sigma_s \mathbb{Z}_\ell[H_0]$ invariance under the action of $\bar{x}_s = \bar{x}_1^{-1} \cdots \bar{x}_{s-1}^{-1}$ is clear. The rank computation follows by changing the basis of H_0 from $\bar{x}_1, \ldots, \bar{x}_{s-1}$ to the basis $\bar{x}_2, \ldots, \bar{x}_s$ and arguing as before. □

The image of the map Q equals to the space generated by elements

$$\begin{pmatrix} \Sigma_1 \beta_1 \\ \Sigma_2 \beta_2 \\ \vdots \\ \Sigma_s \beta_s \end{pmatrix} + \begin{pmatrix} 1 \\ \bar{x}_1 \\ \vdots \\ \bar{x}_1 \cdots \bar{x}_{s-1} \end{pmatrix} \beta_{s+1}.$$

For different choices of $\beta_1, \ldots, \beta_s \in \mathscr{A}^{\mathfrak{F}'_{s-1}, \mathfrak{R}_k}$ the first summand forms a free $\mathbb{Z}_\ell$-module of rank $s(\ell^k)^{s-2}$ and the second summand is a free $\mathbb{Z}_\ell$-module of rank $(\ell^k)^{s-1}$. Also their intersection is just $\mathbb{Z}_\ell$.

Indeed, if for some $\beta_1, \ldots, \beta_{s+1} \in \mathbb{Z}_\ell[H_0]$ we have

$$\beta_{s+1}(1, \bar{x}_1, \ldots, \bar{x}_1 \cdots \bar{x}_{s-1}) = (\Sigma_1 \beta_1, \ldots, \Sigma_s \beta_s)$$

then by comparison of the first coordinates we see that β_{s+1} is invariant under the action of $\bar{x}_1$. So comparison of second coordinate gives us that $\bar{x}_1 \beta_{s+1} = \beta_{s+1}$ is invariant under the action of $\bar{x}_2$. By continuing this way we see that β_{s+1} is invariant under the whole group H_0, that is β_{s+1} belongs to the rank one $\mathbb{Z}_\ell$-module generated by $\Sigma_1\Sigma_2\cdots\Sigma_s$. In this way we see that

LEMMA 4.5

$$\mathrm{Im}(Q)=\left(\bigoplus_{\nu=1}^{s}\Sigma_i\mathbb{Z}_\ell[H_0]\right)\bigoplus\mathbb{Z}_\ell[H_0]/\mathbb{Z}_\ell\Sigma_1\cdots\Sigma_s. \tag{35}$$

Also

$$\mathrm{rank}_{\mathbb{Z}_\ell}Q=s(\ell^k)^{s-2}+(\ell^k)^{s-1}-1.$$

We would like to compute the cokernel of Q as a $\mathbb{Z}_\ell[H_0]$-module. This computation lies within the theory of integral representation theory. This seems a very difficult problem since a complete set of representatives of the classes of indecomposable modules for groups of the form $(\mathbb{Z}/\ell^k\mathbb{Z})^t$ seems to be known only for $t=1$ and $k=1,2$, see [22]. In this article, we will not consider the problem in the integral representation setting and instead we will consider the simpler problem of determination of the H_0-action on the space $H_1(C_{k,s-1},\mathbb{F})$, where $\mathbb{F}$ is a field which contains $\mathbb{Z}_\ell$ and the ℓ^k-roots of unity. Let us fix a primitive ℓ^k root of unity ζ_{ℓ^k}. Set

$$\mathscr{I}:=(\mathbb{Z}\cap[0,\ell^k))^{s-1}.$$

If $\mathbf{i}\in\mathscr{I}$, set $i_s:=i_1+\cdots+i_{s-1}$. Now define

$$z(\mathbf{i}):=\#\{j:1\le j\le s \text{and } i_j\equiv 0 \mod \ell^k\}. \tag{36}$$

Now set

$$c_{\mathbf{i}}:=\begin{cases}1+z(\mathbf{i}) & z(\mathbf{i})<s\\ z(\mathbf{i}) & z(\mathbf{i})=s\end{cases} \tag{37}$$

For an element $\mathbf{i}=(i_1,\ldots,i_{s-1})\in\mathbb{N}^{s-1}$ we define a character $\chi_{\mathbf{i}}$ on H_0 by

$$\chi_{\mathbf{i}}\left(\bar{x}_1^{\nu_1},\ldots,\bar{x}_{s-1}^{\nu_{s-1}}\right)=\zeta^{\sum_{\mu=1}^{s-1}\nu_\mu i_\mu}.$$

We have the following

LEMMA 4.6 *We have the following decomposition*

$$\mathrm{Im}(Q)\otimes\mathbb{F}=\bigotimes_{\mathbf{i}\in\mathscr{I}}\mathbb{F}c_{\mathbf{i}}\chi_{\mathbf{i}},$$

where $c_{\mathbf{i}}\in\mathbb{N}$ is the multiplicity of the corresponding character.

