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Weierstrass semigroups for maximal curves realizable as Harbater–Katz–Gabber covers

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Abstract: We present a necessary and sufficient condition for a maximal curve, defined over the algebraic closure of a finite field, to be realised as an HKG-cover. We use an approach via pole numbers in a rational point of the curve. For this class of curves, we compute their Weierstrass semigroup as well as the jumps of their higher ramification filtrations at this point, the unique ramification point of the cover.

Keywords: Automorphism, maximal curve, Weierstrass semigroup, Weierstrass point, Katz–Gabber cover, higher ramification filtration, zero p-rank.

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1 Introduction

Let X be a projective nonsingular algebraic curve of genus $g \geq 2$ and denote by F its function field. Maximal curves are the curves that attain the Hasse–Weil bound $|X(\mathbb{F}_{q^2})| \leq q^2 + 1 + 2gq$ for their number of rational points over the finite field $\mathbb{F}_{q^2}$ where q is a p-power. The interest in such curves with many rational points was renewed after Goppa's construction of codes with good parameters from such curves, see [16], and many interesting applications in coding theory and cryptography arise from them. For a survey article see [8]; some other sources are [6; 38; 17; 11; 2; 13] as well as the books [33; 18]. Some open problems concerning maximal curves are their classification and the structure of their Weierstrass semigroups at rational points; the latter is also closely connected to the construction of good Algebraic Geometry (AG) codes, see [19; 3].

For curves X defined over $\mathbb{F}_q$, we can also consider the Frobenius linear series $\mathcal{F}$ of X which plays a central role, especially in applications to maximal curves; see [18], [37, Sections 3 & 4] and [35]. The degree of $\mathcal{F}$ arises as a certain value of the product of the $\mathbb{Z}$-irreducible factors of the characteristic polynomial of the Frobenius endomorphism, acting on the Jacobian of X. For maximal curves, $\mathcal{F}$ has the form $\mathcal{F} = |(q+1)P|$ where P is any $\mathbb{F}_{q^2}$-rational point, see [18, Section 10.2]. For the many arithmetic interpretations of this, the reader is referred to [18, Section 9.8] and [37]. Any information on the Frobenius linear series provides insight into the problem of the classification of maximal curves, see for example [4].

Here, we are interested in maximal curves (X, G), where $G \subseteq \mathrm{Aut}(X)$ is a p-group with $p > 3$, viewed over an algebraic closure of their field of definition $\bar{\mathbb{F}}_{q^2}$, that can be realized as Harbater–Katz–Gabber or HKG-covers, i.e. as p-group Galois covers $X \to X/G \simeq \mathbb{P}^1$ with only one fully and wildly ramified point P. These covers are important because of the Katz–Gabber compactification of Galois actions on complete local rings, while they also provide examples of curves with the most automorphisms, see Section 2.1. We discover that *among the maximal curves* (X, G), *the ones that occur as HKG-covers with Galois group G are those with a certain pole number at their Weierstrass semigroup at their unique wildly ramified point* (Theorem 2).

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Moreover, for such curves we know

- a lower bound for the order of G, see [15, Theorem 1.1],
- the jumps for the higher ramification filtrations at P, see [21, Theorem 14],
- the Weierstrass semigroup at P, see Corollary 3 and Lemma 4 here, and
- that the unique ramification point of the cover is a Weierstrass point, see Corollary 3.

2 Zero p-rank curves

2.1 Where the most automorphisms occur. Let us focus on the case of curves X with zero p-rank. More precisely, in this case every p-subgroup G of the automorphism group $\mathrm{Aut}(X)$ of our curve X can be realized as the stabilizer of a *unique* place, see for example [18, paragraph 11.13]. Thus we can suppose that $G = G_0(P) = G_1(P)$ for some $P \in X$. This means that the Galois cover

$$\pi : X \longrightarrow X/G_1(P) \tag{1}$$

is wildly ramified at the *unique* point P. The case where $X/G_1(P)$ is rational naturally leads to HKG-covers. This is also the case where the most automorphisms occur: if not, it is well known that $|G_1(P)|$ is upper bounded by the genus of the curve, see [18, Theorem 11.78 (i)]. A useful criterion for the cover (1) to be an HKG-cover is $|G_1(P)|$ to be a pole number at the point P, see [21, Corollary 18]. On the other hand, it is well known that every curve X that admits an HKG-cover has zero p-rank; we summarize [21, Theorem 40]:

Theorem 1. *The following conditions are equivalent:*

(1) *the curve X has zero p-rank and $|G|$ is a pole number at the unique point $P \in X$ that G stabilizes;*
(2) *the cover $X \to X/G$ is an HKG-cover.*

The above theorem is valid for any curve X defined over any algebraically closed field of positive characteristic $p > 3$. For the study of the automorphism group in the case where X has zero 2-rank, see [14]. Well known examples of HKG-covers with many automorphisms are the curves equipped with big actions: recall that a curve X together with a subgroup G of the automorphism group of X is called a big action if G is a p-group and $\frac{|G|}{g} > \frac{2p}{p-1}$. All big actions have the following property, see [25]: if (X, G) is a big action, then there is a unique point P of X such that $G_1(P) = G$, the group $G_2(P)$ is not trivial and strictly contained in $G_1(P)$ and the quotient $X/G_2(P)$ is isomorphic to $\mathbb{P}^1$. The reader should keep in mind that the jumps of the higher ramification filtrations (see [31] for an introduction) for HKG-covers affect the structure of the Weierstrass semigroup at P; for a big action the latter is given in [21, Corollary 39].

2.2 Over finite fields. One class of zero p-rank curves are the maximal curves. All known families of maximal curves with $|G_1(P)|$ a pole number can be also described as HKG-covers. We show in Theorem 2 that this condition for the maximal curves viewed over $\bar{\mathbb{F}}_{q^2}$ is equivalent to $q \le |G_1(P)|$. Note that this last condition is true for all the “generic" families of maximal curves that we know: the Hermitian curve, the (generalized) Giulietti–Korchmáros curve (see [13], [9] and [17]), the Garcia–Stichtenoth curve [10]; while for all maximal curves is true that $q, q+1 \in H(P)$ for a $\mathbb{F}_{q^2}$-rational point P. Finally, when $m_r = q+1$ then the linear series $|m_r P|$ that naturally arises from [23, Proposition 2.3] is called the Frobenius linear series and it is an invariant of the curve at a rational point, see [18] and Remark 6.

Although these curves are naturally defined over $\mathbb{F}_{q^2}$, here we view them over some algebraic closure $\bar{\mathbb{F}}_{q^2}$.

For an abelian $G_1(P)$, maximal curves and curves with many automorphisms are connected via the theory of global ray class fields, see [24] and [1], and through the identification of “many rational points” with “many automorphisms”, see [26].

Here, we compute explicitly the Weierstrass semigroups $H(P)$ for maximal curves satisfying the condition $q \le |G_1(P)|$. Our motivation are the many connections of these semigroups with the construction of good AG (Algebraic Geometric) codes, see [19]. These numerical semigroups have a special structure: all their minimal generators but the last one are divisible by the characteristic of the field. Moreover, they are all *symmetric* by [21, Corollary 43], and in some cases *telescopic*, see Lemma 4, with their minimal generators explicitly given by Corollary 3; for some basic facts about telescopic numerical semigroups see [19, Section 5.4] or [28]. The symmetry is expressed in the semigroup in the following way: $m \in H$ if and only if $2g - 1 - m \notin H$. It is a direct consequence of the symmetry of $H(P)$ that P is a Weierstrass point with respect to the canonical linear series, see Corollary 2. For some interesting geometric properties concerning the symmetric class of numerical semigroups, the reader can look at [28, p. 142] and the references there.

