Automorphisms of Algebraic Curves

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Στους γονείς μου, Αναστασία και Παναγιώτη
και στη γιαγιά Βασιλική

Contents

Synopsis

What follows is a dissertation in the subject of algebraic geometry and more specifically in the theory of algebraic curves in positive characteristic. It aims to present the three peer-reviewed published i.e. [34] and [33] or accepted [32] works of the author. These papers revolve around two subjects and in particular the first two are concerned with the theory of Harbater-Katz-Gabber curves, while the last one deals with automorphisms of curves. This discrimination forces the thesis to be divided into two parts, each one dealing with the appropriate area.

In the preliminaries several results which were known before our work are presented. The first section constitutes a quick introduction to the general theory of Harbater-Katz-Gabber curves. The second section is dedicated on the formulation of Petri's theorem and on a recent result regarding its computation.

After the introduction our work becomes the focus. The first part consists of chapters 2 and 3. They include the following results:

- The determination of the irreducible polynomials of the generating elements of such a curve, see section 2.1,
- a cohomological condition on a particular class of elements of the function field of the curve, also in section 2.1
- an application of the above for the determination of specific elements in the Nottingham group, 2.2,
- the determination of the canonical ideal of an HKG curve, see chapter 3.

The second part deals with automorphisms of algebraic curves. Namely we put together the theory of syzygies of the canonical embedding and the theory of automorphisms of curves.

For a non-singular complete algebraic curve X over an algebraically closed field of characteristic $p \geq 0$, if the genus g of the curve X is $g \geq 2$ then the automorphism group $G = \mathrm{Aut}(X)$ of the curve X is finite. For the theory of automorphisms of curves we refer to the survey articles [1], [8].

On the other hand the theory of syzygies which originates in the work of Hilbert and Sylvester has attracted a lot of researchers and it seems that

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a lot of geometric information can be found in the minimal free resolution of the ring of functions of an algebraic curve. For an introduction we refer to [15].

Throughout chapters 4 and 5, X will be a non-hyperelliptic, non-trigonal and also not a non-singular quintic of genus $\neq 6$. These conditions are needed for Petri's theorem to hold.

In section 4.1 we employ the machinery of Petri's theorem along with proposition 1.12 in order to give a necessary and sufficient condition for an element in $\mathrm{GL}(H^0(X, \Omega_X))$ to act as an automorphism of our curve. In this way we prove that the automorphism group of a curve X as a finite set can be seen as a subset of the $g^2(g+1)^2 - 1$-dimensional projective space and can be described by explicit quadratic equations.

In section 5 we show that the automorphism group G of the curve acts linearly on a minimal free resolution $\mathbf{F}$ of the ring of regular functions S_X of the curve X canonically embedded in $\mathbb{P}^{g-1}$. Notice that an action of a group G on a graded module M gives rise to a series of linear representations $\rho_d : G \to M_d$ to all linear spaces M_d of degree d for $d \in \mathbb{Z}$. For the case of the free modules F_i of the minimal free resolution $\mathbf{F}$ we relate the actions of the group G in both F_i and in the dual F_{g-2-i} in terms of an inner automorphism of G.

This information is used in order to show that the action of the group G on generators of the modules F_i sends generators of degree d to linear combinations of generators of degree d. Let $S = \mathrm{Sym}(H^0(X, \Omega_X))$ be the symmetric algebra of $H^0(X, \Omega_X)$.

The degree d-part of $\mathrm{Tor}_i^S(k, S_X)$ will be denoted by $\mathrm{Tor}_i^S(k, S_X)_d$, which is a vector space of dimension $\beta_{i,d}$. We can use our computation in order to show that all $\mathrm{Tor}_i^S(k, S_X)_d$ are acted on by the group G. Notice that this also follows by Koszul cohomology, see [2].

To elaborate briefly on this, one starts with the vector space $V = H^0(X, \Omega_X)$, $\dim V = g$, $S = \mathrm{Sym}(V)$ and considers the exact Koszul complex

$$0 \to \wedge^g V \otimes S(-g) \to \wedge^{g-1} V \otimes S(-g+1) \to \cdots$$

$$\cdots \to \wedge^2 V \otimes S(-2) \to v \otimes S(-1) \to S \to k \to 0.$$

The symmetry property of the Tor functor implies that one can calculate $\mathrm{Tor}_i^S(k, S_X)$ by using the Koszul resolution of k instead of the Koszul resolution of S_X. Since the Koszul resolution of k is a complex of G-modules and all differentials are G-module morphisms the $\mathrm{Tor}_i^S(k, S_X)_d$ are naturally G-modules. On the other hand the passage to the action on generators is not explicit since the isomorphism between the graded components of the terms in the minimal resolution and Koszul cohomology spaces is not explicit, as it comes from the spectral sequence that ensures the symmetry of Tor functor.

Finally, the representations to the d graded space of each F_i, $\rho_{i,d} : G \to \mathrm{GL}(F_{i,d})$ can be expressed as a direct sum of the G-modules $\mathrm{Tor}_i^S(k, S_X)_d$.

We conclude by showing that the G-module structure of all F_i is determined by knowledge of the G-module structure of $H^0(X, \Omega_X)$ and the G-module structure of each $\mathrm{Tor}_i^S(k, S_X)$ for all $0 \leq i \leq g-2$.

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Σύνοψη

Το παρόν αποτελεί μια διατριβή στον κλάδο της αλγεβρικής γεωμετρίας και ειδικότερα στη θεωρία των αλγεβρικών καμπυλών στη θετική χαρακτηριστική. Επιδιώκει την παρουσίαση των τριών υποβεβλημένων και δημοσιευμένων (βλ. [34], [33]) ή υπο δημοσίευση (βλ. [32]) εργασιών του συγγραφέα. Οι εργασίες αυτές περιστρέφονται γύρω από δύο θεματικές ενότητες και συγκεκριμένα οι δύο πρώτες πραγματεύονται τη θεωρία των καμπυλών Harbater-Katz-Gabber ενώ η τρίτη τους αυτομορφισμούς καμπυλών. Η διάκριση αυτή οδηγεί στη διάσπαση της διατριβής σε δύο μέρη, εκ των οποίων το καθένα εξετάζει την αντίστοιχη περιοχή.

Στην εισαγωγή παρατίθεται μια σειρά από αποτελέσματα τα οποία χρησιμοποιούνται αργότερα και ήταν γνωστά πριν την πραγματοποίηση της παρούσας έρευνας. Η πρώτη παράγραφος εισάγει την έννοια των καμπυλών Harbater-Katz-Gabber και των βασικών τους ιδιοτήτων. Κατόπιν η δεύτερη παράγραφος αφιερώνεται στο κανονικό ιδεώδες, το θεώρημα του Petri και ένα πρόσφατο αποτέλεσμα σχετικό με τον υπολογισμό του.

Στα επόμενα κεφάλαια η συνεισφορά μας έρχεται στο προσκήνιο. Το πρώτο μέρος αποτελείται από τα κεφάλαια 2 και 3, τα οποία περιλαμβάνουν τα εξής αποτελέσματα:

- Τον καθορισμό των ελάχιστων πολυωνύμων των γεννητόρων μιας καμπύλης Harbater-Katz-Gabber, βλέπε παράγραφο 2.1,
- μία συνομολογιακή συνθήκη επί μιας συγκεκριμένης συλλογής στοιχείων του function field της καμπύλης, επίσης στην παράγραφο 2.1,
- μια εφαρμογή των παραπάνω για τον καθορισμό συγκεκριμένων στοιχείων της ομάδας Nottingham, βλέπε 2.2,
- τον καθορισμό του κανονικού ιδεώδους της καμπύλης, βλέπε κεφάλαιο 3.

Το δεύτερο μέρος καταπιάνεται με αυτομορφισμούς καμπυλών αξιοποιώντας τη θεωρία συζυγιών της κανονικής εμφύτευσης μιας καμπύλης στον προβολικό χώρο.

Αφενώς για μία non-singular πλήρη αλγεβρική καμπύλη X επί ενός αλγεβρικά κλειστού σώματος χαρακτηριστικής $p \geq 0$, αν το γένος g είναι

Λέξεις κλειδιά: Καμπύλες Harbater-Katz-Gabber, Κανονικό ιδεώδες, θεωρία Παραμόρφωσης.

μεγαλύτερο ή ίσο του 2 τότε η ομάδα αυτομορφισμών $G = \mathrm{Aut}(X)$ της καμπύλης είναι πεπερασμένη. Στη θεωρία των αυτομορφισμών καμπυλών αναφέρονται οι επισκοπήσεις [1], [8].

Αφετέρου η θεωρία των συζυγιών, η οποία εκκινεί από τις εργασίες των Hilbert και Sylvester, έχει προσελκύσει πολλούς ερευνητές και διαφαίνεται ότι μεγάλο μέρος της γεωμετρικής πληροφορίας περιλαμβάνεται στην minimal free resolution του δακτυλίου συναρτήσεων μιας αλγεβρικής καμπύλης. Για μία εισαγωγή παραπέμπουμε στο [15].

Στα κεφάλαια 4 και 5, η X θα είναι μία μη υπερελλειπτική, μη τριγωνική καθώς επίσης δεν είναι non-singular quintic γένους 6. Οι συνθήκες αυτές επιβάλλονται από το θεώρημα του Petri.

Στην παράγραφο 4.1 αξιοποιούμε το θεώρημα Petri και την πρόταση 1.12 για να δώσουμε μια ικανή και αναγκαία συνθήκη ώστε ένα στοιχείο της $\mathrm{GL}(H^0(X, \Omega_X))$ να δρα σαν αυτομορφισμός της καμπύλης. Έτσι αποδεικνύουμε ότι η ομάδα αυτομορφισμών της καμπύλης X σας πεπερασμένο σύνολο μπορεί να θεωρηθεί υποσύνολο του προβολικού χώρου διάστασης $g^2(g+1)^2 - 1$ και να περιγραφεί από συγκεκριμένες δευτεροβάθμιες εξισώσεις.

Στην παράγραφο 5 αποδεικνύουμε ότι η ομάδα αυτομορφισμών G της καμπύλης δρα στη minimal free resolution $\mathbf{F}$ του δακτυλίου των regular συναρτήσεων S_X της εμφυτευμένενης στο $\mathbb{P}^{g-1}$ καμπύλης X. Επιπλέον ισχύει ότι η δράση μιας ομάδας G σε ένα graded πρότυπο M οδηγεί σε μια σειρά από γραμμικές αναπραστάσεις $\rho_d : G \to M_d$ στους γραμμικούς χώρους M_d βαθμού d για $d \in \mathbb{Z}$. Στην περίπτωση των ελεύθερων προτύπων F_i της minimal free resolution $\mathbf{F}$ συσχετίζουμε τη δράση της G στο F_i και στο δϋικό του F_{g-2-i} μέσω ενός εσωτερικού αυτομορφισμού της G.

Η συνθήκη αυτή χρησιμοποιείται για να αποδείξουμε ότι η δράση της ομάδας στους γεννήτορες των προτύπων F_i απεικονίζει γεννήτορες βαθμού d σε γραμμικούς συνδυασμούς γεννητόρων βαθμού d. Έστω $S = \mathrm{Sym}(H^0(X, \Omega_X))$ η συμμετρική άλγεβρα του $H^0(X, \Omega_X)$.

Τα στοιχεία βαθμού d του $\mathrm{Tor}_i^S(k, S_X)$ συμβολίζονται με $\mathrm{Tor}_i^S(k, S_X)_d$, το οποίο είναι ένας γραμμικός χώρος διάστασης $\beta_{i,d}$. Μπορούμε να χρησιμοποιήσουμε τον υπολογισμό μας για να δείξουμε ότι η ομάδα G δρα επί κάθε $\mathrm{Tor}_i^S(k, S_X)_d$. Οφείλει κανείς να επισημάνει ότι αυτό επάγεται και από τη συνομολογία Koszul, βλέπε [2].

Πιο συγκεκριμένα, εκκινώντας από τον γραμμικό χώρο $V = H^0(X, \Omega_X)$, $\dim V = g$, $S = \mathrm{Sym}(V)$ εξετάζουμε το ακριβές σύμπλοκο Koszul

$$0 \to \wedge^g V \otimes S(-g) \to \wedge^{g-1} V \otimes S(-g+1) \to \cdots$$

$$\cdots \to \wedge^2 V \otimes S(-2) \to v \otimes S(-1) \to S \to k \to 0.$$

Η συμμετρική ιδιότητα του συναρτητή Tor συνεπάγεται τον υπολογισμό του $\mathrm{Tor}_i^S(k, S_X)$ χρησιμοποιώντας την Koszul resolution του k αντί για την Koszul resolution του S_X. Αφού η Koszul resolution του k είναι ένα σύμπλεγμα G-προτύπων και όλα τα διαφορικά είναι μορφισμοί G-προτύπων, οι $\mathrm{Tor}_i^S(k, S_X)_d$ είναι με φυσικό τρόπο G-πρότυπα. Ωστόσο το πέρασμα στη δράση επί των γεννητόρων δεν είναι συγκεκριμένο αφού ο ισομορφισμός μεταξύ των graded μερών των όρων της minimal resolution και των χώρων συνομολογίας Koszul δεν είναι συγκεκριμένος, αφού έρχεται από τη φασματική ακολουθία που διασφαλίζει τη συμμετρία του συναρτητή Tor.

Τέλος οι αναπαραστάσεις στον χώρο βαθμού d του κάθε F_i, $\rho_{i,d} : G \to \mathrm{GL}(F_{i,d})$ μπορούν να διατυπωθούν ως ευθύ άθροισμα των G-προτύπων $\mathrm{Tor}_i^S(k, S_X)_d$. Κλείνοντας αποδεικνύουμε ότι η δομή G-προτύπου των F_i καθορίζεται από τη γνώση της δομής G-προτύπου του $H^0(X, \Omega_X)$ και της δομής G-προτύπου του κάθε $\mathrm{Tor}_i^S(k, S_X)$ για κάθε $0 \leq i \leq g-2$.

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Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης

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Chapter 1

Preliminaries

1.1 Introduction to the theory of HKG curves

In this chapter we provide the general framework for this thesis. We will work over an algebraically closed field k of characteristic $p \geq 5$.

Definition 1.1. *A Harbater-Katz-Gabber cover (HKG-cover for short) is a Galois cover $X_{\mathrm{HKG}} \to \mathbb{P}^1$, such that there are at most two branched k-rational points $P_1, P_2 \in \mathbb{P}^1$, where P_1 is tamely ramified and P_2 is totally and wildly ramified. All other geometric points of $\mathbb{P}^1$ remain unramified. In this article we are mainly interested in p-groups so our HKG-covers have a unique ramified point, which is totally and wildly ramified.*

Harbater-Katz-Gabber curves grew out mainly due to work of Harbater [21] and of Katz and Gabber [26]. They, are important because of the Harbater-Katz-Gabber compactification theorem of Galois actions on complete local rings and they proved to be an important tool in the study of local actions, in the deformation theory of curves with automorphisms and to the celebrated proof of Oort conjecture, see [5, 40, 41, 33, 14, 12, 13, 43]. The interested reader can also consult the paper by Karanikolopoulos and Kontogeorgis on HKG curves, [25].

In the w,ork of Harbater, Katz and Gabber it is shown that any finite subgroup G of $\mathrm{Aut}(k[[t]])$ can be associated with an HKG-curve X. More precisely, G is the semi-direct product of a cyclic group of order prime to p (the maximal tamely ramified quotient) by a normal p-subgroup (the wild inertia group). We are interested in the latter group, so from now on we will replace the initial group G with the finite p-subgroup of $\mathrm{Aut}(k[[t]])$.

Working with the HKG-curve X allows us to use several global tools like the genus, the p-rank of the Jacobian etc to the study of $k[[t]]$. One can employ the Weierstrass semigroup attached to the unique ramified point P and use the results of [25] on relating the structure of the Weierstrass semigroup to the jumps of the ramification filtration.

1.1.1 Ramification filtration

Let $X \to \mathbb{P}^1$ be a HKG-cover, that is, a Galois cover with Galois group a p-group G fully ramified over one point $P \in \mathbb{P}^1$. In the associated HKG-curve X, the group G will coincide with the inertia group of the curve at the unique ramified point, $G_T(P) = \{\sigma \in G(P) : v_p(\sigma(t) - t) \geq 1\}$, where t is a local uniformizer at P and v_P is the corresponding valuation. For more information on ramification filtration the reader is referred to [45]. We define $G_i(P)$ to be the subgroup of $\sigma \in G(P)$ that acts trivially on $\mathscr{O}_p/m_P^{i+1}$, obtaining the following filtration;

$$G_T(P) = G_0(P) = G_1(P) \supseteq G_2(P) \supseteq \cdots \supseteq \{1\}. \tag{1.1}$$

Let us call an integer i a *jump of the ramification filtration* if $G_i(P) \gneqq G_{i+1}(P)$ and denote by

$$G_0(P) = G_1(P) = \cdots = G_{b_1}(P) \gneqq G_{b_1+1}(P) = \cdots = G_{b_2}(P) \gneqq \cdots \gneqq G_{b_\mu}(P) \gneqq \{1\} \tag{1.2}$$

the filtration of the jumps, assuming that there are exactly μ jumps.

1.1.2 The Weierstrass semigroup

The Weierstrass semigroup $H(P)$ is the semigroup consisting of all pole numbers, i.e. $m \in \mathbb{N}$, such that there is a function f on X with $(f)_\infty = mP$. For the Weierstrass semigroup $H(P)$ we consider all pole numbers m_i forming an increasing sequence

$$0 = m_0 < \ldots < m_{r-1} < m_r,$$

where m_r is the first pole number not divisible by the characteristic. If $g \geq 2$ and $p \geq 5$ we can prove that $m_r \leq 2g - 1$, see [31, lemma 2.1].

Let $F = k(X)$ be the function field of the HKG-curve X. For every m_i, $0 \leq i \leq r$ in the Weierstrass semigroup we denote by $f_i \in F$ an element of F that has a unique pole at P of order m_i, i.e. $(f_i)_\infty = m_i P$. For each $i \in \{0, \ldots, r\}$ the set $\{f_0, \ldots, f_i\}$ forms a basis for the Riemann-Roch space $L(m_i P)$. The spaces

$$k = L(m_0 P) \subsetneq L(m_1 P) \subsetneq \cdots \subsetneq L(m_r P) \tag{1.3}$$

give rise to a natural flag of vector spaces corresponding to the Weierstrass semigroup. Notice that if μ is a pole number in $H(P)$ we have $\mu = m_{\dim L(\mu P) - 1}$.

1.1.3 Representation filtration

For each $0 \leq i \leq r$ we consider the representations

$$\rho_i : G_1(P) \to \mathrm{GL}(L(m_i P)) \tag{1.4}$$

which give rise to a decreasing sequence of groups

$$G_1(P) = \ker\rho_0 \supseteq \ker\rho_1 \supseteq \ker\rho_2 \supseteq \ldots \supseteq \ker\rho_r = \{1\}. \tag{1.5}$$

Recall that r is the index of m_r, the first pole number not divisible by p. In [31] A. Kontogeorgis proved that ρ_r is faithful hence the last equality $\ker\rho_r = \{1\}$.

We shall call the last filtration the *representation filtration of* G.

Definition 1.2. *An index i is called a jump of the representation filtration if and only if* $\ker\rho_i \gneqq \ker\rho_{i+1}$.

