GALOIS ACTION ON HOMOLOGY OF THE HEISENBERG CURVE.

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Abstract. The Heisenberg curve is defined topologically as a cover of the Fermat curve and corresponds to an extension of the projective line minus three points by the non-abelian Heisenberg group modulo n. We compute its fundamental group and investigate an action from Artin's Braid group to the curve itself and its homology. We also provide a description of the homology in terms of irreducible representations of the Heisenberg group over a field of characteristic 0.

1. Introduction

A significant portion of research in modern number theory revolves around the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the various ways to extract information from it. A particularly interesting exact sequence containing $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the following

$$1 \to \widehat{F_2} \cong \pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}) \to \pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{Q}} - \{0,1,\infty\}) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1,$$

where $\widehat{F_2}$ denotes the free profinite group on two elements. As the middle term is the étale fundamental group of the projective line with three ramified points, ramified covers $X \to \mathbb{P}^1$ has become an attractive research topic. Specifically, cyclic covers of $\mathbb{P}^1$ have received a lot of attention in the mathematical community. Furthermore, a well-studied generalization of them are the so-called generalized Fermat curves, being extensions of $\mathbb{P}^1$ by a direct product of cyclic groups. Our motivation is to proceed a step further and study a curve which arises as an extension of the projective line by a semi-direct product, as well as study its homology. The motivation about the homology of a curve of genus g comes from that it naturally arises as the lattice Λ, when considering the Jacobian of the curve over $\mathbb{C}$ as $\mathbb{C}^g/\Lambda$. A standard technique of obtaining Galois representations is through the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the ℓ^k-torsion points of the Jacobian of the curve as an abelian variety, that is the Galois action on the Tate module $\Lambda \otimes \mathbb{Z}_\ell$ for a prime number ℓ.

In terms of computations and previous work we rely on, in [12], [11] the fundamental group of an open abelian Galois cover $X \to \mathbb{P}^1$ of the projective line was computed and used in order to study the actions of the automorphism group $\mathrm{Aut}(X)$, the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the braid group on the homology

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of the curve X. In this article we follow the same approach in order to study the n-level Heisenberg curve X_{H_n}, which is a Galois extension of the projective line with Galois group isomorphic to the non-abelian discrete Heisenberg group H_n, see definition 8. The automorphism group of Heisenberg curves was studied in [1].

Our main theorem in this paper is the following.

Theorem 1. *The homology group of the Heisenberg curve X_{H_n} as a $\mathbb{F}[H_n]$-module, over a field $\mathbb{F}$ of characteristic zero which contains the n-th roots of unity, has the following decomposition in terms of irreducible representations of H_n.*

$$H_1(X_H, \mathbb{F}) = \bigoplus_{j=0}^{n-1} \bigoplus_{i,s=0}^{\gcd(n,j)-1} h_{ijs}\chi_{ijs},$$

where the integer h_{ijs} denotes the multiplicity of the irreducible representation associated with the character χ_{ijs}. The explicit formulas for the multiplicites h_{ijs} are given in eq. (14) and the irreducible characters χ_{ijs} are discussed in the appendix.

The approach we use is as follows. From the ramified cover $X \to \mathbb{P}^1$ we can remove the branched points $0, 1, \infty$ and obtain an open cover $X^0 \to X_3 := \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Covering space theory then provides that the open curve X^0 can be described as a quotient of the universal covering space $\tilde{X_3}$ by the fundamental group of the open curve $\pi_1(X^0, x') < \pi_1(X_3, x)$ for a fixed point $x \in X_3$ and an arbitrarily chosen fixed preimage x' in X^0. The fundamental group $\pi_1(X_3, x)$ is the free group F_2 of rank 2 and can be presented as

$$\pi_1(X_3, x) = \langle x_1, x_2, x_3 | x_1x_2x_3 = 1 \rangle,$$

with each x_i representing the homotopy class of loops on x around the punctures $0, 1, \infty$ respectively.

This setting fits the framework of Y. Ihara as in [9], [10]. Firstly, the braid group B_3 can be realized as a subgroup of $\mathrm{Aut}(F_2)$ generated by the elements σ_i for $i = 1, 2$ given by

$$\sigma_i(x_k) = \begin{cases} x_k & k \neq i, i+1, \\ x_i x_{i+1} x_i^{-1} & k = i, \\ x_i & k = i+1. \end{cases}$$

where we use that $x_3 = (x_1x_2)^{-1}$. There is a natural surjection $B_3 \to S_3$ with its kernel being the so called pure braid group P_n, where every element $\sigma \in P_n$, satisfies

$$\sigma(x_k) \sim x_k,$$

and by $\sim$ we denote conjugation. According to Ihara the pure braid group is a discrete analogue of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in the following sense. Fixing a prime ℓ, by considering the étale pro-ℓ fundamental group of $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{P_1, P_2, \infty\}$, with $P_1, P_2 \in \mathbb{Q}$, he introduced the monodromy representation

$$\mathrm{Ih}_2 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(\mathfrak{F}_2),$$

with $\mathfrak{F}_2$ being the pro-ℓ completion of F_2. The group $\mathfrak{F}_2$ can be a considered as a quotient in the pro-ℓ category of the free pro-ℓ group $\mathfrak{F}_3$ and it admits a similar

presentation $\mathfrak{F}_2 = \langle x_1, x_2, x_3 | x_1x_2x_3 = 1\rangle$. Ihara's representation has image inside the group

$$\left\{\sigma \in \mathrm{Aut}(\mathfrak{F}_2) :\ \sigma(x_i) \sim x_i^{N(\sigma)}, (1 \leq i \leq 3) \text{ for some } N(\sigma) \in \mathbb{Z}_\ell^*\right\}, \tag{1}$$

where $N(\sigma)$ does not depend on x_i and the composition $N \circ \mathrm{Ih}_2 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^*$ coincides with the ℓ-cyclotomic character.

We begin by considering the perhaps most famous curve in number theory, that is the Fermat curve in projective coordinates $x^n + y^n = z^n$, which has a fundamental group denoted by R_{Fer_n}. By considering a subgroup of R_{Fer_n} that is the kernel of the surjection $F_2 \to H_n$ we obtain the Heisenberg curve through the Galois correspondence between subcovers and subgroups of $\pi_1(X_3, x)$. This kernel denoted by R_{Heis_n} is precisely the fundamental group of the Heisenberg curve. We then use the free generators of R_{Fer} as stepping stones and employ Schreier's lemma on them to compute the free generators of R_{Heis}, in order to investigate the braid group action.

What can occur with the Heisenberg curve is also interesting in the field of moduli versus field of definition point of view as in the work of Debes, Douai in [5] and Debes, Emsalem in [6]. Let K be a field and K_S a separable closure, then a curve defined a priori on K_S might not always be definable over K. Their work provide a cohomological measure to when this is possible, considering reductions to covers and their automorphisms while descending from K_S to K.

In that framework, similar to how Ihara derives the monodromy representation, an action of $\mathrm{Gal}(K_S/K)$ is lifted on the geometric (étale) fundamental group $\Pi_{K_S}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \cong \widehat{F_2}$ (canonically) via the exact sequence

$$1 \longrightarrow \Pi_{K_S}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \longrightarrow \Pi_K(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \longrightarrow \mathrm{Gal}(K_S/K) \to 1,$$

where the notation Π_K represents the arithmetic fundamental group over K and is consistent with the notation in [5, 6] and forthcoming work of the authors [16]. This exact sequence induces an (outer) action on the fundamental groups of covers of $\mathbb{P}^1$ and the field of moduli for a cover X is defined to be the fixed field of the automorphisms in $\mathrm{Gal}(K_S/K)$ that produce a cover isomorphic to X.

In contrast to this arithmetic action of $\mathrm{Gal}(K_S/K)$ as outer automorphisms on $\mathfrak{F}_2$, in this paper we consider the geometric in nature action of the braid group– realized as the mapping class group of the punctured projective line with some constraint on ∞–and its induced action on the covers. Heisenberg curves turns out to be an interesting example in this setting, whereas Fermat curves stay invariant under the braid group action by having fundamentals groups that are characteristic in F_2 . We provide a case in Remark 19 where the Heisenberg curve under the braid action gets mapped to an entirely new non-isomorphic curve.

It is well-known that both the absolute Galois group and the mapping class group behave similarly, since both act as outer automorphisms of fundamental groups. Because of this analogy, the geometric non-invariance of the Heisenberg curve carries a direct "definability" consequence. Specifically, there is a part of the mapping class group of the punctured projective line under which the curve does not remain invariant, meaning it does not remain "definable" over the entire group. This precisely mirrors the arithmetic case, where a curve is defined over a finite extension L/K if and only if it remains invariant under the action of the

entire group $\mathrm{Gal}(K_S/L)$. This analogy is also investigated more thoroughly by the authors in [16].

Structure of the paper. In §2 we provide the preliminaries in §2.1 and define Heisenberg curves as cyclic covers of Fermat curves in §2.2. Then in §2.3 we compute its fundamental group and describe the Galois action in §2.3.1. Afterwards, in §2.4 we describe the elements that correspond to lifts of homotopy classes of loops around ∞ in terms of our previously established generators. Moreover, in §2.5 we discuss all the elements corresponding to classes of loops around the punctures made by removing the ramified points. Then, we define the homology of the compactified curve after taking the quotient by these elements. In §2.6 we investigate the braid action on the curve and its homology and discuss the Burau representation in §2.6.1. In §3 we discuss the theory of Alexander modules leading up to the proof of our main theorem. We provide an appendix A of the irreducible characters of H_n, as many computations in §3 rely on them. Finally, we provide the following small case example, highlighting many parts of the paper.

Example 2. For $n = 3$ the Fermat curve X_{F_3} is given by the affine equation $x^3 + y^3 = 1$ and it has a projective canonical weierstrass equation $zy^2 = x^3 - 432z^3$ with genus 1. The Heisenberg curve X_{H_3} is also of genus 1; thus we have an isogeny of elliptic curves $X_{H_3} \to X_{F_3}$ and an equation of X_{H_3} has been computed in [1] to be $y^2 = x^3 + 2^4 \cdot 3^6$.

In terms of group theory, the open Fermat and Heisenberg curves can be defined as quotients of the universal covering space $\tilde{X}_3$ of $X_3 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. More precisely, if $F_2 = \langle a, b\rangle$ is the Galois group of $\tilde{X}_3$ over X_3, then the open curves $X^\circ_{F_3}, X^\circ_{H_3}$ are defined by the fundamental groups

$$\langle a^3, b^3, [a, b]\rangle,\ \langle a^3, b^3, [a, [a, b]], [b, [a, b]]\rangle$$

respectively, as subgroups of F_2. Taking the quotient of these by $\Gamma = \langle a^3, b^3, (ab)^3\rangle$ which are the third powers of the classes of loops around the branched points $\{0, 1, \infty\}$, we obtain the fundamental groups of the curves X_{F_3}, X_{H_3} which we expect to be isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Indeed, in [11] the free generators of the fundamental group of the Fermat curve have been computed and for $n = 3$ the generators of $H_1(X_{F_3}, \mathbb{Z})$ are $[a, b] \bmod \Gamma$ and $a[a, b]a^{-1} \bmod \Gamma$. We provide a SageMath [15] script[1] which verifies the word problem that $[a, b], a[a, b]a^{-1}$ indeed commute $\mathrm{mod}\Gamma$.