Maximal curves and curves equipped with a big action have zero p-rank, see for example [12, Corollary 2.5] and [25, first lines of the proof of Proposition 2.5]. Thus for such a curve X, given a p-subgroup G of $\mathrm{Aut}(X)$, whenever $X \to X/G$ is an HKG-cover, see Theorem 1, the lower ramification jumps of G are given by [21, Theorem 13]. All big actions are HKG-covers by [21, Corollary 39], while for maximal curves we give in next section a simple necessary and sufficient condition for this to happen.

2.3 Main result. Let X a maximal curve over $\mathbb{F}_{q^2}$ and fix a p-subgroup G of $\mathrm{Aut}(X)$. We now see an equivalent form of Theorem 1 for a maximal X together with G, viewed over $\bar{\mathbb{F}}_{q^2}$, to admit an HKG-cover in terms of the Weierstrass semigroup at the unique point P that is stabilized by G.

Theorem 2. *Let X be a maximal curve defined over $\bar{\mathbb{F}}_{q^2}$, where q is a p-power and $G \subseteq \mathrm{Aut}(X)$. Let $P \in X$ the unique point stabilized by $G_1 := G_1(P) \subseteq G_0(P) = G$. Then the cover $\pi : X \longrightarrow X/G_1(P)$ is an HKG cover if and only if $q \leq |G_1|$.*

Proof. For the forward direction, by Theorem 1 it is enough to show that $|G_1|$ is a pole number at P iff $q \leq |G_1|$. Indeed, by [18, Proposition 10.6 (XII)] q is a pole number for every point P. Thus if $q \leq |G_1(P)|$ then q divides $|G_1(P)|$ and $|G_1(P)|$ is a pole number.

For the opposite direction note that $q = p^s$ for some s and that this s is the rank of nilpotency of the Cartier operator, see [12] (for basic definitions concerning the Cartier operator the reader can also look at [36, Section 2] or at the introductory section of [29]). This means that if $|G_1(P)|$ is a pole number then it cannot be less that q, as a consequence of the minimality of the rank of nilpotency of the Cartier operator. Indeed, according to [29, Proposition 2.3 and Section 4 pp. 90–91], [34, Corollary 2.7] the rank of the Cartier operator is the smallest s such that $p^s \in H(P)$; if we had $|G_1(P)| < p^s$ and $|G_1(P)|$ was a pole number, then $|G_1(P)|$ divides p^s and thus the rank would be strictly less than s, which is a contradiction. □

With the notation used in [21] we can now compute explicitly the Weierstrass semigroup at the ramification point of the above class of maximal HKG-covers. Denote by $0 = m_0 \leq m_1 \leq \cdots \leq m_n$ all pole numbers at P in increasing sequence (we consider the *natural* partial ordering of the semigroup: a is smaller than b if $b = a + c$ for some element c in the semigroup) up to m_n, the first pole number at P not divisible by the characteristic. Recall that for HKG-covers of Equation (1), the minimal generators of $H(P)$ are of the form $\bar{m}_i := m_{c_i+1} = p^{h_i}\lambda_i$, where all but the last one of the $\bar{m}_j$'s are divisible by p, the c_i are the representation jumps, i.e. $\ker\rho_{c_i} < \ker\rho_{c_i+1}$, the λ_i with $(\lambda_i, p) = 1$ are the lower ramification jumps of G_1; moreover, $p^{h_i} := |\ker\rho_{c_i+1}|$ for $1 \leq i \leq n-1$ and $p^{h_0} = |G_1(P)|$. Each of the generators $\bar{m}_i$ above can also viewed as the image of the minimal generator λ_i of $H(Q_{i+1})$ prime to the characteristic, where $Q_i = F_i \cap P$ for $1 \leq i \leq n$, that lies below P by the multiplication by $|\ker\rho_{c_i+1}|$, i.e. the push forward map (see [21, Lemma 16]) induced by Equation (1)

$$\pi_{i+1}^* : H(Q_{i+1}) \overset{\times|\ker\rho_{c_i+1}|}{\hookrightarrow} H(P)$$

applied at all the n intermediate fixed subfields of the Galois tower of the representation filtration $F_i := F^{\ker\rho_{c_i}}$ of G; see [21, Lemma 24]. It is known that the above fixed fields of the representaion filtration concide with the fixed fields of the ramification filtration of G; see [21, Theorem 14, (2)].