We will denote the jumps in the representation filtration by

$$c_1 < c_2 < \ldots < c_{n-1} < c_n = r - 1,$$

that is

$$\ker\rho_{c_i} > \ker\rho_{c_i+1}.$$

The last equality $c_n = r - 1$ is proved in [25, rem. 9]. We have now a sequence of decreasing groups

$$G_1(P) = \ker\rho_0 = \ldots = \ker\rho_{c_1} > \ldots \ker\rho_{c_{n-1}} > \ker\rho_{c_n} > \{1\} \tag{1.6}$$

which gives rise to the following sequence of extensions;

$$F^{G_1(P)} = F^{\ker\rho_{c_1}} \subset F^{\ker\rho_{c_2}} \subset \cdots \subset F^{\ker\rho_{c_n}} \subset F. \tag{1.7}$$

1.1.4 A relation of the two filtrations in the case of HKG-covers

Following the exposition in [25] one can relate the filtrations defined in eq. (1.2), (1.6) and the Weierstrass semigroup in the following way:

Theorem 1.3. *We distinguish the following two cases:*

• *If $G_1(P) > G_2(P)$ then the Weierstrass semigroup is minimally generated by $m_{c_i+1} = p^{h_i}\lambda_i$, $(\lambda_i, p) = 1$, $1 \leq i \leq n$ and the cover $F/F^{G_2(P)}$ is an HKG-cover as well. In this case $|G_2(P)| = m_1$.*

• *If $G_1(P) = G_2(P)$ then the Weierstrass semigroup is minimally generated by $m_{c_i+1} = p^{h_i}\lambda_i$, $(\lambda, p) = 1$, $1 \leq i \leq n$ and by an extra generator $p^h = |G_1(P)|$, which is different by all m_{c_i+1} for all $1 \leq i \leq n$.*

Especially when $X \to \mathbb{P}^1$ is an HKG-cover, the number of ramification jumps μ coincides with the number of representation jumps n, i.e. $n = \mu$. The integers λ_i, which appear as factors of the integers m_{c_i+1}, $1 \leq i \leq n$ are the jumps of the ramification filtration, i.e. $\lambda_i = b_i$ and $G_{b_i} = \ker\rho_{c_i}$ for $2 \leq i \leq n$. Summing up we have the following options for the ramification filtration

$$G_1(P) = \cdots = G_{\lambda_1} \gneqq G_{\lambda_1+1} = \cdots = G_{\lambda_2} \gneqq G_{\lambda_2+1} = \cdots = G_{\lambda_n} \gneqq \{1\}$$

or

$$G_1(P) > G_2(P) = \cdots = G_{\lambda_1} \gneqq G_{\lambda_1+1} = \cdots = G_{\lambda_2} \gneqq G_{\lambda_2+1} = \cdots = G_{\lambda_n} \gneqq \{1\}$$

Proof. See [25, th. 13,th. 14]. □

Remark 1.4. *The reader should notice that* $\ker \rho_{c_1} = \ker \rho_0 = G_1(P) = G_{b_1}(P)$ *by definition, hence* $G_{b_i} = \ker \rho_{c_i}$ *for every* $i \in \{1, \dots, n = \mu\}$.

Theorem 1.3 allows us to use the well known fact that the quotients $G_{b_i}/G_{b_{i+1}}$ are elementary abelian p-groups, hence the quotients $\ker \rho_{c_i}/\ker \rho_{c_{i+1}}$ are elementary abelian too, and the corresponding sequence of fields in (1.7) is in fact, a sequence of elementary abelian p-group extensions.

Also in [25, prop. 27] the authors observed that for a $\sigma \in \ker \rho_{c_i} - \ker \rho_{c_{i+1}}$ the following hold;

$$\sigma(f_\nu) = f_\nu \text{ for all } \nu \leq c_i$$

$$\sigma(f_{c_i+1}) = f_{c_i+1} + C(\sigma) \text{ for some } C(\sigma) \in k^*.$$

They also proved (prop. 20 & rem. 21) that for each $i \in \{1, \dots, n\}$ we have $F^{\ker \rho_{c_{i+12}}} = F^{\ker \rho_{c_i}}(f_{c_i+1})$.

In order to simplify the notation we set $F_i := F^{\ker \rho_{c_i}}$, $\bar{m}_i := m_{c_i+1}$ and $\bar{f}_i := f_{c_i+1}$, see also eq. (1.10).

Example 1.5. *In the Artin-Schreier extension* $F = k(x)(y)$ *where* $y^p - y = x^m$ *only the place* $P = \infty$ *is ramified with the following ramification filtration:*

$$\mathbb{Z}/p\mathbb{Z} = G_0 = \cdots = G_m > \{1\},$$

i.e. the first and unique ramification jump is at m*, see [46, prop. 3.7.8]. The representation filtration is given by*

$$G_0 = \ker\rho_0 = \cdots = \ker\rho_{m-1} > \{1\},$$

that is, the first representation jump is at $c_1 = m - 1$ *and* $\bar{f}_1 = f_{c_1+1} = y$*, where* $c_1 = m - 1$ *and* $c_1 + 1 = m$*. Thus* $F = F_2 = F_1(\bar{f}_1)$*, and* $\bar{f}_0$ *is the generator* x *of the rational function field* $k(x)$*.*

1.1.5 The action on the representation filtration

An automorphism of a curve acts on all "invariants" of the curve including the Weierstrass semigroup of the unique ramified point. Usually this action on invariants provides useful information about the action. Unfortunately the action of the group G on the semigroup $H(P)$ is trivial. This is not the case when we move to the action to appropriate flags of vector spaces. More precisely we will consider flags of k-vector spaces

$$\bar{V} : k = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_m \subsetneq \cdots$$

where $V_i = L(iP)$. We will say that a group G is acting on a flag $\bar{V}$, if there is a homomorphism

$$\rho : G \to \mathrm{Aut}(\bar{V}),$$

i.e. when $\rho(g)$ is an isomorphism such that $\rho(g)(V_i) = V_i$ for all V_i in the flag.

Remark 1.6. *Since the representation ρ_r is faithful it makes sense to consider the representation not on the whole flag but only up to $L(m_r P)$. The natural isomorphisms on this truncated flag are given by invertible upper triangular matrices.*

Recall that s is the the greatest index of $\bar{m}_i$ such that $\bar{m}_i < m$. For every $1 \leq i \leq s$ and for every $1 \leq j \leq r$ we have

$$\sigma(f_i) = f_i + C_i(\sigma), \text{ where } C_i(\sigma) \in L\big((m_i - 1)P\big)$$
$$\sigma(\bar{f}_i) = \bar{f}_i + \bar{C}_i(\sigma), \text{ where } \bar{C}_i(\sigma) \in L\big((\bar{m}_i - 1)P\big).$$

As will be explained in the next chapter, in section 2.1, if $\bar{f}_1, \ldots, \bar{f}_s$ are fixed, then the values $\bar{C}_i$ for $1 \leq i \leq s$ determine the action completely.

Also notice that for each $i \in \{1, \ldots, r\}$, f_i is a polynomial expression of the $\bar{f}_1, \ldots, \bar{f}_s$. By proposition 1.10 we have $\bar{C}_i \in L\big((\bar{m}_i - 1)P\big) = k_{\mathbf{n},\bar{m}_i}[\bar{f}_0, \ldots, \bar{f}_{i-1}]$. The functions $\sigma \mapsto C_i(\sigma)$ and $\sigma \mapsto \bar{C}_i(\sigma)$ are cocycles, i.e.

$$\bar{C}_i(\sigma\tau) = \bar{C}_i(\sigma) + \sigma\bar{C}_i(\tau).$$

Remark 1.7. *The selection of the generators $\bar{f}_i$ for $0 \leq i \leq s$ is not unique. Every element $a \in k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]$ gives rise to a new generator $\bar{f}_i + a$.*

The new cocycle $\bar{C}_i'$ which is defined in terms of the generator $\bar{f}_i + a$ is given by

$$\sigma(\bar{f}_i + a) = \sigma(\bar{f}_i) + \sigma(a) = \bar{f}_i + a + \bar{C}_i(\sigma) + \sigma(a) - a = \bar{f}_i + a + \bar{C}_i'(\sigma).$$

Therefore

$$\bar{C}_i'(\sigma) = \bar{C}_i(\sigma) + (\sigma - 1)a.$$

Also instead of selecting the generator $\bar{f}_i$, which has pole order $\bar{m}_i$ at P we can select $\lambda\bar{f}_i$ for any $\lambda \in k^*$. This change leads to cocycle $\lambda\bar{C}_i$. Therefore selecting the generator amounts to giving an element in the projective space

$$\mathbb{P}H^1\left(\frac{G}{\ker \rho_{i-1}}, k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]\right)$$

This gives us the following

Lemma 1.8. *The cocycles $\bar{C}_i, \bar{C}_i'$ corresponding to different generators $\bar{f}_i, \bar{f}_i'$ with the same pole number $\bar{m}_i$, that is $\bar{f}_i' = \lambda\bar{f}_i + a$, $a \in k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]$ satisfy the relation*

$$\bar{C}_i'(\sigma) = \lambda\bar{C}_i(\sigma) + (\sigma - 1)\lambda a$$

and a generator free description of the action is determined by a series of classes $\tilde{C}_i$ in

$$\begin{array}{ccc} H^1\left(\frac{G}{\ker\rho_{i-1}}, k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]\right) & \xhookrightarrow{\text{inf}} & H^1(G, k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]) \\ \downarrow & & \downarrow \\ \mathbb{P}H^1\left(\frac{G}{\ker\rho_{i-1}}, k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]\right) & \xhookrightarrow{\text{inf}} & \mathbb{P}H^1(G, k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]) \end{array} \quad (1.8)$$

These cocycles satisfy certain conditions which will be given in eq. (2.3) and theorem 2.4. The monomorphism inf is the inflation map in group cohomology, see [52, II.2-3, p. 64], while $\overline{\inf}[C]$ of the projective class $[C]$ of the cocycle C is given by

$$\overline{\inf}[C] = [\inf(C)].$$

Remark 1.9. *The vector space $k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]$ has as base the space of monomials $\bar{f}_0^{\nu_0} \bar{f}_1^{\nu_1} \ldots \bar{f}_{i-1}^{\nu_{i-1}}$, of degree smaller than m, where $\nu_i < p^{n_i}$. The action on them can be described in terms of the binomial theorem, i.e.*

$$\bar{f}_0^{\nu_0} \bar{f}_1^{\nu_1} \cdots \bar{f}_{i-1}^{\nu_{i-1}} \xrightarrow{\sigma} \bar{f}_0^{\nu_0} \sum_{\mu_1}^{\nu_1} \cdots \sum_{\mu_{i-1}}^{\nu_{i-1}} \binom{\mu_1}{\nu_1} \cdots \binom{\mu_{i-1}}{\nu_{i-1}} \bar{f}_1^{\mu_1} \cdots \bar{f}_{i-1}^{\mu_{i-1}} \bar{C}_1^{\nu_1 - \mu_1} \cdots \bar{C}_{i-1}^{\nu_{i-1} - \mu_{i-1}}. \tag{1.9}$$

The following proposition should be evident:

Proposition 1.10. *For a given $m \in H(P)$, in the case of HKG-covers we have*

$$L\big((m-1)P\big) = k_{\mathbf{n},m}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_s],$$

where

$$k_{\mathbf{n},m}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_s] = \left\langle \begin{array}{c} \bar{f}_0^{a_0} \bar{f}_1^{a_1} \cdots \bar{f}_s^{a_s} : 0 \le a_i < p^{n_i} \text{ for all } 1 \le i \le s, \\ \text{and } \deg(\bar{f}_0^{a_0} \bar{f}_1^{a_1} \cdots \bar{f}_s^{a_s}) = \sum_{\nu=0}^{s} a_\nu \bar{m}_\nu < m \end{array} \right\rangle_k . \tag{1.10}$$

In the above equation $\deg(\bar{f}_i)$ is the pole order of $\bar{f}_i$ at P. The integer s is determined uniquely; it is the greatest index of $\bar{m}_i$ such that $\bar{m}_i < m$ holds. The quantity $\mathbf{n} = (n_1, \ldots, n_s) \in \mathbb{N}^s$ depends on the ramification filtration, specifically n_i is the number of $\mathbb{Z}/p\mathbb{Z}$ components in each elementary abelian group G_i/G_{i+1} obtained by quotients of the lower ramification filtration.

1.2 The canonical ideal

The study of the canonical embedding and the determination of the canonical ideal is a classical subject in algebraic geometry, see [3, III.3], [44], [39, p. 20], [47] for a modern account.

It is expected that a lot of information of the deformation of the action is hidden in the canonical ideal, see also [24], [11].

Consider a complete non-singular non-hyperelliptic curve of genus $g \ge 3$ over the algebraically closed field k. Let Ω_X denote the sheaf of holomorphic differentials on X. The canonical ideal is defined as I_X in the following theorem:

Theorem 1.11 (Noether-Enriques-Petri)**.** *There is a short exact sequence*

$$0 \to I_X \to \mathrm{Sym} H^0(X, \Omega_X) \to \bigoplus_{n=0}^{\infty} H^0(X, \Omega_X^{\otimes n}) \to 0,$$

where I_X is generated by elements of degree 2 and 3. Also if X is not a non-singular quintic of genus 6 or X is not a trigonal curve, then I_X is generated by elements of degree 2.

For a proof of this theorem we refer to [44], [20]. The ideal I_X is called *the canonical ideal* and it is the homogeneous ideal of the embedded curve $X \to \mathbb{P}_k^{g-1}$.

The following is a recent result by Charalampous et al. [11], which provides a computational criterion for the determination of the canonical ideal. It roughly states that in order to show that a set of quadratic differentials generates the canonical ideal, it suffices to show that the “initial terms” of the differentials generate a large enough subspace of the degree 2 part of the polynomial ring of symmetric differentials.

Proposition 1.12. *Let J be a set of homogeneous polynomials of degree 2 containing the elements G_0 and an extra set of generators G' and let I be the canonical ideal. Assume that the hypotheses imposed by Petri's theorem in order for the canonical ideal to be generated by polynomials of degree two are fulfilled. If $\dim_L \left(S/\langle \mathrm{in}_{\prec} J\rangle\right)_2 \leq 3(g-1)$, then $I = \langle J \rangle$, where $S = \mathrm{Sym}(H^0(X, \Omega_X)$ is the symmetric algebra of $H^0(X, \Omega_X)$.*

For a proof see [11].

Chapter 2

Describing an HKG-cover as a sequence of Artin-Schreier extensions

2.1 The description

It is known, see [18], that every elementary abelian field extension L/K, with Galois group $(\mathbb{Z}/p\mathbb{Z})^n$, is given as an Artin-Schreier extension of the form

$$L = K(y): \qquad y^{p^n} - y = b,\ b \in K.$$

In our case, the elementary abelian field extension F_{i+1}/F_i can be generated by an element $y \in F_{i+1}$ but this element might not be the semigroup generator $\bar{f}_i$. We can give a description of the Artin-Schreier extension F_{i+1}/F_i using a monic polynomial

$$A_i(X) = X^{p^{n_i}} + a_{n_i-1}X^{p^{n_i-1}} + \cdots + a_1X^p + a_0X - D_i,$$

which can be computed in terms of the Moore determinant [19]. Notice that this polynomial is an additive polynomial minus a constant term. Let $\{\sigma_1, \ldots, \sigma_{n_i}\}$ be a basis of the Galois group Gal$(F_{i+1}/F_i) \cong (\mathbb{Z}/p\mathbb{Z})^{n_i}$, seen as an $\mathbb{F}_p$-vector space, and let $w_1, \ldots, w_{n_i}$ be elements of k^* such that $\sigma_j(\bar{f}_i) = \bar{f}_i + w_j$. Let W be the $\mathbb{F}_p$-subspace of k spanned by the w_j, $j = 1, \ldots, n_i$. We have $\dim_{\mathbb{F}_p} W = n_i$.

Let $P_i(X) = \prod_{a\in W}(X - a)$. Since every w_i is an element of k, Gal(F_{i+1}/F_i) acts trivially on $P_i(X)$ and we consider the polynomial

$$A_i(X) := P_i(X) - P_i(\bar{f}_i).$$

Notice that, for a $\sigma \in$ Gal(F_{i+1}/F_i), we can write $\sigma = \sigma_1^{\nu_1} \circ \cdots \circ \sigma_{n_i}^{\nu_{n_i}}$ and

$$\sigma(\bar{f}_i + a) = \bar{f}_i + \nu_1 w_1 + \cdots + \nu_{n_i} w_{n_i} + a, \text{ for all } a \in W \subset k.$$

This means that $P_i(\bar{f}_i)$ is Gal(F_{i+1}/F_i) invariant, i.e. belongs to F_i. Therefore, the polynomial $A_i(X)$ belongs to $F_i[X]$, is monic of degree $p^{n_i} = [F_{i+1} :$

$F_i]$ and vanishes at $\bar{f}_i$ hence it is the irreducible polynomial of $\bar{f}_i$ over F_i. The polynomial $P_i(X)$ is given by

$$P_i(X) = \frac{\Delta(w_1, w_2, \ldots, w_{n_i}, X)}{\Delta(w_1, w_2, \ldots, w_{n_i})}, \tag{2.1}$$

where $\Delta(w_1, \ldots, w_n)$ is the Moore determinant;

$$\Delta(w_1, \ldots, w_n) = \det \begin{bmatrix} w_1 & w_2 & \ldots & w_n \\ w_1^p & w_2^p & \ldots & w_n^p \\ \vdots & \vdots & & \vdots \\ w_1^{p^{n_i-1}} & w_2^{p^{n_i-1}} & \ldots & w_{n_i}^{p^{n_i-1}} \end{bmatrix}.$$

It is an additive polynomial of the form

$$P_i(X) = X^{p^{n_i}} + a_{n_i-1}X^{p^{n_i-1}} + \cdots + a_1X^p + a_0X,$$

where $a_i \in k \subset F_i$. We have proved that the generator $\bar{f}_i$ of the extension F_{i+1}/F_i satisfies an equation of the form

$$\bar{f}_i^{\,p^{n_i}} + a_{n_i-1}\bar{f}_i^{\,p^{n_i-1}} + \cdots + a_1\bar{f}_i^{\,p} + a_0\bar{f}_i = D_i, \tag{2.2}$$

for some $a_{n_i-1}, \ldots, a_0 \in k$, $D_i = P_i(\bar{f}_i) \in F_i$.

Remark 2.1. *Instead of $\bar{f}_i$ one can use $\lambda\bar{f}_i$. The additive polynomial corresponding to $\lambda\bar{f}_i$ is equal to $\lambda^{p^{n_i-1}}P_i(X)$, where $P_i(X)$ is the additive polynomial corresponding to $\bar{f}_i$. Indeed, when we change $\bar{f}_i$ to $\lambda\bar{f}_i$ the $\mathbb{F}_p$-vector space W is changed to $\lambda \cdot W$, that is the basis elements w_i are changed to λw_i. Hence, the Moore determinant in the numerator of eq. (2.1) defining $P_i(\lambda X)$ is multiplied by $\lambda^{1+p+\cdots+p^{n_i-1}}$ while the denominator is multiplied by $\lambda^{1+p+\cdots+p^{n_i-2}}$. Therefore $P_i(\lambda X) = \lambda^{p^{n_i-1}}P_i(X)$.*

We have the following:

Theorem 2.2. *The cocycles $\bar{C}_i \in H^1(\mathrm{Gal}(F_{i+1}/F_1), k_{\mathbf{n},\bar{m}_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}])$, when restricted to the elementary abelian group $\mathrm{Gal}(F_{i+1}/F_i) < \mathrm{Gal}(F_{i+1}/F_1)$ describe fully the elementary abelian extension F_{i+1}/F_i given by the equation*

$$P_i(Y) = D_i.$$

Moreover the element $D_i = P_i(\bar{f}_i)$ is described by the additive polynomial $P_i(Y)$ and by the selection of $\bar{f}_i$. A different selection of $\bar{f}'_i$, i.e. $\bar{f}'_i = \lambda\bar{f}_i + a$, for some $a \in k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]$, $\lambda \in k^*$ gives rise to the same polynomial $\lambda^{p^{n_i-1}}P_i$ and to a different D'_i given by $D'_i = \lambda^{p^{n_i-1}}D_i + \lambda^{p^{n_i-1}}P_i(a)$. The two extensions $F_i(\bar{f}_i)$ and $F_i(\bar{f}'_i)$ are equal.

Proof. The only part we didn't prove is the dependence of the additive polynomial to the selection of the generator $\bar{f}_i$. We have seen that changing $\bar{f}_i$ adds a coboundary to $\bar{C}_i$.