Let $T = [a, b]$ and let yxy^{-1} be denoted by x^y for elements x, y in F_2. Using the results from this paper, we compute that the free generators of the fundamental group of $X^\circ_{H_3}$ are

$$(a^3)^{b^i T^j},\ (b^3)^{a^i T^j}, \quad 0 \le i, j \le 2,$$

$$((ab)^3)^{a^i T^j}, \quad 0 \le i, j \le 2,\ (i, j) \ne (0, 0),$$

$$[a, T],\ [a, T]^a.$$

Thus, $H_1(X_{H_3}, \mathbb{Z})$ is generated by the classes of $[a, T]$ and $[a, T]^a$ modulo Γ and SageMath can again verify that these commute and do not belong in the same class. Lastly, for a field $\mathbb{F}$ of characteristic 0 which contains the roots of $x^3 - 1$, if χ_{ijs} are

[1]The script can be found at `https://github.com/noulasd/HeisenbergCurve`

the irreducible characters of the discrete Heisenberg group H_3 as described in the appendix, the regular representation is

$$\mathbb{F}[H_n] = \bigoplus_{i,s=0}^{2} \chi_{i,0,s} \oplus 3\chi_{0,1,0} \oplus 3\chi_{0,2,0},$$

of 9 one-dimensional and 2 three-dimensional irreducible representations. The homology over $\mathbb{F}$ as a subrepresentation has only the non-trivial characters χ_{ijs} appearing $3/\gcd(3,j) - z_j(i,s)$ times, for $z_j(i,s)$ as defined in §3, that is $z_1(0,0) = z_2(0,0) = 3$ and $z_0(1,1) = z_0(2,2) = 2$. We thus recover

$$H_1(X_{H_3}, \mathbb{F}) = \mathbb{F}\chi_{1,0,1} \oplus \mathbb{F}\chi_{2,0,2},$$

as the two-dimensional vector space of the torus that corresponds to the elliptic curve.

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2. Heisenberg Curves

2.1. Group theory preliminaries.

Definition 3. The commutator $[x, y]$ of two elements x, y in a group is defined as $[x, y] = xyx^{-1}y^{-1}$.

Definition 4. The exponent x^y of two elements x, y in a group G is defined as $x^y = yxy^{-1}$. Note that because our notation has the action on the right, we have $x^{y_1 y_2} = (x^{y_2})^{y_1}$, for y_1, y_2 in G.

Definition 5. For a group G the subgroup G' is generated by the commutators $[g, h]$ for g, h in G. The abelianization is defined as $G^{\mathrm{ab}} := G/G'$.

Lemma 6. *With the above definitions, for three elements x, y, z in a group, the following identity holds.*

$$[xy, z] = [y, z]^x \cdot [x, z].$$

Proof.

$$\begin{aligned}[xy, z] &= xyz \cdot y^{-1}x^{-1}z^{-1} = xyz \cdot (y^{-1}z^{-1}) \cdot (zy) \cdot y^{-1}x^{-1}z^{-1} \\ &= x[y, z]zx^{-1}z^{-1} = x[y, z]x^{-1}[x, z] = [y, z]^x \cdot [x, z].\end{aligned}$$

□

Lemma 7. *For two elements x, y in a group and for a positive integer j*

$$[x^j, y] = [x, y]^{x^{j-1}} \cdot [x, y]^{x^{j-2}} \cdots [x, y]^x \cdot [x, y],$$
$$[x, y^j] = [x, y] \cdot [x, y]^y \cdots [x, y]^{y^{j-2}} \cdot [x, y]^{y^{j-1}}.$$

Proof. See [7, 0.1, p.1] □

Definition 8. The (Discrete) Heisenberg group modulo n is defined as the group of matrices of the form

$$H_n := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{Z}/n\mathbb{Z} \right\}.$$

It is generated by

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that,

$$[a, b] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

According to [3], H_n admits a presentation:

$$H_n = \langle a, b \mid a^n, b^n,\ a[a, b] = [a, b]a,\ b[a, b] = [b, a]b \rangle.$$

We also have the relation $[a, b]^n = 1$. Indeed, we can derive it from the other relations as below:

$$1 = a^n b^n = b \cdot a^n [a, b]^n b^{n-1} = b[a, b]^n b^{n-1} = [a, b]^n. \tag{2}$$

2.2. **The Heisenberg curve as a cover of the projective line and the Fermat curve.** In [1] the Heisenberg curve is defined as a cover of the projective line minus three points $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with deck group H_n. The Heisenberg curve is a $\mathbb{Z}/n\mathbb{Z}$-cover of the Fermat curve, which in turn has deck group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ over the punctured projective line. We denote as F_2 the free group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, x_0)$ with generators by a and b, which are the classes of loops around 0 and 1 respectively. The whole data is depicted below in the following exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_{\mathrm{Heis}_n} & \longrightarrow & F_2 & \longrightarrow & H_n & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & R_{\mathrm{Fer}_n} & \longrightarrow & F_2 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \end{array}$$

where $R_{\mathrm{Heis}_n}, R_{\mathrm{Fer}_n}$ are the normal closures of the subgroups generated by the elements

$$\begin{aligned} R_{\mathrm{Fer}_n} &:= \langle a^n, b^n,\ [a, b] \rangle \\ R_{\mathrm{Heis}_n} &:= \langle a^n, b^n,\ a[a, b]a^{-1}[a, b]^{-1},\ b[a, b]b^{-1}[a, b]^{-1} \rangle \end{aligned}$$

and $[a, b]^n \in R_{\mathrm{Heis}_n}$. We will omit the n and write R_{Heis} whenever it is clear in the given context, although because of the following results from [1] we will have to keep track of it at various times.

Lemma 9. *The Heisenberg curve is an unramified cover of the Fermat curve if n is odd and a ramified one if n is even. Moreover, in the ramified case the points above ∞ have ramification index $2n$.*

Proof. See [1, Lemma 11] for the first statement and [1, proof of Lemma 14] for the second statement. □

Using the Riemann-Hurwitz genus formula we obtain the following

Lemma 10. *The genus g of the (closed) Heisenberg curve is*

$$g = \begin{cases} \frac{n^2(n-3)}{2} + 1 & \text{if } \gcd(n,2) = 1, \\ \frac{n^2(n-3)}{2} + \frac{n^2}{4} + 1 & \text{if } 2 \mid n. \end{cases}$$

Proof. See [1, lemma 15]. □

2.3. **Fundamental group of the Heisenberg curve.** In this section we will describe the fundamental group of the Heisenberg curve. To do this, we will make use of the Schreier lemma [4, chap. 2, sec. 8] which given a free group and a generating set of it, it can provide a generating set for a given subgroup under some conditions.

More precisely, for a free group F with basis $X = \{x_1, \dots, x_s\}$ a Schreier Transversal of a subgroup H is a set T of reduced words such that all initial segments of a word in T is also in T and every coset of H in F contains a unique word of T. We will denote this unique word by $\overline{g}$ for every g in F. The Schreier's lemma concludes that H has a freely generating set consisting of the elements $\gamma(t,x) := tx\overline{tx}^{-1}, t \in T, x \in X$, whenever tx is not in T and $\gamma(t,x)$ does not reduce to 1.

We will use the description of the fundamental group of the Fermat curve R_{Fer} as it is given in [11], and compute R_{Heis} using the Schreier's lemma. The group R_{Fer} has the description

$$R_{\text{Fer}} = \langle a_1, b_1, \dots, a_g, b_g, \gamma_1, \dots, \gamma_{3n} \ \mid \ \gamma_1\gamma_2\cdots\gamma_{3n} \cdot [a_1,b_1][a_2,b_2]\cdots[a_g,b_g] = 1 \rangle,$$

where $g = \frac{(n-1)(n-2)}{2}$ is the genus of the Fermat curve. The $3n$ elements γ_m are the elements $(a^n)^{b^i}, (b^n)^{a^i}$ and $((ab)^n)^{a^i}$ for $0 \le i \le n-1$ and the $2g$ elements remaining are $[b,a]^{a^ib^j}$ for $0 \le i \le n-2$ and $0 \le j \le n-3$.

Initially in [11, §2.1], a set of free generators for R_{Fer} is computed via the Schreier process. However, this initial set is not convenient and a change of generators takes place in the abelianization of R_{Fer}. We note that while this transformation is written in additive notation, the operations themselves do not actually require commutativity and can be applied directly to the free group R_{Fer}. Thus, we can use the following basis from [11, Lemma 2.6] as a generating set of R_{Fer}:

$$(a^n)^{b^i}, (b^n)^{a^i}, \ 0 \le i \le n-1, \ \ [a,b]^{a^ib^j}, \ 0 \le i,j, \le n-2,$$

which will be our initial input in the Schreier process.

As the Heisenberg curve is a $\mathbb{Z}/n\mathbb{Z}$-cover of the Fermat curve, $R_{\text{Fer}_n}/R_{\text{Heis}_n}$ is cyclic, generated by the commutator $T := [a,b]$. Therefore we choose $T^k, 0 \le k \le n-1$ as a Schreier transversal for $R_{\text{Fer}_n}/R_{\text{Heis}_n}$. Before computing the generators in Schreier's algorithm, the following elementary lemma will be beneficial in capturing the information about the ramification throughout this paper.

Lemma 11. *In H_n we have that $(ab)^n = T^{\frac{n}{2}} = T^{-\frac{n}{2}}$ if $2 \mid n$, otherwise it is 1. Moreover, $(ab)^n$ is in R_{Heis_n} if and only if 2 does not divide n.*

Proof. In H_n the elements a and b commute with T, thus we can compute

$$\begin{aligned}(ab)^n &= Tb\underbrace{a^2b}_{\text{swap}}(ab)^{n-2} = T^{1+2}b^2\underbrace{a^3b}_{\text{swap}}(ab)^{n-3} = \cdots \\ &= T^{1+\cdots+(n-1)}b^{n-1}a^nb \\ &= T^{\frac{n(n-1)}{2}} \\ &= \begin{cases} 0, & \text{if } \gcd(n,2)=1 \\ T^{\frac{n}{2}}, & \text{if } 2 \mid n. \end{cases}\end{aligned}$$

For the second assertion, note that $H_n \cong F_2/R_{\mathrm{Heis}_n}$. □

Now we compute the elements $\overline{tx}$ for t in the transversal set $\langle T\rangle$ and x in R_{Fer}:

$$\begin{aligned}\overline{T^k(b^n)^{a^i}} &= T^k, \\ \overline{T^k(a^n)^{b^i}} &= T^k, \\ \overline{T^k[a,b]^{a^ib^j}} &= \begin{cases} T^{k+1}, & k \le n-1, \\ 1, & k = n-1. \end{cases}\end{aligned}$$

The above computations are straightforward, since they happen in H_n where T commutes with a and b. We compute now the free generators $\gamma(t,x)$ which do not reduce to the trivial element.

$$T^k(b^n)^{a^i}\cdot\left(\overline{T^k(b^n)^{a^i}}\right)^{-1} = T^k(b^n)^{a^i}T^{-k}, \quad 0 \le i,k \le n-1$$

$$T^k(a^n)^{b^i}\cdot\left(\overline{T^k(a^n)^{b^i}}\right)^{-1} = T^k(a^n)^{b^i}T^{-k}, \quad 0 \le i,k \le n-1$$

$$T^k[a,b]^{a^ib^j}\cdot\left(\overline{T^k[a,b]^{a^ib^j}}\right)^{-1} = \begin{cases} T^{n-1}[a,b]^{a^ib^j}, & 0 \le i,j \le n-2, \\ & k = n-1 \\ T^k[a,b]^{a^ib^j}T^{-(k+1)}, & 0 \le k,i,j \le n-2, \\ & (i,j) \ne (0,0), \\ 1, & (i,j) = (0,0), 0 \le k \le n-2. \end{cases}$$