Proof. Consider the decomposition given in Lemma 4.5. The module $\mathbb{F}[H_0]$ contains once every possible character, therefore

$$\mathbb{F}[H_0] = \bigotimes_{\mathbf{i}\in\mathscr{I}} \mathbb{F}\chi_{\mathbf{i}}$$

On the other hand the modules $\Sigma_i\mathbb{F}[H_0]$ for $0 \le i \le s-1$ are trivially acted on by elements $\bar{x}_i$. This means that

$$\Sigma_i\mathbb{F}[H_0] = \bigoplus_{\substack{\mathbf{i}\in\mathscr{I} \\ \mathbf{i}=(\nu_1,\ldots,\nu_{i-1},0,\nu_{i+1},\ldots,\nu_{s-1})}} \mathbb{F}\chi_{\mathbf{i}}.$$

Also the module $\Sigma_s\mathbb{F}[H_0]$ contains elements which are invariant by elements of the group generated by $\bar{x}_1\cdots\bar{x}_{s-1}$, since $\bar{x}_s = \bar{x}_1^{-1}\cdots\bar{x}_{s-1}^{-1}$. This means that all characters which appear in the decomposition of $\Sigma_s\mathbb{F}[H_0]$ on $\bar{x}_1^{\nu}\cdots\bar{x}_{s-1}^{\nu}$ should give 1, which is equivalent to

$$\chi_{\mathbf{i}}(\bar{x}_1^{\nu}\cdots\bar{x}_{s-1}^{\nu}) = \zeta_{\ell^k}^{\nu\sum_{\mu=1}^{s-1} i_\mu} = 1 \Rightarrow \sum_{\mu=1}^{s-1} i_\mu \equiv 0 \mod \ell^k.$$

Therefore, the decomposition into characters is given by

$$\Sigma_s\mathbb{F}[H_0] = \bigoplus_{\substack{\mathbf{i}\in\mathscr{I} \\ i_1,\ldots,i_{s-1}=0 \\ i_1+\cdots+i_{s-1}=0}} \mathbb{F}\chi_{\mathbf{i}}.$$

Given a character $\chi_{\mathbf{i}}$ we now count the number of times it appears. It appears on the summands $\Sigma_j\mathbb{F}[H_0]$ for $0 \le j \le s-1$ when $i_j = 0$ and in the summand $\Sigma_s\mathbb{F}[H_0]$ when $i_1 + \cdots + i_{s-1} \equiv 0 \mod \ell^k$. Also it appears on $\mathbb{F}[H_0]/\Sigma_1\cdots\Sigma_s$ only if $(i_1,\ldots,i_{s-1}) \neq (0,\ldots,0)$. □

LEMMA 4.7 *We have*

$$\operatorname{rank}_{\mathbb{Z}_\ell} \mathscr{A}_{\psi}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} = (s-1)(\ell^k)^{s-1} - s(\ell^k)^{s-2} + 1 \tag{38}$$

and

$$\mathscr{A}_{\psi}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} \otimes \mathbb{F} = \bigoplus_{\mathbf{i}\in\mathscr{I}} (s - c_{\mathbf{i}})\chi_{\mathbf{i}}. \tag{39}$$

Proof. The rank computation follows since $\mathscr{A}_{\psi}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ is the cokernel of Q, so

$$\operatorname{rank}_{\mathbb{Z}_\ell} \mathscr{A}_{\psi}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} = s(\ell^k)^{s-1} - s(\ell^k)^{s-2} - (\ell^k)^{s-1} + 1 = (s-1)(\ell^k)^{s-1} - s(\ell^k)^{s-2} + 1.$$

Similarly the decomposition in equation (39) follows by the decomposition of $\mathbb{F}[H_0]$ into characters. □

PROPOSITION 4.8

$$H_1(C_{\ell^k,s-1},\mathbb{F}) = \bigoplus_{\mathbf{i}\in\mathscr{I}} \mathbb{F}C(\mathbf{i})\chi_{\mathbf{i}}$$

where

$$C(\mathbf{i}) = \begin{cases} s-c_{\mathbf{i}}-1 = s-z(\mathbf{i})-2 & \textit{if } \mathbf{i}\neq(0,\dots,0) \\ s-c_{\mathbf{i}} = s-z(\mathbf{i}) & \textit{if } \mathbf{i}=(0,\dots,0) \end{cases} \tag{40}$$

Moreover

$$\mathrm{rank}_{\mathbb{Z}_\ell} H_1(C_{\ell^k,s-1},\mathbb{Z}_\ell) = (s-1)\left(\ell^k\right)^{s-1} + 2 - s\left(\ell^k\right)^{s-2}.$$

Proof. From the exact sequence given in equation (29) and the rank computation given in equation (38) in example 4.3 we have:

$$\begin{aligned} \mathrm{rank}\left(R_{\ell^k}/(\mathfrak{R}_k\cap R_{\ell^k})\right)^{\mathrm{ab}} &= \mathrm{rank}_{\mathbb{Z}_\ell}\mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} - \mathrm{rank}_{\mathbb{Z}_\ell}\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}+1 \\ &= (s-2)(\ell^k)^{s-1}+2-s(\ell^k)^{s-2}. \end{aligned} \tag{41}$$

The above abelianization corresponds to the $\mathbb{Z}_\ell$-homology of the generalized Fermat curves of type $(k,s-1)$. The above rank coincides with the genus computation given in equation (19).

Let us write

$$H_1(C_{\ell^k,s-1},\mathbb{F}) = \bigoplus_{\mathbf{i}\in\mathscr{I}} \mathbb{F}C(\mathbf{i})\chi_{\mathbf{i}}$$

for some integers $C(\mathbf{i})$. By Lemma 4.7 and the short exact sequence given in (29) we obtain equation (40). □

REMARK 4.9 For the case of classical Fermat curves we have $s=3$. The character $\chi_{0,0,0}$ has $z(0,0,0)=3$ and $C_{0,0,0}=0$. Similarly the characters $\chi_{0,i,i},\chi_{i,0,i}$ for $1\leq i\leq \ell^k-1$ and the character $\chi_{i,j,i+j}$ with $i+j\equiv 0 \mod \ell^k$ have $z(0,i,i)=z(i,0,i)=z(i,j,i+j)=1$ so their contribution is $C(0,i,i)=C(i,0,i)=C(i,j,i+j)=0$. All other characters $\chi_{(i,j,i+j)}$ have $z(i,j,i+j)=0$ and their contribution is $C(i,j,i+j)=1$. In this way we arrive to the same result as in equation (2).