But when σ belongs to $\mathrm{Gal}(F_{i+1}/F_i)$, $\bar{C}_i(\sigma)$ belongs to k, and k admits the trivial action. Therefore, all coboundaries are zero and the result follows by lemma 1.8. □

The additive polynomial $P_i(Y)$, which depends on the values of $\bar{C}_i(\sigma)$ with $\sigma \in \mathrm{Gal}(F_{i+1}/F_i)$ gives also compatibility conditions for the cocycle $\bar{C}_i$ on all elements of $\mathrm{Gal}(F_{i+1}/F_1)$. Namely, by application of σ to eq. (2.2) we obtain the following

$$\boxed{P_i(\bar{C}_i(\sigma)) = (\sigma - 1)D_i \text{ for all } \sigma \in \mathrm{Gal}(F_i/F_1).} \tag{2.3}$$

So if σ keeps D_i invariant, for instance when $\sigma \in \mathrm{Gal}(F/F_i)$, then $\bar{C}_i(\sigma) \in \mathbb{F}_{p^n} \subset k$.

Equation (2.3) is essentially a relation among the cocycles $\bar{C}_i(\sigma)$ and $\bar{C}_\nu(\sigma)$ for $\nu < i$. Indeed, the element $D_i \in k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]$ is a polynomial expression on the elements $\bar{f}_0, \ldots, \bar{f}_{i-1}$, and the action is given in terms of the elements $\bar{C}_\nu(\sigma)$ for $\nu < i$ and $\bar{f}_i$ as given in eq. (1.9).

Lemma 2.3. *An additive polynomial $P \in k[Y]$ defines a map*

$$\begin{aligned} H^1(G, k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]) &\longrightarrow H^1(G, k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]) \\ d &\longmapsto P(d), \end{aligned} \tag{2.4}$$

Proof. Notice first that elements in the space $L(\nu P)$, for some $\nu \in \mathbb{N}$, can be multiplied as elements of the ring $\mathbf{A}$, so a polynomial expression $P(d)$ of a cocycle d makes sense. One has to be careful since the multiplication of two elements in $L(\nu P)$, is not in general an element of $L(\nu P)$, since it can have a pole order greater than ν. Therefore the value $P(d)$ is an element in $L(\mu P)$ for some $\mu \in \mathbb{N}$ for big enough μ. However notice that eq. (2.3) implies that $P(\bar{C}_i(\sigma)) \in k_{\mathbf{n},\bar{m}_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]$ so that $P_i(\bar{C}_i) \in H^1(G, k_{\mathbf{n},\bar{m}_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}])$.

Finally observe now that if d is a cocycle, i.e. $d(\sigma\tau) = d(\sigma) + \sigma d(\tau)$, then

$$P(d(\sigma\tau)) = P(d(\sigma) + \sigma d(\tau)) = P(d(\sigma)) + P(\sigma d(\tau)) = P(d(\sigma)) + \sigma P(d(\tau)).$$

On the other hand if $d(\sigma) = (\sigma - 1)b$ is a coboundary, then

$$P(d(\sigma)) = P\big((\sigma - 1)b\big) = (\sigma - 1)P(b)$$

is a coboundary as well. □

This allows us to give a cohomological interpretation of eq. (2.3):

Theorem 2.4. *The cocycles $\bar{C}_i$ given in eq. (1.8) are in the kernel of the map P_i acting on cohomology as defined in lemma 2.3. The corresponding element D_i is then the element expressing $P(C_i)$ as a coboundary. The elementary abelian extension is determined by a series of cocycles $\bar{C}_i \in H^1(\mathrm{Gal}(F_{i+1}/F_i), k_{\mathbf{n},\bar{m}_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}])$, which define a series of additive polynomials P_i and extend to cocycles in $\bar{C}_i \in H^1(\mathrm{Gal}(F_{i+1}/F_1), k_{\mathbf{n},\bar{m}_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}])$ so that each $\bar{C}_i$ is in the kernel of P_i.*

Remark 2.5. *In remark 2.1 we have seen that by changing the generator $\bar{f}_0$ to $\lambda\bar{f}_0$ the additive polynomial is changed from P_i to $\lambda^{p^{n_i - 1}}P_i$. The corresponding map*

$$\mathbb{P}H^1(G, k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]) \longrightarrow \mathbb{P}H^1(G, k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}])$$

is not affected.

2.2 Application to Nottingham groups

An automorphism σ of the complete local algebra $k[[t]]$ is determined by the image $\sigma(t)$ of t, where $\sigma(t) = \sum_{i=1}^{\infty} a_i t^i \in k[[t]]$. We consider the subgroup of normalised automorphisms that is, automorphisms of the form

$$\sigma : t \mapsto t + \sum_{i=2}^{\infty} a_i t^i.$$

S. Jennings [23] proved that the set of latter automorphisms forms a group under substitution, denoted by $\mathscr{N}(k)$, called the Nottingham group. This group has many interesting properties, for instance R. Camina proved in [9] that every countably based pro-p group can be embedded, as a closed subgroup, in the Nottingham group. We refer the reader to [10] for more information regarding $\mathscr{N}(k)$. We would like to provide an explicit way to describe the elements of $\mathscr{N}(k)$. It is proved in [28, prop. 1.2] and [37, sec. 4, th. 2.2], that each automorphism of order p is conjugate to the automorphism given by

$$t \mapsto t(1 + ct^m)^{-1/m} = t \left(\sum_{\nu=0}^{\infty} \binom{-1/m}{\nu} c^\nu t^{\nu m} \right) \tag{2.5}$$

for some $c \in k^\times$ and some positive integer m prime to p.

In [5] F. Bleher, T. Chinburg, B. Poonen and P. Symonds, studied the extension $L/k(t)$, where $L := k(\{\sigma(t) : \sigma \in G\})$, where G is a finite subgroup of Aut$k[[t]]$. Notice here that each automorphism of order p^n is conjugate to $t \mapsto \sigma(t)$, where $\sigma(t) \in k[[t]]$ is algebraic over $k(t)$. Also in [5] the notion of almost rational automorphism is defined: an automorphism $\sigma \in \mathrm{Aut}(k[[t]])$ is called almost rational if the extension $L/k(t)$ is Artin-Schreier.

The rational function field $k(t)$, despite its simple form, is not natural with respect to the group G acting on the HKG-cover. For example the determination of the algebraic extension $L/k(t)$ and the group of the normal closure seems very difficult.

Here we plan to give another generalization, by using the fact that the "natural" rational function field with respect to the Harbater-Katz-Gabber cover is X^{G_1} and not $k(t)$.

In [31, p. 473] A. Kontogeorgis proposed the following explicit form for an automorphism of an HKG-cover of order p^n:

$$\sigma(t) = t \left(1 + \sum_{i=1}^{r} c_i(\sigma) u_i t^{m-m_i} \right)^{-1/m},$$

where m is the first pole number which is not divisible by the characteristic p, u_i/t^{m_i} for $1 \leq i \leq r$ are functions in $L(mP)$ (u_i is a unit) and $1/t^m$ is the function corresponding to m (t being the local uniformizer). In the latter function the unit is absorbed by Hensel's lemma.

A canonical selection of uniformizer

In an attempt to describe in explicit form automorphisms of $k[[t]]$ let us quote here some results from [31]. We will work with the corresponding HKG-cover $X \xrightarrow{G} \mathbb{P}^1$ corresponding to a finite subgroup $G \subset \mathrm{Aut}(k[[t]])$. Again let m_r denote the first pole number not divisible by the characteristic and f_i, $i = 1, \ldots, \dim L(m_r P) = r$ a basis for the space $L(m_r P)$, such that

$$(f_i)_\infty = m_i. \tag{2.6}$$

As we have seen this basis is not unique but eq.(2.6) implies that if the element f_i is selected, then $f_i' = \lambda_i f_i + a_i$, where $a_i \in L\big((m_i - 1)P\big)$ is also a basis element of valuation m_r.

This means that the base change we will consider, corresponds to invertible upper triangular matrices, i.e. to linear maps which keep the flag of the vector spaces $L(m_i P)$.

Recall that $m = m_r$ is the first pole number not divisible by p. Let us focus on the element f_r. This element is of the form $f_r = u_m/t^m$, where u_m is a unit. Since $(m,p) = 1$ we know by Hensel's lemma that u_m is an m-th power so by a change of uniformizer we can assume that $f_r = 1/t^m$. When changing from a uniformizer t to a uniformizer $t' = \phi(t) = tu(t)$ ($u(t)$ is a unit in $k[[t]]$), the automorphism $\sigma \in k[[t]]$ expressed as an element in $k[[t']]$ is a conjugate of the initial automorphism, i.e. $\phi\sigma\phi^{-1}$. By selecting the canonical uniformizer with respect to f_r we see that the expression of an arbitrary σ can take a simpler representation after conjugation. Also this result is in accordance with (and can be seen as a generalization of) the result of Klopsch and Lubin, [28], [37]. The selection of uniformizer $t = t_{f_r}$ is unique once f_r is selected.

Definition 2.6. *We will call the uniformizer $t_{f_r} = f_r^{-1/m}$ the canonical uniformizer corresponding to f_r.*

What happens if we change the function f_r to $f_r' = f_r + a$, where $a \in L\big((m-1)P\big)$? Then $a = u/t^\mu$, with $0 \leq \mu < m$ and in this case the new uniformizer is given by

$$t_{f_r'} = \left(f_r + \frac{u}{t^\mu}\right)^{-1/m} = t\left(1 + ut^{m-\mu}\right)^{-1/m} = t\left(1 + at^m\right)^{-1/m}.$$

Keep in mind that the set of uniformizers for the local ring $k[[t]]$ equals to $tu(t)$, where u is a unit of the ring $k[[t]]$.

Let $\bar{m}_1, \ldots, \bar{m}_s$ be the generators of the Weierstrass semigroup $H(P)$. These elements correspond to a successive sequence of function fields $F_i = F_{i-1}(\bar{f}_{i-1})$ so that $v(\bar{f}_{i-1}) = p^{|\mathrm{Gal}(F/F_i)|}\lambda_{h-1} = \bar{m}_i$. It is not clear that $\bar{m}_i \geq \bar{m}_j$ for $j < i$. However if for some j we have $\bar{m}_j < \bar{m}_i$ for some $i < j$ then

$$\sigma(\bar{f}_j) = \sigma(\bar{f}_j) + \bar{C}_j(\sigma), \text{ where } \bar{C}_j \in k[\bar{f}_0, \ldots, \widehat{\bar{f}_i}, \ldots, \bar{f}_{j-1}],$$

that is, $\bar{f}_i$ does not appear in any term of the polynomial expression of $\bar{C}_j(\sigma)$, for all $\sigma \in G$. This means that we can generate an HKG-cover with corresponding function field generated by fewer elements than the initial one.

If we assume that among all HKG-covers which correspond to a local action of G on $k[[t]]$ we select one whose function field is minimally generated then $\bar{m}_1 < \bar{m}_2 < \ldots < \bar{m}_s$.

Lemma 2.7. *Let $m = m_r$ be the first pole number not divisible by the characteristic p. Then $m = \bar{m}_s$, that is the pole number corresponding to the last generator $\bar{f}_s$.*

Proof. It is clear that not all pole numbers are divisible by p since $m \in H(P)$, $p \nmid m$. So at least one generator must be prime to p. On the other hand $F_i = F_{i-1}(\bar{f}_{i-1})$, thus the pole numbers $\bar{m}_i$ of elements $\bar{f}_i$ for $i < s$ are divisible by p, see also [25, eq. (6)]. Therefore only the last generator can be not divisible by p. □

Theorem 2.8. *Let $\bar{C}_s \in H^1(G, k_{\mathbf{n},m}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{s-1}])$ be the cocycle corresponding to $m = m_s$, where m is the first pole number not divisible by p, see lemma 2.7. We choose as uniformizer the canonical uniformizer $t = \bar{f}_s^{-1/m}$. We define the representation:*

$$\begin{aligned} \Phi : G &\longrightarrow \mathrm{Aut}(k[[t]]) \\ \sigma &\longmapsto \left(t \mapsto t(1 + \bar{C}_s(\sigma)t^m)^{-1/m}\right). \end{aligned} \tag{2.7}$$

The expression $1 + \bar{C}_s(\sigma)t^m)^{-1/m}$ can be expanded as a powerseries using the binomial theorem and determines uniquely an automorphisms of $k[[t]]$. We have that for all $\sigma, \tau \in G$

$$\Phi(\tau\sigma) = \Phi(\sigma)\Phi(\tau).$$

Furthermore Φ is a monomorphism.

Proof. We begin by noticing that $\sigma(\bar{f}_s) = \bar{f}_s + \bar{C}_s(\sigma)$ and we can select t so that $t^{-m} = \bar{f}_s$. Using the above expression we can determine the value of $\sigma(t)$ using

$$\frac{1}{\sigma(t)^m} = \frac{1}{t^m} + \bar{C}_s(\sigma),$$

see also [31, eq. 4]. In this way σ coincides with the image of $\Phi(\sigma) \in \mathrm{Aut}(k[[t]])$ in eq. (2.7).

Recall that $\sigma \in G$ acts on the elements $\bar{f}_0, \ldots, \bar{f}_{s-1}$ by definition in terms of the cocycles $\bar{C}_i(\sigma)$. This was defined to be a left action. Also this action is by construction assumed to be compatible with the action of G on $k[[t]]$ in the sense that when we see the elements $\bar{f}_i$ as elements in $k[[t]][t^{-1}]$, then $\sigma(\bar{f}_i) = \Phi(\sigma)(\bar{f}_i)$, that is the action of σ on $\bar{f}_i$ as elements in $k_{\mathbf{n},m_{i+1}}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{s-1}]$ coincides with the action of σ on f_i seen as an element in the quotient field of $k[[t]]$. In other words we have

$$\sigma(f_i(t)) = f_i(\sigma(t)) = f_i(t) + C_i(\sigma).$$

We will prove first that this is a homomorphism i.e.

$$t(1+\bar{C}_s(\tau\sigma)t^m))^{-1/m} = t(1+\bar{C}_s(\sigma)t^m)^{-1/m} \circ t(1+\bar{C}_s(\tau)t^m)^{-1/m}, \tag{2.8}$$

where $\circ$ denotes the composition of two powerseries. The right hand side of the above equation equals

$$t\big(1+\bar{C}_s(\tau)t^m\big)^{\frac{-1}{m}} \left(1+\frac{\tau(\bar{C}_s(\sigma))t^m}{1+\bar{C}_s(\tau)t^m)}\right)^{\frac{-1}{m}} = t\left(1+\big(\bar{C}_s(\tau)+\tau\bar{C}_s(\sigma)\big)t^m\right)^{-1/m}$$

so eq. (2.8) holds by the cocycle condition for $\bar{C}_s$.

The kernel of the homomorphism Φ, consists of all elements $\sigma \in G$ such that $\bar{C}_s(\sigma) = 0$. But if $\bar{C}_s(\sigma) = 0$ then $\sigma(t) = t$ and σ is the identity. $\square$

Remark 2.9. *The above construction behaves well when we substitute f_m with $f'_m = f_m + a$. In any case the representation given in eq. (2.7) is given in terms of the canonical uniformizer t_{f_r} corresponding to the element $\bar{f}_s = f_r$ which gives rise to the cocycle $\bar{C}_s$.*

Remark 2.10. *Equation (2.7) implies that the knowledge of the cocycle $\bar{C}_s$ implies the knowledge of $\sigma(t)$, which in turn gives us how σ acts on all other elements $\bar{f}_i$ for all $0 \le i \le s-1$. Subsequently one may be led to believe that $\bar{C}_s$ can determine all other cocycles $\bar{C}_\nu$ for all $1 \le \nu \le s-1$. This is not entirely correct. Indeed, $\bar{C}_s$ is a cocycle with values on the G-module $k_{\mathbf{n},\bar{m}_s}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{s-1}]$, therefore the action of G on $\bar{f}_i$ for $0 \le i \le s-1$ is assumed to be known and is part of the definition of the cocycle $\bar{C}_s$. That means that $\bar{C}_i$ are assumed to be known and part of the definition of $\bar{C}_s$.*

Proposition 2.11. *If $\sigma \in G$, $\sigma \neq 1$, then*

$$v_P(\sigma(t)-t) = m - v_P\big(\bar{C}_s(\sigma)\big) + 1 = I(\sigma),$$

where $-I(\sigma)$ is the Artin character since k is algebraically closed, see [45, VI.2]. Therefore $\sigma \in G_{I(\sigma)} - G_{I(\sigma)+1}$.

Proof. The valuation of $\sigma(t) - t$ comes from the binomial expansion of eq. (2.7). The rest is the definition of the ramification group. $\square$

2.2.1 Elements of order p in the Nottingham group

It is known that every element of order p in $\mathrm{Aut}(k[[t]])$ is conjugate to the automorphism

$$t \mapsto t(1+ct^m)^{-1/m}, \qquad \text{where } c \in k,$$

for some m prime to p, see [28, prop. 1.2] and [37, th. 2.2].

We can obtain this result using theorem 2.8. Let σ be an automorphism of $k[[t]]$ of order p. Let $X \to \mathbb{P}^1$ be the corresponding HKG-cover. The sequence of higher ramification groups equals $\langle\sigma\rangle = G_0 = G_1 = \cdots = G_m > \{1\}$, i.e. there is only one jump in the ramification filtration. If $m = 1$ then $G_i(P) = \{1\}$ for $i \ge 2$ and in this case the genus $g_X = 0$. This is a trivial case so we can assume that $m > 1$. From theorem 1.3 we know that the Weierstrass semigroup is generated by $p = |G_1(P)|$ and m_r. If m_i is a pole number less than m_r then m_i is a multiple of p, hence the corresponding elements f_i with pole order m_i at P will be powers of f_0 where $(f_0)_\infty = pP$.

Since the ramification filtration jumps only once, the same holds for the representation filtration, i.e.

$$G_1(P) = \ker \rho_{c_1} > \{1\}$$

So if σ is not the identity then by [25, prop.27] we have that

$$\sigma(f_0^i) = f_0^i \text{ for } i = 0, 1, \ldots, \lfloor m_r/p \rfloor \text{ and}$$
$$\sigma(f_{c_1+1}) = \sigma(f_r) = f_r + C(\sigma) \text{ where } C(\sigma) \in k^\times.$$

Compare also with the computation of proposition 2.11. To obtain the result we notice the following; changing the local uniformizer to a canonical one imposes the substitution of σ by a conjugate which, by theorem 2.8, maps t to the desired form.

2.2.2 Elements of order p^h in the Nottingham group

Let us now consider an element σ of order p^h. As before the cyclic group

$$G_0(P) = G_1(P) = \cdots = G_{b_1}(P) \gneqq G_{b_1+1}(P) = \cdots = G_{b_2}(P) \gneqq \cdots \gneqq G_{b_\mu}(P) \gneqq \{1\}$$

Since a cyclic group has only cyclic subgroups and all quotients of cyclic groups are cyclic, while $G_{b_i}/G_{b_{i+1}}$ is elementary abelian, we see that the number of gaps μ is equal to h and p^{h-i} is the exact power of p dividing each $\bar{m}_i$.

Observe that all intermediate elementary abelian extensions $F_{i+1}/F_i = F_i(\bar{f}_i)/F_i$ are cyclic. The additive polynomial describing the extension $F_i(\bar{f}_i)/F_i$ is given by

$$Y^p - \bar{C}_i^{p-1} Y = \bar{f}_i^p - \bar{C}_i^{p-1} \bar{f}_i,$$

by computation of the Moore determinant $\det \begin{pmatrix} C_i & Y \\ C_i^p & Y^p \end{pmatrix}$, where $\bar{C}_i$ is computed at a generator σ^{p^i} of the cyclic group $\mathrm{Gal}(F_{i+1}/F_i) = G_{b_{i+1}}/G_{b_i}$, (i.e. $\sigma^{p^i}(\bar{f}_i) = \bar{f}_i + \bar{C}_i(\sigma^{p^i})$). Since $\bar{C}_i \in k$, if we rescale $\bar{f}_i$ by $\bar{f}_i/C_i$, we can assume without loss of generality that the equation is an Artin-Schreier one:

$$Y^p - Y = \bar{f}_i^p - \bar{f}_i = D_i, \text{ where } D_i \in F_i.$$

Let g be an automorphism of the HKG-cover X. Since $g(\bar{f}_\nu) = \bar{f}_\nu + \bar{c}_\nu(g)$ and $\bar{c}_\nu(g) \in F_{\nu-1}$, the automorphism g gives rise to an automorphism $g : F_\nu \to F_\nu$ for all ν. We have that

$$\bar{C}_i(g)^p - \bar{C}_i(g) = (g-1)(\bar{f}_i^p - \bar{f}_i) = (g-1)D_i. \qquad (2.9)$$

Notice that eq. (2.9) has many solutions $\bar{C}_i(g)$ for a fixed g, which differ by an element $\bar{c}_i(\sigma)$ for some $\sigma \in \mathrm{Gal}(F_{i+1}/F_i)$, since $(g\sigma - 1)(D_i) = (g - 1)(D_i)$.

