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On the tangent space of the deformation functor of curves with automorphisms

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We provide a method to compute the dimension of the tangent space to the global infinitesimal deformation functor of a curve together with a subgroup of the group of automorphisms. The computational techniques we developed are applied to several examples including Fermat curves, p-cyclic covers of the affine line and to Lehr–Matignon curves.

The aim of this paper is the study of equivariant equicharacteristic infinitesimal deformations of a curve X of genus g, admitting a group of automorphisms. This paper is the result of my attempt to understand the work of J. Bertin and A. Mézard [2000] and of G. Cornelissen and F. Kato [2003].

Let X be a smooth projective algebraic curve, defined over an algebraically closed field of characteristic $p \geq 0$. The infinitesimal deformations of the curve X, without considering compatibility with the group action, correspond to directions on the vector space $H^1(X, \mathcal{T}_X)$ which constitutes the tangent space to the deformation functor of the curve X [Harris and Morrison 1998]. All elements in $H^1(X, \mathcal{T}_X)$ give rise to unobstructed deformations, since X is one-dimensional and the second cohomology vanishes.

In the study of deformations together with the action of a subgroup of the automorphism group, a new deformation functor can be defined. The tangent space of this functor is given by Grothendieck's [1957] equivariant cohomology group $H^1(X, G, \mathcal{T}_X)$; see [Bertin and Mézard 2000, 3.1]. In this case the wild ramification points contribute to the dimension of the tangent space of the deformation functor and also pose several lifting obstructions, related to the theory of deformations of Galois representations.

Bertin and Ménard [2000], after proving a local-global principle, focused on infinitesimal deformations in the case G is cyclic of order p and considered liftings to characteristic zero, while Cornelissen and Kato [2003] considered the case of deformations of ordinary curves without putting any other condition on the automorphism group. The ramification groups of automorphism groups acting on

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ordinary curves have a special ramification filtration, i.e., the p-part of every ramification group is an elementary abelian group, and this makes the computation possible, since elementary abelian group extensions are given explicitly in terms of Artin–Schreier extensions.

In this paper we consider an arbitrary curve X with automorphism group G. By the theory of Galois groups of local fields, the ramification group at every wild ramified point can break to a sequence of extensions of elementary abelian groups [Serre 1979, IV]. We will use this decomposition together with the spectral sequence of Lyndon–Hochschild–Serre in order to reduce the computation to one involving elementary abelian groups.

We are working over an algebraically closed field of positive characterstic and for the sake of simplicity we assume that $p \geq 5$.

The dimension of the tangent space of the deformation functor depends on the group structure of the extensions that appear in the decomposition series of the ramification groups at wild ramified points. We are able to give lower and upper bounds of the dimension of the tangent space of the deformation functor.

In particular, if the decomposition group G_P at a wild ramified point P is the semidirect product of an elementary abelian group with a cyclic group such that there is only a lower jump at the i-th position in the ramification filtration, then we are able to compute exactly the dimension of the local contribution $H^1(G_P, \mathcal{T}_{\mathcal{O}})$ (Proposition 2.9 and Section 3.1).

We begin our exposition in Section 1 by surveying some of the known deformation theory. Next we proceed to the most difficult task, namely the computation of the tangent space of the local deformation functor, by employing the low terms sequence stemming from the Lyndon–Hochschild–Serre spectral sequence.

The dimension of equivariant deformations that are locally trivial, i.e., the dimension of $H^1(X/G, \pi_*^G(\mathcal{T}_X))$ is computed in Section 3. The computational techniques we developed are applied in the case of Fermat curves that are known to have large automorphism group, in the case of p-covers of $\mathbb{P}^1(k)$ and in the case of Lehr–Matignon curves. Moreover, we are able to recover the results of [Cornelissen and Kato 2003] concerning deformations of ordinary curves. Finally, we try to compare our result with the results of R. Pries [2002; 2004] concerning the computation of unobstructed deformations of wild ramified actions on curves.

1. Some deformation theory

There is nothing original in this section, but for the sake of completeness, we present some of the tools we will need for our study. This part is essentially a review of [Bertin and Mézard 2000; Cornelissen and Kato 2003; Mazur 1997].

Let k be an algebraic closed field of characteristic $p \geq 0$. We consider the category $\mathscr{C}$ of local Artin k-algebras with residue field k.

Let X be a nonsingular projective curve defined over the field k, and let G be a fixed subgroup of the automorphism group of X. We will denote by (X, G) the couple of the curve X together with the group G.

A deformation of the couple (X, G) over the local Artin ring A is a proper, smooth family of curves

$$\mathscr{X} \to \mathrm{Spec}(A)$$

parametrized by the base scheme $\mathrm{Spec}(A)$, together with a group homomorphism $G \to \mathrm{Aut}_A(\mathscr{X})$ such that there is a G-equivariant isomorphism ϕ from the fibre over the closed point of A to the original curve X:

$$\phi : \mathscr{X} \otimes_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \to X.$$

Two deformations $\mathscr{X}_1, \mathscr{X}_2$ are considered to be equivalent if there is a G-equivariant isomorphism ψ, making the diagram

$$\begin{tikzcd} \mathscr{X}_1 \arrow[rr, "\psi"] \arrow[dr] & & \mathscr{X}_2 \arrow[dl] \\ & \mathrm{Spec}\, A & \end{tikzcd}$$

commutative. The global deformation functor is defined as

$$D_{\mathrm{gl}} : \mathscr{C} \to \mathrm{Sets}, \mathrm{A} \mapsto \left\{ \begin{array}{l} \text{Equivalence classes} \\ \text{of deformations of} \\ \text{couples } (X, G) \text{ over } A \end{array} \right\}$$

Let D be a functor such that $D(k)$ is a single element. If $k[\epsilon]$ is the ring of dual numbers, then the Zariski tangent space t_D of the functor is defined by $t_D := D(k[\epsilon])$. If the functor D satisfies the "Tangent Space Hypothesis", i.e., when the mapping

$$h : D(k[\epsilon] \times_k k[\epsilon]) \to D(k[\epsilon]) \times D(k[\epsilon])$$

is an isomorphism, then the $D(k[\epsilon])$ admits the structure of a k-vector space [Mazur 1997, p. 272]. The tangent space hypothesis is contained in the hypothesis (H3) of Schlessinger, which holds for all the functors in this paper, since all the functors admit versal deformation rings [Schlessinger 1968; Bertin and Mézard 2000, Section 2].

The tangent space $t_{D_{gl}} := D_{\mathrm{gl}}(k[\epsilon])$ of the global deformation functor is expressed in terms of Grothendieck's equivariant cohomology [1957], which combines the construction of group cohomology and sheaf cohomology.

We recall quickly the definition of equivariant cohomology theory: We consider the covering map $\pi : X \to Y = X/G$. For every sheaf F on X we denote by $\pi_*^G(F)$ the sheaf

$$V \mapsto \Gamma(\pi^{-1}(V), F)^G, \text{ where } V \text{ is an open set of } Y.$$

The category of $(G, \mathcal{O}_X)$-modules is the category of $\mathcal{O}_X$-modules with an additional G-module structure. We can define two left exact functors from the category of $(G, \mathcal{O}_X)$-modules, namely

$$\pi_*^G \text{ and } \Gamma^G(X, \cdot),$$

where $\Gamma^G(X, F) = \Gamma(X, F)^G$. The derived functors $R^q\pi_*^G(X, \cdot)$ of the first functor are sheaves of modules on Y, and the derived functors of the second are groups $H^q(X, G, F) = R^q\Gamma^G(X, F)$.

Theorem 1.1 [Bertin and Mézard 2000]. *Let $\mathcal{T}_X$ be the tangent sheaf on the curve X. The tangent space $t_{D_{\text{gl}}}$ to the global deformation functor, is given in terms of equivariant cohomology as $t_{D_{\text{gl}}} = H^1(X, G, \mathcal{T}_X)$. Moreover the following sequence is exact*:

$$0 \to H^1(X/G, \pi_*^G(\mathcal{T}_X)) \to H^1(X, G, \mathcal{T}_X) \to H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) \to 0. \quad (1)$$

For a local ring $k[[t]]$ we define the local tangent space $\mathcal{T}_{\mathcal{O}}$, as the $k[[t]]$-module of k-derivations. The module $\mathcal{T}_{\mathcal{O}} := k[[t]]\, d/dt$, where $\delta = d/dt$ is the derivation such that $\delta(t) = 1$. If G is a subgroup of $\text{Aut}(k[[t]])$, then G acts on $\mathcal{T}_{\mathcal{O}}$ in terms of the adjoint representation. Moreover by [Cornelissen and Kato 2003] there is a bijection

$$D_\rho(k[\epsilon]) \xrightarrow{\cong} H^1(G, \mathcal{T}_{\mathcal{O}}).$$

In order to describe the tangent space of the local deformation space we will compute first the space of tangential liftings, i.e., the space $H^1(G, \mathcal{T}_{\mathcal{O}})$.

This problem was solved in [Bertin and Mézard 2000] when G is a cyclic group of order p, and in [Cornelissen and Kato 2003] when the original curve is ordinary.

We will apply the classification of groups that can appear as Galois groups of local fields in order to reduce the problem to elementary abelian group case.

1.1. ***Splitting the branch locus.*** Let P be a wild ramified point on the special fibre X, and let $\sigma \in G_j(P)$ where $G_j(P)$ denotes the j-ramification group at P. Assume that we can deform the special fibre to a deformation $\mathcal{X} \to A$, where A is a complete local discrete valued ring that is a k-algebra. Denote by m_A the maximal ideal of A and assume that $A/m_A = k$. Moreover assume that σ acts fibrewise on $\mathcal{X}$. We will follow [Green and Matignon 1998] in expressing the expansion

$$\sigma(T) - T = f_j(T)u(T),$$

where $f_j(T)=\sum_{\nu=0}^{j} a_i T^i$ ($a_i \in m_A$ for $\nu=0,\dots,j-1$, $a_j=1$) is a distinguished Weierstrass polynomial of degree j [Bourbaki 1989, VII, 8, Proposition 6] and $u(T)$ is a unit of $A[[T]]$. The reduction of the polynomial f_j modulo m_A gives the automorphism σ on $G_j(P)$ but σ when lifted on $\mathscr{X}$ has in general more than one fixed points, since $f_j(T)$ might be a reducible polynomial. If $f_j(T)$, gives rise to only one horizontal branch divisor then we say that the corresponding deformation does not split the branch locus.

Moreover, if we reduce $\mathscr{X} \times_A \operatorname{Spec} A/m_A^2$ we obtain an infinitesimal extension that gives rise to a cohomology class in $H^1(G(P), \mathscr{T}_{\mathbb{O}})$ by [Cornelissen and Kato 2003, Proposition 2.3].

On the other hand cohomology classes in $H^1(X/G, \pi_*^G(\mathscr{T}_X))$ induce trivial deformations on formal neighbourhoods of the branch point P [Bertin and Mézard 2000, 3.3.1] and do not split the branch points. In the special case of ordinary curves, the distinction of deformations that do or do not split the branch points does not occur since the polynomials f_j are of degree 1.

1.2. ***Description of the ramification group.*** The finite groups that appear as Galois groups of a local field $k((t))$, where k is algebraically closed of characteristic p are known [Serre 1979].

Let L/K be a Galois extension of a local field K with Galois group G. We consider the ramification filtration of G,

$$G = G_0 \supset G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supset G_{n+1} = \{1\}. \tag{2}$$

The quotient G_0/G_1 is a cyclic group of order prime to the characteristic, G_1 is p-group and for $i \geq 1$ the quotients G_i/G_{i+1} are elementary abelian p-groups. If a curve is ordinary, we know by [Nakajima 1987] that the ramification filtration is short, i.e., $G_2=\{1\}$, and this gives that G_1 is an elementary abelian group.

We are interested in the ramification filtrations of the decomposition groups acting on the completed local field at wild ramified points. To study this question, we introduce some notation: Consider the set of jumps of the ramification filtration $1=t_f<t_{f-1}<\cdots<t_1=n$, such that

$$G_1=\ldots=G_{t_f} > G_{t_f+1}=\ldots=G_{t_{f-1}} > G_{t_{f-1}+1} \geq \ldots \geq G_{t_1}=G_n > \{1\}, \tag{3}$$

i.e., $G_{t_i} > G_{t_i+1}$. For this sequence it is known that $t_\mu \equiv t_\nu \bmod p$ for all $\mu, \nu \in \{1,\dots,f\}$; see [Serre 1979, Proposition 10, p. 70].

1.3. ***Lyndon–Hochschild–Serre spectral sequences.*** Hochschild and Serre [1953] considered the following problem: Given the short exact sequence of groups

$$1 \to H \to G \to G/H \to 1, \tag{4}$$

and a G-module A, how are the cohomology groups $H^i(G, A)$, $H^i(H, A)$ and $H^i(G/H, A^H)$ related? They gave an answer to the above problem in terms of a spectral sequence. For small values of i this spectral sequence gives us the low-degree exact sequence

$$0 \longrightarrow H^1(G/H, A^H) \xrightarrow{\text{inf}} H^1(G, A) \xrightarrow{\text{res}} H^1(H, A)^{G/H} \\ \xrightarrow{\text{tg}} H^2(G/H, A^H) \xrightarrow{\text{inf}} H^2(G, H), \quad (5)$$

where res, tg, inf denote the restriction, transgression and inflation maps.

Lemma 1.2. *Let H be a normal subgroup of G, and let A be a G-module. The group G/H acts on the cohomology group $H^1(H, A)$ in terms of the conjugation action given explicitly on the level of 1-cocycles as follows: Let $\bar{\sigma} = \sigma H \in G/H$. The cocycle*

$$\begin{aligned} d : H &\to A \\ x &\mapsto d(x) \end{aligned}$$

is sent by the conjugation action to the cocycle

$$\begin{aligned} d^{\bar{\sigma}} : H &\to A \\ x &\mapsto \sigma d(\sigma^{-1} x \sigma), \end{aligned}$$

where $\sigma \in G$ is a representative of $\bar{\sigma}$.

Proof. This explicit description of the conjugation action on the level of cocycles is given in Propostion 2-5-1 (p. 79) of [Weiss 1969]. The action is well defined by Corollary 2-3-2 of the same reference. □

Our strategy is to use Equation (5) in order to reduce the problem of computation of $H^1(G, \mathcal{T}_{\mathcal{O}})$ to an easier computation involving only elementary abelian groups.

Lemma 1.3. *Let A be a k-module, where k is a field of characteristic p. For the cohomology groups we have $H^1(G_0, A) = H^1(G_1, A)^{G_0/G_1}$.*

Proof. Consider the short exact sequence

$$0 \to G_1 \to G_0 \to G_0/G_1 \to 0.$$

Equation (5) implies the sequence

$$0 \to H^1(G_0/G_1, A^{G_1}) \to H^1(G_0, A) \to H^1(G_1, A)^{G_0/G_1} \to H^2(G_0/G_1, A^{G_1}).$$

But the order of G_0/G_1 is not divisible by p, and is an invertible element in the k-module A. Thus the groups $H^1(G_0/G_1, A^{G_1})$ and $H^2(G_0/G_1, A^{G_1})$ vanish and the result follows from [Weibel 1994, Corollary 6.59]. □

Lemma 1.4. *If $G = G_i$, $H = G_{i+1}$ are groups in the ramification filtration of the decomposition group at some wild ramified point, and $i \geq 1$ then the conjugation action of G on H is trivial.*

Proof. Let L/K denote a wild ramified extension of local fields with Galois group G, let $\mathcal{O}_L$ denote the ring of integers of L and let m_L be the maximal ideal of $\mathcal{O}_L$. Moreover we will denote by L^* the group of units of the field L. We can define [Serre 1979, Proposition 7, p. 67; Proposition 9, p. 69] injections

$$\theta_0 : \frac{G_0}{G_1} \to L^* \quad \text{and} \quad \theta_i : \frac{G_i}{G_{i+1}} \to \frac{m_L^i}{m_L^{i+1}},$$

with the property

$$\theta_i(\sigma\tau\sigma^{-1}) = \theta_0(\sigma)^i \theta_i(\tau) \quad \text{for all } \sigma \in G_0 \text{ and } \tau \in G_i/G_{i+1}.$$

If $\sigma \in G_{t_j} \subset G_1$ then $\theta_0(\sigma) = 1$ and since θ_i is an injection, the above equation implies that $\sigma\tau\sigma^{-1} = \tau$. Therefore, the conjugation action of an element τ in G_i/G_{i+1} on G_j is trivial, and the result follows. □

1.4. ***Description of the transgression map.*** In this section we will try to determine the kernel of the transgression map. The definition of the transgression map given in (5) is not suitable for computations. We will give an alternative description, following [Neukirch et al. 2000].