Lemma 12. *The generators of the free group R_{Heis_n} are listed below, as a union of the following sets:*

$$\begin{aligned}A_1 &= \{T^k(a^n)^{b^i}T^{-k}, \quad 0 \le i,k \le n-1\}, \quad \#A_1 = n^2, \\ A_2 &= \{T^k(b^n)^{a^i}T^{-k}, \quad 0 \le i,k \le n-1\}, \quad \#A_2 = n^2, \\ A_3 &= \{T^{n-1}T^{a^ib^j}, \quad 0 \le i,j \le n-2\}, \quad \#A_3 = (n-1)^2, \\ A_4 &= \{T^kT^{a^ib^j}T^{-(k+1)}, \quad 0 \le k,i,j \le n-2,\ (i,j) \ne (0,0)\}, \\ \#A_4 &= (n-1)^3 - (n-1).\end{aligned}$$

Proof. This is a direct consequence of the Schreier lemma. Notice that the above given sets add up to n^3+1 generators, as predicted by the Schreier index formula:

$$\begin{aligned}\operatorname{rank}(R_{\mathrm{Heis}_n}) &= [F_2 : R_{\mathrm{Heis}_n}](2-1)+1\\ &= [R_{\mathrm{Fer}_n} : R_{\mathrm{Heis}_n}](\operatorname{rank}(R_{\mathrm{Fer}_n})-1)+1\\ &= n^3+1.\end{aligned}$$

Indeed, we compute

$$\sum \#A_i = n^3+1.$$

□

This basis will have a convenient form once Galois action is introduced, although it lacks ramification data. Specifically, we would like to be able to describe generators as homotopy classes of loops on the punctured tori R_{Heis} consists of, which would mean to have conjugates of $(ab)^n$ in the basis. Lemma 11 tells us that $(ab)^n$ will be in R_{Heis_n} for odd n and we can expect $(ab)^{2n}$ to be in R_{Heis_n} for even n. We will provide this in detail in a later section.

2.3.1. *Galois action.* Suppose we have a group G with a normal subgroup N, which G acts on by conjugation. We would like to define a conjugation action of G/N on N, induced from the action of G, but this can only be well-defined modulo inner automorphisms. Considering this problem, we can have a well-defined action of G/N on the abelianization N/N'. We will use this on the exact sequence

$$1 \to R_{\mathrm{Heis}} \to F_2 \to H_n \to 1$$

to obtain a Galois action on $R^{\mathrm{ab}}_{\mathrm{Heis}}$. Consider the two generators $\alpha = aR_{\mathrm{Heis}}$, $\beta = bR_{\mathrm{Heis}}$ as well as $\tau = [a,b]R_{\mathrm{Heis}}$ of the group H_n. Then, there exists a well-defined action of these three on $R_{\mathrm{Heis}_n}/R'_{\mathrm{Heis}_n}$ given by conjugation, that is

$$x^\alpha = x^a = axa^{-1}, \quad x^\beta = x^b = bxb^{-1}, \quad x^\tau = x^T = TxT^{-1}$$

for all $x \in R_{\mathrm{Heis}_n}/R'_{\mathrm{Heis}_n}$. Notice that this is a well-defined action, which implies that

$$\begin{aligned}(x^\alpha)^\tau &= x^{\tau\alpha} = x^{\alpha\tau} = (x^\tau)^\alpha\\ \left(x^\beta\right)^\tau &= x^{\tau\beta} = x^{\beta\tau} = (x^\tau)^\beta,\end{aligned}$$

that is the actions of α and β commute with τ. In general, the actions of α and β do not commute in our setting, in contrast to the setting in [11] regarding Fermat curves. We will still use a, b, T as exponents to denote the conjugation in the free group F_2 as it was initially defined and we will use α, β, τ when the base of the exponent is in R_{Heis}, if we can realize this as an action from an element of H_n.

For example, to illustrate why this distinction is necessary, consider the commutator $[a^n, T]$ in R_{Heis}. Because a^n is also in R_{Heis}, its projection to the abelianization is well-defined, and we can write the commutator in additive module notation as $a^n - (a^n)^\tau$. In the free group F_2, however, we can also factor this same commutator multiplicatively as $T^{a^n} \cdot T^{-1}$. As $T \notin R_{\mathrm{Heis}}$, if we were to erroneously translate this second factoring into the abelianization of R_{Heis}, it would be written additively as $T^{\alpha^n} - T^{-1}$. If this latter expression could be realized as an action of H_n, then because a^n projects to the identity in H_n, the action would be trivial. The element $[a^n, T]$ would incorrectly reduce to $T - T = 0$. This highlights why the base element must be in R_{Heis} in order to apply the properties of the H_n-action.

Lemma 13. *The elements $T^n, T^{-n}[a^ib^j, T]$ are in R_{Heis}. In $R_{\mathrm{Heis}}/R'_{\mathrm{Heis}}$ they are decomposed as follows:*

$$T^n = a^n - (a^n)^\beta - \sum_{k=0}^{n-2}\sum_{i=0}^{n-2-k} [a,T]^{\tau^k\alpha^i} \tag{3}$$

$$T^{-n} = b^n - (b^n)^\alpha - \sum_{k=0}^{n-2}\sum_{j=0}^{n-2-k} [b,T^{-1}]^{\tau^{-k}\beta^j} \tag{4}$$

$$[a^ib^j,T] = [b,T]^{\alpha^i\sum_{\lambda=0}^{j-1}\beta^\lambda} + [a,T]^{\sum_{\lambda=0}^{i-1}\alpha^\lambda}. \tag{5}$$

Proof. The element T^n happens to be in A_3 for $(i,j)=(0,0)$, also we expected it to be in R_{Heis} because it is trivial in $H_n \cong F_2/R_{\mathrm{Heis}}$ from equation (2). Using Lemmas 6 and 7 we can write

$$[a^ib^j,T] = [b^j,T]^{a^i}[a^i,T] = [b,T]^{\alpha^i(\beta^{j-1}+\beta^{j-2}+\cdots+\beta+1)}\cdot[a,T]^{(\alpha^{i-1}+\alpha^{i-2}+\cdots+\alpha+1)},$$

which is in R_{Heis} and the claimed decomposition in (5) follows in the abelianization. We also compute:

$$a^n - (a^n)^\beta = [a^n,b] = T^{a^{n-1}}\cdot T^{a^{n-2}}\cdots T^a\cdot T,$$

by Lemma 6. Adding $T^{-1}T$ between each conjugate of T and $T^{-n}T^n$ at the end, the above element becomes

$$[a^{n-1},T]T\cdot[a^{n-2},T]T\cdots[a,T]T\cdot T\cdot T^{-n}T^n =$$
$$[a^{n-1},T]T\cdot[a^{n-2},T]T\cdots[a,T]\cdot T^{-(n-2)}\cdot T^n.$$

Notice that each T that is not inside the commutator in $[*,T]T$ contributes as a conjugation–by utilizing one T^{-1} from the element $T^{-(n-2)}$–to every element that comes after it, since conjugation is distributive in any group. That implies the last commutator $[a,T]$ is conjugated by T^{n-2}, then $[a^2,T]$ is conjugated by T^{n-3} and so on. The process finishes at $[a^{n-2},T]$ being conjugated by T and $[a^{n-1},T]$ with no conjugation at all.

As we will use this computation trick again, we will refer to it as *nested conjugations*.

We can now rewrite the above computation with elements in $R^{\mathrm{ab}}_{\mathrm{Heis}}$ admitting action by H_n. We have in additive notation:

$$a^n - (a^n)^\beta = [a^{n-1},T] + [a^{n-2},T]^\tau + \cdots + [a,T]^{\tau^{n-2}} + T^n,$$

and the result follows. Similarly, we can prove eq. (4). □

We rewrite now the sets A_i with the H_n action:

$$A_1 = \{(a^n)^{\tau^k\beta^i},\quad 0\le i,k\le n-1\},$$
$$A_2 = \{(b^n)^{\tau^k\alpha^i},\quad 0\le i,k\le n-1\},$$
$$A_3 = \{T^n + [a^ib^j,T]^{\tau^{n-1}},\quad 0\le i,j\le n-2\},$$
$$A_4 = \{[a^ib^j,T]^{\tau^k},\quad 0\le i,j,k\le n-2, (i,j)\ne(0,0)\}.$$

Applying an invertible transformation on A_3, we rewrite our basis as:

$$(a^n)^{\tau^k\beta^i}, (b^n)^{\tau^k\alpha^i},\ 0\le i\le n-1,\quad T^n,\quad [a^ib^j,T]^{\tau^k},\ 0\le i,j\le n-2, (i,j)\ne(0,0),$$

for $0 \leq k \leq n-1$. Using Lemma 13 we decompose in $R_{\mathrm{Heis}}/R'_{\mathrm{Heis}}$ even further:

$$\begin{aligned} T^k T^{a^i b^j} T^{-(k+1)} &= [a^i b^j, T]^{\tau^k} = \\ &= [b,T]^{\tau^k \alpha^i \sum_{\lambda=0}^{j-1} \beta^\lambda} + [a,T]^{\tau^k \sum_{\lambda=0}^{i-1} \alpha^\lambda}, \end{aligned}$$

and

$$\begin{aligned} T^{n-1} T^{a^i b^j} &= T^n T^{-1} T^{a^i b^j} T^{-1} T = T^n T^{-1} [a^i b^j, T] T \\ &= T^n + [b,T]^{\tau^{n-1} \alpha^i \sum_{\lambda=0}^{j-1} \beta^\lambda} + [a,T]^{\tau^{n-1} \sum_{\lambda=0}^{i-1} \alpha^\lambda}. \end{aligned}$$

So far we have proved the following, which will be of use when investigating the braid action on the homology and the so called Burau representation.

Proposition 14. *For all n, the free $\mathbb{Z}[H_n]$-module $R^{\mathrm{ab}}_{\mathrm{Heis}}$ is generated by the elements:*

$$a^n, b^n, [a,T], [b,T].$$

As a $\mathbb{Z}$-module, it can be generated by the n^3+1 elements:

$$\begin{aligned} (a^n)^{\beta^i, \tau^k}, (b^n)^{\alpha^i \tau^k}, &\quad 0 \leq i \leq n-1, \\ [a,T]^{\alpha^i \tau^k}, &\quad 0 \leq i \leq n-3, (i,k) = (n-2, 0), \\ [b,T]^{\alpha^i \beta^j \tau^k}, &\quad 0 \leq i \leq n-2, 0 \leq j \leq n-3, \end{aligned}$$

for $0 \leq k \leq n-1$.