EXAMPLE 4.10 Let us now compute $\mathscr{A}_\psi^{R_{\ell^k},\mathfrak{R}_k}$ and R_{ℓ^k} is the the pro-ℓ completion of the group generated by

$$\{x_1^i x_j x_1^{-i-1} : 2\leq j\leq s-1, 0\leq i\leq \ell^k-2\}\cup\{x_1^{\ell^k-1}x_j : 1\leq j\leq s-1\}.$$

This group corresponds to the open cyclic cover of order ℓ^k of $\mathbb{P}^1$ ramified fully above s-points of the projective line, see [**15**, lemma 11]. Let $\mathfrak{R}_k=\Gamma$ be the smallest closed normal subgroup of $\mathfrak{F}_{s-1}$ generated by $x_1^{\ell^k},\dots,x_{s-1}^{\ell^k}$. We have the short exact sequence

$$1 \to R_{\ell^k}/R_{\ell^k}\cap \mathfrak{R}_k \to \mathfrak{F}_{s-1}/\mathfrak{R}_k \to \mathbb{Z}/\ell^k\mathbb{Z} \to 0.$$

We compute $\mathscr{A}^{R_{\ell^k},\mathfrak{R}_k} = \mathbb{Z}_\ell[\mathbb{Z}/\ell^k\mathbb{Z}]$, which is an $\mathbb{Z}_\ell$-module of rank ℓ^k. On the other hand observe that the $\mathbb{Z}_\ell$-module $\mathscr{A}_\psi^{R_{\ell^k},\mathfrak{R}_k}$ is given by exactly the same cokernel as the module $\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$. The only difference is that $\mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ is a $\mathbb{Z}_\ell[(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}]$-module while $\mathscr{A}^{R_{\ell^k},\mathfrak{R}_k}$ is a $\mathbb{Z}_\ell[\mathbb{Z}/\ell^k\mathbb{Z}]$-module.
So following exactly the same method as in example 4.3 we conclude that

$$\operatorname{rank}_{\mathbb{Z}_\ell}\mathscr{A}_\psi^{R_{\ell^k},\mathfrak{R}_k} = s\cdot\ell^k - s - \ell^k + 1 = (s-1)\ell^k - s + 1.$$

Also we compute the rank

$$\operatorname{rank}\big(R_{\ell^k}/(R_{\ell^k}\cap\mathfrak{R}_k)\big)^{\mathrm{ab}} = \operatorname{rank}_{\mathbb{Z}_\ell}\mathscr{A}_\psi^{R_{\ell^k},\mathfrak{R}_k} - \operatorname{rank}_{\mathbb{Z}_\ell}\mathscr{A}^{R_{\ell^k},\mathfrak{R}_k} + 1 = (s-2)(\ell^k-1).$$

The module $\big(R_{\ell^k}/R_{\ell^k}\cap\mathfrak{R}_k\big)^{\mathrm{ab}}$ corresponds to the $\mathbb{Z}_\ell$-homology of the above curves, corresponding to R_{ℓ^k} and its rank is twice the genus of the curve, in accordance with the genus formula given in [**15**, equation 21].

5. Galois modules in terms of the Magnus embedding

5.1. *The group $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ as an $\mathscr{A}$-module*

In this section we will study the $\mathscr{A}$-module structure of $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$. This is the arithmetic analogon of the Gassner representation, as Ihara points out in [**12**]. This consideration leads to the Galois representation of the Tate module, see Section 5.1.4. Finally in Section 5.1.5 we will study the passage from the Gassner representation to the Burau by seeing the generalized Fermat curve as a cover of the projective line.

Application to generalized Fermat curves
Consider the the smallest closed normal subgroup $\mathfrak{R}_k$ of $\mathfrak{F}_{s-1}$ containing all $x_i^{\ell^k}$ for $1\le i\le s-1$. Define also

$$\mathfrak{F}_{s-1,k} = \mathfrak{F}_{s-1}/\mathfrak{R}_k.$$

Set $\bar\lambda = \{0,1,\infty,\lambda_1,\dots,\lambda_{s-3}\}$ and let $\mathcal{M}$ be the maximum pro-ℓ extension of $K=\bar{\mathbb{k}}(t)$ unramified outside the set of points $\bar\lambda$. Consider the function field of the generalized Fermat curves

$$K_k := K\left(t^{\frac{1}{\ell^k}}, (t-1)^{1/\ell^k}, (t-\lambda_1)^{1/\ell^k},\dots,(t-\lambda_{s-3})^{1/\ell^k}\right).$$

Let K_k^{ur} and K_k^{unrab} be the maximal unramified and maximal abelian unramified extensions of K_k, respectively. Also let K' be the maximum abelian unramified extension of K and K'' be the maximum abelian unramified extension of K'. By covering space theory, the fields K', K'' correspond to the groups $\mathfrak{F}'_{s-1}$ and $\mathfrak{F}''_{s-1}$, respectively. The function field K_k corresponds to the group $\mathfrak{F}'_{s-1}\mathfrak{R}_k$ and is equal to the function field of the generalized Fermat curve.

The aim of this section is the following characterization of the maximal unramified abelian extension K_k^{unrab} of the function field K_k of the generalized Fermat curve. This is a generalization of a similar construction by Ihara for the classical Fermat curves, see [**11**, sec. II, p. 63]

THEOREM 5.1 *We have that* $\mathrm{Gal}(K_k^{\text{unrab}}/K_k) \cong \mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k}$.