Let A be a k-algebra that is acted on by G so that the G action is compatible with the operations on A. Let $\bar{A}$ be the set $\mathrm{Map}(G, A)$ of set-theoretic maps of the finite group G to the G-module A. The set $\bar{A}$ can be seen as a G-module by defining the action $f^g(\tau) = gf(g^{-1}\tau)$ for all $g, \tau \in G$. We observe that $\bar{A}$ is projective. The submodule A can be seen as the subset of constant functions. Notice that the induced action of G on the submodule A seen as the submodule of constant functions of $\bar{A}$ coincides with the initial action of G on A. We consider the short exact sequence of G-modules

$$0 \to A \to \bar{A} \to A_1 \to 0. \tag{6}$$

Let $H \lhd G$. By applying the functor of H-invariants to the short exact sequence (6) we obtain the long exact sequence

$$0 \to A^H \to \bar{A}^H \to A_1^H \overset{\psi}{\to} H^1(H, A) \to H^1(H, \bar{A}) = 0, \tag{7}$$

where the last cohomology group is zero since $\bar{A}$ is projective.

We split this four-term sequence into two short exact sequences

$$\begin{gathered} 0 \to A^H \to \bar{A}^H \to B \to 0, \\ 0 \to B \to A_1^H \xrightarrow{\psi} H^1(H, A) \to 0, \end{gathered} \tag{8}$$

where we have defined $B = \ker\psi$. Now we apply the G/H-invariant functor to these two short exact sequences in order to obtain

$$H^i(G/H, B) = H^{i+1}(G/H, A^H),$$

$$0 \to B^{G/H} \to A_1^G \to H^1(H, A)^{G/H} \xrightarrow{\delta} H^1(G/H, B) \xrightarrow{\phi} H^1(G/H, A_1^H) \cdots \quad (9)$$

It can be proved (see [Neukirch et al. 2000, Exercise 3, p. 71]) that the composition

$$H^1(H, A)^{G/H} \xrightarrow{\delta} H^1(G/H, B) \xrightarrow{\cong} H^2(G/H, A^H)$$

is the transgression map.

Lemma 1.5. *Assume that G is an abelian group. If the quotient G/H is a cyclic group isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and the group G can be written as a direct sum $G = G/H \times H$ then the transgression map is identically zero.*

Proof. Notice that if $A^H = A$ then this lemma can be proved by the explicit form of the transgression map as a cup product; see [Neukirch et al. 2000, Exercise 2, p. 71; Hochschild and Serre 1953].

The study of the kernel of the transgression is reduced to the study of the kernel of δ in (9). We will prove that the map ϕ in (9) is 1-1, and then the desired result will follow by exactness.

Let σ be a generator of the cyclic group $G/H = \mathbb{Z}/p\mathbb{Z}$. We denote by $N_{G/H}$ the norm map $A \to A$, sending

$$A \ni a \mapsto \sum_{g\in G/H} ga = \sum_{\nu=0}^{p-1} \sigma^\nu a.$$

By $I_{G/H}A$ we denote the submodule $(\sigma-1)A$ and by ${}_{N_{G/H}}A = \{a \in A : N_{G/H}a = 0\}$. Since G/H is a cyclic group we know that

$$H^1(G/H, B) = \frac{{}_{N_{G/H}}B}{I_{G/H}B} \quad \text{and} \quad H^1(G/H, A_1^H) = \frac{{}_{N_{G/H}}A_1^H}{I_{G/H}A_1^H}; \quad (10)$$

see [Serre 1979, VIII 4] and [Weibel 1994, Theorem 6.2.2]. Observe that the map ϕ defined in (9) can be given in terms of (10) as the map sending

$$b \bmod I_{G/H}B \mapsto b \bmod I_{G/H}A_1^H.$$

The map ϕ is well defined since $I_{G/H}B \subset I_{G/H}A_1^H$. The kernel of ϕ is computed:

$$\ker\phi = \frac{{}_{N_{G/H}}B \cap I_{G/H}A_1^H}{I_{G/H}B}.$$

The short exact sequence in (8) is a short exact sequence of $k[G/H]$-modules. This sequence seen as a short exact sequence of k-vector spaces is split, i.e., there is a section $s : H^1(H, A) \to A_1^H$ so that $\psi \circ s = \mathrm{Id}_{H^1(H,A)}$. This section map is only a k-linear map and not apriori compatible with the G/H-action.

Let us study the map ψ more carefully. An element $x \in A_1^H$ is a class $a \bmod A$ where $a \in \overline{A}$, and since $x \in A_1^H$ we have

$$a^h - a = ha - a = c[h] \in A.$$

It is a standard argument that $c[h]$ is an 1-cocycle $c[h] : H \to A$ and the class of this cocycle is defined to be $\psi(x)$. Since the image of $c[h]$ seen as a cocycle $c[h] : H \to \overline{A}$ is trivial, $c[h]$ is a coboundary i.e. we can select $\bar{a}_c \in \overline{A}$ so that

$$c[h] = \bar{a}_c^h - \bar{a}_c. \tag{11}$$

Obviously $\bar{a}_c \bmod A$ is H-invariant and we define one section as

$$s(c[h]) = \bar{a}_c \bmod A.$$

We have assumed that the group G can be written as $G = H \times G/H$ therefore we can write the functions $\bar{a}_c$ as functions of two arguments

$$\begin{aligned} \bar{a}_c : H \times G/H &\to A \\ (h, g) &\mapsto \bar{a}_c(h, g) \end{aligned}$$

Notice that (11) gives us that the for every $h, h_1 \in H$ the quantity $\bar{a}_c(h_1, g_1)^h - \bar{a}_c(h_1, g_1)$ does not depend on $g_1 \in G/H$. Using, this independence of $\bar{a}_c$ on the second argument we can compute that

$$s(c[h]^\sigma) = s(c[h])^\sigma,$$

i.e., the function s is compatible with the G/H-action. But every element $a \in A_1^H$ can be written as $a = b_a + s(\psi(a))$, where $b_a := a - s\psi(a) \in B$, since $\psi(b_a) = 0$. An arbitrary element in $I_{G/H} A_1^H$ is therefore written as

$$(\sigma - 1)a = (\sigma - 1)b_a + s(\sigma \cdot \psi(a) - \psi(a)). \tag{12}$$

If $(\sigma - 1)a \in_{N_{G/H}} B \cap I_{G/H} A_1^H$ we have, since $\mathrm{Im}(s) \cap B = \{0\}$,

$$s(\sigma \cdot \psi(a) - \psi(a)) = 0 \quad \text{if and only if} \quad (\sigma - 1)a = (\sigma - 1)b_a \in I_{G/H} B.$$

Therefore, ϕ is an injection and the desired result follows. □

1.5. ***The G-module structure of*** $\mathcal{T}_{\mathbb{O}}$**.** Our aim is to compute the first order infinitesimal deformations, i.e., the tangent space $D_\rho(k[\epsilon])$ to the infinitesimal deformation functor D_ρ [Mazur 1997, p. 272]. This space can be identified with $H^1(G, \mathcal{T}_{\mathbb{O}})$. The conjugation action on $\mathcal{T}_{\mathbb{O}}$ is defined as follows:

$$\left(f(t)\frac{d}{dt}\right)^\sigma = f(t)^\sigma \sigma \frac{d}{dt} \sigma^{-1} = f(t)^\sigma \sigma \left(\frac{d\sigma^{-1}(t)}{dt}\right)\frac{d}{dt}, \tag{13}$$

where $d\sigma^{-1}/dt$ denotes the operator sending an element $f(t)$ to $(d/dt)f^{\sigma^{-1}}(t)$, i.e. we first compute the action of σ^{-1} on f and then we take the derivative with respect to t. We will approach the cohomology group $H^1(G, \mathcal{T}_{\mathcal{O}})$ using the filtration sequence given in (2) and the low degree terms of the Lyndon–Hochschild–Serre spectral sequence.

The study of the cohomology group $H^1(G, \mathcal{T}_{\mathcal{O}})$ can be reduced to the study of the cohomology groups $H^1(V, \mathcal{T}_{\mathcal{O}})$, where V is an elementary abelian group. These groups can be written as a sequence of Artin–Schreier extensions that have the advantage of the extension and the corresponding actions having a relatively simple explicit form:

Lemma 1.6. *Let L be a an elementary abelian p-extension of the local field $K := k((x))$, with Galois group $G = \bigoplus_{\nu=1}^{s} \mathbb{Z}/p\mathbb{Z}$, such that the maximal ideal of $k[[x]]$ is ramified completely and the ramification filtration has no intermediate jumps i.e. is given by $G = G_0 = \cdots = G_n > \{1\} = G_{n+1}$. Then the extension L is given by $K(y_1, \dots, y_s)$ where $1/y_i^p - 1/y_i = f_i(x)$, where $f_i \in k((x))$ with a pole at the maximal ideal of order n.*

Proof. The desired result follows by the characterization of abelian p-extensions in terms of Witt vectors [Jacobson 1989, 8.11]. Notice that the exponent of the group G is p and we have to consider the image of $W_1(k((x)) = k((x))$, where $W_\lambda(\cdot)$ denotes the Witt ring of order λ as is defined in [Jacobson 1989, 8.26]. □

Lemma 1.7. *Every $\mathbb{Z}/p\mathbb{Z}$-extension $L = K(y)$ of the local field $K := k((x))$, with Galois group $G = \mathbb{Z}/p\mathbb{Z}$, such that the maximal ideal of $k[[x]]$ is ramified completely, is given in terms of an equation $f(1/y) = 1/x^n$, where $f(z) = z^p - z$ is in $k[z]$. The Galois group of the above extension can be identified with the $\mathbb{F}_p$-vector space V of the roots of the polynomial f, and the correspondence is given by*

$$\sigma_v : y \to \frac{y}{1+vy} \quad \textit{for } v \in V. \tag{14}$$

Moreover, we can select a uniformization parameter of the local field L such that the automorphism σ_v acts on t as follows:

$$\sigma_v(t) = \frac{t}{(1+vt^n)^{1/n}}.$$

Finally, the ramification filtration is given by $G = G_0 = \cdots = G_n > \{1\} = G_{n+1}$, and $n \not\equiv 0 \bmod p$.

Proof. By the characterization of abelian extensions in terms of Witt vectors we have $f(1/y) = 1/x^n$, where $f(z) = z^p - z \in k[z]$ (see also [Stichtenoth 1993, A.13]). Moreover the Galois group can be identified with the one dimensional $\mathbb{F}_p$-vector space V of roots of f, sending $\sigma_v : y \to y/(1+vy)$.

The filtration of the ramification group G is given by $G \cong G_0 = G_1 = \cdots G_n$, $G_i = \{1\}$ for $i \geq n+1$ [Stichtenoth 1993, Proposition III.7.10, p. 117]. Computation yields

$$x^n = ((1/y)^p - 1/y)^{-1} = \frac{y^p}{1 - y^{p-1}}, \tag{15}$$

hence $v_L(y) = n$, i.e., $y = \epsilon t^n$, where ϵ is a unit in $\mathcal{O}_L$ and t is the uniformization parameter in $\mathcal{O}_L$. Moreover, the polynomial f can be selected so that p does not divide n; see [Stichtenoth 1993, III. 7.8]. Since k is an algebraically closed field, Hensel's lemma implies that every unit in $\mathcal{O}_L$ is an n-th power, therefore we might select the uniformization parameter t such that $y = t^n$, and the desired result follows by (14). □

Lemma 1.8. *Let $H = \bigoplus_{\nu=1}^{s} \mathbb{Z}/p\mathbb{Z}$ be an elementary abelian group with ramification filtration*

$$H = H_0 = \cdots = H_n > H_{n+1} = \{1\} \quad \textit{and} \quad H_\kappa = \{1\} \textit{ for } \kappa \geq n+1.$$

The upper ramification filtration in this case coincides with the lower ramification filtration.

Proof. Let m be a natural number. We define the function $\phi : [0, \infty] \to \mathbb{Q}$ so that for $m \leq u < m+1$,

$$\phi(u) = \frac{1}{|H_0|} \sum_{i=1}^{m} |H_i| + (u - m) \frac{|H_{m+1}|}{|H_0|},$$

and since $H_{n+1} = \{1\}$ we compute

$$\phi(u) = \begin{cases} u & \text{if } m+1 \leq n, \\ n + (u - n - 1)/|H_0| & \text{if } m+1 > n. \end{cases}$$

The inverse function ψ is computed by

$$\psi(u) = \begin{cases} u & \text{if } u \leq n, \\ |H_0|u + (-n|H_0| + n + 1) & \text{if } u > n. \end{cases}$$

Therefore, by the definition of the upper ramification filtration we have $H^i = H_{\psi(i)} = H_i$ for $i \leq n$, while for $u > n$ we compute $\psi(u) = |H_0|u - n|H_0| + n \geq n$, thus $H^u = H_{\psi(u)} = \{1\}$. □

Lemma 1.9. *Let $a \in \mathbb{Q}$. Then for every prime p and every $\ell \in \mathbb{N}$ we have*

$$\left\lfloor \left\lfloor \frac{a}{p^\ell} \right\rfloor / p \right\rfloor = \left\lfloor \frac{a}{p^{\ell+1}} \right\rfloor.$$

Proof. This result follows by expressing a as a Laurent p-adic expansion in p, $a = \sum_{\nu=\lambda}^{-1} a_\nu p^\nu + \sum_{\nu=0}^{\infty} a_\nu p^\nu$ and by noticing that $\lfloor a/p^\ell \rfloor$ is the power series $\sum_{\nu=0}^{\infty} a_{\nu+\ell}\, p^\nu$. □

The arbitrary $\sigma_v \in \mathrm{Gal}(L/K)$ sends $t^n \mapsto t^n/(1+vt^n)$, so by computation

$$\frac{d\sigma_v(t)}{dt} = \frac{1}{(1+vt^n)^{(n+1)/n}}.$$

Lemma 1.10. *We consider an Artin–Schreier extension $L/k((x))$ and we keep the notation from Lemma 1.7. Let $\sigma_v \in \mathrm{Gal}(L/K)$. The corresponding action on the tangent space $\mathcal{T}_{\mathcal{O}}$ is given by*

$$\left(f(t)\frac{d}{dt}\right)^{\sigma_v} = f(t)^{\sigma_v}(1+vt^n)^{(n+1)/n}\frac{d}{dt}.$$

Proof. We have

$$\frac{d\sigma_v^{-1}(t)}{dt} = \frac{d\sigma_{-v}(t)}{dt} = \frac{1}{(1-vt^n)^{(n+1)/n}}$$

and by computation

$$\sigma_v\left(\frac{d\sigma_{-v}(t)}{dt}\right) = (1+vt^n)^{(n+1)/n}.$$ □

Letting $\mathcal{O} = \mathcal{O}_L$, we will now compute the space of "local modular forms"

$$\mathcal{T}_{\mathcal{O}}^{G_{t_i}} = \{f(t) \in \mathcal{O} : f(t)^{\sigma_v} = f(t)(1+vt^n)^{-(n+1)/n} \text{ for all } \sigma_v \in G_{t_i}\},$$

for $i \geq 1$. First we do the computation for a cyclic p-group.