Proof. This follows by counting the indices that are possible to appear from the previous decomposition of $[a^i b^j, T]$. This provides the entire new basis, except for the single element $[a,T]^{\alpha^{n-2}}$. To recover this, we look at the decomposition of T^n in equation (3). The element $[a,T]^{\alpha^{n-2}}$ is the only new generator appearing in this decomposition. Since T^n is included in our generating set, we can use it to substitue for $[a,T]^{\alpha^{n-2}}$, which completes the generating set. □

2.4. **Decomposition of $(ab)^n$.** The element $(ab)^n$ corresponds to the n-lift of the path in $\mathbb{P}^1 \setminus \{0,1,\infty\}$ around ∞, thus it is interesting to see how it decomposes in R_{Heis} in terms of our generators. From Lemma 11 this will only be possible for odd n, and for even n we will be able to decompose the elements $(ab)^n T^{-\frac{n}{2}}, T^{\frac{n}{2}} (ab)^n$ in R_{Heis_n}, from which we can form conjugates of $(ab)^{2n}$.

Lemma 15. *For all $n \in \mathbb{N}$, we can write the elements $(ab)^n$ as follows:*

$$(ab)^n = \prod_{i=1}^{n-1} \left(\prod_{j=0}^{i-1} [a^i b^{i-1-j}, T^{-1}] T^{-1} \right) \cdot a^n b^n.$$

Proof. We begin by computing the $n=2$ case:

$$\begin{aligned} (ab)^2 &= abab^{-1}a^{-1}abb = a[b,a]a^{-1}aabb \\ &= a[b,a]a^{-1}aba^{-1}b^{-1}bab^{-1} \cdot abb \\ &= aT^{-1}a^{-1}Tbab^{-1} \cdot abb \\ &= [a,T^{-1}]T^{-1}a^2b^2, \end{aligned}$$

and the $n = 3$ case:

$$\begin{aligned}(ab)^3 = (ab)^2ab &= [a, T^{-1}]T^{-1}a^2b^2ab \\ &= [a, T^{-1}]T^{-1}a^2b \cdot ba \cdot (b^{-1}a^{-1}ab) \cdot b \\ &= [a, T^{-1}]T^{-1}a^2bT^{-1} \cdot ab^2 \\ &= [a, T^{-1}]T^{-1}a^2bT^{-1} \cdot (b^{-1}a^{-2}TT^{-1}a^2b) \cdot ab^2 \\ &= [a, T^{-1}]T^{-1}[a^2b, T^{-1}]T^{-1} \cdot a^2bab^2\end{aligned}$$

and

$$a^2bab^2 = a^2ba \cdot (b^{-1}a^{-1}a^{-2}TT^{-1}a^2ab) \cdot b^2 = [a^2, T^{-1}]T^{-1}a^3b^3,$$

thus

$$(ab)^3 = [a, T^{-1}]T^{-1} \cdot [a^2b, T^{-1}]T^{-1} \cdot [a^2, T^{-1}]T^{-1}a^3b^3,$$

and with the nested conjugations trick

$$(ab)^3 = [a, T^{-1}] \cdot [a^2b, T^{-1}]^{T^{-1}} \cdot [a^2, T^{-1}]^{T^{-2}} \cdot T^{-3} \cdot a^3b^3.$$

Let

$$E(k) := \prod_{i=1}^{k-1} \left(\prod_{j=0}^{i-1} [a^ib^{i-1-j}, T^{-1}]T^{-1} \right),$$

so that

$$(ab)^k = E(k)a^kb^k.$$

We prove the following claim, for positive integers μ, λ:

$$a^\mu b^\lambda ab = \prod_{j=0}^{\lambda-1} [a^\mu b^{\lambda-1-j}, T^{-1}]T^{-1}a^{\mu+1}b^{\lambda+1}.$$

Indeed:

$$\begin{aligned}a^\mu b^\lambda ab &= a^\mu b^{\lambda-1}ba(b^{-1}a^{-1}ab)b \\ &= a^\mu b^{\lambda-1}T^{-1}ab^2 \\ &= a^\mu b^{\lambda-1}T^{-1}(b^{-(\lambda-1)}a^{-\mu}TT^{-1}a^\mu b^{\lambda-1})ab^2 \\ &= [a^\mu b^{\lambda-1}, T^{-1}]T^{-1}a^\mu b^{\lambda-1}ab^2,\end{aligned}$$

and the result follows from recursion. Thus,

$$\begin{aligned}(ab)^{k+1} = (ab)^kab &= E(k)a^kb^kab \\ &= E(k) \cdot \prod_{j=0}^{k-1} [a^kb^{k-1-j}, T^{-1}]T^{-1} \cdot a^{k+1}b^{k+1} \\ &= E(k+1)a^{k+1}b^{k+1}.\end{aligned}$$

□

We have written $(ab)^n$ into a succession of elements of the form $[*, T^{-1}]T^{-1}$, so the nested conjugations trick will be applied. Notice that $T^{n(n-1)/2}$ is $\frac{n-1}{2}T^n$ in additive notation for odd n, but we cannot write this as an element in R_{Heis} for

even n. Using the above and that $[*,T^{-1}] = -[*,T]^{\tau^{n-1}}$, we have in $R^{\mathrm{ab}}_{\mathrm{Heis}_n}$ for odd n:

$$(ab)^n = -\sum_{i=1}^{n-1}\sum_{j=0}^{i-1}[a^ib^{i-1-j},T]^{\tau^{\left(n-1-\frac{i(i-1)}{2}-j\right)}} + \frac{(n-1)}{2}T^{-n} + a^n + b^n, \tag{6}$$

and for even n we make a slight adjustment to the nested conjugations trick

$$\begin{aligned}(ab)^nT^{-\frac{n}{2}} &= \prod_{i=1}^{n-1}\left(\prod_{j=0}^{i-1}[a^ib^{i-1-j},T^{-1}]T^{-1}\right)\cdot T^{\frac{n(n-1)}{2}}\cdot T^{-\frac{n(n-1)}{2}}\cdot T^{-\frac{n}{2}}T^{\frac{n}{2}}\cdot a^nb^n\cdot T^{-\frac{n}{2}}\\ &= -\sum_{i=1}^{n-1}\sum_{j=0}^{i-1}[a^ib^{i-1-j},T]^{\tau^{\left(n-1-\frac{i(i-1)}{2}-j\right)}} + \left(\frac{n}{2}-1\right)T^{-n} + (a^n)^{\tau^{\frac{n}{2}}} + (b^n)^{\tau^{\frac{n}{2}}}.\end{aligned}$$

Using the above formulas and the decomposition of $[a^ib^j,T]$ one could, after some computations, derive a basis of $R^{\mathrm{ab}}_{\mathrm{Heis}}$ with the conjugates of $(ab)^n$ included as generators, but we will not be performing this task. Notice that, in the even case even though $(ab)^n$ is not in R_{Heis}, we expect from the ramification index over the point ∞ for $(ab)^{2n}$ to be included, which happens as

$$(ab)^nT^{-\frac{n}{2}} + ((ab)^nT^{-\frac{n}{2}})^{\tau^{\frac{n}{2}}} = (ab)^{2n}, \tag{7}$$

and this is stabilized under the action of $\tau^{\frac{n}{2}}$, which implies we have the elements $((ab)^{2n})^{\alpha^i\tau^k}$, $0 \le i \le n-1, 0 \le k \le \frac{n}{2}-1$ in R_{Heis_n} for even n.

2.5. **Elements stabilized.** In this section we will partially describe the fundamental group of the open Heisenberg curve as

$$R_{\mathrm{Heis}_n} = \langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_m \mid \gamma_1\gamma_2\cdots\gamma_m\cdot[a_1,b_1][a_2,b_2]\cdots[a_g,b_g] = 1\rangle,$$

where g is the genus of the Heisenberg curve, as in Lemma 10 and m is the number of ramified points of the cover. Combining the Schreier index formula with the above description we have that m satisfies the following equation:

$$2g + m - 1 = n^3 + 1,$$

from which we expect that m is $3n^2$ in the unramified case and $\frac{5}{2}n^2$ in the ramified case. In the unramified case, the decomposition of $(ab)^n$ in equation (6) establishes the rules for a change of generators. Although the explicit formulas are complicated, this decomposition dictates how to remove certain free generators—specifically, certain conjugates of $[a,T]$ and $[b,T]$—and replace them with the conjugates of $(ab)^n$; however, due to the single relation $\gamma_1\cdots\gamma_{3n^2}[a_1,b_1]\cdots[a_g,b_g] = 1$ in the fundamental group of the punctured surface, we only need to swap n^2-1 of them. (In the ramified case, we have to add to our generating set the $\frac{n^2}{2}-1$ conjugates of $(ab)^nT^{\frac{n}{2}}$ and then use equation (7) to swap them with the conjugates of $(ab)^{2n}$.)

While we do not explicitly execute this complicated base change here, the operational rules to do so have been established above. Carrying out this exchange theoretically realizes the standard presentation of R_{Heis}: the genus generators a_i, b_i correspond to conjugates of $[a,T]$ and $[b,T]$ that remain after the base change (using the indices from Proposition 14), while the ramification elements γ_i correspond to the newly swapped-in conjugates. These elements γ_i and their respective stabilizers are depicted in the following table:

Invariant element γ_m	Index m	Fixed by
Unramified case $(n,2)=1$:		
$(a^n)^{\tau^k\beta^i}$, $1\le k,i\le n$	$1\le m\le n^2$	$\langle\alpha\tau^i\rangle$
$(b^n)^{\alpha^i\tau^k}$, $1\le k,i\le n$	$n^2+1\le m\le 2n^2$	$\langle\beta\tau^{n-i}\rangle$
$((ab)^n)^{\alpha^i\tau^k}$, $1\le k,i\le n$	$2n^2+1\le m\le 3n^2$	$\langle\alpha\beta\tau^{n-i}\rangle$
Ramified case $(n,2)=2$:		
$(a^n)^{\tau^k\beta^i}$, $1\le k,i\le n$	$1\le m\le n^2$	$\langle\alpha\tau^i\rangle$
$(b^n)^{\alpha^i\tau^k}$, $1\le k,i\le n$	$n^2+1\le m\le 2n^2$	$\langle\beta\tau^{n-i}\rangle$
$((ab)^{2n})^{\alpha^i\tau^k}$, $\substack{1\le i\le n\\ 1\le k\le \frac{n}{2}}$	$2n^2+1\le m\le \frac{5}{2}n^2$	$\langle\alpha\beta\tau^{n-i}\rangle, \langle\tau^{\frac{n}{2}}\rangle$

Definition 16. Let Γ be the normal closure in the free group F_2 of the elements $\gamma_1,\dots\gamma_{3n^2}$ in the unramified case (or $\gamma_1,\dots,\gamma_{\frac{5}{2}n^2}$ in the ramified case). Equivalently, Γ is the normal closure in F_2 of the subgroup $R_{\text{Heis}}\cap\langle a^n,b^n,(ab)^n\rangle$ in both cases.

We denote by X_H the Heisenberg curve as the ramified cover $X_H\to\mathbb{P}^1$ and we have that the homology group of the curve is

$$H_1(X_H,\mathbb{Z})=\frac{R_{\text{Heis}}}{\Gamma\cdot R'_{\text{Heis}}}.$$

Having the $\mathbb{Z}[H_n]$-basis of $R^{\text{ab}}_{\text{Heis}}$ in mind, the natural question now is if both $[a,T]$ and $[b,T]$ generate the homology as a $\mathbb{Z}[H_n]$-module. Although the decomposition of $(ab)^n$ is very complicated and contains many non-free elements, we can extract from it enough information to answer this.