The representation filtration has the following form (the filtrations are collectively depicted in the diagrams below)

$$F^{G_1(P)} = F_0 = F^{\ker\rho_0} \subset F_1 = F^{\ker\rho_1} \subset \cdots \subset F_r = F^{\ker\rho_r} = F.$$

We have $p^{h-i} = |\ker\rho_{c_{i+1}}|$ for $0 \leq i \leq n-1$ and $p^h = |G_1(P)|$. The generators of the Weierstrass semigroup are $p^h, p^{h-1}\lambda_1, \ldots, p\lambda_{\mu-1}, \lambda_\mu$. We have the following tower of fields:

$$\begin{tikzcd}
F = F_{h+1} = F_h(\bar{f}_h) \arrow[d, dash, "\mathbb{Z}/p\mathbb{Z}"] \arrow[dddd, dash, bend right=60, "\mathbb{Z}/p^h\mathbb{Z}"'] & \{1\} \arrow[d, dash, "p"] & 1 \arrow[d, dash, "p"] \\
F_h = F_{h-1}(\bar{f}_{h-1}) \arrow[d, dash, "\mathbb{Z}/p\mathbb{Z}"] & G_{b_h} = \ker\rho_{c_h} \arrow[d, dash, "p"] & \langle \sigma^{p^{h-1}} \rangle \text{ (order=}p) \arrow[d, dash, "p"] \\
F_{h-1} \arrow[d, dash, dotted] & G_{b_{h-1}} = \ker\rho_{c_{h-1}} \arrow[d, dash, dotted] & \langle \sigma^{p^{h-2}} \rangle \text{ (order=}p^2) \arrow[d, dash, dotted] \\
F_2 = F_1(\bar{f}_1) \arrow[d, dash, "\mathbb{Z}/p\mathbb{Z}"] & G_{b_2} = \ker\rho_{c_2} \arrow[d, dash, "p"] & \langle \sigma^{p} \rangle \text{ (order=}p^{h-1}) \arrow[d, dash, "p"] \\
F_1 = k(\bar{f}_0) \arrow[d, equal] & G_{b_1} = \ker\rho_{c_1} \arrow[d, equal] & \langle \sigma \rangle \arrow[d, equal] \\
F_0 = F^{G_1(P)} & G_1(P) & G_1(P)
\end{tikzcd}$$

For every $g \in \mathrm{Gal}(F/F_1)$ we have

$$g(\bar{f}_{r-1}) - \bar{f}_{r-1} = \bar{C}_{r-1}(g).$$

For a cyclic group $\mathbb{Z}/p^i\mathbb{Z}$ the cohomology is given by:

$$H^1(\mathbb{Z}/p^i\mathbb{Z}, A) = \frac{\{a \in A : N(a) = 0\}}{(\sigma_i - 1)A},$$

where σ_i is a generator of the cyclic group $\mathbb{Z}/p^i\mathbb{Z}$ and $N = 1 + \sigma + \cdots + \sigma^{p^i-1}$ is the norm, see [51, th. 6.2.2, p. 168]. In view of theorem 2.4 we will consider the groups $\mathrm{Gal}(F_{i+1}/F_1)$, which are generated by the generator σ of the cyclic group $\mathrm{Gal}(F_{h+1}/F_1)$ modulo the subgroup $\mathrm{Gal}(F_{h+1}/F_{i+1})$. Thus in the group $\mathrm{Gal}(F_{i+1}/F_1)$ the order of σ equals p^i.

Observe now that $\tau = \sigma^{p^{i-1}}$ acts trivially on $A = k_{\mathbf{n},m_i}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{i-1}]$. We now compute the norm for $\mathrm{Gal}(F_{i+1}/F_1)$:

$$\begin{aligned}1+\sigma+\cdots+\sigma^{p^i-1} &= \sum_{\nu=0}^{p^i-1}\sigma^{\nu} = \sum_{\pi=0}^{p-1}\sum_{\upsilon=0}^{p^{i-1}-1}\sigma^{\pi p^{i-1}}\sigma^{\upsilon}\\ &= \sum_{\pi=0}^{p-1}\tau^{\pi}\sum_{\upsilon=0}^{p^{i-1}-1}\sigma^{\upsilon},\end{aligned}$$

where $\tau := \sigma^{p^{i-1}}$, and observe that the above equation restricted on A gives

$$1+\sigma+\cdots+\sigma^{p^i-1} = p\cdot\sum_{\upsilon=0}^{p^{i-1}-1}\sigma^{\upsilon}$$

which is zero on A. So we finally arrive at the computation:

$$H^1\left(\mathbb{Z}/p^i\mathbb{Z}, k_{\mathbf{n},\bar{m}_i}[\bar{f}_0,\bar{f}_1,\ldots,\bar{f}_{i-1}]\right) = k_{\mathbf{n},\bar{m}_i}[\bar{f}_0,\bar{f}_1,\ldots,\bar{f}_{i-1}]_{\mathbb{Z}/p^i\mathbb{Z}},$$

where the latter space is the space of $\mathbb{Z}/p^i\mathbb{Z}$-coinvariants.

Proposition 2.12. *A cyclic group of the Nottingham group is described by a series of elements* $\bar{C}_i \in k_{\mathbf{n},\bar{m}_i}[\bar{f}_0,\ldots,\bar{f}_{i-1}]_{\mathbb{Z}/p^i\mathbb{Z}}$ *so that* $\bar{C}_i^p - \bar{C}_i$ *is zero in the space* $k_{\mathbf{n},\bar{m}_i}[\bar{f}_0,\ldots,\bar{f}_{i-1}]_{\mathbb{Z}/p^i\mathbb{Z}}$*.*

In order to ensure that the element σ *has order* p^h *we should have,* $\bar{C}_s(\sigma^{p^{\nu}}) \neq 0$*, for all* $0 \leq \nu < h$ *i.e.*

$$\left(1+\sigma+\cdots+\sigma^{p^{\nu}-1}\right)\bar{C}(\sigma) \neq 0.$$

Chapter 3

On the canonical ideal of an HKG curve

3.1 Introduction

In this chapter our aim is to calculate the canonical ideal of an HKG-curve X/k. In order to do so we use proposition 1.12. Additionally we employ the breakdown process of an HKG-curve into Artin-Schreier extensions as described in the previous chapter while also expanding our understanding of the generating elements (section 3.2). In this chapter we will again assume that the Galois group of the HKG-cover $X \to \mathbb{P}^1$ is a p-group.

We define a set of possible generators of the canonical ideal (i.e. $\mathbf{A}+\mathbf{A}$) and then define an equivalence relation (def. 3.3) appropriately which throws away the non-generators, a result in the spirit of the first isomorphism theorem (section 3.3). There is a bijection (check eq. 3.12)

$$\psi : H_2 \longrightarrow \mathbf{A} + \mathbf{A}/\sim .$$

where H_2 can be identified with a basis of the space of holomorphic differentials. In this way we are allowed to associate elements of a basis with sums of elements of $\mathbf{A}$ and we use these sums instead, since they are easier to manipulate. The bijection ψ also allows us to work interchangeably between the space $\mathbf{A}+\mathbf{A}$ and the space of 2-differentials. Then in section (3.4) we interpret the equations of the intermediate Artin-Schreier extensions as equations of quadratic differentials defining a set of relations K_0 and $K_{\bar{v},i}$, which we prove that are part of the canonical ideal, see proposition 3.6 and 3.10. Of these two K_0 is the "trivial" part, imposed by the definition of the canonical map while $K_{\bar{v},i}$ is slightly less trivial and is derived from the tower of Artin-Schreier equations giving an HKG-curve. Notice that in order to be able to generate the canonical ideal by quadratic polynomials we have to assume that all intermediate extensions satisfy the assumptions of Petri's theorem, see lemma 3.8.

In section (3.5) we prove that the aforementioned sets generate the canonical ideal, using the bijection of the previous paragraph, by induction on the number of intermediate extensions of the function field.

In the last section (3.6) we give several examples illustrating our construction. These examples are used to demonstrate the fact that, despite the possibly complicated definition of the generating sets (along with the proof), computations can be done efficiently in specific situations.

3.2 Preliminaries

As before suppose X is an HKG curve over the algebraically closed field k (the same assumptions as in the introduction). The canonical ideal I of X was described in section 1.2.

Extra assumption: We are going to assume that our curves are non-trigonal, so that the third condition of Petri's theorem (thm. 1.11) is satisfied. In lemma 3.8 the reasons for this demand become apparent.

Summarizing the results of the previous chapters we have seen that an HKG curve is defined by a series of extensions $F_{i+1} = F_i(\bar{f}_i)$, where the irreducible polynomials of $\bar{f}_i$ are of the form

$$X^{p^{n_i}} + a^{(i)}_{n_i-1} X^{p^{n_i-1}} + \cdots + a^{(i)}_0 X - D_i. \qquad (3.1)$$

$$\begin{tikzcd}
F = F_{\xi+1} = F_\xi(\bar{f}_\xi) \arrow[d, dash, "p^{n_\xi}"] \arrow[dddd, dash, bend right=60, "|G_0|"'] \\
F_\xi = F_{\xi-1}(\bar{f}_{\xi-1}) \arrow[d, dash, "p^{n_{\xi-1}}"] \\
F_{\xi-1} \arrow[d, dotted, dash] \\
F_2 = F_1(\bar{f}_1) \arrow[d, dash, "p^{n_1}"] \\
F_1 = k(\bar{f}_0) \arrow[d, equal] \\
F_0 = F^{G_1(P)}
\end{tikzcd}$$

The coefficients $a^{(i)}_{n_i-j} \in k$, $j = 1, \ldots, n_1$ and $D_i \in F_i$ has pole divisor $p^{n_i}\bar{m}_i P$.
The Weierstrass semigroup H is generated by the elements $\{|G_0|, \bar{m}_1, \ldots, \bar{m}_\xi\}$ where $\bar{m}_i = p^{n_{i+1}+\cdots+n_\xi} b_i$. Notice that the ramification groups are given by $|G_{b_{i+1}}| = p^{n_{i+1}+\cdots+n_\xi}$ and they form the following filtration sequence

$$G_0(P) = G_1(P) = \cdots = G_{b_1}(P) \gneqq G_{b_1+1}(P) = \cdots$$

$$\cdots = G_{b_2}(P) \gneqq \cdots \gneqq G_{b_\mu}(P) \gneqq \{1\}.$$

We know that $(b_i, p) = 1$ and $|G_0| = p^{n_1+\cdots+n_\xi}$, see [25], [33].
The above subset of the Weierstrass semigroup might not be the minimal set of generators, since this depends on whether $G_1(P)$ equals $G_2(P)$, see [25, thm. 13]. We will denote by

$$H_s = \{h : h \in H, h \leq s(2g-2)\} \qquad (3.2)$$

the part of the Weierstrass semigroup bounded by $s(2g-2)$. We will also denote by $\mathbf{A}$ the set

$$\mathbf{A} = \{(i_0, \ldots, i_\xi) \in \mathbb{N}^{\xi+1} : i_0|G_0| + \sum_{\nu=1}^{\xi} i_\nu \bar{m}_\nu \leq 2g-2\}. \qquad (3.3)$$

For each $h \in H_1$ there is a fixed element $\bar{f}_h$ with unique pole at P of order h. These elements are the field generators, such that $F_{i+1} = F_i(\bar{f}_i)$. The sets H_1 and $\mathbf{A}$ have the same cardinality and moreover the map

$$H_s \ni h \longmapsto f_h df_0^{\otimes s}, \tag{3.4}$$

gives rise to a basis of $H^0(X, \Omega^s)$, see [25, proposition 42]. We will also denote $f_h df_0^{\otimes s}$ by ω_h and since each element of $\mathbf{A}$ corresponds to an element $L \in H_1$ we will define $\omega_L := \omega_h$. This implies that the cardinality of H_s is given by

$$\#H_s = \begin{cases} g & \text{if } s = 1 \\ (2s-1)(g-1) & \text{if } s > 1. \end{cases}$$

We will denote by $\mathbb{T}^2$ the monomials of $\mathrm{Sym} H^0(X, \Omega_X)$ of degree two (i.e. of the form $\omega_L \omega_K$). For a graded ring S we will use $(S)_2$ to denote elements of degree 2.

The information of the successive extensions is encoded in the coefficients $a_j^{(i)}$ of the additive left part of eq. (3.1) *and* in the elements $D_i \in F_i$. Eq. (3.1) vanishes at $\bar{f}_i$, yielding the equality

$$\bar{f}_i^{p^{n_i}} + a_{n_i-1}^{(i)} \bar{f}_i^{p^{n_i-1}} + \cdots + a_0^{(i)} \bar{f}_i = D_i$$

where, taking valuations on both sides, yields that the valuation of D_i is $-p^{n_i} \bar{m}_i$. Notice that the minus sign comes from the fact that $\bar{f}_i$ has a *pole* at P and since it is of order $\bar{m}_i$, one has $v_P(D_i) = v_P(\bar{f}_i^{p^{n_i}}) = -p^{n_i} \bar{m}_i$. Since D_i belongs to $F_i = F^{G_1(P)}(\bar{f}_1, \ldots, \bar{f}_{i-1})$ and $F^{G_1(P)} = k(\bar{f}_0)$ (see [25, remark 21]), one can express D_i as

$$D_i(\bar{f}_0, \ldots, \bar{f}_{i-1}) = \sum_{(\ell_0, \ldots, \ell_{i-1}) \in \mathbb{N}^i} \alpha^{(i)}_{\ell_0, \ldots, \ell_{i-1}} \bar{f}_0^{\ell_0} \cdots \bar{f}_{i-1}^{\ell_{i-1}}. \tag{3.5}$$

where $\alpha^{(i)}_{\ell_0, \ldots, \ell_{i-1}} \in k$ are some coefficients, not to be confused with the coefficients in eq. (3.1). We will need the following:

Lemma 3.1. *Assume that* $(\ell_0, \ldots, \ell_{i-1}), (w_0, \ldots, w_{i-1}) \in \mathbb{N}^i$ *such that*

$$1 \le \ell_\lambda, w_\lambda < p^{n_\lambda} \text{ for all } 1 \le \lambda \le i-1 \tag{3.6}$$

and

$$\ell_0 |G_0| + \ell_1 \bar{m}_1 + \cdots + \ell_{i-1} \bar{m}_{i-1} = w_0 |G_0| + w_1 \bar{m}_1 + \cdots + w_{i-1} \bar{m}_{i-1}. \tag{3.7}$$

Then $(\ell_0, \ldots, \ell_{i-1}) = (w_0, \ldots, w_{i-1})$.

Proof. Assume that $(\ell_0, \ldots, \ell_{i-1}) \neq (w_0, \ldots, w_{i-1})$. We have by assumption, after cancelling $p^{n_i + \cdots + n_\xi}$ from both sides,

$$\begin{aligned} \ell_0 p^{n_1 + \cdots + n_{i-1}} + \sum_{v=1}^{i-2} \ell_v p^{n_{v+1} + \cdots + n_{i-1}} b_v + \ell_{i-1} b_{i-1} = \\ = w_0 p^{n_1 + \cdots + n_{i-1}} + \sum_{v=1}^{i-2} w_v p^{n_{v+1} + \cdots + n_{i-1}} b_v + w_{i-1} b_{i-1}. \end{aligned} \tag{3.8}$$

By the coprimality of b_{i-1} and p we get that $p^{n_{i-1}}$ divides $w_{i-1}-\ell_{i-1}$. Suppose that the last difference is not zero and assume without loss of generality that it is positive i.e.

$$w_{i-1} - \ell_{i-1} = \lambda p^{n_{i-1}},\ \lambda > 0.$$

Then w_{i-1} is strictly greater than $p^{n_{i-1}}$ which contradicts the inequality (3.6) so we must have $w_{i-1} = \ell_{i-1}$. Cancelling the corresponding terms on either side of eq. 3.7 allows us to perform the same procedure yielding $w_{i-2} = \ell_{i-2}$. Proceeding with induction we get $w_1 = \ell_1$ which means that also w_0 equals ℓ_0, a contradiction since the elements were assumed different. □

The following lemma allows us to manipulate the elements D_i:

Lemma 3.2. *Let $F = F_{\xi+1}$ be the top field, with generators $\bar{f}_i$, $i = 0, \ldots, \xi$ and associated irreducible polynomials A_i as in equation (3.1):*

$$A_i(X) = X^{p^{n_i}} + a^{(k)}_{n_i-1} X^{p^{n_i-1}} + \cdots + a^{(k)}_0 X - D_i,$$

where D_i is given in equation (3.5),

$$D_i(\bar{f}_0, \ldots, \bar{f}_{i-1}) = \sum_{(\ell_0,\ldots,\ell_{i-1})\in\mathbb{N}^i} a^{(i)}_{\ell_0,\ldots,\ell_{i-1}} \bar{f}_0^{\ell_0} \cdots \bar{f}_{i-1}^{\ell_{i-1}}.$$

Then one of the monomials $\bar{f}_0^{\ell_0} \ldots \bar{f}_{i-1}^{\ell_{i-1}}$ has also pole divisor $p^{n_i}\bar{m}_i P$ and this holds for all $i = 1, \ldots, \xi$.

Proof. Recall that $D_i \in F_i$, $\bar{f}_i \in F_{i+1} - F_i$ and the pole divisor of D_i is $p^{n_i}\bar{m}_i P$. Suppose on the contrary (for D_i) that, none of the monomial summands of D_i has pole divisor of the desired order, $p^{n_i}\bar{m}_i P$. In other words,

$$\ell_0|G_0| + \ell_1\bar{m}_1 + \cdots + \ell_{i-1}\bar{m}_{i-1} \neq p^{n_i}\bar{m}_i$$

for all $\ell_0, \ldots, \ell_{i-1}$ appearing as exponents. We can assume that ℓ_λ, w_λ satisfy the inequality of eq. (3.6) for all exponents of all monomial summands of D_i since, otherwise, we can substitute the corresponding element $\bar{f}_\lambda^{\ell_\lambda}$ with terms of smaller exponents because of its irreducible polynomial, see also eq. (3.1).