Lemma 1.11. *Let $L/k((x))$ be an Artin–Schreier extension with Galois group $H = \mathbb{Z}/p\mathbb{Z}$ and ramification filtration*

$$H_0 = H_1 = \ldots = H_n > \{1\}.$$

Let t be the uniformizer of L and denote by $\mathcal{T}_{\mathcal{O}}$ the set of elements of the form $f(t)\,d/dt$, $f(t) \in k[[t]]$, equipped with the conjugation action defined in (13). The space $\mathcal{T}_{\mathcal{O}}^G$ is G-equivariantly isomorphic to the $\mathcal{O}_K$-module consisting of elements of the form

$$f(x)x^{n+1-\lfloor (n+1)/p \rfloor}\frac{d}{dx}, \quad f(x) \in \mathcal{O}_K.$$

Proof. Using the description of the action in Lemma 1.10 we see that $\mathcal{T}_{\mathcal{O}}$ is isomorphic to the space of Laurent polynomials of the form $\{f(t)/t^{n+1} : f(t) \in \mathcal{O}\}$, and the isomorphism is compatible with the G-action. Indeed, we observe first that $t^{n+1}\,d/dt$ is a G-invariant element in $\mathcal{T}_{\mathcal{O}}$. Then, for every $f(t)\,d/dt \in \mathcal{T}_{\mathcal{O}}$, the map sending

$$f(t)\frac{d}{dt} = \frac{f(t)}{t^{n+1}}t^{n+1}\frac{d}{dt} \mapsto \frac{f(t)}{t^{n+1}},$$

is a G-equivariant isomorphism.

We have

$$\{f(t)/t^{n+1}, f(t) \in \mathbb{O}\}^G = \{f(t)/t^{n+1}, f(t) \in \mathbb{O}\} \cap k((x)),$$

so the G-invariant space consists of elements $g(x)$ in K such that g seen as an element in L belongs to $\mathcal{T}_{\mathbb{O}}$, i.e., $v_L(g) \geq -(n+1)$. Consider the set of functions $g(x) \in K$ such that $v_L(g) = pv_K(g) \geq -(n+1)$, i.e., $v_K(g) \geq -(n+1)/p$. Since $v_K(g)$ is an integer the last inequality is equivalent to $v_K(g) \geq -\lfloor (n+1)/p \rfloor$.

Now a simple computation with the defining equation of the Galois extension L/K shows that

$$t^{n+1}\frac{d}{dt} = x^{n+1}\frac{d}{dx},$$

and the desired result follows. □

Similarly one can prove the following more general lemma:

Lemma 1.12. *We are using the notation of Lemma 1.11. Let A be the fractional ideal $k[[t]]t^a\, d/dt$, where a is a fixed integer. The G-module A is G-equivariantly isomorphic to $t^{a-(n+1)}k[[t]]$. Moreover, the space A^G is the space of elements of the form*

$$f(x)x^{n+1-\lfloor (n+1-a)/p \rfloor}\frac{d}{dx}.$$

Next we proceed to the more difficult case of elementary abelian p-groups.

Lemma 1.13. *Let $G = \bigoplus_{i=1}^{s} \mathbb{Z}/p\mathbb{Z}$ be the Galois group of the fully ramified elementary abelian extension $L/k((x))$ and assume that the ramification filtration is of the form*

$$G = G_0 = G_1 = \cdots = G_n > \{1\}.$$

Let t denote the uniformizer of L. Denote by $\mathcal{T}_{\mathbb{O}}$ the set of elements of the form $f(t)\, d/dt$, $f(t) \in k[[t]]$, equipped with the conjugation action defined in (13). The space $\mathcal{T}_{\mathbb{O}}^G$ is G-equivariantly isomorphic to the $\mathbb{O}_K$-module consisted of elements of the form

$$f(x)x^{n+1-\lfloor n+1/p^s \rfloor}\frac{d}{dx}, \quad f(x) \in \mathbb{O}_K,$$

where $p^s = |G|$.

Proof. We will break the extension $L/k((x))$ to a sequence of extensions $L = L_0 > L_1 > \ldots L_s = k((x))$, such that L_i/L_{i+1} is a cyclic p-extension. Denote by π_i the uniformizer of L_i. According to Lemma 1.8 the ramification extension L_i/L_{i+1} is of conductor n, i.e. the conditions of Lemma 1.11 are satisfied. We will prove the result inductively. For the extension L/L_1 the statement is true by Lemma 1.11. Assume that the lemma is true for L/L_i so a $k[[\pi_i]]$ basis of

$$\mathcal{T}_{\mathbb{O}}^{\bigoplus_{\nu=1}^{i} \mathbb{Z}/p\mathbb{Z}}$$

is given by the element $\pi_i^{n+1-\lfloor n+1/p^i \rfloor} d/d\pi_i$. Then Lemma 1.11 implies that a $k[[\pi_{i+1}]]$ basis for

$$\mathcal{T}_{\mathbb{O}}^{\bigoplus_{\nu=1}^{i+1}\mathbb{Z}/p\mathbb{Z}} = \left(\mathcal{T}_{\mathbb{O}}^{\bigoplus_{\nu=1}^{i}\mathbb{Z}/p\mathbb{Z}}\right)^{\mathbb{Z}/p\mathbb{Z}}$$

is given by the element

$$\pi_{i+1}^{n+1-\left\lfloor \left(n+1-(n+1-\lfloor n+1/p^i \rfloor)\right)/p \right\rfloor} \frac{d}{d\pi_{i+1}}.$$

The desired result follows by Lemma 1.9. □

Similarly one can prove the following more general lemma:

Lemma 1.14. *We are using the notation of Lemma 1.13. Let A be the fractional ideal $k[[t]]t^a\, d/dt$, where a is a fixed integer. The G-module A is G-equivariantly isomorphic to $t^{a-(n+1)}k[[t]]$. The space A^G is the space of elements of the form*

$$f(x)x^{n+1-\lfloor (n+1-a)/p^s \rfloor} \frac{d}{dx}.$$

By induction, this computation can be extended to yield:

Proposition 1.15. *Let $L = k((t))$ be a local field acted on by a Galois p-group G with ramification subgroups*

$$G_1 = \ldots = G_{t_f} > G_{t_f+1} = \ldots = G_{t_{f-1}} > G_{t_{f-1}+1} \geq \ldots \geq G_{t_1} = G_n > G_{t_0} = \{1\}.$$

We consider the tower of local fields

$$L^{G_0} = L^{G_1} \subseteq L^{G_i} \subseteq \ldots L^{\{1\}} = L.$$

Let us denote by π_i a local uniformizer for the field L^{G_i}, i.e. $L^{G_i} = k((\pi_i))$. The extension $L^{G_{t_i+1}}/L^{G_{t_i}}$ is Galois with Galois group the elementary abelian group $H(i) := G_{t_i}/G_{t_i+1}$. The ramification filtration of the group $H(i)$ is given by

$$H(i)_0 = H(i)_1 = \ldots = H(i)_{t_i} > H(i)_{t_i+1} = \{1\}$$

and the conductor of the extension is t_i. Let $\mathbb{O}$ be the ring of integers of L. The invariant space $\mathcal{T}_{\mathbb{O}}^{G_{t_i}}$ is the $\mathbb{O}^{G_{t_i}}$-module generated by

$$\pi_i^{\mu_i} \frac{d}{d\pi_i}, \tag{16}$$

where $\mu_0 = 0$ and $\mu_i = t_i + 1 - \left\lfloor \dfrac{-\mu_{i-1} + t_i + 1}{|G_{t_i}|/|G_{t_{i-1}}|} \right\rfloor.$

Proof. The first statements are clear from elementary Galois theory. What needs a proof is the formula for the dimensions μ_i. For $i = 1$, the group $G_{t_1} = G_n$

is elementary abelian and Lemma 1.13 applies, under the assumption $G_{t_0} = \{1\}$. Hence

$$\mathcal{T}_{\mathcal{O}}^{G_{t_1}} = \pi_1^{\,n+1-\left\lfloor \frac{n+1}{|G_{t_1}|/|G_{t_0}|} \right\rfloor} \frac{d}{d\pi_1}.$$

Assume that the formula is correct for i, so

$$\mathcal{T}_{\mathcal{O}}^{G_{t_i}} = \pi^{\mu_i} \frac{d}{d\pi_i}.$$

Then Lemma 1.14 implies that

$$\mathcal{T}_{\mathcal{O}}^{G_{t_{i+1}}} = \left(\mathcal{T}_{\mathcal{O}}^{G_{t_i}}\right)^{G_{t_{i+1}}/G_{t_i}} = \pi_{i+1}^{\mu_{i+1}} \frac{d}{d\pi_{i+1}},$$

where $\mu_{i+1} = n_{i+1} + 1 - \left\lfloor (n_{i+1} + 1 - \mu_i)/|G_{t_{i+1}}/G_{t_i}| \right\rfloor$ and the inductive proof is complete. $\square$

Let $k((t))/k((x))$ be a cyclic extension of local fields of order p, such that the maximal ideal $xk[[x]]$ is ramified completely. For the ramification groups G_i we have

$$\mathbb{Z}/p\mathbb{Z} = G = G_0 = \cdots = G_n > G_{n+1} = \{1\}.$$

Hence, the different exponent is computed $d = (n+1)(p-1)$. Let $E = t^a k[[t]]$ be a fractional ideal of $k((t))$. Let $N(E)$ denote the images of elements of E under the norm map corresponding to the group $\mathbb{Z}/p\mathbb{Z}$. It is known that $N(E) = x^{\lfloor (d+a)/p \rfloor} k[[x]]$, and $E \cap k[[x]] = x^{\lceil a/p \rceil} k[[x]]$. The cohomology of cyclic groups is 2-periodic and by [Bertin and Mézard 2000, Proposition 4.1.1] we have

$$\dim_k H^1(G, E) = \dim_k H^2(G, E) = \frac{E \cap k[[x]]}{N(E)} = \left\lfloor \frac{d+a}{p} \right\rfloor - \left\lceil \frac{a}{p} \right\rceil. \tag{17}$$

Remark 1.16. The proposition just quoted actually contains the following formula instead of (17):

$$\dim_k H^1\Big(G, k[[x]]\frac{d}{dx}\Big) = \left\lfloor \frac{2d}{p} \right\rfloor - \left\lceil \frac{d}{p} \right\rceil.$$

But $k[[x]]\dfrac{d}{dx} \cong x^{-n-1}k[[x]]$, and $d = (n+1)(p-1)$; thus

$$\begin{aligned}\left\lfloor \frac{2d}{p} \right\rfloor - \left\lceil \frac{d}{p} \right\rceil &= \left\lfloor \frac{d+(n+1)p-n-1}{p} \right\rfloor - \left\lceil \frac{(n+1)p-n-1}{p} \right\rceil \\ &= \left\lfloor \frac{d-n-1}{p} \right\rfloor - \left\lceil \frac{-n-1}{p} \right\rceil,\end{aligned}$$

and the two formulas coincide.

Corollary 1.17. *Let G be an abelian group that can be written as a direct product $G = H_1 \times H_2$ of groups H_1, H_2, and suppose that $H_2 = \mathbb{Z}/p\mathbb{Z}$. The following sequence is exact*:

$$0 \to H^1(H_2, A^{H_1}) \to H^1(H_1 \times H_2, A) \to H^1(H_1, A)^{H_2} \to 0$$

Proof. The group H_2 is cyclic of order p so the transgression map is identically zero by Lemma 1.5 and the desired result follows. □

Remark 1.18. It seems that the result of J. Bertin and A. Mézard, solves the problem of determining the dimension of the k-vector spaces $H^1(\mathbb{Z}/p\mathbb{Z}, A)$ for fractional ideals of $k[[x]]$. But in what follows we have to compute the G/H-invariants of the above cohomology groups, therefore an explicit description of these groups and of the G/H-action is needed.

2. Computing $H^1(\mathbb{Z}/p\mathbb{Z}, A)$

We will need the following

Lemma 2.1. *Let a be a p-adic integer. The binomial coefficient $\binom{a}{i}$ is defined for a, as usual, by*

$$\binom{a}{i} = \frac{a(a-1)\cdot(a-i+1)}{i!}$$

and it is also a p-adic integer [Gouvêa 1997, Lemma 4.5.11]. Moreover, the binomial series is defined

$$(1+t)^a = \sum_{i=0}^{\infty} \binom{a}{i} t^i. \tag{18}$$

Let i be an integer and let $\sum_{\mu=0}^{\infty} b_\mu p^\mu$ and $\sum_{\mu=0}^{\infty} a_\mu p^\mu$ be the p-adic expansions of i and a respectively. The p-adic integer $\binom{a}{i} \not\equiv 0 \bmod p$ if and only if every coefficient $a_i \geq b_i$.

Proof. The only thing that needs a proof is the criterion of the vanishing of the binomial coefficient mod p. If a is a rational integer, then this is a known theorem due to Gauss [Eisenbud 1995, Proposition 15.21]. When a is a p-adic integer we compare the coefficients mod p of the expression

$$(1+t)^a = (1+t)^{\sum_{\mu=0}^{\infty} a_\mu p^\mu} = \prod_{\mu=1}^{\infty} (1+t^{p^\mu})^{a_\mu}$$

and of the binomial expansion in (18) and the result follows. □

Lemma 2.2 (Nakayama map). *Let $G = \mathbb{Z}/p\mathbb{Z}$ be a cyclic group of order p and let $A = t^a k[[t]]$ be a fractional ideal of $k[[t]]$. Let x be a local uniformizer of the* *field* $k((t))^{\mathbb{Z}/p\mathbb{Z}}$. *Let* $\alpha \in H^2(G, A)$, *and let* $u[\sigma, \tau]$ *be any cocycle representing the class* α. *The map*

$$\begin{aligned} \phi : H^2(G, A) &\to \frac{x^{\lceil a/p \rceil} k[[x]]}{x^{\lfloor ((n+1)(p-1)+a)/p \rfloor} k[[x]]} \\ \alpha &\mapsto \sum_{\rho \in G} u[\rho, \tau], \quad \tau \in G, \end{aligned} \tag{19}$$

is well defined and is an isomorphism.

Proof. Let A be a G-module. Denote by $\hat{H}^0(G, A)$ the zeroth Tate cohomology. We use Remark 4-5-7 and theorem 4-5-10 from [Weiss 1969] to prove that the map $H^2(G, A) \ni \alpha \mapsto \sum_{\rho \in G} u[\rho, \tau] \in \hat{H}^0(G, A)$ is well defined and an isomorphism. This map was introduced by T. Nakayama [1936] and used to give an explicit formula for the reciprocity law isomorphism for local class field theory.