Proposition 17. *For $n=3$ the homology of the Heisenberg curve X_{H_n} is generated by $[a,T]$ as a $\mathbb{Z}[H_n]$-module. For $n\ge 4$ it is generated by $\{[a,T],[b,T]\}$ as a $\mathbb{Z}[H_n]$-module, and the two elements belong in different classes* mod Γ.

Proof. Observe in the decomposition of $(ab)^n$ from equation (6) that for $n=3$ the only non-zero power of b is 1, which means only one conjugate of $[b,T]$ appears, say $[b,T]^x$. We can apply the action of x^{-1} such that $[b,T]$ appears and by reducing mod Γ there will be linear relation between $[b,T]$ and conjugates of $[a,T]$. The exact computation is as follows. Combining the decompositions of $(ab)^3,[a^ib^j,T]$ and T^3 we are left with

$$(ab)^3=-[a,T]^{\tau^2}-[b,T]^{\alpha^2\tau}-[a,T]^{\alpha\tau}-(a^3)^\beta+b^3,$$

and applying the action of $\alpha\tau^2$ and reducing modΓ we get

$$[b,T]\equiv-[a,T]^{\alpha\tau}-[a,T]^{\alpha^2}\bmod\Gamma.$$

For $n\ge 4$, the same reasoning provides that $[b,T]$ will be linearly dependant to conjugates of $[a,T]$ and to at least one of its own conjugates modulo Γ. □

2.6. **Braid group action.** In this section we will describe the action of the braid group on $H_1(X_H, \mathbb{Z})$. The braid group B_3 in this setting will be realized as an automorphism group of F_2 in terms of the faithful Artin representation. See [12, 2.1] for a more detailed description and an application to cyclic covers of the projective line, also [11] for the action of B_3 on the closed Fermat curve. The group B_3 is generated by σ_1 and σ_2 such that

$$\sigma_1(a) = aba^{-1} \quad \sigma_1(b) = a \quad \sigma_2(a) = a \quad \sigma_2(b) = a^{-1}b^{-1}.$$

We would like to have a well-defined action of B_3 on the fundamental group of the Heisenberg curve, as we have for the Fermat curve. However, there is an obstruction.

Lemma 18. *The group R_{Heis_n} is characteristic for odd n, which means every automorphism in $\mathrm{Aut}(F_2)$ keeps R_{Heis_n} invariant.*

Proof. The group $R_{\mathrm{Heis}} \subset F_2 = \langle a, b\rangle$ is the normal closure of the elements a^n, b^n and $[a, T], [b, T]$. The automorphism group of F_2 is generated by the Nielsen transformations below, see [4, Th. 1.5, p.125]

$$\mathrm{A}ut(F_2) = \langle n_a, n_b, n_{ab}, n_{ba}\rangle$$

where

$$\begin{array}{llll} n_a(a) = a^{-1} & n_a(b) = b & n_b(a) = a & n_b(b) = b^{-1} \\ n_{ab}(a) = ab & n_{ab}(b) = b & n_{ba}(a) = a & n_{ba}(b) = ba \end{array}$$

We compute

$$\begin{aligned} n_a(a^n) &= a^{-n}, \\ n_a([a,T]) &= [a^{-1}, [a^{-1}, b]] = [a, T]^{a^{-2}T^{-1}}, \\ n_a([b,T]) &= [b, [a^{-1}, b]] = [b, [b, a]^{a^{-1}}] = [T, b]^{a^{-1}}, \\ n_{ab}(a^n) &= (ab)^n \in R_{\mathrm{Heis}_n} \text{ in the unramified case}, \\ n_{ab}([a,T]) &= [ab, [ab, b]] = [ab, [a, b]] = [b, T]^a \cdot [a, T], \\ n_{ab}([b,T]) &= [b, [ab, b]] = [b, T]. \end{aligned}$$

and the action of n_b, n_{ba} is symmetrical. □

Notice that because of the anomaly in $n_{ab}(a^n)$, R_{Heis_n} cannot be characteristic for even n. In contrast, in the unramified case the previous lemma tells us that the braid group B_3 keeps the Heisenberg curve invariant, this means it has a well-defined action on R_{Heis} which induces an action on $H_1(X_H, \mathbb{Z})$.

Remark 19. There are two things worth pointing out about the ramified case here. One is that B_3 does not keep Γ and R_{Heis} invariant. Thus an action of B_3 on $H_1(X_{H_n}, \mathbb{Z})$ for even n does not exist. The second thing is that the braid group B_3 sends the Heisenberg curve to its conjugate curves defined by the orbit of σ_2 on R_{Heis}, since $\sigma_2(b^n) = ((ab)^{-n})^b \notin R_{\mathrm{Heis}}$. This is an interesting example in the field of moduli versus field of definition perspective, as discussed in [5], [6]. In this context, a curve X defined a priori over a separable closure K_s might or might not be definable over a base field K. To determine this, one must take into account the curves arising from the Galois action on the coefficients of X and whether these new curves are isomorphic to X. The braid group B_3 mimics exactly this setting by not keeping R_{Heis} invariant.

To expand further on the above remark, suppose that we work over $\overline{\mathbb{Q}}$ for simplicity. Following the definitions in [5], we have that $\Pi_{\overline{\mathbb{Q}}}(B^*)$ is the geometric fundamental group of $B^* = \mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}$. The group $\Pi_{\overline{\mathbb{Q}}}(B^*)$ is canonically isomorphic to the profinite free group $\mathfrak{F}_2$ and fits the following split exact sequence

$$1 \to \Pi_{\overline{\mathbb{Q}}}(B^*) \to \Pi_{\mathbb{Q}}(B^*) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1,$$

with the middle term $\Pi_{\mathbb{Q}}(B^*)$ being the arithmetic fundamental group of B^*. Through Galois theory, G-covers of B^* correspond to surjective homomorphims $\Psi : \Pi_{\mathbb{Q}}(B^*) \to G$ and also the well-defined action up to inner automorphisms and sections of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\Pi_{\overline{\mathbb{Q}}}(B^*)$ induces an action on these covers.

As the braid group B_3 can act on the topological generators of $\mathfrak{F}_2$ it has a similar action on the covers. Let σ denote $\sigma_2 \in B_3$ and consider the homomorphisms

$$\Psi, \Psi^\sigma : \mathfrak{F}_2 \to H_n,$$

corresponding to the quotient maps by the profinite groups $\hat{R}_{\mathrm{Heis}_n}, \sigma(\hat{R}_{\mathrm{Heis}_n})$ respectively. We provide an interesting fact above these two covers in the following proposition.

Proposition 20. *Assume n to be even. The Heisenberg curve X_{H_n} and its σ-conjugate are not isomorphic over $\overline{\mathbb{Q}}$.*

Proof. According to [5], the curves are isomorphic if and only if Ψ, Ψ^σ are conjugate by some element in H_n. That is there is an element h in H_n such that

$$\Psi(x) = h\Psi^\sigma(x)h^{-1}, \quad \text{for all } x \in \Pi_{\overline{\mathbb{Q}}}(B^*).$$

Set x to be ab, then $\Psi(ab) = \alpha\beta$ which is of order $2n$ in H_n, however, $\Psi^\sigma(ab)$ is of order n since $(ab)^n$ is contained in $\sigma_2(R_{\mathrm{Heis}_n})$. Thus the above criterion is not satisfied and the result follows. □

We describe now the action of B_3 on the $\mathbb{Z}[H_n]$-generators of $H_1(X_{H_n}, \mathbb{Z})$, for odd n, according to Proposition 17, which will be useful for describing the Burau representation. Firstly, we have that

$$\sigma_1(T) = (T^{-1})^a, \quad \sigma_2(T) = [b^{-1}, a^{-1}] = (T^{-1})^{b^{-1}a^{-1}},$$

and with a repeated use of Lemma 7 we compute the following.

$$\begin{aligned}
\sigma_1[a, T] &= -[b, T]^{\alpha\tau^{-1}} \\
\sigma_1[b, T] &= -[a, T]^{\alpha\tau^{-1}} \\
\sigma_2[a, T] &= [b, T]^{\beta^{-1}\tau^{-1} - \alpha^{-1}\beta^{-1}\tau^{-1}} - [a, T]^{\alpha^{-1}\tau^{-1}} \\
\sigma_2[b, T] &= [b, T]^{(\alpha^{-1}\beta^{-1})^2\tau^{-1}} + [a, T]^{\alpha^{-1}\beta^{-1}\alpha^{-1}\tau^{-1}}.
\end{aligned}$$

2.6.1. *Burau Representation.* We will see that the action of B_3 on the generators of the homology of the Heisenberg curve X_{H_n}, for odd n, induces a representation:

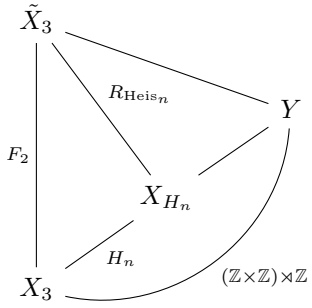
$$\rho : B_3 \longrightarrow \mathrm{GL}(2, A), \tag{8}$$

where

$$A = \mathbb{Z}\langle a^{\pm 1}, b^{\pm 1}\rangle / \langle [a, t], [b, t], t = [a, b]\rangle,$$

and $\mathbb{Z}\langle a^{\pm 1}, b^{\pm 1}\rangle$ is the noncommutative Laurent polynomial ring in two variables over $\mathbb{Z}$. For a similar construction of the Burau representation for cyclic covers of $\mathbb{P}^1$ see [12].

Let $X_3 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and denote by $\tilde{X}_3$ its universal covering space. We can obtain the Heisenberg curves X_{H_n} as quotients of an "abstract curve" Y defined as $\tilde{X}_3/I$ where I corresponds to $\langle [a,t],[b,t]\rangle$. Then X_{H_n} is the quotient Y/J, where J corresponds to $\langle a^n, b^n, (ab)^n\rangle$ as depicted in the following diagram.



The homology group $H_1(Y, \mathbb{Z})$ through similar techniques from this paper has a conjugation action by the discrete Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\} \cong (\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z} = \langle a, t\rangle \rtimes \langle b\rangle$$

and is thus an A-module. Repeating the braid action computations from the previous section we get that this action introduces the representation ρ given in eq. (8) which is given by

$$\rho(\sigma_1) = \begin{pmatrix} 0 & -at^{-1} \\ -at^{-1} & 0 \end{pmatrix},$$
$$\rho(\sigma_2) = \begin{pmatrix} -a^{-1}t^{-1} & -a^{-1}b^{-1}a^{-1}t^{-1} \\ (1-a^{-1})b^{-1}t^{-1} & (a^{-1}b^{-1})^2 t^{-1} \end{pmatrix}.$$

Similar to [12] we define

$$\mathbb{Z}_\ell[\overline{H}] = \varprojlim_n \mathbb{Z}_\ell[H_{\ell^n}].$$

Then, we have that

$$H_1(Y, \mathbb{Z}_\ell) = \widehat{F_2(3)}/\widehat{F_2(3)'},$$

where $F_2(3) = [F_2, [F_2, F_2]]$ is the third term of the lower central series of F_2 and by $\widehat{\ }$ we denote the respective pro-ℓ completions. Note that Y is not in fact an algebraic curve, but an inverse limit of Heisenberg curves. As it is of infinite index in $\widehat{F_2}$, it can not correspond to an algebraic curve that is a finite cover of $\mathbb{P}^1$ unramified outside $0, 1, \infty$. However, it still is an infinite topological cover of X_3 with fundamental group $F_2(3)$.