By the strict triangle inequality there will be at least two different monomials $\bar{f}_0^{\ell_0} \ldots \bar{f}_{i-1}^{\ell_{i-1}}$, $\bar{f}_0^{w_0} \ldots \bar{f}_{i-1}^{w_{i-1}}$ in the sum of D_i sharing the same valuation and the contradiction follows from lemma 3.1. □

3.3 Preparation for the main theorem

Define the Minkowski sum (recall the definition of $\mathbf{A}$ given in eq. (3.3))

$$\mathbf{A} + \mathbf{A} = \{L + K : L, K \in \mathbf{A}\},$$

where $L+K=(i_0+j_0,\dots,i_\xi+j_\xi)$ for $L=(i_0,\dots,i_\xi)$, $K=(j_0,\dots,j_\xi)$. There is a natural map

$$\mathbb{N}^{\xi+1}\ni(i_0,i_1,\dots,i_\xi)=\bar{h}\longmapsto||\bar{h}||=i_0|G_0|+\sum_{\nu=1}^{\xi}i_\nu\bar{m}_\nu\in\mathbb{N}, \tag{3.9}$$

which restricts to the map

$$\begin{aligned}&\mathbf{A}+\mathbf{A}\xrightarrow{||\cdot||}H_2 \qquad\qquad (3.10)\\ &L+K\longmapsto\big(L+K\big)\begin{pmatrix}|G_0|\\ \bar{m}_1\\ \vdots\\ \bar{m}_\xi\end{pmatrix}=(i_0+j_0)|G_0|+\sum_{v=1}^{\xi}(i_v+j_v)\bar{m}_v.\end{aligned}$$

The map given in eq. (3.10) is not one to one. In order to bypass this we introduce a suitable equivalence relation $\sim$ on $\mathbf{A}+\mathbf{A}$ so that there is a bijection

$$\psi:(\mathbf{A}+\mathbf{A})/\sim\quad\longrightarrow H_2':=\mathrm{Im}\psi\subset H_2.$$

Definition 3.3. *Define the equivalence relation* $\sim$ *on* $\mathbf{A}+\mathbf{A}$*, by the rule*

$$(L+K)\sim(L'+K')\text{ if and only if }||L+K||=||L'+K'||.$$

The function ψ together with eq. (3.4) allows us to express a quadratic differential ω_h corresponding to an element $h\in H_2'$ as an element in $\mathbf{A}+\mathbf{A}$ by selecting a representative $L+K\in\mathbf{A}+\mathbf{A}$ of the class of $\psi(f)$. That is for every element $h\in H_2'$ we can write

$$\psi([L_h+K_h])=h\text{ for certain elements }L_h,K_h\in\mathbf{A}. \tag{3.11}$$

It is clear by our definitions that the following equality holds.

$$\left|\frac{\mathbf{A}+\mathbf{A}}{\sim}\right|=|H_2'|\le|H_2|=3g-3. \tag{3.12}$$

as we mentioned in the introduction, the reasons for the definition of the equivalence relation will be clear later but the curious reader may check proposition 1.12. We will need the following:

Lemma 3.4. *The equivalence class of the element* $L+K=(i_0+j_0,\dots,i_\xi+j_\xi)\in\mathbf{A}+\mathbf{A}$ *corresponds under the assignment*

$$A+B\in\mathbf{A}+\mathbf{A}\mapsto\omega_A\omega_B$$

to the following set of degree 2 *monomials*

$$\Gamma_{L+K}:=\left\{\begin{array}{l}\omega_A\omega_B\in\mathrm{Sym}H^0(X,\Omega_X):\textit{ for }A=(a_0,\dots,a_\xi),B=(b_0,\dots,b_\xi)\\ \textit{such that:}\\ ((a_0+b_0)-(i_0+j_0))|G_0|+\displaystyle\sum_{v=1}^{\xi-1}\big(a_v+b_v-(i_v+j_v)\big)\bar{m}_v=\lambda\bar{m}_\xi p^{n_\xi}\\ \textit{and }a_\xi+b_\xi-(i_\xi+j_\xi)=-\lambda p^{n_\xi}\textit{ for some }\lambda\in\mathbb{Z}\end{array}\right\},$$

Proof. The equivalence class of $L+K$ is a subset of $\mathbf{A}+\mathbf{A}$ which corresponds to holomorphic differentials as described below: Notice first that two equivalent elements $L+K$, $L'+K'$ satisfy

$$(i_0+j_0-(i_0'+j_0'))|G_0|+\sum_{v=1}^{\xi}(i_v+j_v-(i_v'+j_v))\bar{m}_v=0$$

which, combined with the facts that $(\bar{m}_\xi,p)=1$ and $\bar{m}_i=p^{n_{i+1}+\cdots+n_\xi}b_i$ yields that there is an integer λ such that

$$\big(i_0+j_0-(i_0'-j_0')\big)\frac{|G_0|}{p^{n_\xi}}+\sum_{v=1}^{\xi-1}(i_v+j_v-(i_v'+k_v'))\frac{\bar{m}_v}{p^{n_\xi}}=\lambda\bar{m}_\xi \tag{3.13}$$

$$\text{and } i_\xi'+j_\xi'-(i_\xi+j_\xi)=\lambda p^{n_\xi}. \tag{3.14}$$

□

Remark 3.5. *By Petri's theorem the canonical map ϕ (check eq. (1.11)) maps a degree 2 polynomial in the symmetric algebra of $H^0(X,\Omega_X)$ to $f_h df_0^{\otimes 2}\in H^0(X,\Omega_X^{\otimes 2})$, that is*

$$\phi\left(\sum_\nu a_\nu\omega_{L_\nu}\omega_{K_\nu}\right)=f_h df_0^{\otimes 2},\qquad a_\nu\in k. \tag{3.15}$$

It is not correct that a holomorphic 2-differential $f_h df_0^{\otimes 2}$ is the image of a single element $\omega_L\omega_K$. Indeed, for the genus 9 Artin-Schreier curve

$$y^7-y=x^4$$

a basis for the set of holomorphic differentials corresponds to the set

$$\mathbf{A}=\{[0,0],[0,1],[0,2],[0,3],[0,4],[1,0],[1,1],[1,2],[2,0]\}$$

$$\omega_{0,0}=x^0y^0dx,\quad \omega_{0,1}=x^0y^1dx,\quad \omega_{0,2}=x^0y^2dx,\quad \omega_{0,3}=x^0y^3dx,\quad \omega_{0,4}=x^0y^4dx,$$
$$\omega_{1,0}=x^1y^0dx,\quad \omega_{1,1}=x^1y^1dx,\quad \omega_{1,2}=x^1y^2dx,\quad \omega_{2,0}=x^2y^0dx$$

while the holomorphic 2-differential $x^4ydx^{\otimes 2}$ cannot be expressed as a single monomial of the above differentials, but as the following linear combination

$$\omega_{0,4}^2-\omega_{0,2}^2=y(y^7-y)dx^{\otimes 2}=x^4ydx^{\otimes 2}.$$

If the 2-differential $f_0^{i_0}\cdots f_\xi^{i_\xi}df_0^{\otimes 2}$ is the image of a single monomial $\omega_K\omega_L$ with $K+L=(i_0,\dots,i_\xi)$, then it is clear that the element $h=|G_0|i_0+\sum_{\nu=1}^{\xi}\bar{m}_\nu i_\nu$ in H_2 is the image of $L+K\in\mathbf{A}+\mathbf{A}$.

3.4 The generating sets of the canonical ideal

For any element $K=(i_0,\dots,i_\xi)\in\mathbb{N}^{\xi+1}$ we will denote by f_K the element $f_0^{i_0}\cdots f_\xi^{i_\xi}$.

Proposition 3.6. *Consider the sets of quadratic holomorphic differentials:*

$$K_0 := \{\omega_L\omega_K - \omega_{L'}\omega_{K'} \in \mathrm{Sym}H^0(X,\Omega_X) : L+K = L'+K', L,K,L',K' \in \mathbf{A}\}.$$

Then K_0 is contained in the canonical ideal.

Proof. For the canonical map $\phi : \mathrm{Sym}(H^0(X,\Omega_X)) \to \bigoplus_{n\geq 0} H^0(X,\Omega_X^n)$ one has;

$$\phi(\omega_K\omega_L - \omega_{K'}\omega_{L'}) = f_{K+L}df_0^{\otimes 2} - f_{K'+L'}df_0^{\otimes 2} = 0.$$

□

Remark 3.7. *Since K_0 is included in the canonical ideal we have that*

$$\omega_{K_h}\omega_{L_h} = \omega_{K_{h'}}\omega_{L_{h'}}$$

modulo the canonical ideal for any selection of $K_h+L_h, K_{h'}+L_{h'}$ representing $h,h' \in \mathbf{A}+\mathbf{A}$ such that $K_h + L_h = K_{h'} + L_{h'}$. Therefore, we will denote 2-differentials by $\omega_h^{\otimes 2}$.

Using this notation we can rewrite the summands of D_i in eq. (3.5) as 2-differentials as explained below:

Lemma 3.8. *The elements $D_i \in F_i$ have degree less than $4g-4$, yielding that $D_i \cdot df_0^{\otimes 2}$ are 2-holomorphic differentials in F. In particular every monomial summand $\bar{f}_0^{\ell_0}\cdots\bar{f}_{i-1}^{\ell_{i-1}}$ that appears in the expression of D_i given in eq. (3.5) can be given as an element*

$$(0,\ldots,0) + (\ell_0,\ldots,\ell_{i-1},0,\ldots,0) \in \mathbf{A}+\mathbf{A}.$$

and the element D_i can be written as a 2-differential as

$$D_i \cdot df_0^{\otimes 2} = \sum_{\substack{\bar{\lambda}=(\ell_0,\ldots,\ell_{i-1},0,\ldots,0)\in\mathbf{A}+\mathbf{A} \\ ||\bar{\lambda}||\leq p^{n_i}\bar{m}_i}} a_{\bar{\lambda}}^{(i)}\omega_{\bar{\lambda}}^{\otimes 2}, \tag{3.16}$$

Proof. By equation (3.1) we have that the absolute value of the valuation of D_i in F_{i+1} is $p^{n_i}b_i$. We will first show that $p^{n_i}b_i \leq 4g_{F_{i+1}} - 4$.

According to the Riemann-Hurwitz formula the genera of F_{i+1} and F_i are related by

$$2(g_{F_{i+1}} - 1) = p^{n_i}2(g_{F_i} - 1) + (b_i+1)(p^{n_i} - 1). \tag{3.17}$$

Therefore

$$\begin{aligned} 4(g_{F_{i+1}} - 1) - p^{n_i}b_i &= 2p^{n_i}2(g_{F_i} - 1) + p^{n_i}b_i - 2b_i + 2p^{n_i} - 2 \\ &= 2p^{n_i}2(g_{F_i} - 1) + (p^{n_i} - 2)b_i + 2(p^{n_i} - 1). \end{aligned} \tag{3.18}$$

If $g_{F_i} \geq 1$ then we have the desired inequality. Suppose that $g_{F_i} = 0$. This can only happen for $i = 1$ since $p^{n_i} > 1$ and $b_i > 1$. Therefore we need to show that

$$b_1 p^{n_1} - 2p^{n_1} - 2b_1 - 2 \geq 0$$

and we are working over the rational function field. The assumption on our curve being non-hyperelliptic implies that $p^{n_i} > 2$ as well as $b_i > 2$ and the last inequality becomes

$$b_i \geq \frac{2p^{n_i} + 2}{p^{n_i} - 2}, \tag{3.19}$$

which is satisfied for $p^n > 7$. Also the remaining cases, i.e. $p^{n_i} = 5, 7$ require b_i to be ≥ 4 which is also true since $b_i = 2$ is exluded by non-hyperellipticity and $b_i = 3$ by non-trigonality.

Now the rest can be proved by induction as follows; We showed that

$$p^{n_i} b_i \leq 4g_{F_{i+1}} - 4 \tag{3.20}$$

When we move from F_{i+1} to F_{i+2} the absolute value of the valuation of D_i becomes $p^{n_{i+1}+n_i} b_i$ and we need to show that

$$p^{n_{i+1}+n_i} b_i \leq 4g_{F_{i+2}} - 4$$

By 3.20 it suffices to show that $p^{n_{i+1}}(4g_{F_{i+1}} - 4) \leq 4g_{F_{i+2}} - 4$ which by the Riemann-Hurwitz formula (stated above) is equivalent to $(b_{i+1}+1)(p^{n_{i+1}}-1)$ being non-negative, which holds.

□

Remark 3.9. *If we assume that F_i is neither trigonal nor hyperelliptic then the same holds for all fields F_k for $k \geq i$, see [42, Appendix].*

The set K_0 does not contain all elements of the canonical ideal. For instance, it does not contain the information of the defining equation of the Artin-Schreier extension and also the canonical ideal is not expected to be binomial.

Before the definition of the other generating sets of the canonical ideal, let us provide some insight into the process used to construct the elements of these sets.

Equation (3.1) is satisfied by the element $\bar{f}_i$, i.e,

$$\bar{f}_i^{p^{n_i}} + a_{n_i-1}^{(i)} \bar{f}_i^{p^{n_i-1}} + \cdots + a_0^{(i)} \bar{f}_i - D_i = 0.$$

This equation can be multiplied by elements of the form $\bar{f}_0^{\,v_0} \cdots \bar{f}_\xi^{\,v_\xi}$ for any $v_0, \ldots, v_\xi$, giving rise to

$$\bar{f}_0^{\,v_0} \cdots \bar{f}_\xi^{\,v_\xi} \left(\bar{f}_i^{p^{n_i}} + a_{n_i-1}^{(i)} \bar{f}_i^{p^{n_i-1}} + \cdots + a_0^{(i)} \bar{f}_i - D_i \right) = 0,$$

which equals

$$\bar{f}_0^{\,v_0} \cdots \bar{f}_i^{v_i+p^{n_i}} \cdots \bar{f}_\xi^{\,v_\xi} + \ldots + a_0^{(i)} \bar{f}_0^{\,v_0} \cdots \bar{f}_i^{v_i+1} \cdots \bar{f}_\xi^{\,v_\xi} - \bar{f}_0^{\,v_0} \cdots \bar{f}_i^{v_i} \cdots \bar{f}_\xi^{\,v_\xi} D_i = 0.$$

If the exponents $(v_0, \ldots, v_\xi)$ are selected so that each summand in the last equation is an element in $\mathbf{A}+\mathbf{A}$, then the equation gives rise to an element in the canonical ideal.

Proposition 3.10. *Set*

$$\bar{v} := (v_0, \dots, v_\xi) \in \mathbb{N}^{\xi+1}$$
$$\bar{\gamma}_{\bar{v},i,\nu} := (v_0, \dots, v_i + p^{n_i-\nu}, v_{i+1}, \dots, v_\xi),\ 0 \le \nu \le n_i$$

such that $||\bar{\gamma}_{\bar{v},i,0}|| \le 4g-4$. *Also set*

$$\Lambda_i = \{\bar{\lambda} = (\ell_0, \dots, \ell_{i-1}) \in \mathbb{N}^i : 0 \le \ell_\nu < p^{n_\nu} \text{ for } 1 \le \nu \le i\}$$
$$\bar{\beta}_{\bar{v},i,\bar{\lambda}} := (\ell_0, \dots, \ell_{i-1}, 0, \dots, 0) + \bar{v} \in \mathbf{A} + \mathbf{A}.$$

Define

$$K_{\bar{v},i} := \left\{ \omega^{\otimes 2}_{\bar{\gamma}_{\bar{v},i,0}} + \sum_{\nu=1}^{n_i} a_\nu^{(i)} \omega^{\otimes 2}_{\bar{\gamma}_{\bar{v},i,\nu}} - \sum_{\substack{\bar{\lambda}\in\Lambda_i \\ ||\bar{\lambda}|| \le p^{n_i}\bar{m}_i}} a_{\bar{\lambda}}^{(i)} \omega^{\otimes 2}_{\bar{\beta}_{\bar{v},i,\bar{\lambda}}} \right\} \tag{3.21}$$

Then $K_{\bar{v},i}$ *is contained in the canonical ideal for* $1 \le i \le \xi$.

Notice here that $\bar{v}$ is fixed while $\bar{\lambda}$ is running.

Proof. Again consider $\phi : \mathrm{Sym}(H^0(X, \Omega_X)) \to \bigoplus_{n\ge 0} H^0(X, \Omega_X^n)$. Then

$$\phi\left(\omega^{\otimes 2}_{\bar{\gamma}_{\bar{v},i,0}} + \sum_{\nu=1}^{i} a_\nu^{(i)} \omega^{\otimes 2}_{\bar{\gamma}_{\bar{v},i,\nu}} - \sum_{\substack{\bar{\lambda}\in\Lambda_i \\ ||\bar{\lambda}|| \le p^{n_i}\bar{m}_i}} a_{\bar{\lambda}}^{(i)} \omega^{\otimes 2}_{\bar{\beta}_{\bar{v},i,\bar{\lambda}}} \right) =$$
$$= \Bigg(f_{(v_0,\dots,v_i+p^{n_i},\dots,v_\xi)} + \sum_{\nu=1}^{i} a_\nu^{(i)} f_{(v_0,\dots,v_i+p^{n_i-\nu},\dots,v_\xi)} -$$
$$- \sum_{\substack{\bar{\lambda}\in\Lambda_i \\ ||\bar{\lambda}|| \le p^{n_i}\bar{m}_i}} a_{\bar{\lambda}}^{(i)} f_{(\ell_0+v_0,\dots,\ell_{i-1}+v_{i-1},v_i,\dots,v_\xi)} \Bigg) df_0^{\otimes 2} =$$
$$= f_{(v_0,\dots,v_\xi)} \left(\bar{f}_i^{p^{n_i}} + \sum_{\nu=1}^{i} a_v^{(i)} \bar{f}_i^{p^{n_i-\nu}} - \sum_{\substack{\bar{\lambda}\in\Lambda_i \\ ||\bar{\lambda}|| \le p^{n_i}\bar{m}_i}} a_{\bar{\lambda}}^{(i)} f_{(\ell_0 \cdots + \ell_{i-1},0,\dots,0)} \right) df_0^{\otimes 2},$$

which equals 0 due to the relation satisfied by the irreducible polynomial of $\bar{f}_i$. $\square$

3.5 The main theorem

We define a term order which compares products of differentials as follows: Let $\omega_{I_1}\omega_{I_2}\cdots\omega_{I_d}, \omega_{I'_1}\omega_{I'_2}\cdots\omega_{I'_{d'}}$ be two such products and consider the $(k+1)-$tuples $I_1 + \cdots + I_d = (v_0, \dots, v_\xi)$, $I'_1 + \cdots + I'_{d'} = (v'_0, \dots, v'_\xi)$.

Define

$$\omega_{I_1}\omega_{I_2}\cdots\omega_{I_d} \prec \omega_{I'_1}\omega_{I'_2}\cdots\omega_{I'_{d'}} \Leftrightarrow (v_0,\ldots,v_\xi) <_{\text{colex}} (v'_0,\ldots,v'_\xi)$$

that is

- $v_\xi < v'_\xi$ or
- $v_\xi = v'_\xi$ and $v_{\xi-1} < v'_{\xi-1}$ or
- $\vdots$
- $v_i = v'_i$ for all $i = k,\ldots,1$ and $v_0 < v'_0$.

We are going to work with the initial terms of the sets defined in the last two propositions where, by "initial term" we mean a maximal term with respect to the colexicographical order. We denote initial terms with $\mathrm{in}_\prec(\cdot)$.

Lemma 3.11. *For the element $K_{\bar{v},i}$ of proposition 3.10 we have that*

$$\mathrm{in}_\prec(K_{\bar{v},i}) = \omega_{\bar{\gamma}_{\bar{v},i,0}}.$$

and also, in the polynomial $K_{\bar{v},i}$ there is another summand which is smaller colexicographically than $\omega_{\bar{\gamma}_{\bar{v},i,0}}$ but has the same $||\cdot||$-value.

Proof. Indeed, in eq. (3.21) there are two elements of maximal value in terms of $||\cdot||$. Namely $\omega_{\bar{\gamma}_{\bar{v},i,0}}$ and $a^{(i)}_{\bar{\lambda}}\omega^{\otimes 2}_{\bar{\beta}_{\bar{v},i,\bar{\lambda}}}$, for the $\bar{\lambda} = (\ell_0,\ldots,\ell_{i-1},0,\ldots,0) \in \mathbf{A}+\mathbf{A}$ corresponding to the monomial $\bar{f}_0^{\ell_0}\cdots\bar{f}_{i-1}^{\ell_{i-1}}$ of minimum valuation which exists due to lemma 3.2. Of these two elements, $\omega_{\bar{\gamma}_{\bar{v},i,0}}$ is bigger since it corresponds to the element $(v_0,\ldots,v_i+p^{n_i},\ldots,v_\xi)$, while the other corresponds to the smaller element $(v_0+l_0,\ldots,v_{i-1}+l_{i-1},v_i,\ldots,v_\xi)$, with respect to the colexicographical order. $\square$

We are now ready to state our main result. Recall that we have assumed throughout this article that X is a Harbater-Katz-Gabber cover which is non-elliptic of genus ≥ 3 over k. We also have assumed that X is non-trigonal so that the canonical ideal is generated by elements of degree 2 (see also theorem 1.11).