We will express $\hat{H}^0(G, A)$ for $G = \mathbb{Z}/p\mathbb{Z}$ with generator σ and $A = t^a k[[t]]$. We know that

$$\hat{H}^0(\mathbb{Z}/p\mathbb{Z}, t^a k[[t]]) = \frac{\ker(\delta)}{N_{\mathbb{Z}/p\mathbb{Z}}(t^a k[[t]])},$$

where $\delta = \sigma - 1$ and $N_{\mathbb{Z}/p\mathbb{Z}} = \sum_{i=0}^{p-1} \sigma^i$. We compute that

$$\begin{aligned} \ker(\delta) &= t^a k[[t]] \cap k((x)) = x^{\lceil a/p \rceil} k[[x]], \\ N_{\mathbb{Z}/p\mathbb{Z}}(t^a k[[t]]) &= x^{\left\lfloor \frac{a+(n+1)(p-1)}{p} \right\rfloor} k[[x]]. \end{aligned}$$

This completes the proof. $\square$

Let $A = t^a k[[t]]$ be a fractional ideal of $k[[t]]$. We consider the fractional ideal $t^{a+n+1} k[[t]]$, and we form the short exact sequence

$$0 \to t^{a+n+1} k[[t]] \to t^a k[[t]] \to M \to 0, \tag{20}$$

where M is an $(n+1)$-dimensional k-vector space with basis

$$\left\{ \frac{1}{t^{-a}}, \frac{1}{t^{-a-1}}, \ldots, \frac{1}{t^{-a-n}} \right\}.$$

Let σ_v be the automorphism $\sigma_v(t) = t/(1 + vt^n)^{1/n}$, where $v \in \mathbb{F}_p$. The action of σ_v on $1/t^\mu$ is given by

$$\sigma_v : \frac{1}{t^\mu} \mapsto \frac{(1 + vt^n)^{\mu/n}}{t^\mu} = \frac{1}{t^\mu} \left(\sum_{\nu=0}^{\infty} \binom{\mu/n}{\nu} v^\nu t^{\nu n} \right). \tag{21}$$

The action of $\mathbb{Z}/p\mathbb{Z}$ on the basis elements of M is given by

$$\sigma_v(1/t^\mu) = \begin{cases} 1/t^\mu & \text{if } -a < \mu, \\ 1/t^{-a} - \dfrac{a}{n} v 1/t^{-a-n} & \text{if } \mu = -a. \end{cases} \tag{22}$$

We consider the long exact sequence obtained by applying the G-invariants functor to (20):

$$
\begin{aligned}
0 \to t^{a+n+1}k[[t]]^G \to t^a k[[t]]^G \to M^G \xrightarrow{\delta_1} H^1(G, t^{a+n+1}k[[t]]) \\
\to H^1(G, t^a k[[t]]) \to H^1(G, M) \xrightarrow{\delta_2} H^2(G, t^{a+n+1}k[[t]]) \to \cdots
\end{aligned} \tag{23}
$$

Lemma 2.3. *Assume that the group $G = \mathbb{Z}/p\mathbb{Z}$ is generated by σ_v. The map δ_1 in (23) is onto.*

Proof. By (22) we have

$$\dim_k M^{\mathbb{Z}/p\mathbb{Z}} = \begin{cases} n+1 & \text{if } p \mid a \\ n & \text{if } p \nmid a. \end{cases}$$

Now, if x is a local uniformizer of the field $k((t))^{\mathbb{Z}/p\mathbb{Z}}$, then

$$\left(\frac{1}{t^{-a-(n+1)}}k[[t]]\right)^{\mathbb{Z}/p\mathbb{Z}} = x^{\lceil (a+(n+1))/p\rceil} k[[x]] = \left(\frac{1}{x}\right)^{\lfloor(-a-(n+1))/p\rfloor} k[[x]],$$

and similarly

$$\left(\frac{1}{t^{-a}}k[[t]]\right)^{\mathbb{Z}/p\mathbb{Z}} = \left(\frac{1}{x}\right)^{\lfloor -a/p\rfloor} k[[x]].$$

The image of δ_1 has dimension $\dim_k M^{\mathbb{Z}/p\mathbb{Z}} - \lfloor -a/p\rfloor + \lfloor -a-(n+1)/p\rfloor$. Moreover for the dimension of $H^1(\mathbb{Z}/p\mathbb{Z}, (1/(t^{-a-(n+1)}))k[[t]])$ we compute

$$h := \dim_k H^1(\mathbb{Z}/p\mathbb{Z}, \frac{1}{t^{-a-(n+1)}}k[[t]]) = (n+1) - \lceil -a/p\rceil + \lfloor -a-(n+1)/p\rfloor.$$

We now observe that $\dim_k \mathrm{Im}(\delta_1) = h$ by studying separately the cases $p \mid a$ and $p \nmid a$. This finishes the proof. ☐

Proposition 2.4. *The cohomology group $H^1(\mathbb{Z}/p\mathbb{Z}, M)$ is isomorphic to*

$$H^1(\mathbb{Z}/p\mathbb{Z}, M) \cong \begin{cases} \bigoplus_{i=-a-n}^{-a} \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, k) & \text{if } p \mid a, \\ \bigoplus_{i=-a-n+1}^{-a} \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, k) & \text{if } p \nmid a. \end{cases}$$

Proof. Assume that the arbitrary automorphism $\sigma_v \in G = \mathbb{Z}/p\mathbb{Z}$ is given by $\sigma_v(t) = t/(1+vt^n)^{1/n}$ where $v \in \mathbb{F}_p$. Let us write a cocycle d as

$$d\sigma_v = \sum_{i=-a-n}^{-a} \alpha_i(\sigma_v)\frac{1}{t^i}. \tag{24}$$

By computation,

$$d(\sigma_v)^{\sigma_v^\mu} = \sum_{i=-a-n}^{-a} \alpha_i(\sigma_v)\frac{1}{t^i} + \alpha_{-a}(\sigma_v)\frac{-a}{n}\mu v\frac{1}{t^{-a-n}}.$$

We apply the cocycle condition $d(\sigma_v + \sigma_w) = d(\sigma_w) + d(\sigma_v)^{\sigma_w}$ for $d(\sigma_v)$ given in (24) and we obtain the following conditions on the coefficients $\alpha_i(\sigma_v)$:

$$\alpha_i(\sigma_w + \sigma_v) = \alpha_i(\sigma_w) + \alpha_i(\sigma_v) \quad \text{for } i \neq -a-n,$$
$$\alpha_{-a-n}(\sigma_w + \sigma_v) = \alpha_{-a-n}(\sigma_w) + \alpha_{-a-n}(\sigma_v) + \alpha_{-a}(\sigma_v)\frac{-a}{n}w.$$

The last equation allows us to compute the value of α_{-a-n} on any power σ_v^ν of the generator σ_v of $\mathbb{Z}/p\mathbb{Z}$. Indeed, we have

$$\alpha_{-a-n}(\sigma_v^\nu) = \nu\alpha_{-a-n}(\sigma_v) + (\nu - 1)\alpha_{-a}(\sigma_v)\frac{-a}{n}v.$$

This proves that the function α_{-a-n} depends only on the selection of $\alpha_{-a-n}(\sigma_v) \in k$.

We will now compute the coboundaries. Let $b = \sum_{i=-a-n}^{-a} b_i/t^i$, $b_i \in k$ be an element in M. By computation,

$$b^{\sigma_v} - b = b_{-a}\frac{-a}{n}v\frac{1}{t^{-a-n}}.$$

For the computation of the cohomology groups we distinguish two cases:

- If $p \mid a$, the $\mathbb{Z}/p\mathbb{Z}$-action on M is trivial, so

$$H^1(\mathbb{Z}/p\mathbb{Z}, M) = \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, M) = \bigoplus_{i=-a-n}^{-a} \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, k).$$

  The dimension of $H^1(\mathbb{Z}/p\mathbb{Z}, M)$ in this case is $n+1$.

- If $p \nmid a$, the coboundary kills the contribution of the cocycle on the $1/t^{-a-n}$ basis element and the cohomology group is

$$H^1(\mathbb{Z}/p\mathbb{Z}, M) = \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, M) = \bigoplus_{i=-a-n+1}^{-a} \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, k). \qquad \square$$

Lemma 2.5. *Assume that $p \geq 3$. Let $e = 1$ if $p \nmid a$ and $e = 0$ if $p \mid a$. If $n \geq 2$ then an element*

$$\sum_{i=-a-n+e}^{-a} a_i(\,\cdot\,)\frac{1}{t^i} \in H^1(\mathbb{Z}/p\mathbb{Z}, M)$$

is in the kernel of δ_2 if and only if $a_i(\,\cdot\,)\binom{i/n}{p-1} = 0$ for all i. If $n = 1$ then an element

$$\sum_{i=-a-n+e}^{-a} a_i(\,\cdot\,)\frac{1}{t^i} \in H^1(\mathbb{Z}/p\mathbb{Z}, M)$$

is in the kernel of δ_2 *if and only if* $a_i(\cdot)\binom{i/n}{p-1} = 0$ *for all* $-a-n+e \le i \le -a$ *and* $a_i(\cdot)\binom{i/n}{2p-2} = 0$ *for all* $-a-n+e \le i \le -a$ *such that* $2(p-1)n - i < (n+1)p + p\lfloor a/p \rfloor$.

Proof. A derivation $a_i(\sigma_v)(1/t^i)$, $-a-n+e \le i \le -a$ representing a cohomology class in $H^1(\mathbb{Z}/p\mathbb{Z}, M)$ is mapped to

$$\begin{aligned}\delta_2\big(a_i(\cdot)\frac{1}{t^i}\big)[\sigma_w, \sigma_v] &= a_i(\sigma_v)\frac{1}{t^i}^{\sigma_w} - a_i(\sigma_v+\sigma_w)\frac{1}{t^i} + a_i(\sigma_w)\frac{1}{t^i} \\ &= \frac{a_i(\sigma_v)}{t^i}\Big(\sum_{\nu=1}^{\infty}\binom{i/n}{\nu} w^\nu t^{n\nu}\Big). \end{aligned} \tag{25}$$

We now consider the map ϕ defined in (19) in the proof of Lemma 2.2. The map $\delta_2 : H^1(G, M) \to H^2(G, t^{a+n+1}k[[t]])$ is composed with ϕ and the image of $\phi \circ \delta_2$ in $x^{\lceil (a+n+1)/p \rceil} k[[x]]/x^{\lfloor ((n+1)p+a)/p \rfloor} k[[x]]$ is given by

$$\phi \circ \delta_2\big(a_i(\cdot)\frac{1}{t^i}\big) = \sum_{w \in \mathbb{Z}/p\mathbb{Z}} \frac{a_i(\sigma_v)}{t^i}\Big(\sum_{\nu=1}^{\infty}\binom{i/n}{\nu} w^\nu t^{n\nu}\Big).$$

Now recall that

$$\sum_{w \in \mathbb{Z}/p\mathbb{Z}} w^\nu = \begin{cases} 0 & \text{if } p-1 \nmid \nu, \\ -1 & \text{if } p-1 \mid \nu, \end{cases}$$

and every homomorphism $a_i : (\mathbb{Z}/p\mathbb{Z}, \cdot) \to (k, +)$ is given by $a_i(\sigma_w) = \lambda_i w$, where $\lambda_i \in k$. Therefore,

$$\phi \circ \delta_2\big(a_i(\cdot)\frac{1}{t^i}\big) = \sum_{\nu=1,\, p-1 \mid \nu}^{\infty} \binom{i/n}{\nu}(-1)a_i(\sigma_v)t^{n\nu - i}. \tag{26}$$

Observe that $(p-1) \mid \nu$ is equivalent to $\nu = \mu p - \mu$, and since $\nu \ge 1$, we have $\mu \ge 1$. Thus (26) becomes

$$\begin{aligned}&\sum_{\mu=1}^{\infty}\binom{i/n}{\mu p - \mu}(-1)\lambda_i t^{(\mu p - \mu)n - i} \\ &= \binom{i/n}{p-1}(-1)\lambda_i t^{(p-1)n-i} + \binom{i/n}{2p-2}(-1)\lambda_i t^{(2p-2)n-i} + \text{higher order terms.}\end{aligned}$$

Claim 2.6. If $n \ge 2$ and $p \ge 3$ then for all $a \le -i \le a+n$ and for $\mu \ge 2$

$$\mu(p-1)n - i \ge p\Big\lfloor \frac{(n+1)p+a}{p} \Big\rfloor. \tag{27}$$

If $n=1$ and $p\geq 3$ then (27) holds for $a\leq -i\leq a+n$ and for $\mu\geq 3$. Moreover

$$(p-1)n-i<p\left\lfloor\frac{(n+1)p+a}{p}\right\rfloor,$$

for $a\leq -i\leq a+n$. Indeed, the inequality

$$\mu\geq\frac{n+1}{n}\frac{p}{p-1}\tag{28}$$

holds for $p\geq 3, n\geq 2$ and $\mu\geq 2$ or for $p\geq 3$, $n=1$, $\mu\geq 3$. Therefore, (28) implies that

$$n+1+\left\lfloor\frac{a}{p}\right\rfloor-\frac{a}{p}\leq n+1\leq\mu\frac{p-1}{p}n$$

and therefore

$$(n+1)p+\left\lfloor\frac{a}{p}\right\rfloor p\leq\mu(p-1)n+a\leq\mu(p-1)n-i$$

and the first assertion is proved. For the second assertion, we compute

$$\frac{a}{p}<1+\left\lfloor\frac{a}{p}\right\rfloor\quad\Rightarrow\quad\frac{a}{p}+n<n+1+\left\lfloor\frac{a}{p}\right\rfloor$$

and thus

$$(p-1)n-i<a+pn<p(n+1)+p\left\lfloor\frac{a}{p}\right\rfloor.$$

Since for elements $g\in k[[x]]\subset k[[t]]$ we have $pv_x(g)=v_t(g)$ we observe that all elements in $k[[t]]$ that have valuation greater or equal to $(n+1)p+\lfloor a/p\rfloor$ are zero in the lift of the ideal $x^{(n+1+\lfloor a/p\rfloor}k[[x]]$ on $k[[t]]$. Therefore Claim 2.6 gives us that for $p\geq 3, n\geq 2$,

$$\phi\circ\delta_2(a_i(\,\cdot\,)\frac{1}{t^i})=\binom{i/n}{p-1}(-1)\lambda_i t^{(p-1)n-i}$$

so $\sum_{i=-a-n}^{-a}a_i(\,\cdot\,)(1/t^i)$ is in the kernel of δ_2 if and only if

$$\binom{i/n}{p-1}(-1)\lambda_i=0\ \text{ for all } i.$$

The case $n=1$ follows by a similar argument. □

Proposition 2.7. *The cohomology group $H^1(\mathbb{Z}/p\mathbb{Z},t^a k[[t]])$ is isomorphic to the k-vector space generated by*

$$\left\{\frac{1}{t^i},\,b\leq i\leq -a,\ \textit{such that}\ \binom{i/n}{p-1}=0\right\},$$

where $b=-a-n$ if $p\mid a$ and $b=-a-n+1$ if $p\nmid a$.

Proof. If $n \geq 2$ and $p \geq 3$ then the result is immediate by the exact sequence (23), Lemma 2.3 and Lemma 2.5 and by the computation of $H^1(\mathbb{Z}/p\mathbb{Z}, M)$ given in Proposition 2.4.

Assume that $n = 1$, and let $a = a_0 + a_1 p + a_2 p^2 + \cdots$ be the p-adic expansion of a. Then the inequality

$$(n+1)p + p\left\lfloor \frac{a}{p} \right\rfloor \leq 2(p-1)n + a \tag{29}$$

holds if $a_0 \neq 0, 1$. Indeed, in this case we have $2/p \leq a/p - \lfloor a/p \rfloor < 1$ and (29) holds. Therefore, for the case $p \mid a$ and $a = 1 + pb$, $b \in \mathbb{Z}$ we have to check the binomial coefficients $\binom{i/n}{2p-2}$ as well. We will prove that in these cases if $\binom{i/n}{p-1} = 0$ then $\binom{i/n}{2p-2} = 0$ and the proof will be complete.

Assume, first that $p \mid a$ and $n = 1$. Then,$-a-1 \leq i \leq -a$, i.e. $i = -a-1$ or $i = -a$. We compute that $\binom{-a}{p-1} = 0$ since there is no constant term in the p-adic expansion of $-a$. Moreover the p-adic expansion of $2p-2$ is computed $2p-2 = p-2+p$, and since $p \neq 2$ we have $\binom{-a}{2p-2} = 0$ as well. For $i = -a-1$ we have $i = p-1+pb$ for some $b \in \mathbb{Z}$; therefore by comparing the p-adic expansions of $-a-1, p-1$ we obtain that $\binom{-a-1}{p-1} \neq 0$, and this value of i does not contribute to the cohomology.