Proposition 21. *The group $H_1(Y, \mathbb{Z}_\ell)$ is generated by both $[a,t],[b,t]$ as a $\mathbb{Z}_\ell[\bar{H}]$-module.*

Therefore, each element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $H_1(Y, \mathbb{Z}_\ell)$, using Ihara's action given in eq. (1), on the generators of $H_1(Y, \mathbb{Z}_\ell)$ given in Proposition 21.

3. Alexander modules

Let $\mathbb{F}$ be a field of characteristic 0 containing a primitive n-th root of unity. In this section we will use the theory of Alexander modules and the Crowell exact sequence, as described in Chapter 10 from [13], to describe the homology $H_1(X_H, \mathbb{Z})$ of the closed Heisenberg curve as an $\mathbb{F}[H_n]$-module in terms of the characters of the Heisenberg group.

Let F_2 be the free group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, x_0)$ as before with generators a, b and R_{Heis} as defined previously. Let Γ be $\langle a^n, b^n, (ab)^n \rangle$. We can describe $G := F_2/R_{\text{Heis}} \cap \Gamma$ as

$$G = \langle a, b, ab \mid \ a^{e_1} = b^{e_2} = (ab)^{e_3} = a \cdot b \cdot (ab)^{-1} = 1 \rangle.$$

where $e_1 = e_2 = n$ and $e_3 = n$ or $2n$ if n is odd or even respectively. Let $\overline{R}_{\text{Heis}} := R_{\text{Heis}}/R_{\text{Heis}} \cap \Gamma \cong R_{\text{Heis}} \cdot \Gamma/\Gamma$. This reduces to R_{Heis}/Γ in the unramified case. We have a quotient map

$$\psi : F_2/\Gamma \to F_2/R_{\text{Heis}} \cong H_n.$$

Set also $\varepsilon : \mathbb{Z}[H_n] \to \mathbb{Z}$ to be the augmentation map $\sum a_g g \mapsto \sum a_g$.

We consider $\mathscr{A}_\psi$ to be the *Alexander module*, a free $\mathbb{Z}$-module

$$\mathscr{A}_\psi = \left(\bigoplus_{g \in F_2/\Gamma} \mathbb{Z}[H_n] dg \right) / \langle d(g_1 g_2) - dg_1 - \psi(g_1) dg_2 : \ g_1, g_2 \in F_2/\Gamma \rangle_{\mathbb{Z}[H_n]},$$

where $\langle \cdots \rangle_{\mathbb{Z}[H_n]}$ is considered to be a $\mathbb{Z}[H_n]$-module generated by the elements appearing inside. The elements $\{dg : g \in F_2/\Gamma\}$ are formal symbols representing differentials and the quotient ensures they satisfy the twisted derivation identity $d(g_1 g_2) = dg_1 + \psi(g_1) dg_2$, with respect to multiplication by elements of H_n.

By the above definitions, $\overline{R}^{\text{ab}}_{\text{Heis}}$ is $H_1(X_H, \mathbb{Z})$. Define the map $\theta_1 : \overline{R}^{\text{ab}}_{\text{Heis}} \to \mathscr{A}_\psi$ given by

$$\overline{R}^{\text{ab}}_{\text{Heis}} \ni n \mapsto dn$$

and the map $\theta_2 : \mathscr{A}_\psi \to \mathbb{Z}[H_n]$ to be the homomorphism induced by

$$dg \mapsto \psi(g) - 1 \quad \text{for } g \in G.$$

It is not immediate to see why θ_1 is well-defined, in fact it arises as the boundary map $H_1(\overline{R}_{\text{Heis}}, \mathbb{Z}) \to H_0(\overline{R}_{\text{Heis}}, I_{\mathbb{Z}[G]})$ in group homology by taking the $\overline{R}_{\text{Heis}}$-homology sequence of the short exact sequence

$$1 \longrightarrow I_{Z[G]} \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 1,$$

of $\mathbb{Z}[\overline{R}_{\text{Heis}}]$-modules, where $I_{\mathbb{Z}[G]}$ is the augmentation ideal. For more details see the proof in [13, Thm. 10.2.1].

We have the following exact sequence

$$1 \longrightarrow \overline{R}_{\text{Heis}} \longrightarrow G \xrightarrow{\psi} H_n \longrightarrow 1$$

from which we obtain the Crowell exact sequence of $\mathbb{Z}[H_n]$-modules [13, sec. 9.2]

$$1 \longrightarrow \overline{R}^{\text{ab}}_{\text{Heis}} = H_1(X_H, \mathbb{Z}) \xrightarrow{\theta_1} \mathscr{A}_\psi \xrightarrow{\theta_2} \mathbb{Z}[H_n] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 1. \tag{9}$$

The above exact sequence relates the homology of the projective curve X_H with the Alexander module $\mathscr{A}_\psi$; thus, we need to understand the latter. We will do this via the theory of Fox free differential calculus. We briefly review the main tool we need, the Fox derivative. For a more detailed exposition of the theory, we refer to [2, sec. 3.1], [13, Chap. 9].

Let F be a free group with free generators $x_1, \dots, x_r$ and denote by $\varepsilon_{\mathbb{Z}[F]}$ the augmentation map $\mathbb{Z}[F] \to \mathbb{Z}$. The Fox derivative $\partial/\partial x_j$, for $j = 1, \dots, r$, is a $\mathbb{Z}$-linear map $\mathbb{Z}[F] \to \mathbb{Z}[F]$ satisfying the following properties:

$$(1) \quad \frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

$$(2) \quad \frac{\partial(\alpha+\beta)}{\partial x_j} = \frac{\partial \alpha}{\partial x_j} + \frac{\partial \beta}{\partial x_j}, \quad \frac{\partial(c\alpha)}{\partial x_j} = c\frac{\partial \alpha}{\partial x_j} \quad (\alpha, \beta \in \mathbb{Z}[F], c \in \mathbb{Z}).$$

$$(3) \quad \frac{\partial(\alpha\beta)}{\partial x_j} = \frac{\partial \alpha}{\partial x_j}\varepsilon_{\mathbb{Z}[F]}(\beta) + \alpha\frac{\partial \beta}{\partial x_j} \quad (\alpha, \beta \in \mathbb{Z}[F]).$$

$$(4) \quad \frac{\partial f^{-1}}{\partial x_j} = -f^{-1}\frac{\partial f}{\partial x_j} \quad (f \in F).$$

By the next proposition, the Alexander module $\mathscr{A}_\psi$ can be understood as the cokernel of a "Jacobian" matrix. This is done by taking the Fox derivatives of all the relations of G (where ψ is defined over), over all the generators of G. Let F_3 be the free group generated by x_1, x_2, x_3. We use that there is a natural epimorphism $\pi : F_3 \to G$ mapping $x_1 \mapsto a, x_2 \mapsto b$ and $x_3 \mapsto (ab)^{-1}$.

Proposition 22. *The module $\mathscr{A}_\psi$ admits a free resolution as a $\mathbb{Z}[H_n]$-module:*

$$(10) \qquad \mathbb{Z}[H_n]^4 \xrightarrow{Q} \mathbb{Z}[H_n]^3 \longrightarrow \mathscr{A}_\psi \longrightarrow 0$$

where 4 and 3 appear as the number of relations and generators of G respectively. The map Q is expressed in form of Fox derivatives as follows, let $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z}[H_n]$. Then

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \mapsto \begin{pmatrix} \psi\pi(\frac{\partial x_1^{e_1}}{\partial x_1}) & \psi\pi(\frac{\partial x_2^{e_2}}{\partial x_1}) & \psi\pi(\frac{\partial x_3^{e_3}}{\partial x_1}) & \psi\pi(\frac{\partial x_1 \cdot x_2 \cdot x_3}{\partial x_1}) \\ \psi\pi(\frac{\partial x_1^{e_1}}{\partial x_2}) & \psi\pi(\frac{\partial x_2^{e_2}}{\partial x_2}) & \psi\pi(\frac{\partial x_3^{e_3}}{\partial x_2}) & \psi\pi(\frac{\partial x_1 \cdot x_2 \cdot x_3}{\partial x_2}) \\ \psi\pi(\frac{\partial x_1^{e_1}}{\partial x_3}) & \psi\pi(\frac{\partial x_2^{e_2}}{\partial x_3}) & \psi\pi(\frac{\partial x_3^{e_3}}{\partial x_3}) & \psi\pi(\frac{\partial x_1 \cdot x_2 \cdot x_3}{\partial x_3}) \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

where π is the natural epimorphism $F_3 \to G$ and e_1, e_2, e_3 are the orders of the three generators in G, as defined previously.

Proof. See [13, Cor. 10.1.6]. □

We apply the exact sequence (9) and the functor $\otimes_{\mathbb{Z}}\mathbb{F}$ to get the exact sequence of $\mathbb{F}[H_n]$-modules

$$(11) \quad 1 \longrightarrow H_1(X_H, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F} \xrightarrow{\theta_1 \otimes 1} \mathscr{A}_\psi \otimes_{\mathbb{Z}} \mathbb{F} \xrightarrow{\theta_2 \otimes 1} \mathbb{F}[H_n] \xrightarrow{\varepsilon} \mathbb{F} \longrightarrow 1$$

In general, it is well-known that the tensor functor is not left exact, but since $\operatorname{char}\mathbb{F} = 0$ we have that $\mathbb{F}$ is a flat $\mathbb{Z}$-module and this provides the left exactness. By counting dimensions, we have that

$$\dim_{\mathbb{F}} \mathscr{A}_\psi \otimes_{\mathbb{Z}} \mathbb{F} = 2g + n^3 - 1$$

where g is the genus as in Lemma 10. We will describe now the regular representation $\mathbb{F}[H_n]$ in terms of the irreducible characters χ_{ijs} of H_n, see appendix A for their definition and a short survey of them. Recall that every irreducible representation χ appears $\deg\chi$ times in the decomposition of $\mathbb{F}[H_n]$.

$$\mathbb{F}[H_n] = \bigoplus_{j=0}^{n-1} \bigoplus_{i,s=0}^{\gcd(n,j)-1} \frac{n}{\gcd(n,j)} \chi_{ijs}.$$

The method now is to use the free resolution given in eq. (10) in order to describe $\mathscr{A}_\psi \otimes_{\mathbb{Z}} \mathbb{F}$ as an $\mathbb{F}[H_n]$-module and then use the Crowell exact sequence to understand the homology. The $\mathbb{Z}[H_n]$-module $\mathscr{A}_\psi$ is the cokernel of the map Q. We will denote as previously α, β, τ the images of $a, b, T = [a, b]$ in H_n. We compute:

$$\frac{\partial x_i^{e_i}}{\partial x_j} = \delta_{ij}\left(1 + x_i + x_i^2 + \cdots + x_i^{e_i - 1}\right),$$

$$\frac{\partial x_1 \cdot x_2 \cdot x_3}{\partial x_1} = 1,$$

$$\frac{\partial x_1 \cdot x_2 \cdot x_3}{\partial x_2} = x_1,$$

$$\frac{\partial x_1 \cdot x_2 \cdot x_3}{\partial x_3} = x_1 x_2.$$

Set the following:

$$\Sigma_1 = 1 + \alpha + \cdots + \alpha^{n-1},$$
$$\Sigma_2 = 1 + \beta + \cdots + \beta^{n-1},$$
$$\Sigma_3 = 1 + (\alpha\beta) + \cdots + (\alpha\beta)^{n-1}.$$