Theorem 3.12. *The canonical ideal is generated by K_0 and by $K_{\bar{v},i}$, for $1 \leq i \leq \xi$ and for the $\bar{v} \in \mathbb{N}^{\xi+1}$ satisfying the inequality $||\bar{\gamma}_{\bar{v},i,0}|| \leq 4g-4$.*

Remark 3.13. *In the above theorem the condition $||\bar{\gamma}_{\bar{v},i,0}|| \leq 4g-4$ implies the condition $||\bar{\gamma}_{\bar{v},i,\nu}|| \leq 4g-4$ for $0 \leq \nu \leq n_i$. We will prove in lemma 3.14 that it also implies the condition $||\bar{\beta}_{\bar{v},i,\bar{\lambda}}|| \leq 4g-4$. This means that the condition $||\bar{\gamma}_{\bar{v},i,\nu}|| \leq 4g-4$ for $0 \leq \nu \leq n_i$ guarantees that, in $K_{\bar{v},i}$, not only the first term (i.e. $\omega^{\otimes 2}_{\bar{\gamma},i,0}$), but also all the others correspond to 2-differentials.*

Lemma 3.14. *The condition $||\bar{\gamma}_{\bar{v},i,0}|| \leq 4g-4$, or in other words,*

$$v_0|G_0| + \sum_{\nu=1}^{\xi} v_\nu \bar{m}_\nu + p^{n_i}\bar{m}_i \leq 4g-4 \tag{3.22}$$

implies that $\bar{\beta}_{\bar{v},i,\bar{\lambda}}$ lies in $\mathbf{A}+\mathbf{A}$, that is, it is also a 2-differential, for all $\bar{\lambda}$ associated with the monomials of D_i.

Proof. For $\bar{\lambda} \in \Lambda_i$ let

$$\bar{\beta}_{\bar{v},i,\bar{\lambda}} = (v_0+\ell_0, \ldots, v_{i-1}+\ell_{i-1}, v_i, \ldots, v_\xi).$$

We need to show that

$$(v_0+\ell_0)|G_0| + \sum_{\nu=1}^{\xi} v_\nu \bar{m}_\nu + \sum_{\nu=1}^{i-1} \ell_\nu \bar{m}_\nu \leq 4g-4.$$

By (3.22) we need to show that

$$\ell_0|G_0| + \sum_{\nu=1}^{i-1} \ell_\nu \bar{m}_\nu \leq p^{n_i}\bar{m}_i.$$

Note that $\bar{\lambda}$ is the exponents of a monomial summand of D_i and, by the valuation's strict triangle inequality one has;

$$v(f_{\bar{\lambda}}) \geq v(D_i) \Leftrightarrow$$
$$-(\ell_0|G_0| + \sum_{\nu=1}^{i-1} \ell_\nu \bar{m}_\nu) \geq -p^{n_i}\bar{m}_i$$

as expected, where $f_{\bar{\lambda}}$ is $\bar{f}_0^{\ell_0} \cdots \bar{f}_{i-1}^{\ell_{i-1}}$ □

Definition 3.15. *Define J to be the set of elements in the canonical ideal consisting of the elements $K_0, K_{\bar{v},i}$ for $1 \leq i \leq \xi$ and for the appropriate $\bar{v} \in \mathbb{N}^{\xi+1}$ satisfying the inequality $||\bar{\gamma}_{\bar{v},i,0}|| \leq 4g-4$.*

In order to prove Theorem 3.12, we need to show that J is the canonical ideal. We will use proposition 1.12, In order to apply proposition 1.12 we will show that

$$\left|\frac{\mathbf{A}+\mathbf{A}}{\sim}\right| = \dim\left(\frac{S}{\langle \mathrm{in}_\prec(J)\rangle}\right)_2, \tag{3.23}$$

where we already know, see eq. (3.12), that the cardinality of the first quotient is $\leq |H_2| = 3g-3$. We identify a k-basis of $(S/\langle \mathrm{in}_\prec\langle J\rangle\rangle)_2$ with $\mathbb{T}^2 - \{\mathrm{in}_\prec(f) : f \in J\}$ and, in order to prove equality (3.23), we define the map

$$\begin{aligned} \Phi : \mathbb{T}^2 - \{\mathrm{in}_\prec(f) : f \in J\} &\longrightarrow \frac{\mathbf{A}+\mathbf{A}}{\sim} \\ \omega_L\omega_K &\longmapsto [L+K] \end{aligned} \tag{3.24}$$

Lemma 3.16. *If* $(u_0,\dots,u_\xi)\in\mathbf{A}+\mathbf{A}$ *then every* $(u'_0,\dots,u'_\xi)$ *with* $0\le u'_\nu\le u_\nu$ *for* $1\le\nu\le\xi$ *is also in* $\mathbf{A}+\mathbf{A}$.

Proof. Since $\bar u=(u_0,\dots,u_\xi)\in\mathbf{A}+\mathbf{A}$ there are $\bar a=(a_0,\dots,a_\xi)$, $\bar b=(b_0,\dots,b_\xi)$ with $\bar u=\bar a+\bar b$ and $\bar a,\bar b\in\mathbf{A}$, that is $||\bar a||,||\bar b||\le 2g-2$. But then every $\bar a'$ (resp. $\bar b'$) with $\bar a'=(a'_0,\dots,a'_\xi)$ (resp. $\bar b'=(b'_0,\dots,b'_\xi)$) such that $0\le a'_\nu\le a_\nu$ (resp. $0\le b'_\nu\le b_\nu$) for $0\le\nu\le\xi$ satisfies $||\bar a'||\le||\bar a||\le 2g-2$ (resp. $||\bar b'||\le||\bar b||\le 2g-2$), that is $\bar a',\bar b'\in\mathbf{A}$. The result follows. $\square$

We start by showing that Φ is one-to-one.

Lemma 3.17. *The map* Φ *is injective.*

Proof. Consider the following elements of $\mathbf{A}$:

$$\begin{aligned} L&=(i_0,i_1,\dots,i_\ell,\dots,i_\xi) & K&=(j_0,j_1,\dots,j_\ell,\dots,j_\xi)\\ L'&=(i'_0,i'_1,\dots,i'_\ell,\dots,i'_\xi) & K'&=(j'_0,j'_1,\dots,j'_\ell,\dots,j'_\xi) \end{aligned}$$

such that, $\omega_K\omega_L,\omega_{L'}\omega_{K'}$ are in $\mathbb{T}^2-\{\mathrm{in}(f):f\in J\}$. Assume that $\Phi(\omega_L\omega_K)=\Phi(w'_Lw'_K)$, i.e. $L+K\sim L'+K'$. Suppose that $i_\xi+j_\xi=i'_\xi+j'_\xi$. Then we have the following equality:

$$(i_0+j_0)|G_0|+\sum_{\ell=1}^{\xi}(i_\ell+j_\ell)\bar m_\ell=(i'_0+j'_0)|G_0|+\sum_{\ell=1}^{\xi}(i'_\ell+j'_\ell)\bar m_\ell$$

from which we cancel the last terms and divide by p^{n_ξ} in order to have

$$(i_0+j_0)p^{n_1+\dots+n_{\xi-1}}+\sum_{\ell=1}^{\xi-1}(i_\ell+j_\ell)\frac{\bar m_\ell}{p^{n_\xi}}=(i'_0+j'_0)p^{n_1+\dots+n_{\xi-1}}+\sum_{\ell=1}^{\xi-1}(i'_\ell+j'_\ell)\frac{\bar m_\ell}{p^{n_\xi}}.$$

By repeating the above process we can assume that there is an $\ell\le\xi$ such that $i'_\nu+j'_\nu=i_\nu+j_\nu$ for $\ell<\nu\le\xi$ and $i'_\ell+j'_\ell\ne i_\ell+j_\ell$ and assume without loss of generality that $i'_\ell+j'_\ell>i_\ell+j_\ell$. Then by lemma 3.4, we would have

$$i'_\ell+j'_\ell-(i_\ell+j_\ell)=\lambda p^{n_\ell} \tag{3.25}$$

for $\lambda>0$. Using this we will show that $\omega_{L'}\omega_{K'}$ belongs to $\mathrm{in}_\prec(J)$. In order to do that, we need to build an element $K_{i,\bar v}$ which has $\omega_{L'}\omega_{K'}$ as its initial term. In other words we look for an element of the following form;

$$\omega^{\otimes 2}_{\bar\gamma_{\bar v,i,0}}+\sum_{\nu=1}^{n_i}a_\nu^{(i)}\omega^{\otimes 2}_{\bar\gamma_{\bar v,i,\nu}}-\sum_{\substack{\lambda\in\Lambda_i\\ ||\bar\lambda||\le p^{n_i}\bar m_i}}a_{\bar h}^{(i)}\omega^{\otimes 2}_{\bar\beta_{\bar v,i,\bar\lambda}}, \tag{3.26}$$

where $\omega^{\otimes 2}_{\bar\gamma_{\bar v,i,0}}=\omega^{\otimes 2}_{L'+K'}$ and everything else should be as defined in proposition 3.10. This comes down to finding $\bar v=(v_0,\dots,v_\xi)\in\mathbb{N}^{\xi+1}$ such that

$$(v_0,\dots,v_\ell+p^{n_\ell},v_{\ell+1}\dots,v_\xi)=(v_0,\dots,v_\ell+p^{n_\ell},i'_{\ell+1}+j'_{\ell+1}\dots,i'_\xi+j'_\xi)=L'+K'.$$

Indeed, recall that if we match our element with an initial term corresponding to $\bar{f}_\ell^{p^{n_\ell}}$ then all the other terms can be defined by the equation of the irreducible polynomial of $\bar{f}_\ell$.

Define $\bar{v}$ as follows:

$$v_s = \begin{cases} i'_s + j'_s & \text{for } s \neq \ell \\ i'_\ell + j'_\ell - p^{n_\ell} & \text{for } s = l \end{cases}$$

The element $(v_0, \ldots, v_\xi)$ lies in $\mathbf{A} + \mathbf{A}$. Indeed, since $L' + K'$ is in $\mathbf{A} + \mathbf{A}$, according to lemma 3.16 we only need to show that $0 \leq v_\nu$ for all $0 \leq \nu \leq \xi$. The only thing that needs to be checked is whether v_ℓ is nonnegative. Equivalently, whether $i'_\ell + j'_\ell \geq p^{n_\ell}$. Now recall that $i'_\ell + j'_\ell = \lambda p^{n_\ell} + (i_\ell + j_\ell)$ and hence $v_\ell = i'_\ell + j'_\ell - p^{n_\ell} = i_\ell + j_\ell + (\lambda - 1)p^{n_\ell}$ by eq. (3.25). Since $\lambda \geq 1$ we get

$$\lambda p^{n_\ell} + (i_\ell + j_\ell) \geq p^{n_\ell}$$

as expected.

This proves that $\omega_{L'+K'}$ is the initial term of $K_{\bar{v},\ell}$ for $\bar{v} = (v_0, \ldots, v_\xi)$, check also lemma 3.11, giving us a contradiction so the map Φ is injective. □

Lemma 3.18. *The map Φ is surjective.*

Proof. Take an equivalence class $[L+K]$ in $(\mathbf{A}+\mathbf{A})/\sim$. Recall the definition of the set Γ_{L+K} given in lemma 3.4. Consider the minimal element of Γ_{L+K} , i.e. $\min\Gamma_{L+K} := \omega_A\omega_B \in \mathbb{T}^2$. There is such a minimal element since Γ_{L+K} is nonempty (for example $\omega_L\omega_K \in \Gamma_{L+K}$) and since our order is a total order. We still need to show that $\omega_A\omega_B$ is not in $\mathrm{in}_\prec(J)$.

Firstly suppose that $\omega_A\omega_B \in \mathrm{in}_\prec(K_0)$. Then there is $\omega_I\omega_J$ such that $\omega_I\omega_J \prec \omega_A\omega_B$ and $A+B = I+J$. By the last equality, $||A+B|| = ||I+J||$ so $A+B \sim I+J$. But this means that $\omega_I\omega_J$ is also in Γ_{L+K} and is colexicographically smaller than $\omega_A\omega_B$, a contradiction.

Suppose now that $\omega_A\omega_B \in \mathrm{in}_\prec(K_{\bar{v},i})$ for some $\bar{v}, i$. Then according to lemma 3.11 there is a second element in the polynomial $K_{\bar{v},i}$ which has the same value when $||\cdot||$ is applied, but is smaller in $\prec$ (a contradiction since, having the same $||\cdot||$–value means that they are equivalent i.e. they both lie in Γ_{L+K}). □

3.6 Examples

We provide here some explicit examples of our method for calculating the canonical ideal of HKG curves.

Artin-Schreier curves

Here we write down the generating sets of the canonical ideal corresponding to Artin-Schreier curves of the form

$$X : y^{p^n} - y = x^m, \qquad (m,p) = 1, \tag{3.27}$$

where the values of m, p are given in the following table. Notice that these curves form an example of an HKG-cover extension for the $k = 1$ case.

m	Petri's theorem requirement
$m > 5$	$p^n > 3$
$m = 4, 5$	$p^n \geq 5$

In this case the genus g of the curve is $g > 6$ and also the curve is not hyperelliptic nor trigonal. Indeed the above given curves have Weierstrass semigroup

$$H := m\mathbb{Z}_+ + p^n\mathbb{Z}_+ \tag{3.28}$$

at the unique ramified point P. Let G be the p^n order Artin-Schreier cover group generated by the automorphism $\tau : y \mapsto y + 1, x \mapsto x$. Assume that there is a degree two covering $X \to \mathbb{P}^1$. This is a Galois covering with Galois group generated by the hyperelliptic involution $j : X \to X$. The hyperelliptic involution cannot be in the p^n order Galois group G of the Artin-Schreier extension, since p is odd. On the other hand it is well known that the hyperelliptic involution is in the center of the automorphism group of X, [7]. Since $\tau(j(P)) = j\tau(P) = P$ we have $j(P) = P$, otherwise the Galois cover $X \to X/G = \mathbb{P}^1$ has two ramified points, a contradiction. But then 2 should be a pole number of the semigroup H, contradicting eq. (3.28).

In order to prove that X is also not trigonal, we can employ the fact that with the assumptions given in the table above we can indeed find a quadratic basis of the canonical ideal. Alternatively we can argue as follows: In characteristic zero we know that at a non ramified point P in the degree 3 cover $X \to \mathbb{P}^1$ of a trigonal curve the first few elements in the Weierstrass semigroup at P are $3n, 3n + 2, 3n + 3$ or $3n, 3n + 1, 3n + 3, 3n + 4$ or $2n + 2$ or $2n + 1, 2n + 3$ for $(g - 1)/n \leq n \leq g/2$, see [27, thm p.172]. On the other hand for a Weierstrass point of the trigonal curve which is not ramified in the degree 3 cover, the Weierstrass semigroup at P is of the form

$$a, a + 1, a + 2, \ldots, a + (s - g), s + 2, s + 3, \ldots$$

for some $g \leq a \leq \lfloor (s + 1)/2 \rfloor + 1$ and $g - 1 < s \leq 2g - 2$, [27, lemma 2.5]. The Lefschetz principle implies that this is the structure of Weierstrass semigroups for a big enough prime p. On the other hand, the ramified point P in the Artin-Schreier cover is a Weierstrass point, see [17, th. 1]. The semigroup structure at P given in eq. (3.28) is not compatible with any of the Weierstrass semigroups of trigonal curves, therefore the curve X is not trigonal at least for big enough p. Unfortunately the bound for the prime p comes from Lefschetz principle and can not be determined.

Recall that H_i denotes the bounded parts of the Weierstrass semigroup (eq. 3.2). For the case at hand we have that

$$|H_1| = g = (m - 1)(p^n - 1)/2$$
$$|H_2| = 3(g - 1).$$

Also $\mathbf{A} = \{L := (i_0, i_1) : i_0 p^n + i_1 m \leq 2(g - 1)\}$ and

$$\mathbf{A} + \mathbf{A} = \{L + K = (i_0 + j_0, i_1 + j_1) \mid L := (i_0, i_1) \in \mathbf{A},\ K := (j_0, j_1) \in \mathbf{A}\}.$$

The equivalence class of $L+K \in \mathbf{A}+\mathbf{A}$, as described in lemma 3.4, corresponds to the following set of degree 2 monomials

$$\Gamma_{L+K} = \{\omega_A\omega_B \in \mathrm{Sym}H^0(X,\Omega_X) : A+B-(L+K) = (\lambda m, -\lambda p^n) \text{ for some } \lambda \in \mathbb{Z}\}.$$

According to proposition 3.6 K_0 is defined by

$$K_0 := \{\omega_L\omega_K - \omega_{L'}\omega_{K'} \in \mathrm{Sym}H^0(X,\Omega_X) : L+K = L'+K', L, K, L', K' \in \mathbf{A}\}.$$

The sets $K_{\bar{v},i}$ containing the information of the Artin-Schreier extension now adopt the following, much simpler form:

$$K_{(v_0,v_1),1} = \left\{\omega^{\otimes 2}_{(v_0,v_1+p^n)} - \omega^{\otimes 2}_{(v_0,v_1+1)} - \omega^{\otimes 2}_{(v_0+m,v_1)}\right\}$$

for the $\bar{v} := (v_0, v_1)$ satisfying $||(v_0, v_1 + p^n)|| \leq 4g - 4$, equivalently,

$$v_0 p^n + v_1 m + p^n m \leq 4g - 4.$$

Notice that if p, n and m are given specific values, the last inequality can be solved explicitly and the generating sets can be written down.

Example 3.19. *Recall that $\omega_{ij} = x^i y^j dx$. Consider the Artin-Schreier curve $y^7 - y = x^4$ of genus 9. The canonical ideal is generated by the set K_0 given by*

$$\begin{gathered}
\{-\omega_{0,4}\omega_{1,0} + \omega_{0,3}\omega_{1,1}, -\omega_{1,0}\omega_{1,1} + \omega_{0,1}\omega_{2,0}, \omega_{0,4}\omega_{1,0} - \omega_{0,3}\omega_{1,1}, \omega_{1,0}\omega_{1,1} - \omega_{0,1}\omega_{2,0}, -\omega_{0,2}^2 + \omega_{0,1}\omega_{0,3}, \omega_{0,2}^2 - \omega_{0,1}\omega_{0,3},\\
-\omega_{0,1}\omega_{1,1} + \omega_{0,0}\omega_{1,2}, \omega_{0,1}\omega_{1,1} - \omega_{0,0}\omega_{1,2}, -\omega_{0,2}\omega_{1,1} + \omega_{0,1}\omega_{1,2}, \omega_{0,2}\omega_{1,1} - \omega_{0,1}\omega_{1,2}, -\omega_{1,1}\omega_{1,2} + \omega_{0,3}\omega_{2,0}, -\omega_{0,2}^2 + \omega_{0,0}\omega_{0,4},\\
\omega_{0,2}^2 - \omega_{0,0}\omega_{0,4}, \omega_{1,1}\omega_{1,2} - \omega_{0,3}\omega_{2,0}, -\omega_{0,1}^2 + \omega_{0,0}\omega_{0,2}, \omega_{0,1}^2 - \omega_{0,0}\omega_{0,2}, -\omega_{0,3}\omega_{1,1} + \omega_{0,2}\omega_{1,2}, -\omega_{1,1}^2 + \omega_{0,2}\omega_{2,0},\\
\omega_{0,3}\omega_{1,1} - \omega_{0,2}\omega_{1,2}, -\omega_{1,0}\omega_{1,2} + \omega_{0,2}\omega_{2,0}, \omega_{1,0}\omega_{1,2} - \omega_{0,2}\omega_{2,0}, \omega_{1,1}^2 - \omega_{0,2}\omega_{2,0}, -\omega_{0,2}\omega_{1,0} + \omega_{0,0}\omega_{1,2}, \omega_{0,2}\omega_{1,0} - \omega_{0,0}\omega_{1,2},\\
\omega_{0,3}\omega_{1,0} - \omega_{0,2}\omega_{1,1}, -\omega_{0,3}\omega_{1,0} + \omega_{0,1}\omega_{1,2}, \omega_{0,3}\omega_{1,0} - \omega_{0,1}\omega_{1,2}, -\omega_{0,2}\omega_{1,0} + \omega_{0,1}\omega_{1,1}, -\omega_{1,0}^2 + \omega_{0,0}\omega_{2,0}, \omega_{0,2}\omega_{1,0} - \omega_{0,1}\omega_{1,1},\\
\omega_{1,0}^2 - \omega_{0,0}\omega_{2,0}, -\omega_{1,1}^2 + \omega_{1,0}\omega_{1,2}, \omega_{1,1}^2 - \omega_{1,0}\omega_{1,2}, -\omega_{0,1}\omega_{0,2} + \omega_{0,0}\omega_{0,3}, -\omega_{0,4}\omega_{1,1} + \omega_{0,3}\omega_{1,2}, \omega_{0,4}\omega_{1,1} - \omega_{0,3}\omega_{1,2},\\
-\omega_{0,2}\omega_{0,3} + \omega_{0,1}\omega_{0,4}, \omega_{0,2}\omega_{0,3} - \omega_{0,1}\omega_{0,4}, -\omega_{0,3}^2 + \omega_{0,2}\omega_{0,4}, \omega_{0,3}^2 - \omega_{0,2}\omega_{0,4}, -\omega_{0,1}\omega_{0,3} + \omega_{0,0}\omega_{0,4}, \omega_{0,1}\omega_{0,3} - \omega_{0,0}\omega_{0,4},\\
-\omega_{0,4}\omega_{1,0} + \omega_{0,2}\omega_{1,2}, \omega_{0,4}\omega_{1,0} - \omega_{0,2}\omega_{1,2}, -\omega_{0,1}\omega_{1,0} + \omega_{0,0}\omega_{1,1}, \omega_{0,1}\omega_{1,0} - \omega_{0,0}\omega_{1,1}, \omega_{1,2}^2 - \omega_{0,4}\omega_{2,0},\\
-\omega_{1,2}^2 + \omega_{0,4}\omega_{2,0}, \omega_{0,1}\omega_{0,2} - \omega_{0,0}\omega_{0,3}, -\omega_{0,3}\omega_{1,0} + \omega_{0,2}\omega_{1,1}\}
\end{gathered}$$