Indeed, we have

$$K' = \bigcup_k K_k, \qquad K' \cap K_k^{\text{ur}} = K_k$$
$$K'' = \bigcup_k K_k^{\text{unrab}}, \qquad K'' \cap K_k^{\text{ur}} = K_k^{\text{unrab}}$$

The Galois correspondence is given as follows:

$$\begin{tikzcd}
& \mathscr{M} \arrow[dl, dash] \arrow[dr, dash] & \\
K'' \arrow[d, dash] \arrow[drr, dash] & & K_k^{\mathrm{ur}} \arrow[d, dash] \\
K' \arrow[dr, dash] & & K_k^{\mathrm{unrab}} \arrow[dl, dash, "\mathrm{Gal}(K_k^{\mathrm{unrab}}/K_k)"] \\
& K_k \arrow[d, dash] & \\
& K &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \{1\} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathfrak{F}''_{s-1} \arrow[d, dash] \arrow[drr, dash] & & \mathfrak{R}_k \arrow[d, dash] \\
\mathfrak{F}'_{s-1} \arrow[dr, dash] & & \mathfrak{F}''_{s-1}\mathfrak{R}_k \arrow[dl, dash] \\
& \mathfrak{F}'_{s-1}\mathfrak{R}_k \arrow[d, dash] & \\
& \mathfrak{F}_{s-1} &
\end{tikzcd}$$

Using standard isomorphism theorems in group theory (see also [**16**, sec. 1.2]) and the definitions we see

$$\mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} \cong \mathfrak{F}'_{s-1}/\left(\mathfrak{F}'_{s-1} \cap \mathfrak{F}''_{s-1}\mathfrak{R}_k\right) \cong \mathfrak{F}'_{s-1}\mathfrak{R}_k/\mathfrak{F}''_{s-1}\mathfrak{R}_k \cong \mathrm{Gal}(K_k^{\text{unrab}}/K_k) \tag{42}$$

is an abelian group, a free $\mathbb{Z}_\ell$-module of rank $2g_{(\ell^k,s-1)}$, where $g_{(\ell^k,s-1)}$ is the genus of the generalized Fermat curve, $F_{\ell^k,s-1}$ so that

$$2g_{(\ell^k,s-1)} = 2 + \ell^{k(s-2)}((s-2)(\ell^k-1)-2). \tag{43}$$

Observe that according to equation (5) we have

$$\mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} \cong H_1(C_{\ell^k,s-1},\mathbb{Z}_\ell).$$

The last genus computation also follows from the following proposition which identifies unramified $\mathbb{Z}/\ell^k\mathbb{Z}$-extensions of a curve X with the group of ℓ^k-torsion points of the Jacobian $J(X)$.

PROPOSITION 5.2 *Let Y be a complete nonsingular algebraic curve defined over a field of characteristic prime to ℓ. The étale Galois covers of Y with Galois group $\mathbb{Z}/\ell^k\mathbb{Z}$ are classified by the étale cohomology group $H^1_{\mathrm{et}}(Y,\mathbb{Z}/\ell^k\mathbb{Z})$ which is equal to the group of ℓ^k-torsion points of* $\mathrm{Pic}(Y)$.

Proof. See [9, Ex. 2.7], [23, sec. 19]. □

Crowell sequence for generalized Fermat curves
We will use the notation of Section 4 where $\bar{R}_0 = \mathfrak{F}'_{s-1}$ and $\Gamma = \mathfrak{R}_k$, $R = \mathfrak{F}'_{s-1}/(\mathfrak{R}_k \cap \mathfrak{F}_{s-1}) \cong \mathfrak{F}'_{s-1}\cdot \mathfrak{R}_k/\mathfrak{R}_k$. Notice also that the commutator identity for quotients imply that $\mathfrak{F}'_{s-1,k} = \mathfrak{F}'_{s-1}\cdot\mathfrak{R}_k/\mathfrak{R}_k = R$. Here we use the presentation $\mathfrak{F}_{s-1} \cong \mathfrak{F}_s/\langle x_1\cdots x_s\rangle$. Let $H_k = (\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}$. We have the short exact sequence

$$1 \to \mathfrak{F}'_{s-1,k} = (\mathfrak{F}_{s-1}/\mathfrak{R}_k)' \to \mathfrak{F}_{s-1,k} = \mathfrak{F}_{s-1}/\mathfrak{R}_k \xrightarrow{\psi} H_k \to 1$$

The Crowell Exact sequence see equation (29) and [20, chap. 9] gives us

$$0 \longrightarrow (\mathfrak{F}'_{s-1,k})^{\mathrm{ab}} = \mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} \xrightarrow{\theta_1} \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} \xrightarrow{\theta_2} \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} \xrightarrow{\varepsilon_{\mathscr{A}_k}} \mathbb{Z}_\ell \longrightarrow 0, \tag{44}$$

where

$$\begin{aligned} \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} &= \mathbb{Z}_\ell[H_k] = \mathbb{Z}_\ell[(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}], \\ \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} &= \operatorname{coker} Q, \qquad \mathbb{Z}_\ell[H_k]^{s+1} \xrightarrow{Q} \mathbb{Z}_\ell[H_k]^s \end{aligned} \tag{45}$$

and $\varepsilon_{\mathscr{A}_k}$ is the augmentation map. The Alexander module for $\mathfrak{F}_{s-1}/\mathfrak{R}_k$ was computed on example 4.3. Notice that $\mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$ and the Crowell sequence know the genus of the generalized Fermat curve, see equation (41).