Corollary 3. *If $X \to X/G_1(P)$ is an HKG-cover and X is a maximal curve, defined over $\bar{\mathbb{F}}_{q^2}$, then the minimal generators of $H(P)$ are given by*

(1) *$H(P) = \langle \bar{m}_j, |G_1(P)| : 1 \leq j \leq n\rangle_{\mathbb{Z}_+}$ when $\lambda_1 \neq 1$, equivalently when $q = |G_1|$, and by*
(2) *$H(P) = \langle \bar{m}_j : 1 \leq j \leq n\rangle_{\mathbb{Z}_+}$ with $\bar{m}_1 = |G_2|$ and $\bar{m}_{n-1} = q$, when $\lambda_1 = 1$, equivalently when $q < |G_1|$.*

Moreover, the unique minimal generator $\bar{m}_n$ of $H(P)$ not divisible by p equals $q+1$ whenever the maximal curve is not *$\bar{\mathbb{F}}_{q^2}$-isomorphic with the curve $y^q + y = x^m$ where m divides $q+1$. In every case, $H(P)$ is a symmetric numerical semigroup and such a P is a Weierstrass point with respect to the canonical linear series.*

Proof. The proofs for the structure of $H(P)$ in both cases come from [21, Proposition 34 and Corollary 32], coupled with the fact that $|G_1(P)|$ is a minimal generator only when $q = |G_1(P)|$, by Theorem 2. Note that $\bar{m}_n$, the first pole number at P not divisible by p, is $q+1$ with one exception up to an $\bar{\mathbb{F}}_{q^2}$ isomorphism; this happens since in any other case $q+1$ is a generator of the Weierstrass semigroup at P, see [6, Theorem 2.3] and keep in mind that for maximal curves $q+1$ is always a pole number at a rational point. Finally the symmetry of the Weierstrass semigroup comes directly from [21, Corollary 43], which implies that the ramification point is Weierstrass point: recall that for a symmetric numerical semigroup H the conductor κ of the semigroup (i.e. the maximum gap plus one) equals $2g$.

Observe that if $2g-1 \notin H(P)$ then the conductor of the semigroup is $\kappa(H(P)) = 2g$. Indeed, since our function field is not hyperelliptic we have $m_i \geq 2i+1$ for $i = 1, \dots, g-2$ and $m_{g-1} \geq 2g-2$ by [37, Lemma 1.25]. This means that there are two cases for m_{g-1}, namely either $m_{g-1} = 2g-2$ or $m_{g-1} = 2g-1$.

Let $\mathcal{K}$ be the canonical linear series and $\mathcal{G}$ the gap sequence of the Weierstrass semigroup. Observe that at a generic point P of a $\mathcal{K}$-classical curve X, we have $m_i(P) = g+i$ for $i \geq 1$. In that case the gaps $\mathcal{G}(P)$ and the generic order sequence $\mathcal{E}(P)$ are classical and they are equal to

$$\mathcal{G}(P) = \{1, \dots, g\} = \{\varepsilon_0^{\mathcal{K}} + 1, \dots, \varepsilon_{g-1}^{\mathcal{K}} + 1\}.$$

In what follows assume that $m_{g-1} = 2g-2$ at the point P of the curve, i.e. $H(P)$ is a symmetric Weierstrass semigroup. Then *at least one* of the following conditions must be satisfied: (1) the curve is not $\mathcal{K}$-classical, (2) P is a Weierstrass point. A closer analysis indicates that if $m_{g-1} = 2g-2$, then the second condition, i.e. P should be a Weierstrass point, is always true. Indeed:

Case a. The curve X is not $\mathcal{K}$-classical and P is ordinary, i.e. $\{1, \dots, g\} \neq \mathcal{G}(P) = \{\varepsilon_0^{\mathcal{K}} + 1, \dots, \varepsilon_{g-1}^{\mathcal{K}} + 1\} = \{j_i(P)^{\mathcal{K}} + 1 \mid 0 \leq i \leq g-1\}$. Thus we have $\varepsilon_{g-1}^{\mathcal{K}} = 2g-2 = j_{g-1}^{\mathcal{K}}(P)$; but this case cannot occur, see [37, Lemma 2.31], [7, p. 235, Section 3].