Assume now that $a = 1+pb$, $b \in \mathbb{Z}$. We have $i = -a$ and $-a = p-1+p(b+1)$. Therefore by comparing the p-adic expansions of $-a$, $p-1$ we obtain that $\binom{-a}{p-1} \neq 0$ and this value of i does not contribute to the cohomology. □

Proposition 2.8. *Let $A = t^a k[[t]]$ be a fractional ideal of the local field $k((t))$. Assume that $H = \bigoplus_{\nu=1}^{s} \mathbb{Z}/p\mathbb{Z}$ is an elementary abelian group with ramification filtration $H = H_0 = \cdots = H_n > H_{n+1} = \{1\}$. Let π_i be the local uniformizer of the local field $k((t))^{\bigoplus_{\nu=1}^{i-1} \mathbb{Z}/p\mathbb{Z}}$, and $a_i = \lceil a_{i-1}/p \rceil$, $a_1 = a$. The cohomology group $H^1(H, A)$ is generated as a k-vector space by the basis elements*

$$\left\{ \bigoplus_{\lambda=1}^{s} \frac{1}{\pi_i^{i_\lambda}}, \quad \begin{matrix} \lambda = 1, \ldots, s \\ b_i \leq i_\lambda \leq -a_i \end{matrix} \quad \text{such that } \binom{i_\lambda/n}{p-1} = 0, \right\}$$

where $b_i = -a_i - n$ if $p \mid a_i$ and $b_i = -a_i - n + 1$ if $p \nmid a_i$. Moreover, let $H(i) := H/\bigoplus_{\nu=1}^{i-1} \mathbb{Z}/p\mathbb{Z}$. The groups $H^1(\bigoplus_{\nu=1}^{i-1} \mathbb{Z}/p\mathbb{Z}, t^a k[[t]])$ are trivial $H(i)$-modules with respect to the conjugation action.

Proof. For $A = t^a k[[t]]$, we compute the invariants

$$t^a k[[t]] \cap k((t))^{\mathbb{Z}/p\mathbb{Z}} = x^{\lceil a/p \rceil} k[[x]],$$

where x is a local uniformizer for the ring of integers of $k((t))^{\mathbb{Z}/p\mathbb{Z}}$. The modules $A^{\bigoplus_{\nu=1}^{i-1} \mathbb{Z}/p\mathbb{Z}}$ can be computed recursively:

$$A^{\bigoplus_{\nu=1}^{i-1} \mathbb{Z}/p\mathbb{Z}} = \pi_i^{a_i} k[[\pi_i]],$$

where π_i is a uniformizer for the local field $k((t))^{\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z}}$ and $a_i = \lceil a_{i-1}/p \rceil$, $a_1 = a$.

To compute the ramification filtration of quotient groups we have to employ the upper ramification filtration for the ramification group [Serre 1979, IV 3, p. 73-74]. But according to Lemma 1.8 the upper ramification filtration coincides with the lower ramification filtration therefore the ramification filtration for the groups $H(i)$ is $H(i)_0 = \cdots = H(i)_n > \{1\}$. For the group $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ Corollary 1.17 implies that

$$H^1(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}, t^a k[[t]]) = H^1\Big(\frac{\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}}{\mathbb{Z}/p\mathbb{Z}}, t^a k[[t]]^{\mathbb{Z}/p\mathbb{Z}}\Big) \oplus H^1(\mathbb{Z}/p\mathbb{Z}, t^a k[[t]])^{H/(\mathbb{Z}/p\mathbb{Z})}.$$

By Lemma 1.8 and by the compatibility [Serre 1979, IV 3, p. 73-74] of the upper ramification filtration with quotients, we obtain that the quotient $H/(\mathbb{Z}/p\mathbb{Z})$ has also conductor n. By Lemma 1.2, Lemma 1.4 and by the explicit description of the group $H^1(\mathbb{Z}/p\mathbb{Z}, t^a k[[t]])$ of Proposition 2.7 and by the fact $H/(\mathbb{Z}/p\mathbb{Z})$ is of conductor n, the action of $H/(\mathbb{Z}/p\mathbb{Z})$ on $H^1(\mathbb{Z}/p\mathbb{Z}, t^a k[[t]])$ is trivial. Thus

$$H^1(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}, t^a k[[t]]) = H^1\Big(\frac{H}{\mathbb{Z}/p\mathbb{Z}}, t^a k[[t]]^{\mathbb{Z}/p\mathbb{Z}}\Big) \oplus H^1(\mathbb{Z}/p\mathbb{Z}, t^a k[[t]]).$$

Moreover the cohomology group $H^1(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}, t^a k[[t]])$ is generated over k by $\langle 1/\pi_1^i \oplus 1/\pi_2^j \rangle$, where $b_1 \le i \le -a$, $b_2 \le j \le -\lceil a/p \rceil$ and $\binom{i/n}{p-1} = \binom{j/n}{p-1} = 0$.

The desired result follows by induction. $\square$

Proposition 2.9. *Let $A = t^a k[[t]]$ be a fractional ideal of the local field $k((t))$. Assume that $H = \bigoplus_{\nu=1}^{s} \mathbb{Z}/p\mathbb{Z}$ is an elementary abelian group with ramification filtration $H = H_0 = \cdots = H_n > H_{n+1} = \{1\}$. The dimension of $H^1(H, A)$ can be computed as*

$$\dim_k H^1(H, A) = \sum_{i=1}^{s} \bigg(\Big\lfloor \frac{(n+1)(p-1)+a_i}{p} \Big\rfloor - \Big\lceil \frac{a_i}{p} \Big\rceil \bigg), \tag{30}$$

where a_i are defined recursively by $a_1 = a$ and $a_i = \lceil a_{i-1}/p \rceil$. In particular, if $A = k[[t]]$, we have

$$\dim_k H^1(H, k[[t]]) = s \Big\lfloor \frac{(n+1)(p-1)}{p} \Big\rfloor. \tag{31}$$

Proof. By induction on the number of direct summands, Corollary 1.17 and Proposition 2.8 we can prove the formula

$$H^1(H, A) = \bigoplus_{i=1}^{s} H^1(\mathbb{Z}/p\mathbb{Z}, A^{\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z}}). \tag{32}$$

To compute the dimensions of the direct summands $H^1\big(\mathbb{Z}/p\mathbb{Z}, A^{\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z}}\big)$, for various i we have to compute the ramification filtration for the groups defined as $H(i) = H/\big(\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z}\big)$, since $\bigoplus_{\nu=1}^{i-1}\mathbb{Z}/p\mathbb{Z} = H/H(i)$. But the upper ramification filtration coincides with the lower ramification filtration by Lemma 1.8. Thus, the dimension of $H^1(H, A)$ can be computed as

$$\dim_k H^1(H, A) = \sum_{i=1}^{s} \left\lfloor \frac{(n+1)(p-1)+a_i}{p} \right\rfloor - \left\lceil \frac{a_i}{p} \right\rceil.$$

In particular if $A = k[[t]]$, then

$$\dim_k H^1(H, k[[t]]) = s \left\lfloor \frac{(n+1)(p-1)}{p} \right\rfloor. \qquad \square$$

Let $\kappa_{t_i} = \dim_k \ker\big(\mathrm{tg} : H^1(G_{t_i+1}, \mathcal{T}_{\mathbb{O}}) \to H^2(G_{t_i}/G_{t_i+1}, \mathcal{T}_{\mathbb{O}}^{G_{t_i+1}})\big)$ be the dimension of the kernel of the transgression map. We have

$$0 \le \kappa_i \le \dim_k H^1(G_{t_i+1}, \mathcal{T}_{\mathbb{O}})^{G_{t_i}/G_{t_i+1}} \le \dim_k H^1(G_{t_i+1}, \mathcal{T}_{\mathbb{O}}). \tag{33}$$

This allows us to compute

Proposition 2.10. *Let G be the Galois group of the extensions of local fields L/K, with ramification filtration G_i and let $(t_\lambda)_{1\le\lambda\le f}$ be the jump sequence in (3). For the dimension of $H^1(G_1, \mathcal{T}_{\mathbb{O}})$ we have the bound*

$$\begin{aligned} H^1(G_1/G_{t_{f-1}}, \mathcal{T}_{\mathbb{O}}^{G_{t_{f-1}}}) &\le \dim_k H^1(G_1, \mathcal{T}_{\mathbb{O}}) \\ &\le \sum_{i=1}^{f} \dim_k H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_{\mathbb{O}}^{G_{t_{i-1}}})^{G_{t_f}/G_{t_i}} \\ &\le \sum_{i=1}^{f} \dim_k H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_{\mathbb{O}}^{G_{t_{i-1}}}), \end{aligned} \tag{34}$$

where $G_{n+1} = \{1\}$. The left bound is best possible in the sense that there are ramification filtrations such that the first inequality becomes an equality.

Proof. Using the low-term sequence in (5) we obtain the following inclusion for $i \ge 1$:

$$\begin{aligned} H^1(G_{t_i}, \mathcal{T}_{\mathbb{O}}) &= H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_{\mathbb{O}}^{G_{t_{i-1}}}) + \ker tg \\ &\subseteq H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_{\mathbb{O}}^{G_{t_{i-1}}}) \oplus H^1(G_{t_{i-1}}, \mathcal{T}_{\mathbb{O}})^{G_{t_i}/G_{t_{i-1}}}. \end{aligned} \tag{35}$$

We start our computation from the end of the ramification groups:

$$H^1(G_{t_2}, \mathcal{T}_{\mathbb{O}}) \subseteq H^1(G_{t_2}/G_{t_1}, \mathcal{T}_{\mathbb{O}}^{G_{t_1}}) \oplus H^1(G_{t_1}, \mathcal{T}_{\mathbb{O}})^{G_{t_2}/G_{t_1}}. \tag{36}$$

Observe here that $\mathcal{T}_{\mathcal{O}}$ is not G_{t_1}-invariant so there in no apriori well defined action of G_{t_2}/G_{t_1} on $\mathcal{T}_{\mathcal{O}}$. But since the group G_{t_1} is of conductor n using the explicit form of $H^1(G_{t_1}, \mathcal{T}_{\mathcal{O}})$ we see that $H^1(G_{t_1}, \mathcal{T}_{\mathcal{O}})$ is a trivial G_1-module. Of course this is also clear from the general properties of the conjugation action [Weiss 1969, Corollary 2-3-2]. We move to the next step:

$$H^1(G_{t_3}, \mathcal{T}_{\mathcal{O}}) \subseteq H^1(G_{t_3}/G_{t_2}, \mathcal{T}_{\mathcal{O}}^{G_{t_2}}) \oplus H^1(G_{t_2}, \mathcal{T}_{\mathcal{O}})^{G_{t_3}/G_{t_2}}. \tag{37}$$

The combination of (36) and (37) gives us

$$\begin{aligned} H^1(G_{t_3}, \mathcal{T}_{\mathcal{O}}) \subseteq H^1\Big(\frac{G_{t_3}}{G_{t_2}}, \mathcal{T}_{\mathcal{O}}^{G_{t_2}}\Big) \oplus H^1\Big(\frac{G_{t_2}}{G_{t_1}}, \mathcal{T}_{\mathcal{O}}^{G_{t_1}}\Big)^{G_{t_3}/G_{t_2}} \\ \oplus \Big(H^1(G_{t_1}, \mathcal{T}_{\mathcal{O}})^{G_{t_2}/G_{t_1}}\Big)^{G_{t_3}/G_{t_2}}. \end{aligned}$$

Using induction based on (35) we obtain

$$H^1(G_1, \mathcal{T}_{\mathcal{O}}) \subseteq \bigoplus_{i=1}^{f} H^1\Big(\frac{G_{t_i}}{G_{t_{i-1}}}, \mathcal{T}_{\mathcal{O}}^{G_{t_{i-1}}}\Big)^{G_{t_f}/G_{t_i}} \subseteq \bigoplus_{i=1}^{f} H^1\Big(\frac{G_{t_i}}{G_{t_{i-1}}}, \mathcal{T}_{\mathcal{O}}^{G_{t_{i-1}}}\Big),$$

and the desired result follows. □

Notice that in the above proposition $G_{t_{i-1}}$ appears in the ramification filtration of G_0 thus the corollary to Proposition IV.1.3 in [Serre 1979] implies that the ramification filtration of $G_{t_i}/G_{t_{i-1}}$ is constant. Namely, if $Q = G_{t_i}/G_{t_{i-1}}$ the ramification filtration of Q is given by $Q_0 = Q_1 = \cdots = Q_{t_i} > \{1\}$. Therefore, $\delta_{t_1} = \dim_k H^1(G_n, \mathcal{T}_{\mathcal{O}})$, and $\dim_k H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_{\mathcal{O}}^{G_{t_{i-1}}})$ can be computed explicitly by Proposition 2.9 since $G_n = G_{t_1}$, $G_{t_i}/G_{t_{i-1}}$, are elementary abelian groups. Namely we will prove:

Proposition 2.11. *Let* $\log_p(\cdot)$ *denote the logarithmic function with base* p. *Let* $s(\lambda) = \log_p |G_{t_\lambda}|/|G_{t_{\lambda-1}}|$ *and let* μ_i *be as in Proposition 1.15. Then*

$$\dim_k H^1\Big(\frac{G_{t_\lambda}}{G_{t_{\lambda-1}}}, \mathcal{T}_{\mathcal{O}}^{G_{t_{\lambda-1}}}\Big) = \sum_{i=1}^{s(\lambda)} \Big(\Big\lfloor \frac{(t_\lambda+1)(p-1)+a_i}{p} \Big\rfloor - \Big\lceil \frac{a_i}{p} \Big\rceil\Big),$$

where $a_1 = -t_\lambda - 1 + \mu_{\lambda-1}$, *and* $a_i = \lceil a_{i-1}/p \rceil$.

Proof. The module $\mathcal{T}_{\mathcal{O}}^{G_{t_{\lambda-1}}}$ is computed in Proposition 1.15 to be isomorphic to $\pi_{\lambda-1}^{\mu_{\lambda-1}}(d/d\pi_{\lambda-1})$, which in turn is $((G_{t_\lambda}/G_{t_{\lambda-1}}))$-equivariantly isomorphic to the module $\pi_{\lambda-1}^{-t_\lambda-1+\mu_{\lambda-1}} k[[\pi_{\lambda-1}]]$. The result follows using Proposition 2.9. □

Remark 2.12. If $n = 1$ (equivalently, if $G_2 = \{1\}$), the left- and right-hand sides of (34) are equal and the bound becomes the formula in [Cornelissen and Kato 2003].