Representation theory on generalized Fermat curves
Let G be one of the groups $G_{\mathbb{Q}}, B_{s-1}$ or B_s. These are representations on the free $\mathbb{Z}_\ell$-modules

$$\rho_k : G \to \mathrm{GL}(H_1(C_{\ell^k,s-1},\mathbb{Z}_\ell)).$$

Let us now combine the two Crowell sequences together.

$$\begin{array}{ccccccccc} 0 \longrightarrow (\mathfrak{F}'_{s-1})^{\mathrm{ab}} = \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} & \longrightarrow & \mathscr{A}^{s-1} & \xrightarrow{d_1} & \mathscr{A} & \longrightarrow & \mathbb{Z}_\ell & \to 0 \\ \downarrow \phi & & \downarrow \phi_3 & & \downarrow \omega & & \| & \\ 0 \to (\mathfrak{F}'_{s-1,k})^{\mathrm{ab}} = \mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} & \xrightarrow{\theta} & \left(\mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}\right) & \xrightarrow{d_{1,k}} & \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} & \to & \mathbb{Z}_\ell & \to 0 \end{array} \tag{46}$$

The top equation is the Blanchfield–Lyndon exact sequence in equation (32). For the vertical arrows: ω is the map induced functorialy by the natural group homomorphism $\mathbb{Z}^{s-1} \to (\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}$:

$$\mathscr{A} = k[[\mathbb{Z}^{s-1}]] \xrightarrow{\omega} \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} = k[\mathbb{Z}/\ell^k\mathbb{Z}]$$

The map ϕ_3 is defined as follows: we begin from the short exact sequence:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathfrak{F}'_{s-1} & \longrightarrow & \mathfrak{F}_{s-1} & \longrightarrow & \dfrac{\mathfrak{F}_{s-1}}{\mathfrak{F}'_{s-1}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \dfrac{\mathfrak{F}'_{s-1}\cdot\mathfrak{R}_k}{\mathfrak{R}_k} & \longrightarrow & \dfrac{\mathfrak{F}_{s-1}}{\mathfrak{R}_k} & \longrightarrow & \dfrac{\mathfrak{F}_{s-1}}{\mathfrak{F}'_{s-1}\cdot\mathfrak{R}_k} & \longrightarrow & 1 \end{array}$$

In the first row, we consider the group $\mathfrak{F}_{s-1}$ as the quotient of the free pro-ℓ group is s generators modulo the relation $x_1x_2\cdots x_s = 1$. In the second row, the group $\mathfrak{F}_{s-1}/\mathfrak{R}_k$ is considered as the quotient of the free group in s-generators modulo the relation $x_1x_2\cdots x_s = 1$ and the r-relations generating $\mathfrak{R}_k$.

$$\begin{array}{ccccccc} \mathscr{A} & \xrightarrow{Q_1} & \mathscr{A}^s & \xrightarrow{\psi_1} & \mathscr{A}^{s-1} & \longrightarrow & 0 \\ & & \downarrow \phi_2 & & \downarrow \phi_3 & & \\ \left(\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}\right)^{r+1} & \xrightarrow{Q_2} & \left(\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}\right)^s & \xrightarrow{\psi_2} & \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} & \longrightarrow & 0 \end{array} \tag{47}$$

where Q_1, Q_2 are the maps appearing in Proposition 4.1. In particular the map Q_1 sends

$$\mathscr{A} \ni \beta \mapsto \beta \cdot (1, x_1, x_1x_2, \ldots, x_1x_2\cdots x_{s-1}).$$

The vertical map ϕ_2 is the reduction modulo Γ and it is onto. The image $\phi_3(a)$ for $a \in \mathscr{A}^{s-1}$ is defined by selecting $b \in \mathscr{A}^s$ such that $\psi_1(b) = a$, and then $\phi_3(a) = \psi_2 \circ \phi_2(b)$ as seen in the diagram below:

$$\begin{tikzcd} b \arrow[r, "\psi_1"] \arrow[d, "\phi_2"] & a \arrow[d, "\phi_3"] \\ \phi_2(b) \arrow[r, "\psi_2"] & \phi_3(a) = \psi_2 \circ \phi_2(b) \end{tikzcd}$$

This definition is independent from the selection of b.

Finally, the map ϕ is naturally defined

$$\begin{aligned} \frac{\mathfrak{F}'_{s-1}}{\mathfrak{F}''_{s-1}} &\longrightarrow \frac{\mathfrak{F}'_{s-1} \cdot \mathfrak{R}_k}{\mathfrak{F}''_{s-1} \cdot \mathfrak{R}_k} \\ a\,\mathfrak{F}''_{s-1} &\longmapsto a\,\mathfrak{F}''_{s-1} \cdot \mathfrak{R}_k \end{aligned}$$

For an explanation of these two combined sequences in terms of the 'cotangent sequence' and a functorial point of view we refer to [16].