In the ramified case we are interested in having the following sum instead of Σ_3

$$\begin{aligned}\Sigma_3' &= 1 + (\alpha\beta) + \cdots + (\alpha\beta)^{2n-1} = \\ &= 1 + (\alpha\beta) + \cdots (\alpha\beta)^{n-1} + \tau^{\frac{n}{2}} + \tau^{\frac{n}{2}}(\alpha\beta) + \cdots \tau^{\frac{n}{2}}(\alpha\beta)^{n-1} = \\ &= (1 + \tau^{\frac{n}{2}})\Sigma_3.\end{aligned}$$

Set Σ_3^* to be Σ_3 or Σ_3' varying based on the ramification as previously. The map Q in the exact sequence (10) is given by the matrix on the left-hand side of the following equation

$$\begin{pmatrix} \Sigma_1 & 0 & 0 & 1 \\ 0 & \Sigma_2 & 0 & \alpha \\ 0 & 0 & \Sigma_3^* & \alpha\beta \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} \Sigma_1 r_1 + r_4 \\ \Sigma_2 r_2 + \alpha r_4 \\ \Sigma_3^* r_3 + \alpha\beta r_4 \end{pmatrix}, \tag{12}$$

where $r_i \in \mathbb{Z}[H_n]$. Observe that

$$\Sigma_1\alpha = \Sigma_1,$$
$$\Sigma_2\beta = \Sigma_2,$$
$$\Sigma_3^*\alpha\beta = \Sigma_3^*,$$

in particular, in the even n case

$$\Sigma_3'\tau^{\frac{n}{2}} = \Sigma_3'. \tag{13}$$

Lemma 23. *For $i = 1, 2$ the following isomorphisms hold*

$$\Sigma_i\mathbb{Z}[H_n] \cong \Sigma_i\mathbb{Z}[\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}],$$

and

$$\Sigma_3^*\mathbb{Z}[H_n] \cong \begin{cases} \Sigma_3\mathbb{Z}[\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}], & \text{in the unramified case,} \\ \Sigma_3'\mathbb{Z}[\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/\frac{n}{2}\mathbb{Z}], & \text{in the ramified case.} \end{cases}$$

Proof. We cannot work as in the proof of Lemma 4.4 in [11], because H_n is not a direct product of groups. We will instead explicitly define the isomorphisms. For $i = 1$, consider the $\mathbb{Z}$-linear map $\phi_1 : \mathbb{Z}[\langle\beta\rangle \times \langle\tau\rangle] \to \Sigma_1\mathbb{Z}[H_n]$ given by left multiplication by Σ_1. Let $\beta^j\alpha^m\tau^k$ be an element in H_n. Then, we have:

$$\Sigma_1\beta^j\alpha^m\tau^k = \Sigma_1\alpha^m\beta^j\tau^{k-mj} = \Sigma_1\beta^j\tau^{k-mj} \in \Sigma_1\mathbb{Z}[\langle\beta\rangle \times \langle\tau\rangle],$$

which proves that ϕ_1 is surjective. To prove injectivity, suppose that $\Sigma_1 f = 0$ for some $f = \sum c_{jk}\beta^j\tau^k$. By the above equation, this yields $\sum c_{jk}\alpha^m\beta^j\tau^k = 0$, for some $m = m(j,k)$. But the elements $\alpha^m\beta^j\tau^k$ form the standard basis of $\mathbb{Z}[H_n]$ (or equivalently the elements $\beta^k\alpha^m\tau^k$ by properly adjusting the power k of τ). Thus $f = 0$.

The $i = 2$ case follows similarly, from the following equation:

$$\Sigma_2\beta^j\alpha^i\tau^k = \Sigma_2\alpha^i\tau^k \in \Sigma_1\mathbb{Z}[\langle\alpha\rangle \times \langle\tau\rangle].$$

The action of Σ_3^* is also done similarly, however, we can use both $\Sigma_3^*[\langle\beta\rangle \times \langle\tau\rangle]$ and $\Sigma_3^*[\langle\alpha\rangle \times \langle\tau\rangle]$. To provide the necessary equations, we have to form pairs of $\alpha\beta$ instead of forcing α^i to commute with β^j. We have:

$$\Sigma_3^*\alpha = \Sigma_3^*\alpha\beta^n = \Sigma_3^*\beta^{n-1},$$

and

$$\Sigma_3^*\beta = \Sigma_3^*\beta\alpha^n = \Sigma_3^*\alpha^{n-1}\tau^{-1}.$$

These generalize to

$$\Sigma_3^*\alpha^i = \Sigma_3^*\beta^{n-i}\tau^{1+2+\cdots+(i-1)},$$
$$\Sigma_3^*\beta^j = \Sigma_3^*\alpha^{n-j}\tau^{-1-2-\cdots-j}.$$

Recall at this point that τ is central in H_n. Thus, $\Sigma_3^*\mathbb{Z}[\langle\alpha\rangle \times \langle\tau\rangle] \subseteq \Sigma_3^*\mathbb{Z}[\langle\beta\rangle \times \langle\tau\rangle]$ and the opposite inlcusion holds as well. Notice also that by equation (13) only the powers $\tau^i, i = 0, \ldots, \frac{n}{2} - 1$ will survive in the ramified case. The result follows. □

We have that $\mathrm{Im}(Q)$ equals the space generated by elements

$$\begin{pmatrix} \Sigma_1 r_1 \\ \Sigma_2 r_2 \\ \Sigma_3^* r_3 \end{pmatrix} + \begin{pmatrix} 1 \\ \alpha \\ \alpha\beta \end{pmatrix} r_4.$$

For $r_1, \ldots, r_4 \in \mathbb{Z}[H_n]$ the first summand forms a free $\mathbb{Z}$-module of rank $3n^2$ (resp. $\frac{5}{2}n^2$) for the unramified (resp. ramified) case and the second summand is a free $\mathbb{Z}$-module of rank n^3. Furthermore, their intersection is a $\mathbb{Z}$-module of rank 1.

Indeed, suppose we have $r_1, \ldots, r_4 \in \mathbb{Z}[H_n]$ such that

$$(\Sigma_1 r_1, \Sigma_2 r_2, \Sigma_3^* r_3) = r_4(1, \alpha, \alpha\beta).$$

By comparing the coordinates, we get $r_4 = \Sigma_1 r_1$, which implies α acts trivially on r_4. By the second coordinates, we get $\Sigma_2 r_2 = \alpha r_4 = r_4$, which implies β acts trivially on r_4. Similarly, by the third coordinates and what we have already established, $\alpha\beta$ acts trivially on r_4. This implies that r_4 is invariant under the action of the whole group H_n, that is r_4 belongs to the rank one $\mathbb{Z}$-module generated by the element $\sum_{g\in H_n} g$. We have proved that

Lemma 24.

$$\mathrm{Im}(Q) = \left(\bigoplus_{\nu=1}^{2} \Sigma_\nu \mathbb{Z}[H_n]\right) \bigoplus \Sigma_3^* \mathbb{Z}[H_n] \bigoplus \mathbb{Z}[H_n]/J,$$

where $J = \langle \sum_{g\in H_n} g \rangle$. *Also,*

$$\mathrm{rank}_{\mathbb{Z}}\, Q = \begin{cases} n^3 + 3n^2 - 1, & \text{in the unramified case,} \\ n^3 + \frac{5}{2}n^2 - 1, & \text{in the ramified case.} \end{cases}$$

Remark 25. If Σ is an element invariant by the action of a subgroup $H < G$, then $\Sigma H = \mathrm{Ind}_H^G \mathbb{F}$, where $\mathbb{F}$ has the trivial H action. Indeed, observe that $\mathrm{Ind}_H^G \mathbb{F} = \mathbb{F} \otimes_{\mathbb{F}[H]} \mathbb{F}[G] = \Sigma\mathbb{F}[G]$.

The module $\Sigma_1\mathbb{F}[H_n]$ is isomorphic to $\Sigma_1\mathbb{F}[\langle\beta\rangle \times \langle\tau\rangle]$ by the proof of Lemma 23. Using Remark 25 we see that as a representation its character χ_{Σ_1} is the character of $\mathrm{Ind}_{\langle\alpha\rangle}^{H_n}\mathbb{F}$. Denote by p_α the trivial character of the group $\langle\alpha\rangle$. For a group G and two complex valued characters χ, χ', we denote by $\langle \chi, \chi' \rangle_G := \sum_{g\in G} \chi(g)\overline{\chi'(g)}$ their inner product in G. Using Frobenius reciprocity we compute:

$$\begin{aligned} \langle \chi_{ijs}, \chi_{\Sigma_1} \rangle_{H_n} = \langle \mathrm{Res}\, \chi_{ijs}, p_\alpha \rangle_{\langle\alpha\rangle} &= \frac{1}{n} \sum_{k=0}^{n-1} \chi_{ijs}(\alpha^k) \\ &= \frac{1}{\gcd(n,j)} \sum_{k=0}^{\gcd(n,j)-1} \zeta^{k\frac{n}{\gcd(n,j)}i} = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0, \end{cases} \end{aligned}$$

where "Res" denotes the restriction of the character in the subgroup of H_n where the inner product takes place. We have computed that

$$\Sigma_1\mathbb{F}[H_n] = \bigoplus_{j=0}^{n-1} \bigoplus_{s=0}^{\gcd(n,j)-1} \chi_{0,j,s}.$$

Notice that the above consists of the n one-dimensional subspaces ($j = 0$) and of $(n-1)\cdot\gcd(n,j)$ subspaces ($j \neq 0$) of dimension $n/\gcd(n,j)$, adding up to the correct dimension n^2.