Proposition 2.13. *We will follow the notation of Proposition 2.11. Suppose that for every i, $G_{t_i}/G_{t_{i-1}}$ is a cyclic p-group. Then*

$$\begin{aligned}\dim_k H^1(G_1,\mathcal{T}_{\mathcal{O}}) &= \sum_{i=1}^{f} \dim_k H^1\Big(\frac{G_{t_i}}{G_{t_{i-1}}},\mathcal{T}_{\mathcal{O}}^{G_{t_{i-1}}}\Big)^{G_{t_f}/G_{t_i}} \\ &\le \sum_{i=1}^{f}\left(\left\lfloor\frac{(t_i+1)(p-1)-t_i-1+\mu_{i-1}}{p}\right\rfloor-\left\lceil\frac{-t_i-1+\mu_{i-1}}{p}\right\rceil\right).\end{aligned}$$

Proof. The kernel of the transgression at each step is by Lemma 1.5 the whole $H^1(G_{t_i}/G_{t_{i-1}},\mathcal{T}_{\mathcal{O}}^{G_{t_{i-1}}})^{G_{t_f}/G_{t_i}}$. Therefore the right inner inequality in Equation (34) is achieved. The other inequality is trivial by the computation done in Proposition 2.11 but it is far from being best possible. □

3. Global computations

We consider the Galois cover of curves $\pi : X \to Y = X/G$, and let $b_1,\dots,b_r$ be the ramification points of the cover. We will denote by

$$e_0^{(\mu)} \ge e_1^{(\mu)} \ge e_2^{(\mu)} \ge \cdots \ge e_{n_\mu}^{(\mu)} > 1$$

the orders of the higher ramification groups at the point b_μ. The ramification divisor D of the above cover is a divisor supported at the ramification points $b_1,\dots,b_r$ and is equal to

$$D=\sum_{\mu=1}^{r}\sum_{i=0}^{n_\mu}(e_i^{(\mu)}-1)b_\mu .$$

Let Ω_X^1, Ω_Y^1 be the sheaves of holomorphic differentials at X and Y respectively. We have

$$\Omega_X^1 \cong \mathcal{O}_X(D)\otimes\pi^*(\Omega_Y^1)$$

(see [Hartshorne 1977, IV. 2.3]), and, by taking duals,

$$\mathcal{T}_X \cong \mathcal{O}_X(-D)\otimes\pi^*(\mathcal{T}_Y).$$

Thus $\pi_*(\mathcal{T}_X)\cong\mathcal{T}_Y\otimes\pi_*(\mathcal{O}_X(-D))$ and $\pi_*^G(\mathcal{T}_X)\cong\mathcal{T}_Y\otimes(\mathcal{O}_Y\cap\pi_*(\mathcal{O}_X(-D)))$. We compute (similarly with [Cornelissen and Kato 2003, Proposition 1.6]):

$$\pi_*^G(\mathcal{T}_X)=\mathcal{T}_Y\otimes\mathcal{O}_Y\left(-\sum_{\mu=1}^{r}\left\lceil\sum_{i=0}^{n_\mu}\frac{(e_i^{(\mu)}-1)}{e_0^{(\mu)}}\right\rceil b_i\right).$$

Therefore, the global contribution to $H^1(G,\mathcal{T}_X)$ is given by

$$H^1(Y,\pi_*^G(\mathcal{T}_X))\cong H^1\Big(Y,\mathcal{T}_Y\otimes\mathcal{O}_Y\Big(-\sum_{\mu=1}^{r}\cdot\Big\lceil\sum_{i=0}^{n_\mu}\frac{(e_i^{(\mu)}-1)}{e_0^{(\mu)}}\Big\rceil b_i\cdot\Big)\Big)$$
$$\cong H^0\Big(Y,\Omega_Y^{\otimes 2}\Big(-\sum_{\mu=1}^{r}\Big\lceil\sum_{i=0}^{n_\mu}\frac{(e_i^{(\mu)}-1)}{e_0^{(\mu)}}\Big\rceil b_i\Big)\Big)$$

and, by the Riemann–Roch formula,

$$\dim_k H^1(Y,\pi_*^G(\mathcal{T}_X))=3g_Y-3+\sum_{\mu=1}^{r}\Big\lceil\sum_{i=0}^{n_\mu}\frac{(e_i^{(\mu)}-1)}{e_0^{(\mu)}}\Big\rceil. \tag{38}$$

The local contribution can be bounded by Proposition 2.10 and by combining the local and global contributions, we arrive at the desired bound for the dimension.

3.1. ***Examples.*** Let $V=\mathbb{Z}/p\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/p\mathbb{Z}$ be an elementary abelian group acted on by the group $\mathbb{Z}/n\mathbb{Z}$. Assume that $G:=V\rtimes\mathbb{Z}/n\mathbb{Z}$ acts on the local field $k((t))$ and assume that the ramification filtration is given by $G_0>G_1=\cdots=G_j>G_{j+1}=\{1\}$. Let $H:=\mathbb{Z}/p\mathbb{Z}$ be the first summand of V. Let σ be a generator of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and assume that $\sigma(t)=\zeta t$, where ζ is a primitive n-th root of one. Let $A=t^{a'}k[[t]](d/dx)$ and let $A^H=x^a k[[x]](d/dx)$. The inflation-restriction sequence implies the short exact sequence

$$0\to H^1(V/H,x^a k[[x]]\tfrac{d}{dx})\to H^1(V,A)\to H^1(H,A)\to 0.$$

The group $\mathbb{Z}/n\mathbb{Z}$ acts on $t^{a'}k[[t]]$ but there is no apriori well defined action of $\mathbb{Z}/n\mathbb{Z}$ on $x^a k[[x]]\frac{d}{dx}=(t^{a'}k[[t]](d/dt))^H$, since the group H might not be normal in G. Here by this action of $\mathbb{Z}/n\mathbb{Z}$ we mean the natural module action and not the conjugation action on cocycles defined in Lemma 1.2. An element $d\in H^1(G/H,x^a k[[x]]\,d/dx)$ is sent by the inflation map on the 1-cocycle $\inf(d)$ that is a map

$$\inf(d):G\to t^{a'}g(t)\frac{d}{dt}\in t^{a'}k[[t]]\frac{d}{dt},$$

and the action of σ can be considered on the image of the inflation map, sending $\inf(d)(g)\mapsto\sigma(\inf(d(g))$. We observe that $\sigma(\inf d(g)))$ is zero for any $g\in H$, by the definition of the inflation map, therefore there is an element $a\in t^{a'}k[[t]]\,d/dt$ such that

$$\sigma(\inf d(g))+a^g-a\in x^a k[[x]]\frac{d}{dx}.$$

We can consider the element $\sigma(\inf(d))+a^g-a=\inf(d')$. This means that although there is no well defined action of $\mathbb{Z}/n\mathbb{Z}$ on $k[[x]]$ we can define $\sigma(d)=d'$ modulo cocycles. In what follows we will try to compute the element $d'\in H^1(G/H,x^a k[[x]]\,d/dx)$.

Assume that the Artin–Schreier extension $k((t))/k((x))$ is given by the equation $1/y^p - 1/y = 1/x^j$. Then, we have computed that if g is a generator if H then

$$g(t) = \frac{t}{(1+t^j)^{1/j}} \quad \text{and} \quad x = \frac{t^p}{(1-t^{j(p-1)})^{1/j}}.$$

The action of σ on x, where x is seen as an element in $k[[t]]$ is given by

$$\sigma(x) = \sigma \frac{t^p}{(1-t^{j(p-1)})^{1/j}} = \zeta^p x \frac{(1-t^{j(p-1)})^{1/j}}{(1-\zeta^{j(p-1)} t^{j(p-1)})^{1/j}} = \zeta^p x u,$$

where

$$u = \frac{(1-t^{j(p-1)})^{1/j}}{(1-\zeta^{j(p-1)} t^{j(p-1)})^{1/j}}$$

is a unit of the form $1+y$, where $y \in t^{2j(p-1)} k[[t]]$. The cohomology group $H^1(V/H, x^a k[[x]])$ is generated by the elements $\{1/x^\mu, b \le \mu \le a, \binom{\mu/n}{p-1} = 0\}$. Each element $1/x^\mu$ is written as $1/x^\mu x^{j+1}\, d/dx$ and it is lifted to

$$1/x^\mu t^{j+1} \frac{d}{dt} \xrightarrow{\sigma} \zeta^{-p\mu+j} 1/x^\mu u^{-\mu} t^{j+1} \frac{d}{dt}.$$

In the above formula we have used the fact that the adjoint action of σ on $t^r\, d/dt$ is given by $\sigma : t^r\, d/dt \to \zeta^{(r-1)} t^r\, d/dt$ [Cornelissen and Kato 2003, 3.7]. Obviously the unit u is not H-invariant but we can add to u a 1-coboundary so that it becomes the H-invariant element $\inf(d')$. This coboundary is of the form $a^g - a$, and obviously $a^g - a$ has to be in $t^{2p-1 j} k[[t]]$. This gives us that $(1/x^\mu)' = \zeta^\mu 1/x^\mu + o$, where o is a sum of terms $1/x^\nu$ with $-a < \nu$ and therefore o is cohomologous to zero. Using induction one can prove:

Lemma 3.1. *Let $1/\pi_i^{i_\lambda}$, $\lambda = 1, \dots, s$, $b_i \le i_\lambda \le -a_i$ so that $\binom{i_\lambda/n}{p-1} = 0$ and $b_i = -a_i - j$ if $p \mid a_i$, $b_i = -a_i - j + 1$ if $p \nmid a_i$ be the basis elements of the cohomology group $H^1(V, \mathcal{T}_{\mathbb{O}})$. Then the action of the generator $\sigma \in \mathbb{Z}/n\mathbb{Z}$ on $\mathcal{T}_{\mathbb{O}}$ induces the following action on the basis elements*:

$$\sigma\left(\frac{1}{\pi_i^\mu}\right) = \zeta^{-p^i \mu + j} \frac{1}{\pi_i^\mu}.$$

The Fermat curve. The curve

$$F : x_0^n + x_1^n + x_2^n = 0$$

defined over an algebraically closed field k of characteristic p, such that $n - 1 = p^a$ is a power of the characteristic is a very special curve. Concerning its automorphism group, the Fermat curve has maximal automorphism group with respect to the genus [Stichtenoth 1973]. Also it leads to Hermitian function fields, that are optimal with respect to the number of $\mathbb{F}_{p^{2a}}$-rational points and Weil's bound.

It is known that the Fermat curve is totally supersingular, i.e., the Jacobian variety $J(F)$ of F has p-rank zero, so this curve cannot be studied by the tools of [Cornelissen and Kato 2003]. The group of automorphism of F was computed in [Leopoldt 1996] to be the projective unitary group $G = PGU(3, q^2)$, where $q = p^a = n - 1$. H. Stichtenoth [Stichtenoth 1973, p. 535] proved that in the extension F/F^G there are two ramified points P, Q and one is wildly ramified and the other is tamely ramified. For the ramification group $G(P)$ of the wild ramified point P, the group $G(P)$ consists of the 3×3 matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & \chi & 0 \\ \gamma & -\chi\alpha^q & \chi^{1+q} \end{pmatrix}, \tag{39}$$

where $\chi, \alpha, \gamma \in \mathbb{F}_{q^2}$ and $\gamma + \gamma^q = \chi^{1+q} - 1 - \alpha^{1+q}$. Moreover Leopoldt proves that the order of $G(P)$ is $q^3(q^2 - 1)$ and the ramification filtration is given by

$$G_0(P) > G_1(P) > G_2(P) = \cdots = G_{1+q}(P) > \{1\},$$

where

$$G_1(P) = \ker(\chi : G_0(P) \to \mathbb{F}_{q^2}^*) \quad \text{and} \quad G_2(P) = \ker(\alpha : G_1(P) \to \mathbb{F}_{q^2}).$$

In this section we will compute the dimension of tangent space of the global deformation functor. Namely, we will prove:

Proposition 3.2. *Let p be a prime number, $p > 3$ let X be the Fermat curve*

$$x_0^{1+p} + x_1^{1+p} + x_2^{1+p} = 0.$$

Then $\dim_k H^1(X, G, \mathcal{T}_X) = 0$.

Proof. By the assumption $q = p$ and by the computations of Leopoldt mentioned above we have $G_2 = \cdots = G_{p+1} = \mathbb{Z}/p\mathbb{Z}$. The different of G_{p+1} is computed $(p+2)(p-1)$. Hence, according to (17),

$$\dim_k H^1(G_{p+1}, \mathcal{T}_{\mathcal{O}}) = \left\lfloor \frac{(p+2)(p-1) - (p+2)}{p} \right\rfloor - \left\lceil \frac{-p-2}{p} \right\rceil = p$$

Proposition 2.7 implies that the set

$$\left\{ \frac{1}{t^i}, 2 \le i \le p+2 \text{ where } \binom{i/(p+1)}{p-1} = 0 \right\}$$

is a k-basis of $H^1(G_{p+1}, \mathcal{T}_{\mathcal{O}})$. Indeed, the group G_{1+p} has conductor $1 + p$ and $\mathcal{T}_{\mathcal{O}}$ is G_{1+p}-equivariantly isomorphic to $t^{-p-2}k[[t]]$. Thus following the notation of Proposition 2.7 $-a = p+2$ and $b = 2$. The rational number $(1+p)^{-1}$ has the following p-adic expansion:

$$\frac{1}{1+p} = 1 + (p-1)p + (p-1)p^3 + (p-1)p^5 + \dots$$

and using Lemma 2.1 we obtain that for $2 \le i \le p+2$ the only integer i such that $\binom{i/(p+1)}{p-1} \neq 0$ is $i = p-1$. Thus, the elements

$$\left\{ \frac{1}{t^i} \,\middle|\, 2 \le i \le p+2, i \neq p-1 \right\}$$

form a k-basis of $H^1(G_{p+1}, \mathcal{T}_{\mathcal{O}})$.

Leopoldt in [Leopoldt 1996, 4.1] proves that the $G_0(P)$ acts faithfully on the k-vector space $L((p+1)P)$ that is of dimension 3 with basis functions $1, v, w$ and the representation matrix is given by (39). Moreover, the above functions have t-expansions $v = 1/t^p u$, where u is a unit in $k[[t]]$ and $w = 1/t^{p+1}$, for a suitable choice of the local uniformizer t at the point P. The functions v, w generate the function field corresponding to the Fermat curve and they satisfy the relation $v^n = w^n - (w+1)^n$, therefore one can compute that the unit u can be written as

$$u = 1 + t^{p+1} g, \quad g \in k[[t]].$$

Let σ be an element given by a matrix as in Equation (39). The action of $\sigma \in G_1 = G_1(P)$ on powers of $1/t$ is given by

$$\frac{1}{t^i} = \frac{(1 + \gamma t^{p+1} - a^q u t)^{i/(p+1)}}{t^i}, \tag{40}$$

and the action on the basis elements $\{1/t^i,\ 2 \le i \le p+2, i \neq p-1\}$ is given by

$$\frac{1}{t^i} \mapsto \frac{1}{t^i} + \sum_{\nu=1}^{i-2} a^{q\nu} \binom{i/(p+1)}{\nu} \frac{1}{t^{i-\nu}}.$$

The matrix of this action is given by

$$A_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{2}{p+1} & 1 & 0 & 0 & 0 \\ & \frac{3}{p+1} & 1 & 0 & 0 \\ * & * & \ddots & 1 & 0 \\ * & * & * & \frac{p+2}{p+1} & 1 \end{pmatrix}.$$

We observe that $\sigma(1/t^2) = 1/t^2$ and $\sigma(1/t^p) = 1/t^p$, and moreover that all elements below the diagonal of the matrix A_σ are $i/(p+1)$ and are nonzero unless

$i = p$. Therefore the eigenspace of the eigenvalue 1 is 2-dimensional,

$$H^1(G_{1+p}, \mathcal{T}_{\mathcal{O}})^{G_1/G_{1+p}} =_k \left\langle \frac{1}{t^2}, \frac{1}{t^p} \right\rangle$$

is a basis for it. To compute $H^1(G_1(P), \mathcal{T}_{\mathcal{O}})$ we consider the exact sequence

$$1 \to G_2 \to G_1 \xrightarrow{\alpha} G_1/G_2 \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to 1$$

and the corresponding low-degree-term Lyndon–Hochschild–Serre sequence. The group G_2 is of conductor $p+1$ thus $\mathcal{T}_{\mathcal{O}}^{G_2} = \mathcal{T}_{\mathcal{O}}^{\mathbb{Z}/p\mathbb{Z}}$ is given by Proposition 1.15 ($p > 2$):

$$\mathcal{T}_{\mathcal{O}}^{G_2} = x^{p+2-\lfloor (p+2)/p \rfloor} k[[x]] \frac{d}{dx} = x^{p+1} k[[x]] \frac{d}{dx},$$

where x is a local uniformizer for $\mathcal{O}^{G_2}$. By [Serre 1979, Corollary p. 64] the ramification filtration for G_2/G_1 is

$$G_0/G_2 > G_1/G_2 > \{1\},$$

hence the different for the subgroup $\mathbb{Z}/p\mathbb{Z}$ of G_2/G_1 is $2(p-1)$, and the conductor equals 1. Lemma 1.14 implies $x^{p+1}k[[x]]\, d/dx$ is G_1/G_2-equivariantly isomorphic to $x^{p+1-2}k[[[x]]$. Therefore,

$$H^1(G_1/G_2, \mathcal{T}_{\mathcal{O}}^{G_2}) = H^1(\mathbb{Z}/p\mathbb{Z}, x^{p-1}k[[x]]) \oplus H^1(\mathbb{Z}/p\mathbb{Z}, (x^{p-1}k[[x]])^{\mathbb{Z}/p\mathbb{Z}})$$

We compute

$$\dim_k H^1(\mathbb{Z}/p\mathbb{Z}, x^{p-1}k[[x]]) = \left\lfloor \frac{2(p-1)+p-1}{p} \right\rfloor - \left\lceil \frac{p-1}{p} \right\rceil = 1.$$