LEMMA 5.3 *The group* $\mathfrak{R}_k$ *is invariant under the action of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Proof. For every generator $x_i^{\ell^k}$ and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we have

$$\sigma(x_i^{\ell^k}) = \sigma(x_i)^{\ell^k} = \left(w_i(\sigma) x_i^{N(\sigma)} w_i(\sigma)^{-1}\right)^{\ell^k}.$$

Let a_n be a sequence of integers such that $a_n \to N(\sigma)$. We have

$$\left(w_i(\sigma) x_i^{a_n} w_i(\sigma)^{-1}\right)^{\ell^k} = \left(w_i(\sigma) x_i^{\ell^k} w_i(\sigma^{-1})\right)^{a_n}.$$

The later element is in $\mathfrak{R}_k$ since by definition $\mathfrak{R}_k$ is normal in $\mathfrak{F}_{s-1}$. The limit $a_n \to N(\sigma)$ is in $\mathfrak{R}_k$ since this group is by definition closed. The result follows. □

It is clear that $\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} = \mathbb{Z}_\ell[H_0]$ can be considered through the vertical map ω as an $\mathscr{A}$-module and inherits an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by ω, by writing $\alpha \in \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ as the image of an element $\alpha' \in \mathscr{A}$, that is $\alpha = \omega(\alpha')$ and defining

$$\sigma(\alpha) = \sigma(\omega(\alpha')) = \omega(\sigma\alpha').$$

By Lemma 5.3 this action is well defined. On the other hand an element $\phi = \overline{[x_i, x_j]} \in \mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} = \mathfrak{F}'_{s-1}/(\mathfrak{F}'_{s-1} \cap \mathfrak{F}_{s-1''}\mathfrak{R}_k)$ is sent to the element

$$\theta(\phi) = d[\bar{x}_i, \bar{x}_j] = -\bar{u}_j dx_i + \bar{u}_i dx_j \in \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}.$$

The module $\mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ is an $\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$-module, described by the sequence given in equation (30) and by the matrix Q given in equation (34) and is naturally acted on by the absolute Galois group. Observe also that the map θ sends the class of $[x_i, x_j]$ to $d[x_i, x_j] = u_i dx_j - u_j dx_i$, and this element is annihilated by the elements $\Sigma_i = \sum_{\nu=0}^{\ell^k-1} \bar{x}_i^\nu$ for $1 \le i \le s$. We can see this by direct computations or by observing that in $\mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ we have

$$\Sigma_i \cdot \beta_i = \beta_{s+1}\bar{x}_1 \cdots \bar{x}_{i-1}.$$

and the image $\theta[x_i, x_j]$ has the $s+1$ coordinate $\beta_{s+1} = 0$. The above observation generalizes the definition of ideal $\mathfrak{a}_n$ in equation (8) in the article of Ihara, [**11**].

Therefore,

$$H_1(C_{\ell^k,s-1}, \mathbb{Z}_\ell) \cong \theta((\mathfrak{F}'_{s-1,k})^{\mathrm{ab}}) \subset \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$$

is acted on by $\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}/\langle \Sigma_i : 1 \le i \le s \rangle$, and $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on it in terms of the action given in equation (26). Indeed, $\mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$ is identified with the cokernel of the matrix Q, that is an element in $\mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$ is the class of an s-tuple which is sent to

$$\sigma : (\beta_1, \ldots, \beta_s) + \mathrm{Im}(Q) \longmapsto (\sigma\beta_1, \ldots, \sigma\beta_s) + \mathrm{Im}(Q).$$

This action is well defined since the space $\mathrm{Im}(Q)$ is left invariant under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Indeed, in the commutative ring $\mathscr{A}^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$, the action $\sigma(\bar{x}_i) = \bar{x}_i^{N(\sigma)}$ so $\sigma(\Sigma_i) = \Sigma_i$, and invariance follows by equation (35).

On Jacobian variety of generalized Fermat curves

Consider the ℓ-adic Tate module $T(\mathrm{Jac}(C_{\ell^k,s-1}))$ of the Jacobian of the generalized Fermat curves $C_{\ell^k,s-1}$:

$$T(\mathrm{Jac}(C_{\ell^k,s-1})) = H_1(C_{\ell^k,s-1}, \mathbb{Z}) \otimes \mathbb{Z}_\ell = \frac{\mathfrak{F}'_{s-1,k}}{\mathfrak{F}''_{s-1,k}}.$$

Following Ihara we consider

$$\mathbb{T} := \lim_{\substack{\leftarrow \\ k}} T(\mathrm{Jac}(C_{\ell^k,s-1})) = \lim_{\substack{\leftarrow \\ k}} \frac{\mathfrak{F}'_{s-1,k}}{\mathfrak{F}''_{s-1,k}}, \tag{48}$$

where the inverse limit is considered with respect to the maps $T(\mathrm{Jac}(C_{\ell^{k+1},s-1})) \to T(\mathrm{Jac}(C_{\ell^k,s-1}))$, which is induced by the map

$$(x_0, \dots, x_{s-1}) \mapsto (x_0^\ell, \dots, x_{s-1}^\ell).$$

Let $\bar{C}_{\ell^k,s-1} = C_{\ell^k,s-1} \otimes_{\mathrm{Spec}\mathbb{Q}} \mathrm{Spec}\bar{\mathbb{Q}}$. Consider also the inverse limit

$$\lim_{\substack{\leftarrow \\ k}} \mathrm{Gal}(\bar{C}_{\ell^k,s-1}/\mathbb{P}^1_{\bar{\mathbb{Q}}}) = \lim_{\substack{\leftarrow \\ k}} (\mathbb{Z}/\ell^k\mathbb{Z})^{s-1} = \mathbb{Z}_\ell^{s-1}.$$

Therefore

$$\lim_{\substack{\leftarrow \\ k}} \mathbb{Z}_\ell[\mathrm{Gal}(\bar{C}_{\ell^k,s-1}/\mathbb{P}^1_{\bar{\mathbb{Q}}})] \cong \mathscr{A}$$

and $\mathbb{T}$ can be considered as an $\mathscr{A}$-module. Using equation (46) we obtain

$$\frac{\mathfrak{F}'_{s-1}}{\mathfrak{F}''_{s-1}} \cong \mathbb{T}. \tag{49}$$

See [1, sec. 13] for the explicit isomorphism in the case of Fermat curves.