Case b. The point P is a Weierstrass point with respect to $\mathcal{K}$ and X is a $\mathcal{K}$-classical curve. In this case $\mathcal{G}(P) = \{j_i^{\mathcal{K}}(P)+1 \mid 0 \leq i \leq g-1\} \neq \{\varepsilon_i^{\mathcal{K}}+1 \mid 0 \leq i \leq g-1\} = \{1, \dots, g\}$, where $j_{g-1}^{\mathcal{K}}(P) = 2g-2$ but $\varepsilon_{g-1}^{\mathcal{K}} = g-1 \neq j_{g-1}^{\mathcal{K}}(P)$.

Case c. The point P is a Weierstrass point with respect to $\mathcal{K}$ and the curve X is not $\mathcal{K}$-classical. That is $g-1 < \varepsilon_{g-1}^{\mathcal{K}} < j_{g-1}^{\mathcal{K}}(P)$. □

Surprisingly enough, the situation where $\lambda_1 = 1$ or when $\lambda_1 \neq 1$ and $|G_1|$ is among the two first minimal generators characterizes the Weierstrass semigroup in another way:

Lemma 4. *When $\lambda_1 \neq 1$ and $p^{h_0} = |G_1(P)|$ is the first or the second generator or when $\lambda_1 = 1$, then $H(P)$ is a telescopic numerical semigroup.*

Recall the notation: $p^{h_i} = |\ker \rho_{c_i+1}|$ and $p^{h_0} = |G_1(P)|$ for $1 \leq i \leq n$, while the λ_i, the prime to p parts of the minimal generators of $H(P)$, are the ramification jumps of G_1; see [21, Definition 11, Theorems 13 and 14].

Note that every telescopic numerical semigroup is symmetric but not vice versa.

Proof. First suppose that p^{h_0} equals the first generator. Set $d_{-1} = 0$ and $d_i = \gcd(p^{h_0}, \bar{m}_1, \dots, \bar{m}_i)$. Then $d_i = p^{h_i}$ since $\gcd(\lambda_i, p) = 1$. Recall than $H(P) = H(Q_{n+1}) = \langle p^{h_0}, \bar{m}_1, \dots, \bar{m}_n \rangle_{\mathbb{Z}_+}$. Then

$$H(Q_{i+1}) = \left\langle \frac{p^{h_0}}{d_i}, \frac{\bar{m}_1}{d_i}, \dots, \frac{\bar{m}_i}{d_i} \right\rangle_{\mathbb{Z}_+} \quad \text{for } 0 \leq i \leq n.$$

Then $H(Q_{n+1})$ is telescopic if $\frac{\bar{m}_i}{d_i} \in H(Q_i)$ for $1 \leq i \leq n$. Recall that for λ_i, the generator of $H(Q_{i+1})$ prime to the characteristic, we have $\frac{\bar{m}_i}{d_i} = \lambda_i \in H(Q_i)$ for $1 \leq i \leq n$ by [21, Lemma 16]. Thus $H(Q_{n+1})$ is telescopic.

If $\lambda_1 = 1$ then the elements $\bar{m}_1, \dots, \bar{m}_n$ generate the whole Weierstrass semigroup, since the same argument can be used on the HKG-cover $F \to F^{G_2(P)}$.