At the same time, if π is a local uniformizer for $k((x))^{\mathbb{Z}/p\mathbb{Z}}$ then

$$(x^{p-1}k[[x]])^{\mathbb{Z}/p\mathbb{Z}} = \pi^{\lceil (p-1)/p \rceil} k[[\pi]] = \pi k[[\pi]]$$

and the dimension of the cohomology group is computed:

$$\begin{aligned} \dim_k H^1(\mathbb{Z}/p\mathbb{Z}, (x^{p-1}k[[x]])^{\mathbb{Z}/p\mathbb{Z}}) &= \dim_k H^1(\mathbb{Z}/p\mathbb{Z}, \pi k[[\pi]]) \\ &= \left\lfloor \frac{2(p-1)+1}{p} \right\rfloor - \left\lceil \frac{1}{p} \right\rceil = 0. \end{aligned}$$

Using the bound for the kernel of the transgression we see that

$$\begin{aligned} 1 = \dim_k H^1\left(\frac{G_1}{G_2}, \mathcal{T}_{\mathcal{O}}^{G_2}\right) &\leq \dim_k H^1(G_1, \mathcal{T}_{\mathcal{O}}) \\ &\leq \dim_k H^1\left(\frac{G_1}{G_2}, \mathcal{T}_{\mathcal{O}}^{G_2}\right) + \dim_k H^1(G_2, \mathcal{T}_{\mathcal{O}})^{G_1/G_2} = 3. \end{aligned} \tag{41}$$

To compute the action of G_0 on G_1/G_2 we observe that

$$\begin{pmatrix} 1 & 0 & 0 \\ a & \chi & 0 \\ * & -\chi a^p & \chi^{1+p} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ * & -b^p & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & \chi^{-1} & 0 \\ * & \chi a^p & \chi^{-1-p} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \chi b & 1 & 0 \\ * & -\chi^{p+1} b^p & 1 \end{pmatrix}. \tag{42}$$

If the middle matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ * & -b^p & 1 \end{pmatrix}$$

is an element of $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p \subset \mathbb{F}_{p^2}$ then $b^p = b$. By looking at the computation of (42) we see that the conjugation action of G_0/G_1 to $\mathbb{F}_p$ is given by multiplication $b \mapsto \chi^{1+q} b$. Observe that $(\chi^{1+p})^{p-1} = \chi^{p^2-1} = 1$, thus $\chi^{1+p} \in \mathbb{F}_p$. The action on the cocycles is given by sending the cocycle $d(\tau)$ to $d(\sigma\tau\sigma^{-1})^{\sigma^{-1}}$ therefore the basis cocycle $1/x^{1-p}$ of the one dimensional cohomology group $H^1(\mathbb{Z}/p\mathbb{Z}, x^{p-1}k[[x]])\, d$ goes to $\chi^{p(p-1)+1+1+p} d = \chi^{-p^2} d$ under the conjugation action, as one sees by applying Lemma 3.1. Lemma 1.3 implies that

$$H^1(G_1/G_2, \mathcal{T}_{\mathbb{O}}^{G_2})^{G_0/G_1} = 0.$$

Similarly the conjugation action of G_0/G_1 on an element of G_2 can be computed to be multiplication of τ by $\chi^{1+p} \in \mathbb{F}_p$, and the same argument shows that $H^1(G_{1+p}, \mathcal{T}_{\mathbb{O}})^{G_1/G_{1+p}\, G_0/G_1} = 0$.

Finally the global contribution is computed by formula (38)

$$\begin{aligned} \dim_k H^1(F^G, \pi_*(\mathcal{T}_F)) &= -3 + \left\lceil \sum_{i=0}^{p+2} \frac{|G(P)_i| - 1}{|G(P)|} \right\rceil + \left\lceil 1 - \frac{1}{|G(Q)|} \right\rceil \\ &= -3 + 2 + 1 = 0. \end{aligned}$$

The fact that the tangent space of the deformation functor is zero dimensional is compatible with the fact that there is only one isomorphism class of curves C such that $|\mathrm{Aut}(C)| \geq 16 g_C^4$ [Stichtenoth 1973]. □

***p–Covers of* $\mathbb{P}^1(k)$.** We consider curves C_f of the form

$$C_f : w^p - w = f(x),$$

where $f(x)$ is a polynomial of degree m. We will say that such a curve is in reduced form if the polynomial $f(x)$ is of the form

$$f(x) = \sum_{i=1, (i,p)=1}^{m-1} a_i x^i + x^m.$$

Two such curves C_f, C_g in reduced form are isomorphic if and only if $f = g$. The group $G := \mathrm{Gal}\left(C_f/\mathbb{P}^1(k)\right) \cong \mathbb{Z}/p\mathbb{Z}$ acts on C_f. The number of independent monomials $\neq x^m$ in the sums above is given by

$$m - \left\lfloor \frac{m}{p} \right\rfloor - 1, \tag{43}$$

since $\#\{1 \le i \le m,\, p \mid i\} = \lfloor m/p \rfloor$.

We will compute the tangent space of the deformation functor of the curve C_f together with the group C_f. Let P be the point above $\infty \in \mathbb{P}^1(k)$. This is the only point that ramifies in the cover $C_f \to \mathbb{P}^1(k)$, and the group G admits the ramification filtration

$$G_0 = G_1 = G_2 = \cdots = G_m > G_{m+1} = \{1\}.$$

The different is computed $(p-1)(m+1)$ and $\mathcal{T}_{\mathbb{O}} \cong t^{-m-1}k[[t]]$. Thus the space $H^1(G, \mathcal{T}_{\mathbb{O}})$ has dimension

$$d = \left\lfloor \frac{(p-1)(m+1)-(m+1)}{p} \right\rfloor - \left\lceil \frac{-(m+1)}{p} \right\rceil = m+1-\left\lceil \frac{2m+2}{p} \right\rceil + \left\lfloor \frac{m+1}{p} \right\rfloor.$$

Let $a_0 + a_1 p + a_2 p^2 + \cdots$ be the p-adic expansion of $m+1$. We observe that

$$\left\lceil \frac{2m+2}{p} \right\rceil - \left\lfloor \frac{m+1}{p} \right\rfloor = \left\lceil \frac{2a_0}{p} + \sum_{i \ge 1} 2a_i p^{i-1} \right\rceil - \sum_{i \ge 1} a_i p^{i-1},$$

therefore, if $p \nmid m+1$

$$\left\lceil \frac{2m+2}{p} \right\rceil - \left\lfloor \frac{m+1}{p} \right\rfloor = \left\lfloor \frac{m+1}{p} \right\rfloor + \delta, \quad \text{where } \delta = \begin{cases} 2 & \text{if } 2a_0 > p, \\ 1 & \text{if } 2a_0 < p. \end{cases}$$

Thus, we have for the dimension

$$d = \begin{cases} m+1-\lfloor (m+1)/p \rfloor & \text{if } p \mid m+1, \\ m - \lfloor (m+1)/p \rfloor - \delta & \text{otherwise.} \end{cases}$$

Finally, we compute that

$$\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X)) = -3 + \left\lceil \frac{(m+1)(p-1)}{p} \right\rceil = m - 2 - \left\lfloor \frac{m+1}{p} \right\rfloor.$$

Lehr–Matignon curves. Consider the curve

$$C : y^p - y = \sum_{i=0}^{m-1} t_i x^{1+p^i} + x^{1+p^m} \tag{44}$$

defined over an algebraically closed field k of characteristic $p > 2$. Such curves were examined in [van der Geer and van der Vlugt 1992] in connection with coding theory, and their automorphism group was studied in [Lehr and Matignon 2005], (The extreme Fermat curves studied in example 1 can be written in this form after a suitable transformation; see [Stichtenoth 1993, VI.4.3, p. 203].)

Let $n = 1 + p^m$ denote the degree of the right-hand side of (44), and set $H = \mathrm{Gal}(C/\mathbb{P}^1(k))$. The automorphism group G of C can be expressed in the form

$$1 \to H \to G \to V \to 1,$$

where V is the vector space of roots of the additive polynomial

$$\sum_{0 \le i \le m} \left(t_i^{p^{m-i}} Y^{p^{m-i}} + t_i^{p^m} Y^{p^{m+i}} \right) \tag{45}$$

[Lehr and Matignon 2005, Proposition 4.15]. Moreover there is only one point $P \in C$ that ramifies in the cover $C \to C^G$, namely the point above $\infty \in \mathbb{P}^1(k)$.

In order to simplify the calculations we assume that $t_0 = \cdots = t_{m-1} = 0$ so the curve is given by

$$y^p - y = x^{p^m+1}. \tag{46}$$

The polynomial in (45) is given by $Y^{p^{2m}} + Y$ and the vector space V of the roots is $2m$-dimensional. Moreover, according to [Lehr and Matignon 2005] any automorphism σ_v corresponding to $v \in V$ is given by

$$\sigma_v(x) = x + v, \quad \sigma(y) = y + \sum_{\kappa=0}^{m-1} v^{p^{m+k}} x^{p^k}.$$

Observe that w and x have a unique pole of order $p^m + 1$ and p, respectively, at the point above ∞, so we can select the local uniformizer π so that

$$y = \frac{1}{\pi^{p^m+1}}, \quad x = \frac{1}{\pi} u,$$

where u is a unit in $k[[\pi]]$. By replacing x, y in (46) we observe that the unit u is of the form $u = 1 + \pi^{p^m}$.

A simple computation based on the basis $\{1, x, \ldots, x^{p^{m-1}}, y\}$, of the vector space $L\big((1 + p^m)P\big)$ given in [Lehr and Matignon 2005, Proposition 3.3]. shows that the ramification filtration of G is $G = G_0 = G_1 > G_2 = \ldots = G_{p^m+1} > \{1\}$, where $G_2 = H$ and $G_1/G_2 = V$. Using Proposition 2.7 we obtain the basis

$$\left\{ \frac{1}{\pi_1^i} \,\middle|\, 2 \le i \le p^m + 2 \text{ and } \binom{i/(p^m+1)}{p-1} = 0 \right\}$$

for $H^1(G_2, \mathcal{T}_{\mathbb{O}})$. We have to study the action of G_1/G_2 on $H^1(G_2, \mathcal{T}_{\mathbb{O}})$. From the action of σ_v on y we obtain that the action on the basis elements of $H^1(G_{p^m+1}, \mathcal{T}_{\mathbb{O}})$

is given by

$$\sigma_v\left(\frac{1}{\pi^i}\right) = \frac{1}{\pi^i} + \frac{\left(\sum_{\kappa=0}^{m-1} u^{p^k} v^{p^{m+\kappa}} \pi^{p^m+1-p^{k+1}}\right)^{i/(p^m+1)}}{\pi^i}$$
$$= \frac{1}{\pi^i} + \frac{i}{p^m+1} v^{p^{2m-1}} \frac{1}{\pi^{i-1}} + \dots .$$

If $p \mid i$, all binomial coefficients $\binom{i/(p^m+1)}{\kappa}$ that contribute a coefficient $1/\pi^\kappa$, $2 \leq \kappa \leq p^m+2$ are zero. Therefore, the elements $1/\pi^2$, $1/\pi^{\nu p}$ are invariant. Moreover, by writing down the action of σ_v as a matrix we see that there are no other invariant elements, so the dimension is computed ($p > 2$):

$$\dim_k H^1(G_{p^m+1}, \mathcal{T}_{\mathcal{O}})^{G_1/G_{p^m+1}} = 1 + \left\lfloor \frac{p^m+2}{p} \right\rfloor = 1 + p^{m-1}.$$

This dimension coincides with the computation done on the Fermat curves $m = 1$.

We proceed by computing $H^1(V, \mathcal{T}_{\mathcal{O}}^H)$. The space $\mathcal{T}_{\mathcal{O}}^H$ is computed by Proposition 2.9

$$x^{p^m+2-\lfloor (p^m+2)/p \rfloor} k[[x]] \frac{d}{dx} = x^{p^m+2-p^{m-1}} k[[x]] \frac{d}{dx}.$$

Thus,

$$\dim_k H^1(V, \mathcal{T}_{\mathcal{O}}^H) = \sum_{\nu=1}^{2m} \left(\left\lfloor \frac{2(p-1)+a_i}{p} \right\rfloor - \left\lceil \frac{a_i}{p} \right\rceil \right),$$

where $a_1 = p^m - p^{m-1}$, and $a_i = \lceil a_{i-1}/p \rceil$. By computation $a_\nu = p^{m-\nu+1} - p^{m-\nu}$ for $1 \leq \nu \leq m$, and $a_\nu = 1$ for $\nu > m$. Moreover, an easy computation shows that

$$\left\lfloor \frac{2(p-1)+a_i}{p} \right\rfloor - \left\lceil \frac{a_i}{p} \right\rceil = \begin{cases} 1 & \text{if } 1 \leq \nu < m, \\ 2 & \text{if } \nu = m, \\ 0 & \text{if } m < \nu, \end{cases}$$

thus the dimension of the tangent space is $m+1$.

We have proved that the dimension of $H^1(G_1, \mathcal{T}_{\mathcal{O}})$ is bounded by

$$m+1 = \dim_k H^1(G_1/G_2, \mathcal{T}_{\mathcal{O}}^{G_2}) \leq H^1(G_1, \mathcal{T}_{\mathcal{O}})$$
$$\leq \dim_k H^1(G_1/G_2, \mathcal{T}_{\mathcal{O}}^{G_2}) + H^1(G_2, \mathcal{T}_{\mathcal{O}})^{G_1/G_2} = 2 + m + p^{m-1}.$$

Unfortunately we cannot be more precise here: an exact computation involves the computation of the kernel of the transgression and such a computation requires new ideas and tools.

To this dimension we must add the contribution of

$$\begin{aligned}\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X)) &= 3g_Y - 3 + \sum_{\kappa=1}^{r} \left\lceil \sum_{i=0}^{n_\kappa} \frac{(e_i^{(\kappa)} - 1)}{e_0^{(\kappa)}} \right\rceil \\ &= -3 + \left\lceil 2\frac{p^{m+1}-1}{p^{m+1}} + m\frac{p^m - 1}{p^{m+1}} \right\rceil = -1 + \left\lceil \frac{m}{p} - \frac{2+m}{p^{m+1}} \right\rceil.\end{aligned}$$

The latter contribution is strictly positive if $m \gg p$.

Elementary abelian extensions of $\mathbb{P}^1(k)$. Consider the curve C so that $G_0 = (\mathbb{Z}/p\mathbb{Z})^s \rtimes \mathbb{Z}/n\mathbb{Z}$ is the ramification group of wild ramified point, and moreover the ramification filtration is given by

$$G_0 > G_1 = \ldots = G_j > G_{j+1} = \{1\}. \tag{47}$$

An example of such a curve is provided by the curve defined by

$$C : \sum_{i=0}^{s} a_i y^{p^i} = f(x), \tag{48}$$

where f is a polynomial of degree j and all monomial summands $a_k x^k$ of f have exponent congruent to j modulo n. Let V be the $\mathbb{F}_p$-vector space of the roots of the additive polynomial $\sum_{i=0}^{s} a_i y^{p^i}$. Assume that the automorphism group of the curve defined by (48) is $G := V \rtimes \mathbb{Z}/n\mathbb{Z}$. Thus $C \to \mathbb{P}^1(k)$ is Galois cover ramified only above ∞, with ramification group G and ramification filtration is computed to be as in (47).