The geometric interpretation of this construction is that for fixed s-number of points we can consider all generalized Fermat curves seen as $(\mathbb{Z}/\ell^k\mathbb{Z})^s$ ramified covers of the projective line, for $k \in \mathbb{N}$. In this way we obtain a curve C_s, which is a $\mathbb{Z}_\ell^{s-1}$ cover of the projective line. The Burau representation and the pro-ℓ Burau representation can be defined in terms of such an infinite Galois cover, see [15].

This construction leads to the definition of a subspace $\mathbb{T}^{\mathrm{prim}} \subset \mathbb{T}$ which is a free $\mathscr{A}$-module of rank $s-2$. Observe that the submodule of a free module is not necessarily a free module and $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ is not necessarily free. For example in the following short exact sequence

$$0 \to (\mathfrak{F}'_{s-1})^{\mathrm{ab}} = \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} \to \mathscr{A}^{s-1} \xrightarrow{d_1} \mathscr{A} \to \mathbb{Z}_\ell \to 0$$

$\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ is contained in the free $\mathscr{A}$-module $\mathscr{A}^{s-1}$, but is free itself. The $\mathscr{A}$-module $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ contains the free module of rank $s-2$ (see [21], [14, Th. 5.39])

$$\mathbb{T}^{\mathrm{prim}} := \left\{ (\lambda_j u_1 \cdots \hat{u}_j \cdots u_{s-1})_{j=1,\dots,s-1} : \lambda_j \in \mathscr{A}, \sum_{j=1}^{s-1} \lambda_j = 0 \right\}.$$

Set $w = u_1 \cdots u_{s-1}$. Using equation (49) we see that a basis of $(\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1})^{\text{prim}}$ is given by

$$v_1 = \left(-\frac{w}{u_1}, \frac{w}{u_2}, 0, \ldots, 0\right), \ldots, v_{s-2} = \left(0, \ldots, 0, -\frac{w}{u_{s-2}}, \frac{w}{u_{s-1}}\right).$$

In the case of Fermat curves, that is $s=2$ we have that $(\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1})^{\text{prim}} = \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ and $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ is a free $\mathscr{A}$-module, generated by $[x_1, x_2]$. Notice that the injective map $d : \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} \xrightarrow{d} \mathscr{A}^{s-1}$ is given by sending a representative

$$\begin{aligned}
[x_i, x_j] \to d([x_i, x_j]) &= (1 - x_j)dx_i - (1 - x_i)dx_j \\
&= -u_j \cdot dx_i + u_i \cdot dx_j.
\end{aligned}$$

PROPOSITION 5.4 *Let G be either $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ or the braid group B_s. An element in $g \in G$ induces an action on both $\mathbb{T}$ and $\mathbb{T}^{\text{prim}}$. In particular the subspace $\mathbb{T}^{\text{prim}}$ is a free $\mathscr{A}$-module. Thus we have a cocycle map*

$$\begin{aligned}
\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) &\to \mathrm{GL}_{s-2}(\mathscr{A}) \\
\sigma &\longmapsto (a_{ij}(\sigma))
\end{aligned}$$

This cocycle can be given in terms of the matrix

$$\sigma(w_{ij} d[x_i, x_j]) = \sum_{\nu < \mu} a_{\nu\mu}(\sigma) w_{\nu\mu} d[x_\nu, x_\mu].$$

REMARK 5.5 In [14, sec. 5.3] this cocycle is identified as the Gassner representation and the relation with the classical definition in terms of Fox derivatives see [3, chap. 3] is studied. The Gassner cocycles when restricted to a certain subgroup $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})[1] \subset \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ give rise to a representation instead of cocycle, see [14].

From generalized Fermat curves to cyclic covers of $\mathbb{P}^1$

We will now relate the Crowell sequences for the generalized Fermat curves and cyclic covers $\bar{Y}_{\ell^k}$ of the projective line as they were defined in [15] using the results of [16]. This will provide the relation of the Gassner representation to the Burau representation. The analogon of the Burau representation was defined in [14, p.675] by reduction of the Gassner representation. Here we also consider this reduction with respect to the curve definition of the Burau representation.

We have the following diagram of ramified coverings of curves

$$\begin{tikzcd}
C_{\ell^k, s-1} \arrow[dr, "(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}"] & & \bar{Y}_{\ell^k} \arrow[dl, "\mathbb{Z}/\ell^k\mathbb{Z}"'] \\
& \mathbb{P}^1 - \{0, 1, \infty, \lambda_1, \ldots, \lambda_{s-3}\} &
\end{tikzcd}$$

The passage for the corresponding representations from $C_{\ell^k, s-1}$ to $\bar{Y}_{\ell^k}$ corresponds to the passage from the Gassner representation to the Burau representation, see [3, prop. 3.12] and [14, sec. 5].