Note that we arrive at the same conclusions whenever p^{h_0} is the second generator of $H(P)$, that is when

$$\lambda_1 < |G_1(P)/\ker \rho_{c_2}| \quad \text{and} \quad |G_1(P)/\ker \rho_{c_3}| < \lambda_2 \quad \text{or} \quad m_{c_1+1} < |G_1(P)| < m_{c_2+1}.$$

The reader should be cautioned here about the greatest common divisors $d_i = \gcd(\bar{m}_1, p^{h_0}, \dots, \bar{m}_i)$: one has $d_{-1} = 0$, $d_0 = \bar{m}_1$, $d_i = p^{h_i}$ for $1 \le i \le n$.

In the cases where p^{h_0} is equal to or greater than the third generator, $H(P)$ is no more telescopic: Suppose that p^{h_0} is the i_0th generator with $3 \le i_0 < n$; then the picture for the greatest common divisors $d_i = \gcd(\bar{m}_1, \dots, \bar{m}_{i_0-1}, p^{h_0}, \bar{m}_{i_0}, \dots, \bar{m}_i)$ is the following: $d_{-1} = 0$, $d_0 = \bar{m}_1$, $d_1 = p^{h_2}$,..., $d_{i_0-2} = p^{h_{i_0-1}}$, $d_{i_0-1} = p^{h_{i_0-1}}$, and $d_i = p^{h_i}$ for $i_0 \le i \le n$. Now note that

$$\left\langle \frac{\bar{m}_1}{d_{i_0-1}}, \dots, \frac{\bar{m}_{i_0-1}}{d_{i_0-1}}, \frac{p^{h_0}}{d_{i_0-1}} \right\rangle_{\mathbb{Z}_+} \ni \frac{p^{h_0}}{p^{h_{i_0-1}}} \notin S_{i_0-1} = \left\langle \frac{\bar{m}_1}{d_{i_0-2}}, \dots, \frac{\bar{m}_{i_0-1}}{d_{i_0-2}} \right\rangle_{\mathbb{Z}_+},$$

since every $\mathbb{Z}_+$ linear combination of $p^{h_0}/p^{h_{i_0-1}}$ on the right hand side would lead to a $\mathbb{Z}_+$ linear combination of p^{h_0} on $H(P)$, which contradicts our hypothesis that p^{h_0} is a minimal generator. □

Note that all the involved semigroups $H(Q_i)$ with $1 \le i \le n+1$ are telescopic.

Examples 5. It is now clear that in order to recognize a Weierstrass semigroup as telescopic it is necessary to determine the ordering of the generator $|G_1(P)|$ among all the generators of $H(P)$. Some examples of HKG-covers with a telescopic $H(P)$ are:

- cyclic p^n HKG-covers; we know that $p^n = p^{h_0} = |G_1(P)|$ is the first generator at P whenever $p < \lambda_1$, see [20, Lemma 26, Remark 27 and Example 28].
- big actions, since $\lambda_1 = 1$ by their definition, see [26].
- the GK curve in [13], since $q = |G_1|$ is the second minimal generator; see [21, Example 23] and use Lemma 4.
- generally, all the maximal curves X defined over $\bar{\mathbb{F}}_{q^2}$ with $G \in \mathrm{Aut}(X)$ and $q < |G_1(P)|$; use Corollary 3.

Another nice property of the telescopic Weierstrass semigroups is that pole numbers can be written in a *unique way*, see [19, Lemma 5.34].

Remark 6. The only minimal Weierstrass generator $\bar{m}_n$ at P prime to the characteristic is called the degree of the Frobenius linear series of the curve; it turns out that for maximal HKG-curves the linear series generated by this minimal generator at P coincides with the Frobenius linear series $\mathcal{F}$ at P. The latter is an invariant of the curve at a rational point. It is interesting that in our case the orders of the Frobenius linear series at P are $\bar{m}_n - m_j$ with $0 \le j \le n$, and among these differences lie also the jumps of the ramification filtration; this was first observed in [23], while later in [21] we found exactly the indices j_* where the actual jumps occur. Note also that the projective map arising from the linear series $|\bar{m}_n P|$ is an embedding, see [18, Theorem 10.7].

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