Let us now return to the general case. Let us denote by V the group $(\mathbb{Z}/p\mathbb{Z})^s$. The group V admits the structure of a $\mathbb{F}_p$ vector space, where $\mathbb{F}_p$ is the finite field with p elements. The conjugation action of $\mathbb{Z}/n\mathbb{Z}$ on V implies a representation

$$\rho : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}(V).$$

Since $(n, p) = 1$, Mascke's Theorem gives that V is the direct sum of simple $\mathbb{Z}/n\mathbb{Z}$-modules, i.e., $V = \bigoplus_{i=1}^{r} V_i$. On the other hand, Lemma 1.2 implies that the conjugation action is given by multiplication by ζ^j, where ζ is an appropriate primitive n-th root of one and j is the conductor of the extension. If $\zeta^j \in \mathbb{F}_p$ then all the V_i are one dimensional. In the more general case one has to consider representations

$$\rho_i : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}(V_i),$$

where $\dim_{\mathbb{F}_p} V_i = d$. The dimension d is the degree of the extension $\mathbb{F}_q/\mathbb{F}_p$, where $\mathbb{F}_q$ is the smallest field containing ζ^j. Let $e_1^{(i)}, \ldots, e_d^{(i)}$ be an $\mathbb{F}_p$-basis of V_i, and denote by $(a_{\mu\nu}^{(i)})$ the entries of the matrix corresponding to $\rho_i(\sigma)$, where σ is a generator of $\mathbb{Z}/n\mathbb{Z}$. The conjugation action on the arbitrary

$$v = \sum_{i=1}^{r} \sum_{\mu=1}^{d} \lambda_{\mu}^{(i)} e_{\mu}^{(i)} \in V \tag{49}$$

is described by

$$\sigma e_{\mu}^{(i)} \sigma^{-1} = \sum_{\nu=1}^{d} a_{\mu\nu}^{(i)} e_{\nu}^{(i)}. \tag{50}$$

For the computation of $H^1(G, \mathcal{T}_{\mathcal{O}})$, we notice first that the group $H^1(V, \mathcal{T}_{\mathcal{O}})$ can be computed using Proposition 2.8 and the isomorphism $\mathcal{T}_{\mathcal{O}} \cong t^{-j-1}k[[t]]$.

Next we consider the conjugation action of $\mathbb{Z}/n\mathbb{Z}$ on $H^1(V, \mathcal{T}_{\mathcal{O}})$, in order to compute $H^1(G, \mathcal{T}_{\mathcal{O}}) = H^1(V, \mathcal{T}_{\mathcal{O}})^{\mathbb{Z}/n\mathbb{Z}}$. By (32) we have

$$H^1(V, \mathcal{T}_{\mathcal{O}}) = \bigoplus_{\lambda=1}^{s} H^1\big(\mathbb{Z}/p\mathbb{Z}, \mathcal{T}_{\mathcal{O}}^{\bigoplus_{\nu=1}^{\lambda-1} \mathbb{Z}/p\mathbb{Z}}\big), \tag{51}$$

i.e., the arbitrary cocycle d representing a cohomology class in $H^1(V, \mathcal{T}_{\mathcal{O}})$ can be written as a sum of cocycles d_ν representing cohomology classes in

$$H^1\big(\mathbb{Z}/p\mathbb{Z}, \mathcal{T}_{\mathcal{O}}^{\bigoplus_{\nu=1}^{i-1} \mathbb{Z}/p\mathbb{Z}}\big).$$

Let us follow a similar to (49) notation for the decomposition of d, and write $d = \sum_{i=1}^{r} \sum_{\nu=1}^{d} b_{\nu}^{(i)} d_{\nu}^{(i)}$, where $d_{\nu}^{(i)}(e_{\mu}^{(j)}) = 0$ if $i \neq j$ or $\nu \neq \mu$. Therefore,

$$d(\sigma e_{\mu}^{(i)} \sigma^{-1}) = d\left(\sum_{\nu=1}^{d} a_{\mu\nu}^{(i)} e_{\nu}^{(i)} \right) = \sum_{\nu=1}^{d} b_{\nu}^{(i)} a_{\mu\nu}^{(i)} d_{\nu}^{(i)}(e_{\nu}^{(i)}). \tag{52}$$

We have now to compute the $\mathbb{Z}/n\mathbb{Z}$-action on $d_k^{(i)}$. By Lemma 3.1 the element σ acts on the basis elements $1/\pi_i^{\mu}$ of $H^1(V, \mathcal{T}_{\mathcal{O}})$ as follows

$$\sigma\left(\frac{1}{\pi_i^{\mu}}\right) = \zeta^{-p^i \mu + j} \frac{1}{\pi^{\mu}}. \tag{53}$$

By the remarks above we arrive at

$$\sigma(d)(e_{\mu}^{(i)}) := d(\sigma e_{\mu}^{(i)} \sigma^{-1})^{\sigma^{-1}} = \sum_{\nu=1}^{d} b_{\nu}^{(i)} a_{\mu\nu}^{(i)} \zeta^{-c(\nu,i)} d_{\nu}^{(i)}(e_{\nu}^{(i)}), \tag{54}$$

where $c(\nu, i)$ is the appropriate exponent, defined in (53). Let us denote by $A^{(i)}$ the $d \times d$ matrix $(a_{\nu\mu}^{(i)})$. By (54) $\sigma(d)(e_{\mu}^{(i)}) = d(e_{\mu}^{(i)})$ if and only if $b := (b_1^{(i)}, \dots, b_d^{(i)})$ is a solution of the linear system

$$(A^{(i)} \cdot \mathrm{diag}^{-1}(\zeta^{c(1,i)}, \zeta^{c(2,i)}, \dots, \zeta^{c(d,i)}) - \mathbb{I}_d) b = 0.$$

This proves that the dimension of the solution space is equal to the dimension of the eigenspace of the eigenvalue 1 of the matrix: $A^{(i)}\,\mathrm{diag}^{-1}(\zeta^{c(1,i)}, \zeta^{c(2,i)}, \dots, \zeta^{c(d,i)})$.

Moreover using a basis of the form $1, \zeta, \zeta^2, \dots, \zeta^{d-1}$ for the simple space $V(i)$, we obtain that

$$A^{(i)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & & -a_1 \\ 0 & 1 & \ddots & & \vdots \\ 0 & & 1 & 0 & -a_{d-1} \\ 0 & & & 1 & -a_d \end{pmatrix}$$

It can be proved by induction that the characteristic polynomial of $A^{(i)}$ is $x^d + \sum_{\nu=0}^{d-1} a_\nu x^\nu$, and under an appropriate basis change $A^{(i)}$ can be a written in the form $\mathrm{diag}(\zeta^j, \zeta^{2j}, \dots, \zeta^{j(d-1)})$. Moreover, the characteristic polynomial of the matrix $A^{(i)}\,\mathrm{diag}^{-1}(\zeta^{c(1,i)}, \zeta^{c(2,i)}, \dots, \zeta^{c(d,i)})$ can be computed inductively to be

$$f_i(x) := x^d + \zeta^{c(d,i)} a_{d-1} x^{d-1} + \zeta^{c(d,i)+c(d-1,i)} a_{d-1} + \cdots + \zeta^{\sum_{\nu=2}^{d} c(\nu,i)} x + \zeta^{\sum_{\nu=1}^{d} c(\nu,i)} a_0.$$

If, $f_i(1) \neq 0$, then we set $\delta(i) = 0$. If $f_i(1) = 0$ we set $\delta(i)$ to be the multiplicity of the root 1. The total invariant space has dimension

$$\dim_k H^1(G, \mathcal{T}_{\mathcal{O}}) = \sum_i \delta(i).$$

Comparison with the work of Cornelissen–Kato. We will apply the previous calculation to the case of ordinary curves $j = 1$ and we will obtain the formulas in [Cornelissen and Kato 2003]. We will follow the notation of Proposition 2.8. The number $a_1 = -j - 1 = -2$. Thus, $a_2 = \lceil -2/p \rceil = -\lfloor 2/p \rfloor = 0$ (recall that we have assumed that $p \geq 5$). Furthermore $a_i = 0$ for $i \geq 2$. For the numbers b_i we have $b_1 = -a_1 - j + 1 = 2$, and $b_1 \leq i_1 \leq -a_1$, so there is only one generator, namely $1/\pi_1^2$. Moreover, for $i \geq 2$ we have $b_i = -a_j - j = -1$ and there are two possibilities for $-1 \leq i_\lambda \leq 0 = -a_i$, namely $-1, 0$. But only $\binom{0/n}{p-1} = 0$, and we finally obtain

$$H^1(V, \mathcal{T}_{\mathcal{O}}) \cong \left\langle \frac{1}{\pi_1^2} \right\rangle_k \times \langle 1 \rangle_k \times \cdots \times \langle 1 \rangle_k,$$

a space of dimension $\log_p |V|$.

Let d be the dimension of each simple direct summand of $H^1(V, \mathcal{T}_{\mathcal{O}})$ considered as a $\mathbb{Z}/n\mathbb{Z}$-module. Of course d equals the degree of the extension $\mathbb{F}_p(\zeta)/\mathbb{F}_p$, where ζ is a suitable primitive root of 1. For the matrix $\mathrm{diag}(\zeta^{c(1,i)}, \dots, \zeta^{c(d,i)})$ we have

$$\mathrm{diag}(\zeta^{c(1,i)}, \dots, \zeta^{c(d,i)}) = \begin{cases} \mathrm{diag}(\zeta^2, \zeta, \dots, \zeta) & \text{if } i = 1, \\ \zeta \cdot \mathbb{I}_d & \text{if } i \geq 2. \end{cases}$$

The characteristic polynomial in the first case is computed to be

$$f_1(x) = x^d + \sum_{\nu=1}^{d-1} \zeta^{d-\nu} a_\nu x^\nu + a_0 \zeta^{1+d}.$$

Setting $x = \zeta y$ this becomes

$$\zeta^d \left(y^d + \sum_{\nu=0}^{d-1} a_\nu y^\nu \right) + (\zeta^{d+1} - \zeta^d) a_0.$$

Therefore $y = \zeta$ for $x = 1$, so $f_1(1) = (\zeta^{d+1} - \zeta^d) a_0 \neq 0$, therefore $\delta(1) = 0$.

In the second case, we observe that

$$f_i(x) = x^d + \sum_{\nu=0}^{d-1} \zeta^{d-\nu} a_\nu x^\nu.$$

If we set $x = y/\zeta$, we obtain that 1 is a simple root of f_i, so $\delta(i) = 1$ for $i \geq 2$.

Thus, only the $s/d - 1$ blocks $i \geq 2$ admit invariant elements and

$$\dim_k H^1(V \rtimes \mathbb{Z}/n\mathbb{Z}, \mathcal{T}_\mathcal{O}) = s/d - 1.$$

The global contribution can be computed in terms of (38) and gives us that

$$\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X)) = 3g_Y - 3 + 2r_w + r_t - \sum_{\mu=1}^{r} \left\lceil \frac{1}{n_i} \left(1 + \frac{1}{p^{s^i}} \right) \right\rceil,$$

where n_i is the order of the prime to p part and p^{s_i} is the order of the p-part of the decomposition group at the i-th ramification point of the cover $X \to X/G = Y$. The numbers r_w, r_t are the number of wild,tame ramified points of the above cover, respectively.

Comparison with the work of R. Pries. Consider the curve

$$C : y^p - y = f(x),$$

where $f(x)$ is a polynomial of degree j, $(j, p) = 1$. This, gives rise to a ramified cover of $\mathbb{P}^1(k)$ with ∞ as the unique ramification point. Moreover if all the monomial summands of the polynomial $f(x)$ have exponents congruent to $j \bmod m$, then the curve C admits the group $G := \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/m\mathbb{Z}$ as a subgroup of the group of automorphisms. R. Pries [2002] constructed a configuration space C of deformations of the above curve and computed the dimension C. More precisely by the term *configuration space* we mean a k-scheme C that represents a contravariant functor F from the category of irreducible k-schemes S to the category of sets so that, there is a morphism $T : \mathrm{Hom}(\cdot, C) \to F(\cdot)$ so that it induces a bijection between the k-points of the configuration space C and $F(\mathrm{Spec}(k))$, and if $\phi_S \in F(S)$ then there is a finite radical morphism $i : S' \to S$ and a unique morphism $f : S' \to C$ such that $T(f) = i^*\phi_S$. Pries considered the functor $F_{G,j}$ from irreducible k-schemes S to sets, defined as follows: $F_{G,j}$ is the set of equivalence classes of G–Galois covers $\phi_S : Y_S \to \underline{\mathrm{Spec}(\mathbb{O}_S[[u^{-1}]])}$ ramified only above the horizontal divisor ∞_S defined by $u^{-1} = 0$ and with constant jump j. Two such covers ϕ_S, ϕ'_S are considered to be equivalent if they are isomorphic after pullback by a finite radical morphism $S' \to S$. We refer the reader to the article of Pries for more information about this configuration space C. Pries [2002] proved that C is of dimension

$$r := \#\{e \in E_0 : \text{ for all } \nu \in \mathbb{N}^+, \, p^\nu e \notin E_0\} \tag{55}$$

where $E_0 := \{e : 1 \leq e \leq j, e \equiv j \bmod m\}$. Notice that by considering equivalence classes of G-covers we state a local version of the G-deformation problem. Moreover since we assume that the jump remains constant we are considering deformations that do not split the branch locus. It would be interesting to compare the result of Pries to our computation of $H^1(G, \mathcal{T}_\mathbb{O})$ at a wild ramified point.

We calculate $\dim_k(G, \mathcal{T}_\mathbb{O})$ as follows: According to Proposition 2.7 the tangent space of the deformation space is generated as a k-vector space by the elements of the form $1/x^i$ where $b \leq i \leq j+1$ and

$$b = \begin{cases} 1 & \text{if } p \mid -j-1, \\ 2 & \text{if } p \nmid -j-1. \end{cases}$$

By Lemma 1.4 the $\mathbb{Z}/m\mathbb{Z}$-action on $\mathbb{F}_p$ is given by multiplication by ζ^j where ζ is an appropriate primitive m-th root of unity. This gives us that $\zeta^{jp} = \zeta^j$, i.e. $jp \equiv j \bmod m$. If d_i is the cocycle corresponding to the element $1/x^i$ then

$$d_i(\sigma\tau\sigma^{-1})^{\sigma^{-1}} = \zeta^j d_i(\tau)^{\sigma^{-1}}.$$

But the element $1/x^i$ corresponds to the element $x^{j+1-i}\, d/dx$. The ζ^{-1}-action is given by

$$x^{j+1-i}\frac{d}{dx} \mapsto \zeta^{i-j} x^{j+1-i}\frac{d}{dx}.$$

Therefore, the action of σ on the cocycle corresponding to $1/x^i$ is given by $1/x^i \mapsto \zeta^i(1/x^i)$. Thus, $\dim_k H^1(G, \mathcal{T}_\mathbb{O}) = \dim_k H^1(\mathbb{Z}/p\mathbb{Z}, \mathcal{T}_\mathbb{O})^{\mathbb{Z}/p\mathbb{Z}}$ is equal to

$$\#\left\{ i : b \leq i \leq j+1, \, \binom{i/j}{p-1} = 0, \, i \equiv 0 \bmod m. \right\} \tag{56}$$

By (38) we have

$$\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X)) = 3g_Y - 3 + \sum_{\kappa=1}^{r} \left\lceil \sum_{i=0}^{n_\kappa} \frac{(e_i^{(\kappa)} - 1)}{e_0^{(\kappa)}} \right\rceil,$$

and by computation we get

$$\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X)) = -3 + \left\lceil 1 + \frac{-1}{mp} + \frac{j(p-1)}{mp} \right\rceil.$$

The dimension formulas of (55) and (56) look quite different, but using Maple[1] we computed the table

p	j	m	r	$\dim_k H^1(G, \mathcal{T}_{\mathcal{O}})$	$\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X))$	$\dim_k D(k[\epsilon])$
13	19	6	3	3	1	4
13	35	6	5	4	9	13
13	51	6	8	8	6	14
13	36	3	12	11	10	21
7	81	3	24	23	22	45
7	90	3	26	26	24	50

We observe that $r+a = \dim_k H^1(G, \mathcal{T}_{\mathcal{O}})$, where $a = 1, 0$, and also the dimension of $H^1(Y, \pi_*^G(\mathcal{T}_X))$ is near the two values above. By the difference of the formulas and by the fact that all infinitesimal deformations in $H^1(Y, \pi_*^G(\mathcal{T}_X))$ are unobstructed we obtain that the difference in the dimensions r and $\dim_k D(k[\epsilon])$ can be explained either as obstructed deformations or as deformations splitting the branch points; see Section 1.1.

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