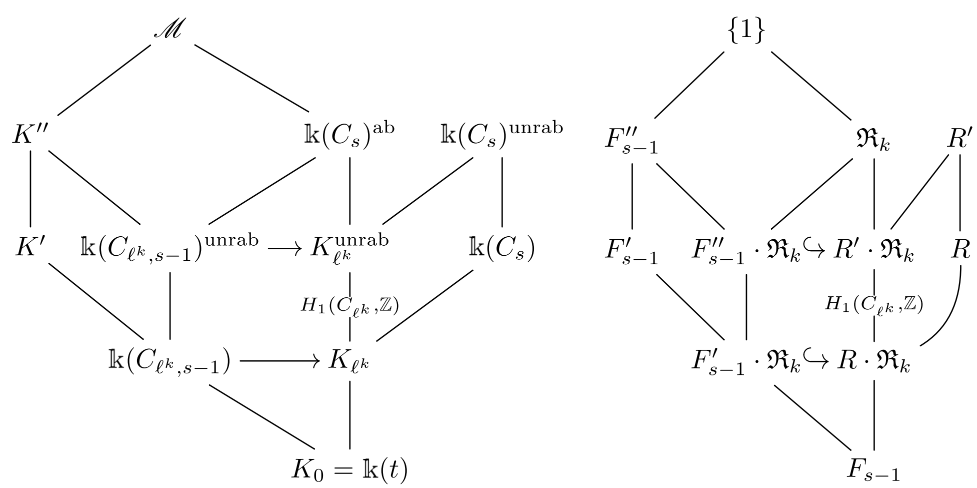
Set $\bar{R}_{\ell^k}$ be the fundamental group of the closed curve $\bar{Y}_{\ell^k}$, which can be computed using Schreier lemma, see [15]:

$$\bar{R}_{\ell^k} = R_{\ell^k}/\Gamma = \left\langle (x_2 x_1^{-1})^{x_1^{\nu}}, \ldots, (x_{s-1} x_1^{-1})^{x_1^{\nu}} : 0 \leq \nu < \ell^k - 1 \right\rangle.$$

Let also C_s be the $\mathbb{Z}$ cover of the projective line ramified over s-points. Let R be its fundamental group, which by [15] equals

$$R = \left\langle \left(x_j x_1^{-1}\right)^{x_1^{\nu}} : \nu \in \mathbb{Z}, 2 \leq j \leq s-1 \right\rangle.$$

The fixed field of $R/\mathfrak{R}_k$ is the function field K_{ℓ^k} of the curve $\bar{Y}_{\ell^k}$ and $\Bbbk(C_s)$ is the function field of the curve C_s. The group R' corresponds to the maximal unramified abelian extension $\Bbbk(C_s)^{\mathrm{ur}}$ of $\Bbbk(C_s)$ while $\mathfrak{R}_k$ corresponds to the maximal unramified extension $\Bbbk(C_s)^{\mathrm{unrab}}$. The group $R' \cdot \mathfrak{R}_k$ corresponds to the maximal abelian unramified $K_{\ell^k}^{\mathrm{unrab}}$ extension of K_{ℓ^k}. The groups $F'_{s-1} \cdot \mathfrak{R}_k$ and $F''_{s-1} \cdot \mathfrak{R}_k$ correspond to the generalized Fermat curve $C_{\ell^k,s-1}$ and the maximal unramified extension $C_{\ell^k,s-1}^{\mathrm{unrab}}$. The groups F'_{s-1}, F''_{s-1} correspond to the maximal abelian unramified extension of K_0 and the maximal abelian unramified extension of K', respectively.



As in the case of generalized Fermat curves we can form the limit

$$\mathbb{T}_R := \varprojlim_k (\mathrm{Jac}(\bar{Y}_{\ell^k})) = \varprojlim_k (R_{\ell^k}/\mathfrak{R}_k)^{\mathrm{ab}} = R^{\mathrm{ab}} = H^1(\bar{Y}_{\ell^k}, \mathbb{Z}_\ell).$$

We now compare the Crowell sequences for the cyclic covers and the Fermat covers, following [16]

$$
\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & R^{\mathrm{ab}} = \mathbb{T}_R \arrow[r] & \mathscr{A}_{\psi}^{R,\mathfrak{R}_k} \arrow[r] \arrow[u] & \mathscr{A}^{R,\mathfrak{R}_k} = \mathbb{Z}_\ell[[\mathbb{Z}/\ell^k\mathbb{Z}]] \arrow[r] \arrow[u] & \mathbb{Z}_\ell \arrow[r] & 0 \\
0 \arrow[r] & (\mathfrak{F}'_{s-1})^{\mathrm{ab}} = \mathbb{T} \arrow[r] \arrow[u, "\phi_1"] & \mathscr{A}_{\psi}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} \arrow[r] \arrow[u, "\phi_2"] & \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} = \mathbb{Z}_\ell[[(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}]] \arrow[r] \arrow[u, "\phi_3"] & \mathbb{Z}_\ell \arrow[r] \arrow[u] & 0
\end{tikzcd}
$$

The map $\phi_1 : \mathbb{T} \to \mathbb{T}_R$ on Tate modules is given by the first vertical map. The action module structure is given by the commutating diagram

$$
\begin{tikzcd}
\mathscr{A} \times \mathbb{T} \arrow[r] \arrow[d] & \mathbb{T} \arrow[d, "\phi_1"] \\
\mathscr{A}^{R,\mathfrak{R}_k} \times \mathbb{T}_R \arrow[r] & \mathbb{T}_R
\end{tikzcd}
$$

where the horizontal maps are the module actions and the first vertical map sends $(a,t) \mapsto (\phi_3(a), \phi_1(t))$. The map ϕ_3 is the reduction identifying the variables $x_1, x_2, \ldots, x_{s-1}$. Let G be as in Proposition 5.4. The map ϕ_2 is defined in a similar way as in equation (47). In particular from the reduction $\mathbb{T} \to \mathbb{T}_R$ we obtain the diagram

$$
\begin{tikzcd}
G \arrow[r] \arrow[dr] & \mathrm{GL}_{s-2}(\mathscr{A}) \arrow[d] \\
 & \mathrm{GL}_{s-2}(\mathscr{A}^{R,\mathfrak{R}_k})
\end{tikzcd}
$$

corresponding to the free parts of $\mathbb{T}$ and $\mathbb{T}_R$, respectively.

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