and one trinomial

$$-\omega_{0,0}\omega_{0,1} + \omega_{0,3}\omega_{0,4} - \omega_{2,0}^2$$

HKG-covers with p-cyclic group

This is a case where all the intermediate subextensions F_i/F_{i-1} are of degree p and the corresponding irreducible polynomials are

$$X^p + a^{(i)}X - D_i$$

In this case the generating sets of the canonical ideal are

$$K_0 := \{\omega_L\omega_K - \omega_{L'}\omega_{K'} \in \mathrm{Sym}H^0(X,\Omega_X) : L+K = L'+K', L, K, L', K' \in \mathbf{A}\}$$

$$K_{\bar{v},i} := \left\{ \omega^{\otimes 2}_{(v_0,\ldots,v_i+p,\ldots,v_\xi)} + a^{(i)}\omega^{\otimes 2}_{(v_0,\ldots,v_i,\ldots,v_\xi)} - \sum_{\substack{\bar{\lambda}\in\mathbf{A}+\mathbf{A}\\ ||\bar{\lambda}||\leq p\bar{m}_i}} a^{(i)}_{\bar{h}}\,\omega^{\otimes 2}_{\bar{\beta}_{\bar{v},i,\bar{\lambda}}} \right\} \tag{3.29}$$

such that $||\bar{\gamma}_{\bar{v},i,0}|| \leq 4g - 4$ where $\bar{\beta}_{\bar{v},i,\bar{\lambda}} = (l_0, \ldots, l_{i-1}, 0, \ldots, 0) + \bar{v}$ as defined before.

Chapter 4

Automorphisms of curves and Petri's theorem

Consider a complete non-singular non-hyperelliptic curve of genus $g \geq 3$ over an algebraically closed field K. The automorphism group of the ambient space $\mathbb{P}^{g-1}$ is known to be $\mathrm{PGL}_g(k)$, [22, example 7.1.1 p. 151]. On the other hand every automorphism of X is known to act on $H^0(X, \Omega_X)$ giving rise to a representation

$$\rho : G \to \mathrm{GL}(H^0(X, \Omega_X)),$$

which is known to be faithful, when X is not hyperelliptic and $p \neq 2$, see [29]. The representation ρ in turn gives rise to a series of representations

$$\rho_d : G \to \mathrm{GL}(S_d),$$

where S_d is the vector space of degree d polynomials in the ring $S := k[\omega_1, \ldots, \omega_g]$.

Let $X \subset \mathbb{P}^r$ be a projective algebraic set. Is it true that every automorphism $\sigma : X \to X$ comes as the restriction of an automorphism of the ambient projective space, that is by an element of $\mathrm{PGL}_k(r)$? For instance such a criterion for complete intersections is explained in [30, sec. 2]. In the case of canonically embedded curves $X \subset \mathbb{P}^{g-1}$ it is clear that any automorphism $\sigma \in \mathrm{Aut}(X)$ acts also on $\mathbb{P}^{g-1} = \mathrm{Proj} H^0(X, \Omega_X)$. In this way we arrive at the following:

Lemma 4.1. *Every automorphism $\sigma \in \mathrm{Aut}(X)$ corresponds to an element in $\mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$ and every element in $\mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$ gives rise to an automorphism of X.*

In the next section we will describe the elements $\sigma \in \mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$.

4.1 Algebraic equations of automorphisms

For now on we will assume that the canonical ideal I_X is generated by polynomials in $k[\omega_1, \ldots, \omega_g] = \mathrm{Sym} H^0(X, \Omega_X)$ of degree 2, that is, the requirements for Petri's theorem hold. Consider such a set of quadratic polynomials $\tilde{A}_1, \ldots, \tilde{A}_r$ generating I_X.

A polynomial $\tilde{A}_i$ of degree two can be encoded in terms of a symmetric $g \times g$ matrix $A_i = (a_{\nu,\mu})$ as follows. Set $\bar{\omega} = (\omega_1, \ldots, \omega_g)^t$. We have

$$\tilde{A}_i(\bar{\omega}) = \bar{\omega}^t A_i \bar{\omega}.$$

The polynomial $\sigma(\tilde{A}_i)$ is still a polynomial of degree two so we write $\sigma(A_i)$ for the symmetric $g \times g$ matrix such that $\sigma(\tilde{A}_i) = \bar{\omega}^t \sigma(A)_i \bar{\omega}$. It is clear that for an element $\sigma \in \mathrm{GL}_g(k)$, $\sigma(I_X) \subset I_X$ holds if and only if for all $1 \leq i \leq r$, $\sigma(A_i) \in \mathrm{span}_k\{A_1, \ldots, A_r\}$. This means that

$$(\sigma_{\mu,\nu})^t A_i (\sigma_{\mu,\nu}) = \sum_{j=1}^{r} \lambda(\sigma)_{ji} A_j \qquad \text{for every } 1 \leq i \leq j. \tag{4.1}$$

4.2 The automorphism group as an algebraic set

Let $A_1, \ldots, A_r$ be a set of linearly independent $g \times g$ matrices such that the $w^t A_i w$ $1 \leq i \leq r$ generate the canonical ideal, and $w^t = (w_1, \ldots, w_g)$ is a basis of the space of holomorphic differentials. By choosing an ordered basis of the vector space of symmetric $g \times g$ matrices we can represent any symmetric $g \times g$ matrix A as an element $\bar{A} \in k^{\frac{g(g+1)}{2}}$, that is

$$\begin{aligned} \bar{\cdot} : \text{Symmetric } g \times g \text{ matrices} &\longrightarrow k^{\frac{g(g+1)}{2}} \\ A &\longmapsto \bar{A} \end{aligned}$$

We can now put together the r elements $\bar{A}_i$ as a $g(g+1)/2 \times r$ matrix $\left(\bar{A}_1 | \cdots | \bar{A}_r\right)$, which has full rank r, since $\{A_1, \ldots, A_r\}$ are assumed to be linearly independent.

Proposition 4.2. *An element $\sigma = (\sigma_{ij}) \in \mathrm{GL}_g(k)$ induces an action on the curve X, if and only if the $g(g+1)/2 \times 2r$ matrix*

$$B(\sigma) = \left[\bar{A}_1, \ldots, \bar{A}_r, \overline{\sigma^t A_1 \sigma}, \ldots, \overline{\sigma^t A_r \sigma}\right]$$

has rank r.

We have that σ is an automorphism if the $g(g+1)/2 \times 2r$-matrix $B(\sigma)$ has rank r, which means that $(r+1) \times (r+1)$-minors of $B(\sigma)$ are zero. This provides us with a description of the automorphism group as a determinantal variety given by explicit equations of degree $(r+1)^2$.

But we can do better. Using Gauss elimination we can find a $\frac{g(g+1)}{2} \times \frac{g(g+1)}{2}$ invertible matrix Q which puts the matrix $\left(\bar{A}_1 | \cdots | \bar{A}_r\right)$ in echelon form, that is

$$Q\left(\bar{A}_1 | \cdots | \bar{A}_r\right) = \left(\frac{\mathbb{I}_r}{\mathbb{O}_{\left(\frac{g(g+1)}{2} - r\right) \times r}} \right).$$

But then for each $1 \leq i \leq r$ eq. (4.1) is satisfied if and only if the lower $(\frac{g(g+1)}{2} - r) \times r$ bottom block matrix of the matrix

$$Q\left(\overline{\sigma^t A_1 \sigma}|\cdots|\overline{\sigma^t A_r \sigma}\right) \tag{4.2}$$

is zero, while the top $r \times r$ block matrix gives rise to the representation

$$\rho_1 : G \to \mathrm{GL}_r(k),$$

defined by equation (4.1). Assuming that the lower $(\frac{g(g+1)}{2} - r) \times r$ bottom block matrix gives us $r(\frac{g(g+1)}{2} - r)$ equations where the entries $\sigma = (\sigma_{ij})$ are seen as indeterminates. In this way we can write down elements of the automorphism group as a zero dimensional algebraic set, satisfying certain quadratic equations.

4.3 An example: the Fermat curve

Consider the projective non singular curve given by equation

$$F_n : x_1^n + x_2^n + x_0^n = 0$$

This curve has genus $g = \frac{(n-2)(n-1)}{2}$. Set $x = x_1/x_0$, $y = x_2/x_0$. For $\omega = \frac{dx}{y^{n-1}} = -\frac{dy}{x^{n-1}}$ we have that the set

$$x^i y^j \omega \text{ for } 0 \leq i + j \leq n - 3 \tag{4.3}$$

forms a basis for holomorphic differentials, [35], [48], [49]. These g differentials are ordered lexicographically according to (i, j), that is

$$\omega_{0,0} < \omega_{0,1} < \cdots < \omega_{0,n-3} < \omega_{1,0} < \omega_{1,1} < \cdots < \omega_{1,n-4} < \cdots < \omega_{n-3,0}.$$

The case $n = 2$ is a rational curve, the case $n = 3$ is an elliptic curve, the case $n = 4$ has genus 3 and gonality 3, the case $n = 5$ has genus 6 and is quintic so the first Fermat curve which has canonical ideal generated by quadratic polynomial is the case $n = 6$ which has genus 10.

Proposition 4.3. *The canonical ideal of the Fermat curve F_n for $n \geq 6$ consists of two sets of relations*

$$G_1 = \{\omega_{i_1,j_1}\omega_{i_2,j_2} - \omega_{i_3,j_3}\omega_{i_4,j_4} : i_1 + i_2 = i_3 + i_4, j_1 + j_2 = j_3 + j_4\}, \tag{4.4}$$

and

$$G_2 = \left\{\omega_{i_1,j_1}\omega_{i_2,j_2} + \omega_{i_3,j_3}\omega_{i_4,j_4} + \omega_{i_5,j_5}\omega_{i_6,j_6} : \begin{smallmatrix} i_1+i_2=n+a, & j_1+j_2=b \\ i_3+i_4=a, & j_3+j_4=n+b \\ i_5+i_6=a, & j_5+j_6=b \end{smallmatrix}\right\} \tag{4.5}$$

where $0 \leq a, b$ are selected such that $0 \leq a + b \leq n - 6$.

We will now prove proposition 4.3 for $n \geq 6$, following the method developed in [11] (i.e. theorem 1.12). Observe that the holomorphic differentials given in eq. (4.3) are in 1-1 correspondence with the elements of the set $\mathbf{A} = \{(i, j) : 0 \leq i + j \leq n - 3\} \subset \mathbb{N}^2$. First we introduce the following term order on the polynomial algebra $S := \mathrm{Sym} H^0(X, \Omega_X)$.

Definition 4.4. *Choose any term order $\prec_t$ for the variables $\{\omega_{N,\mu} : (N,\mu) \in A\}$ and define the term order $\prec$ on the monomials of S as follows:*

$$\omega_{N_1,\mu_1}\omega_{N_2,\mu_2}\cdots\omega_{N_d,\mu_d} \prec \omega_{N'_1,\mu'_1}\omega_{N'_2,\mu'_2}\cdots\omega_{N'_s,\mu'_s} \text{ if and only if} \tag{4.6}$$

- $d < s$ *or*
- $d = s$ *and* $\sum \mu_i > \sum \mu'_i$ *or*
- $d = s$ *and* $\sum \mu_i = \sum \mu'_i$ *and* $\sum N_i < \sum N'_i$
- $d = s$ *and* $\sum \mu_i = \sum \mu'_i$ *and* $\sum N_i = \sum N'_i$ *and*

$$\omega_{N_1,\mu_1}\omega_{N_2,\mu_2}\cdots\omega_{N_d,\mu_d} \prec_t \omega_{N'_1,\mu'_1}\omega_{N'_2,\mu'_2}\cdots\omega_{N'_s,\mu'_s}.$$

By evaluating $\sum_{i=0}^{E}\sum_{j=0}^{E-i} 1$ we can see that

$$\#\{(i,j) \in \mathbb{N}^2 : 0 \le i+j \le E\} = (E+1)(E+2)/2 \tag{4.7}$$

We extend the correspondence between the variables $\omega_{i,j}$ and the points of $\mathbf{A}$ to a correspondence between monomials in S of standard degree 2 and points of the Minkowski sum of $\mathbf{A}$ with itself, defined as

$$\mathbf{A}+\mathbf{A} = \{(i+i', j+j') \mid (i,j),(i',j') \in \mathbf{A}\} \subseteq \mathbb{N}^2. \tag{4.8}$$

Proposition 4.5. *Let $\mathbf{A}$ be the set of exponents of the basis of holomorphic differentials, and let $\mathbf{A}+\mathbf{A}$ denote the Minkowski sum of $\mathbf{A}$ with itself, as defined in (4.8). Then*

$$(\rho,T) \in \mathbf{A}+\mathbf{A} \Leftrightarrow \exists\, \omega_{i,j}\omega_{i',j'} \in S \textit{ such that } \mathrm{mdeg}(\omega_{i,j}\omega_{i',j'}) = (\mathbf{2},\rho,\mathrm{T}).$$

For each $n \in \mathbb{N}$ we write $\mathbb{T}^n$ for the set of monomials of degree n in S and proceed with the characterization of monomials that do not appear as leading terms of binomials in $G_1 \subseteq J$.

Proposition 4.6. *Let σ be the map of sets*

$$\begin{aligned} \sigma : \mathbf{A}+\mathbf{A} &\to \mathbb{T}^2 \\ (\rho,T) &\mapsto \min_{\prec}\{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid (\rho,T) = (i+i', j+j')\}. \end{aligned}$$

Then

$$\sigma(\mathbf{A}+\mathbf{A}) = \{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid \omega_{i,j}\cdot\omega_{i',j'} \ne \mathrm{in}_{\prec}(f),\ \forall\, f \in G_1\}.$$

The above proposition gives a characterization of the monomials that do not appear as initial terms of elements of G_1, therefore they survive in the quotient $(S/\mathrm{in}_{\prec}(J))_2$. Indeed, the minimal of the set $\{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid (\rho,T) = (i+i', j+j')\}$ will never appear as the initial term of an element in G_1. Therefore $\mathbf{A}+\mathbf{A}$ is bijective with a basis of the vector space $(S/\mathrm{in}_{\prec}G_1)_2$. However, some of these monomials appear as initial terms of polynomials in G_2 and these have to be subtracted in order to compute $\dim_L (S/\mathrm{in}_{\prec}(J))_2$

Proposition 4.7. *Let*

$$C = \{(\rho, b) \in \mathbf{A} + \mathbf{A} \mid \rho = n + a, 0 \leq a + b \leq n - 6, a, b \in \mathbb{N}\}.$$

Then

$$\sigma(C) \subseteq \{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid \exists\, g \in G_2 \text{ such that } \omega_{i,j}\omega_{i',j'} = \mathrm{in}_{\prec}(g)\}.$$

Moreover $\#C = \#\sigma(C) = (n-5)(n-4)/2$.

Proof. Observe that elements in G_2 are mapped into elements of the form $x^a y^b (x^n + y^n + 1)\omega^2 \in H^0(X, \Omega_X^{\otimes 2})$. By the form of the initial term of such an element of G_2 we have for $i_1 + i_2 = n + a = \rho, j_1 + j_2 = b$. Therefore

$$i_3 + i_4 = a = \rho - n, j_3 + j_4 = n + b, i_5 + i_6 = a = \rho - n, j_5 + j_6 = b = T$$

We should have $0 \leq a + b \leq n - 6$ and by eq. (4.7) we have that the cardinality of C equals $(n-5)(n-4)/2$. □

We now observe that

$$\mathbf{A} + \mathbf{A} \subset \{i, j \in \mathbb{N} : i + j \leq 2n - 6\}$$

so $\#(\mathbf{A} + \mathbf{A}) \leq (2n-5)(2n-4)/2$ and

$$\begin{aligned}\dim_L (S/\mathrm{in}_{\prec}(J))_2 &= \#\big((\mathbf{A} + \mathbf{A}) \backslash C\big) = \#(\mathbf{A} + \mathbf{A}) - \#C \\ &\leq \frac{(2n-5)(2n-4)}{2} - \frac{(n-5)(n-4)}{2} = 3(g-1).\end{aligned}$$

so by proposition 1.12 we have that $I = J$.

4.3.1 Automorphisms of the Fermat curve

The group of automorphisms of the Fermat curve is given by [50], [36]

$$G = \begin{cases} \mathrm{PGU}(3, p^h), & \text{if } n = 1 + p^h \\ (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \rtimes S_3, & \text{otherwise} \end{cases}$$

The action of the automorphism group is given in terms of a 3×3 matrix A sending

$$x = (x_1/x_0) \mapsto \frac{\sum_{i=0}^{2} a_{1,i}x_i}{\sum_{i=0}^{2} a_{0,i}x_i} \qquad y = (x_2/x_0) \mapsto \frac{\sum_{i=0}^{2} a_{2,i}x_i}{\sum_{i=0}^{2} a_{0,i}x_i},$$

In characteristic 0, the matrix A is a monomial matrix, that is, it has only one non-zero element in each row and column and this element is an n-th root of unity. Two matrices A_1, A_2 give rise to the same automorphism if and only if they differ by an element in the group $\{\lambda\mathbb{I}_3 : \lambda \in k\}$. In any case the group G is naturally a subgroup of $\mathrm{PGL}_3(k)$. Finding the representation matrix of G as an element in $\mathrm{PGL}_{g-1}(k)$ is easy when $n \neq 1 + p^h$ and more complicated in $n = 1 + p^h$ case. We have two different embeddings of the Fermat curve F_n in projective space

$$\mathbb{P}_k^{g-1} \longleftarrow F_n \longrightarrow \mathbb{P}_k^2.$$

In both cases the automorphism group is given as restriction of the automorphism group of the ambient space.

The computation of the automorphism group in terms of the vanishing of the polynomials given in equation (4.2) is quite complicated.

We have performed this computation in magma [6], and it turns out the automorphism group for the $n = 6$ case is described as an algebraic set described by $g^2 = 100$ variables and 756 equations.

> FermatCurve(6,Rationals());
> $x_{7,8} * x_{10,10} - 2 * x_{9,8} * x_{9,10} + x_{10,8} * x_{7,10},$
>
>................756 equations...................
>
> >$x_{7,9} * x_{10,10} - 2 * x_{9,9} * x_{9,10} + x_{10,9} * x_{7,10}$

Chapter 5

Syzygies

5.1 Extending group actions

Recall that $S = k[\omega_1, \ldots, \omega_g]$ is the polynomial ring in g variables. Let M be a graded S-module acted on by the group G, generated by the elements $m_1, \ldots, m_r$ of corresponding degrees $a_1, \ldots, a_r$. We consider the free S-module $F_0 = \bigoplus_{j=1}^r S(-a_j)$ together with the onto map

$$F_0 = \bigoplus_j S(-a_j) \xrightarrow{\pi} M. \tag{5.1}$$

Let us denote by $M_1, \ldots, M_r$ elements of F_0, such that $\pi(M_i) = m_i$, assuming also that deg$(M_i) =$ deg(m_i), for $1 \leq i \leq r$. The action on the generators m_i is given by

$$\sigma(m_i) = \sum_{\nu=1}^r a_{\nu,i} m_i, \text{ for some } a_{\nu,i} \in S. \tag{5.2}$$

Remark 5.1. *We would like to point out here that unlike the theory of vector spaces, an element $x \in F_0$ might admit two different decompositions*

$$x = \sum_{i=1}^r a_i m_i = \sum_{i=1}^r b_i m_i, \text{ that is } \sum_{i=1}^r (a_i - b_i) m_i = 0,$$

and if $a_{i_0} - b_{i_0} \neq 0$ we cannot assume that $a_{i_0} - b_{i_0}$ is invertible, so we can't express m_{i_0} as an S-linear combination of the other elements m_i, for $i_0 \neq i, 1 \leq i \leq r$ in order to contradict minimality. We can only deduce that $\{a_i - b_i\}_{i=1,\ldots,r}$ form a syzygy.

Therefore one might ask if the matrix $(a_{\nu,i})$ given in eq. (5.2) is unique. In proposition 5.4 we will prove that the elements $a_{\nu,i}$ which appear as coefficients in eq. (5.2) are in the field k and therefore the expression is indeed unique.

The natural action of Aut(X) on $H^0(X, \Omega_X)$ can be extended to an action on the ring $S = \text{Sym}H^0(X, \Omega_X)$, so that $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in S$. Therefore if $M = I_X$ then for all $s \in S$, $m \in I_X = M$ we have $\sigma(sm) =$

$\sigma(s)\sigma(m)$. All the actions in the modules we will consider will have this property.

For a free module $F = \bigoplus_{j=1}^{s} S(-a_j)$, generated by the elements M_i, $1 \leq i \leq r$, $\deg(M_i) = a_i$ and a map $\pi : F \to M$ we define the action of G by

$$\sigma\left(\sum_{j=1}^{r} s_j M_j\right) = \sum_{j=1}^{r} \sigma(s_j) \sum_{\nu=1}^{r} a_{\nu,j}(\sigma) M_\nu = \sum_{\nu=1}^{r}\left(\sum_{j=1}^{r} a_{\nu,j}(\sigma)\sigma(s_j)\right) M_\nu,$$

where $\deg_S a_{\nu,j} + a_\nu = \deg_S m_j$. This means that under the action of $\sigma \in G$ the r-tuple $(s_1, \ldots, s_r)^t$ is sent to

$$\begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix} \stackrel{\sigma}{\longmapsto} \begin{pmatrix} a_{1,1}(\sigma) & a_{1,2}(\sigma) & \cdots & a_{1,r}(\sigma) \\ \vdots & \vdots & & \vdots \\ a_{r,1}(\sigma) & a_{r,2}(\sigma) & \cdots & a_{r,r}(\sigma) \end{pmatrix} \begin{pmatrix} \sigma(s_1) \\ \vdots \\ \sigma(s_r) \end{pmatrix}.$$

If $A(\sigma) = \big(a_{i,j}(\sigma)\big)$ is the matrix corresponding to σ then for $\sigma, \tau \in G$ the following cocycle condition holds:

$$A(\sigma\tau) = A(\sigma)A(\tau)^\sigma.$$

If we can assume that G acts trivially on the matrix $A(\tau)$ for every $\tau \in G$ (for instance when $A(\tau)$ is a matrix with entries in k for every $\tau \in G$), then the above cocycle condition becomes a homomorphism condition.

Also if $A(\sigma)$ is a principal derivation, that is there is an $r \times r$ matrix Q, such that

$$A(\sigma) = \sigma(Q) \cdot Q^{-1}$$

then after a basis change of the generators we can show that the action on the coordinates is just given by

$$(s_1, \cdots, s_r)^t \stackrel{\sigma}{\longmapsto} (\sigma(s_1), \cdots, \sigma(s_r))^t,$$

that is the matrix $A(\sigma)$ is the identity. We will call the action on the free resolution $\mathbf{F}$ obtained by extending the action on M the *standard action*.

5.2 Group actions on free resolutions

Recall that $S = k[\omega_1, \ldots, \omega_g]$ is the polynomial ring in g variables. Let M be a graded S-module generated by the elements $m_1, \ldots, m_r$ of corresponding degrees $a_1, \ldots, a_r$. Consider the minimal free resolution

$$0 \longrightarrow F_g \xrightarrow{\phi_g} \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0\ , \tag{5.3}$$

where $\mathrm{coker}(\phi_1) = F_0/\mathrm{Im}\phi_1 = F_0/\ker\pi \cong M$. Let $\mathfrak{m}$ be the maximal ideal of S generated by $\langle \omega_1, \ldots, \omega_g \rangle$. Each free module in the resolution can be written as

$$F_i = \bigoplus_j S(-j)^{\beta_{i,j}},$$

where the integers $\beta_{i,j}$ are the Betti numbers of the resolution. The Betti numbers satisfy

$$\beta_{i,j} = \beta_{g-2-i,g+1-j}. \tag{5.4}$$

as one can see by using the self duality of the above resolution by twisting by $S(-g)$ see [38, prop. 4.1.1], [15, prop. 9.5] or by using Koszul cohomology, see [16, prop. 4.1].

Assume that M and each F_i is acted on by a group G and that the maps δ_i are G-equivariant. We will now study the action of the group G on the generators of F_i. First of all we have that

$$F_i = \bigoplus_{\nu=1}^{r_i} \bigoplus_{\mu=1}^{\beta_{i,\nu}} e_{i,\nu,\mu} S \cong \bigoplus_{\nu=1}^{r_i} S(-d_{i,\nu})^{\beta_{i,\nu}}.$$

In the above formula we assumed that F_i is generated by elements $e_{i,\nu,\mu}$ such that the degree of $e_{i,\nu,\mu} = d_{i,\nu}$ for all $1 \leq \mu \leq \beta_{i,\nu}$. We also assume that

$$d_{i,1} < d_{i,2} < \cdots < d_{i,r_i}.$$

The action of σ is respecting the degrees, so an element of minimal degree $d_{i,1}$ is sent to a linear combination of elements of minimal degree $d_{i,1}$. In this way we obtain a representation

$$\rho_{i,1} : G \to \mathrm{GL}(\beta_{i,1}, k).$$

In a similar way an element $e_{i,2,\mu}$ of degree $d_{i,2}$ is sent to an element of degree $d_{i,2}$ and we have that

$$\sigma(e_{i,2,\mu}) = \sum_{j_1=1}^{\beta_{i,2}} \lambda_{i,2,\mu,j_1} e_{i,2,j_1} + \sum_{j_2=1}^{\beta_{i,1}} \lambda'_{i,2,\mu,j_1} e_{i,1,j_2},$$

where all $\lambda_{i,2,\mu,j_1} \in k$ and all $\lambda'_{i,1,\mu,j_2} \in \mathfrak{m}^{d_{i,2}-d_{i,1}}$. In this case we have a representation with entries in an ring instead of a field, which has the form:

$$\rho_{i,2} : G \to \mathrm{GL}(\beta_{i,1} + \beta_{i,2}, \mathfrak{m}^{d_{i,2}-d_{i,1}}),$$
$$\sigma \mapsto \begin{pmatrix} A_1(\sigma) & A_{1,2}(\sigma) \\ 0 & A_2(\sigma) \end{pmatrix},$$

where $A_1(\sigma) \in \mathrm{GL}(\beta_{i,1}, k)$ and $A_2(\sigma) \in \mathfrak{m}^{d_{i,2}-di,1}\mathrm{GL}(\beta_{i,2}, k)$.

By induction the situation in the general setting gives rise to a series of representations:

$$\rho_{i,j} : G \to \mathrm{GL}(\beta_{i,1} + \beta_{i,2}, \mathfrak{m}^{d_{i,j}-d_{i,1}})$$
$$\sigma \mapsto A(\sigma) = \begin{pmatrix} A_1(\sigma) & A_{1,2}(\sigma) & \cdots & A_{1,j}(\sigma) \\ 0 & A_2(\sigma) & & A_{2,j}(\sigma) \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 0 & A_j(\sigma) \end{pmatrix} \tag{5.5}$$

where $A_\nu(\sigma) \in \mathrm{GL}(\beta_{i,\nu}, k)$ and $A_{\kappa,\lambda}(\sigma)$ is an $\beta_{i,\kappa} \times \beta_{i,\lambda}$ matrix with coefficients in $\mathfrak{m}^{\beta_{i,\lambda}-\beta_{i,\kappa}}$. The representation ρ_{i,r_i} taken modulo $\mathfrak{m}$ reduces to $\mathrm{Tor}_i^S(k, M)$, seen as a $k[G]$-module.

5.3 Unique actions

Let us consider two actions of the automorphisms group G on $H^0(X,\Omega_X)$, which can naturally be extended on the symmetric algebra Sym$H^0(X,\Omega_X)$. We will denote the first action by $g\star v$ and the second action by $g\circ v$, where $g \in G$, $v \in \mathrm{Sym}H^0(X,\Omega_X)$.

Proposition 5.2. *If the curve X satisfies the conditions of faithful action of $G = \mathrm{Aut}(X)$ on $H^0(X,\Omega_X)$, that is X is not hyperelliptic and $p > 2$, [29, th. 3.2] and moreover both actions $\star, \circ$ restrict to actions on the canonical ideal I_X, then there is an automorphism $i : G \rightarrow G$, such that $g \star v = i(g) \circ v$.*

Proof. Both actions of G on $H^0(X,\Omega_X)$ introduce automorphisms of the curve X. That is since $G \star I_X = I_X$ and $G \circ I_X = I_X$, the group G is mapped into $\mathrm{Aut}(X) = G$. This means that for every element $g \in G$ there is an element $g^* \in \mathrm{Aut}(X) = G$ such that $g \star v = g^* v$, where the action on the right is the standard action of the automorphism group on holomorphic differentials. By the definition of the group action for every $g_1, g_2 \in G$ we have $(g_1g_2)^*v = g_1^*g_2^*v$ for all $v \in H^0(X,\omega_X)$ and the faithful action of the automorphism group provides us with $(g_1g_2)^* = g_1^*g_2^*$, i.e. the map $i_* : g \mapsto g^*$ is a homomorphism. Similarly the map corresponding to the $\circ$-action, $i_\circ : g \mapsto g^\circ$ is a homomorphism and the desired homomorphism i is the composition of $i_* i_\circ^{-1}$. □

The map $\mathrm{Hom}_S(F_i, S(-g))$ induces a symmetry of the free resolution $\mathbf{F}$ by sending F_i to F_{g-2-i}. Each free module F_i of the resolution $\mathbf{F}$ is equipped by the extension of the action on holomorphic differentials, according to the construction of section 5.2. On the other hand since $S(-g)$ is a G-module we have that $F_{g-2-i} \cong \mathrm{Hom}_S(F_i, S(-g))$ is equipped by a second action namely every $\phi : F_i \rightarrow S(-g)$ is acted naturally by G in terms of $\phi \mapsto \phi^\sigma = \sigma^{-1}\phi\sigma$. How are the two actions related?

Lemma 5.3. *Denote by $\star$ the action of G on F_i induced by taking the $S(-g)$-dual. The standard and the $\star$-actions are connected in terms of an automorphism ψ_i of G, that is for all $v \in F_i$ $g \star v = \psi_i(g)v$.*

Proof. Assume that $i \leq g-2-i$. Consider the standard action of G on the free resolution $\mathbf{F}$. The module F_{g-2-i} obtains a new action $g \star v$ for $g \in G, v \in F_i$. By 5.2 this $\star$ action is transferred to an action on all F_j for $j \geq g-2-i$, including the final term F_{g-2} which is isomorphic to $S(-1)$. This gives us two actions on $H^0(X,\Omega_X)$ which satisfy the requirements of proposition 5.2. The desired result follows, since the action can be pulled back to all syzygies using either $\mathbf{F}$ or $\mathbf{F}^*$. □

Proposition 5.4. *Under the faithful action requirement we have that all automorphisms $\sigma \in G$ send the direct summand $S(-j)^{\beta_{i,j}}$ of F_i to itself, that is the representation matrix in eq. (5.5) is block diagonal.*

Proof. Consider $F_i = \bigoplus_{\nu=1}^{r_i} M_{i,\nu} S$, where $M_{i,1}, \ldots, M_{i,r_i}$ are assumed to be minimal generators of F_i with descending degrees $a_{i,\nu} = \deg(m_{i,\nu})$, $1 \leq \nu \leq$

r_i. The action of an element σ is given in terms of the matrix $A(\sigma)$ given in equation (5.5). The element $\phi \in \mathrm{Hom}_S(F_i, S(-g))$ is sent to

$$\begin{aligned} h : \mathrm{Hom}_S(F_i, S(-g)) &\xrightarrow{\cong} F_{g-2-i} \\ \phi &\longmapsto \big(\phi(M_{i,1}), \ldots, \phi(M_{i,r_i})\big) \end{aligned} \tag{5.6}$$

Each $\phi(M_{i,\nu})$ can be considered as an element in $S(-g-1+\deg(m_{i,\nu}))$ inside F_{g-2-i}. Observe that the element $\phi \in \mathrm{Hom}_S(F_i, S(-g))$ is known if we know all $\phi(M_{i,\nu})$ for $1 \leq \nu \leq r_i$. From now on we will identify such an element ϕ as a r_i-tuple $\big(\phi(M_{i,\nu})\big)_{1 \leq \nu \leq r_i}$.

Recall that if A, B are G-modules, then there is an natural action on $\mathrm{Hom}(A, B)$, sending $\phi \in \mathrm{Hom}(A, B)$ to ${}^\sigma\phi$, which is the map

$${}^\sigma\phi : A \ni a \mapsto \sigma\phi(\sigma^{-1}a).$$

We have also a second action on the module F_{g-2-i}. We compute ${}^\sigma\phi(M_{i,\nu})$ for all base elements $M_{i,\nu}$ in order to describe ${}^\sigma\phi$:

$$\begin{aligned} \sigma\left(\phi(\sigma^{-1}M_{i,\nu})\right)_{1 \leq \nu \leq \kappa} &= \left(\sum_{\mu=1}^{r_i} \sigma\big(\alpha_{\mu,\nu}(\sigma^{-1})\big)\sigma\phi(M_{i,\mu}) \right)_{1 \leq \nu \leq r_i} \\ &= \left(\sum_{\mu=1}^{r_i} \sigma\big(\alpha_{\mu,\nu}(\sigma^{-1})\big)\chi(\sigma)\phi(M_{i,\mu}) \right)_{1 \leq \nu \leq r_i} \end{aligned}$$

where in the last equation we have used the fact that $\phi(M_i)$ are in the rank one G-module $S(-g) \cong \wedge^{g-1}\Omega^1_X$ hence the action of $\sigma \in G$ is given by multiplication by $\chi(\sigma)$, where $\chi(\sigma)$ is an invertible element is S.

In order to simplify the notation consider i fixed, and denote $M_\nu = M_{i,\nu}$, $r = r_i$, $a_{i,j} = a_j$. We can consider as a basis of $\mathrm{Hom}(F_i, S(-g))$ the morphisms ϕ_μ given by

$$\phi_\mu(M_j) = \delta_{\mu,j} \cdot E, \tag{5.7}$$

where E is a basis element of degree g of the rank 1 module $S(-g) \cong S \cdot E$. This is a different basis than the basis $M_{g-2-i,\nu}$, $1 \leq n \leq r_{g-2-i}$ of F_{g-2-i} we have already introduced.

According to eq. (5.4) if M_j has degree a_j then the element ϕ_j has degree $g + 1 - a_j$. Assume that M_r has maximal degree a_r. Then, ϕ_r has minimal degree. Moreover, in order to describe ${}^\sigma\phi_r$ we have to consider the tuple $({}^\sigma\phi_r(M_1), \ldots, {}^\sigma\phi_r(M_r))$. We have

$$\begin{aligned} \left({}^\sigma\phi_r(M_\nu)\right)_{1 \leq \nu \leq r} &= \left(\sum_{\mu=1}^{r} \sigma\big(\alpha^{(i)}_{\mu,\nu}(\sigma^{-1})\big)\chi(\sigma)\phi_r(M_\mu) \right)_{1 \leq \nu \leq r} \\ &\overset{(5.7)}{=\!=} \left(\sigma\big(\alpha^{(i)}_{r,\nu}(\sigma^{-1})\big)\chi(\sigma)E\right)_{1 \leq \nu \leq r} \end{aligned}$$

and we finally conclude that

$${}^\sigma\phi_r = \sum_{\nu=1}^{r} \sigma^{-1}\big(\alpha^{(i)}_{r,\nu}(\sigma^{-1})\big)\chi(\sigma)\phi_\nu.$$

In this way every element $x \in F_{g-2-i}$ is acted on by σ in terms of the action

$$\sigma \star x = h\big({}^{\sigma}h^{-1}(x)\big),$$

where h is the map given in eq. (5.6). On the other hand the elements $h(\phi_r)$ are in F_{g-2-i} and by lemma 5.3 there is an element $\sigma' \in G$ such that

$$\sigma' h(\phi_r) = \sum_{\nu=1}^{r} \alpha_{\nu,r}^{(g-2-i)}(\sigma') h(\phi_\nu).$$

Since the element ϕ_ν has maximal degree among generators of F_i the element $h(\phi_r)$ has minimal degree. This means that all coefficients

$$\alpha_{\nu,r}^{(g-2-i)}(\sigma') = \sigma\big(\alpha_{r,\nu}^{(i)}(\sigma^{-1})\big)\chi(\sigma)$$

are zero for all ν such that $\deg m_\nu < \deg \mu_r$. Therefore all coefficients $a_{\nu,r}^{(i)}(\sigma)$ for ν such that $\deg m_\nu < \deg m_r$ are zero. This holds for all $\sigma \in G$. By considering in this way all elements $\phi_{r-1}, \phi_{r-2}, \ldots, \phi_1$, which might have greater degree than the degree of ϕ_r the result follows. □

5.4 Representations on the free resolution

Each S-module F_i in the minimal free resolution can be seen as a series of representations of the group G. Indeed, the modules F_i are graded and there is an action of G on each graded part $F_{i,d}$, given by representations

$$\rho_{i,d} : G \to \mathrm{GL}\big(F_{i,d}\big),$$

where $F_{i,d}$ is the degree d part of the S-module F_i. The space $\mathrm{Tor}_i^S(k, S_X))$ is clearly a G-module, and by proposition 5.4 there is a decomposition of G-modules

$$\mathrm{Tor}_i^S(k, S_X) = \bigoplus_{j \in \mathbb{Z}} \mathrm{Tor}_i^S(k, S_X)_j,$$

where $\mathrm{Tor}_i^S(k, S_X)_j$ is the k-vector space generated by generators of F_i that have degree j. This is a vector space of dimension $\beta_{i,j}$.

Denote by $\mathrm{Ind}(G)$ the set of isomorphism classes of indecomposable $k[G]$-modules. If k is of characteristic $p > 0$ and G has no-cyclic p-Sylow subgroup then the set $\mathrm{Ind}(G)$ is infinite, see [4, p.26]. Suppose that each $\mathrm{Tor}_i^S(k, S_X)_j$ admits the following decomposition in terms of $U \in \mathrm{Ind}(G)$:

$$\mathrm{Tor}_i^S(k, S_X)_j = \bigoplus_{U \in \mathrm{Ind}(G)} a_{i,j,U} U \text{ where } a_{i,j,U} \in \mathbb{Z}.$$

We obviously have that

$$\beta_{i,j} = \sum_{U \in \mathrm{Ind}(G)} a_{i,j,U} \dim_k U.$$

The G-structure of F_i is given by

$$\mathrm{Tor}_i^S(k, S_X) \otimes S,$$

that is the G-module structure of $F_{i,d}$ is given by

$$F_{i,d} = \bigoplus_{d \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} \mathrm{Tor}_i^S(k, S_X)_{d-j} \otimes S_j.$$

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