A similar computation yields that

$$\Sigma_2\mathbb{F}[H_n] = \bigoplus_{j=0}^{n-1} \bigoplus_{i=0}^{\gcd(n,j)-1} \chi_{i,j,0}.$$

In the module $\Sigma_3^*\mathbb{F}[H_n]$, we have that $\alpha\beta$ is annihilated, therefore $\Sigma_3^*\mathbb{F}[H_n] = \mathrm{Ind}_{\langle\alpha\beta\rangle}^{H_n}\mathbb{F}$. Moreover, we can expect the one dimensional irreducible characters $\chi_{i,0,s}$ that appear in $\chi_{\Sigma_3^*}$ map $\alpha\beta = \beta\tau\alpha$ to 1, that is $i + s \equiv 0 \bmod n$. We will see how this generalizes to the higher dimensional irreducible characters using Frobenius

reciprocity. Let $p_{\alpha\beta}$ again be the trivial character of $\mathbb{F}[\langle\alpha\beta\rangle]$. We set $d_j = \gcd(n,j)$ and use the fact that $(\alpha\beta)^k = \beta^k\alpha^k\tau^{\binom{k+1}{2}}$. We compute

$$\langle\chi_{ijs},\chi_{\Sigma_3^*}\rangle_{H_n} = \langle\operatorname{Res}\chi_{ijs},p_{\alpha\beta}\rangle_{\langle\alpha\beta\rangle} = \frac{1}{|\langle\alpha\beta\rangle|}\sum_{k=0}^{|\langle\alpha\beta\rangle|-1}\chi_{ijs}((\alpha\beta)^k)$$
$$= \begin{cases} \frac{1}{d_j}\sum_{k=0}^{d_j-1}\zeta^{k\frac{n}{d_j}(i+s)+j\binom{\frac{kn}{d_j}+1}{2}}, & 2\nmid n,\\ \frac{1}{2d_j}\sum_{k=0}^{2d_j-1}\zeta^{k\frac{n}{d_j}(i+s)+j\binom{\frac{kn}{d_j}+1}{2}}, & 2\mid n.\end{cases}$$

For odd n the quantity $j\binom{\frac{kn}{d_j}+1}{2}$ is divisible by n, thus

$$\langle\chi_{ijs},\chi_{\Sigma_3}\rangle_{H_n} = \frac{1}{d_j}\sum_{k=0}^{d_j-1}\zeta^{k\frac{n}{d_j}(i+s)} = \begin{cases}1, & i+s\equiv 0 \bmod d_j,\\ 0, & i+s\not\equiv 0 \bmod d_j.\end{cases}$$

We deal with n being even now. Using that $(ab)^k = (ab)^{k'}\tau^{\frac{n}{2}}$ for $k\geq\frac{n}{2}, k\equiv k' \bmod \frac{n}{2}$, we have that

$$\langle\chi_{ijs},\chi_{\Sigma_3'}\rangle_{H_n} = \frac{1}{2d_j}(1+\zeta^{j\frac{n}{2}})\sum_{k=0}^{d_j-1}\zeta^{k\frac{n}{d_j}(i+s)+j\binom{\frac{kn}{d_j}+1}{2}},$$

which is 0 if j is not even. We thus assume j is even now, it easy to see that

$$j\binom{\frac{kn}{d_j}+1}{2} \equiv \begin{cases}0 \bmod n, & 2\nmid\frac{n}{d_j},\\ k\frac{j}{2}\frac{n}{d_j} \bmod n, & 2\mid\frac{n}{d_j},\end{cases}$$

from which we deduce

$$\langle\chi_{ijs},\chi_{\Sigma_3'}\rangle_{H_n} = \begin{cases}1, & i+s\equiv 0 \bmod d_j,\ 2\nmid\frac{n}{d_j},\\ 1, & i+s\equiv -\frac{j}{2} \bmod d_j,\ 2\mid\frac{n}{d_j},\\ 0, & \text{otherwise.}\end{cases}$$

Let j' be 0 or $-\frac{j}{2}$ according to the previous statement. We have proved that

$$\Sigma_3^*\mathbb{F}[H_n] = \begin{cases}\bigoplus_{j=0}^{n-1}\bigoplus_{\substack{i,s=0\\ i+s\equiv 0 \bmod \gcd(n,j)}}^{\gcd(n,j)-1}\chi_{ijs}, & 2\nmid n,\\ \bigoplus_{\substack{j=0\\ 2|j}}^{n-1}\bigoplus_{\substack{i,s=0\\ i+s\equiv j' \bmod \gcd(n,j)}}^{\gcd(n,j)-1}\chi_{ijs}, & 2\mid n,\end{cases}$$

adding up to the correct dimensions n^2 and $n^2/2$ respectively.

Additionally, the module $\mathbb{F}[H_n]/\langle\sum_{g\in H_n} g\rangle$ has every possible character χ_{ijs} except for the trivial one $\chi_{0,0,0}$. Indeed, it holds as $\mathbb{F}[H_n]$ is semi-simple and $\langle\sum_{g\in H_n}\rangle$ has trivial action by every element of H_n. This implies its character as a subrepresentation is the trivial one.

Counting all the characters that appear previously, we set $z_j(i,s)$ as the number of times the character χ_{ijs} appears on all $\Sigma_1\mathbb{F}[H_n], \Sigma_2\mathbb{F}[H_n]$ and $\Sigma_3^*\mathbb{F}[H_n]$, that is

in compact notation,

$$z_j(i,s) = \begin{cases} [i=0 \text{ or } s=0] + [i=s=0] + [i+s \equiv 0 \bmod \gcd(n,j)], & 2 \nmid n, \\ [i=0 \text{ or } s=0] + [i=s=0] + [i+s \equiv j' \bmod \gcd(n,j)], & 2 \mid n, j \\ [i=0 \text{ or } s=0] + [i=s=0], & 2 \mid n, 2 \nmid j, \end{cases}$$

where $[P]$ is 1 if property P holds and 0 otherwise.

Lemma 26. *We have the following decomposition:*

$$\mathrm{Im}(Q) \otimes \mathbb{F} = \bigoplus c_{ijs} \chi_{ijs},$$

where

$$c_{ijs} = \begin{cases} \frac{n}{\gcd(n,j)} + z_j(i,s), & \text{if } (i,j,s) \neq (0,0,0), \\ \frac{n}{\gcd(n,j)} - 1 + z_j(i,s) = 3, & \text{if } (i,j,s) = (0,0,0). \end{cases}$$

Lemma 27. *We have the decomposition of the Alexander module*

$$\mathscr{A}_\psi \otimes \mathbb{F} = \bigoplus_{j=0}^{n-1} \bigoplus_{i,s=0}^{\gcd(n,j)-1} a_{ijs} \chi_{ijs},$$

where

$$a_{ijs} = \left(3\frac{n}{\gcd(n,j)} - c_{ijs} \right).$$

Also,

$$\mathrm{rank}_{\mathbb{Z}} \mathscr{A}_\psi = \begin{cases} 2n^3 - 3n^2 + 1, & \text{in the unramified case}, \\ 2n^3 - \frac{5}{2}n^2 + 1, & \text{in the ramified case}, \end{cases}$$

agreeing with $2g + n^3 - 1$ that we have already counted from the Crowell exact sequence.

Proof. The Alexander module $\mathscr{A}_\psi$ is the cokernel of Q, thus

$$\begin{aligned} \mathrm{rank}_{\mathbb{Z}} \mathscr{A}_\psi &= \mathrm{rank}_{\mathbb{Z}} \mathbb{Z}[H_n]^3 - \mathrm{rank}_{\mathbb{Z}} Q \\ &= 3n^3 - \begin{cases} n^3 + 3n^2 - 1, \\ n^3 + \frac{5}{2}n^2 - 1, \end{cases} . \\ &= \begin{cases} 2n^3 - 3n^2 + 1, & \text{in the unramified case}, \\ 2n^3 - \frac{5}{2}n^2 + 1, & \text{in the ramified case}, \end{cases} . \end{aligned}$$

and the decomposition of $\mathscr{A}_\psi \otimes \mathbb{F}$ follows from the decomposition of $\mathbb{F}[H_n]$ into characters. □

Theorem 28.

$$H_1(X_H, \mathbb{F}) = \bigoplus_{j=0}^{n-1} \bigoplus_{i,s=0}^{\gcd(n,j)-1} h_{ijs} \chi_{ijs},$$

where

$$h_{ijs} = \begin{cases} \frac{n}{\gcd(n,j)} - z_j(i,s), & \text{if } (i,j,s) \neq (0,0,0), \\ 0, & \text{if } (i,j,s) = (0,0,0). \end{cases} \tag{14}$$

Proof. This follows from Lemma 27 and the Crowell exact sequence as in (11). More specifically, for every non-principal character we have that

$$h_{ijs} = a_{ijs} - \frac{n}{\gcd(n,j)} = 2\frac{n}{\gcd(n,j)} - c_{ijs} = \frac{n}{\gcd(n,j)} - z_j(i,s).$$

□

Appendix A. Irreducible Representations of H_n

In this section we will compute the irreducible representations of the Heisenberg group H_n. This is by no means a new result and it can be found for instance in the work of J. Grassberger and G. Hörmann in [8]. We rely on the general method regarding semi-direct products of groups with the normal group being abelian, as it is described by Serre in §8.2 of his book [14].

Indeed, we can realize H_n as the semi direct product $\langle \alpha, \tau \rangle \rtimes \langle \beta \rangle$ since every element of H_n can be expressed as $\beta^j \tau^k \alpha^i$, $0 \leq i, j, k \leq n-1$ with $\tau = [\alpha, \beta]$ commuting with both α and β. As H_n is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2 \rtimes \mathbb{Z}/n\mathbb{Z}$ we denote by χ_{ij} and χ_s the irreducible characters of the first and the second component respectively, such that

$$\chi_{ij}(\alpha) = \zeta^i,\ \chi_{ij}(\tau) = \zeta^j,\ \chi_s(\beta) = \zeta^s, \quad 0 \leq i, j, s \leq n-1,$$

for a fixed primitive n-th root of unity ζ.

Applying the method in Serre's book, we can conclude that the irreducible representations of H_n are those induced by the product $\chi_{ij} \otimes \chi_s$ for the indices $0 \leq i, s \leq \gcd(n,j) - 1$ and $0 \leq j \leq n-1$. We will denote them by χ_{ijs} and for a g in H_n its image is the linear combination of the values of $\chi_{ij} \otimes \chi_s$ over the elements in the intersection of the conjugacy class of g and the group $G_{ij} = \langle \alpha, \tau \rangle \cdot \langle \beta^{\frac{n}{\gcd(n,j)}} \rangle$. We have explicitly

$$\chi_{ijs}(\beta^m \tau^\lambda \alpha^\mu) = \sum_{\substack{\nu=0 \\ \tau^{\lambda+\nu\mu}\beta^m\alpha^\mu \in G_{ij}}}^{n/\gcd(n,j)-1} \chi_{ij}(\tau^{\lambda+\nu\mu}\alpha^\mu) \cdot \chi_s(\beta^m),$$

for $0 \leq j \leq n-1$ and $0 \leq i, s \leq \gcd(n,j) - 1$, with each χ_{ijs} being of dimension $\frac{n}{\gcd(n,j)}$. Set $d_j = \gcd(n,j)$. The characters χ_{ijs} take values as following:

$$\chi_{ijs}(\beta^m \tau^\lambda \alpha^\mu) = \begin{cases} \frac{n}{d_j}\zeta^{\mu i + ms + \lambda j}, & \frac{n}{d_j} \mid m, \mu, \\ 0, & \text{otherwise}, \end{cases}$$

which we use for computations with the Alexander modules.

As the number of irreducible representations is equal to the number of conjugacy classes, we verify the above by counting the conjugacy classes. The conjugacy class of $\beta^j \tau^k \alpha^i$ consists of the elements $\beta^j \tau^{k+iv-ju} \alpha^i$, or in terms of matrices

$$\begin{pmatrix} 1 & -u & uv - z \\ 0 & 1 & -v \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & i & k \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & u & z \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & i & k + iv - ju \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix}$$

As argued in [8] two elements $\beta^j \tau^k \alpha^i, \beta^{j'} \tau^{k'} \alpha^{i'}$ are in the same class if and only if $i = i', j = j'$ and $k = k' + iv - ju \bmod n$ which has a solution if and only if $k = k' \bmod \gcd(i, j, n)$. Thus, each conjugacy class contains exactly one

element $\beta^j\tau^k\alpha^i$ with $k < \gcd(n,j,i)$. These are in total $\sum_{i,j=0}^{n-1} \gcd(n,j,i)$, which is equal to the sum $\sum_{j=0}^{n-1} \gcd(n,j)^2$ we counted according to Serre, as both are equal to $\sum_{d|n} d^2\phi(\frac{n}{d})$, where ϕ is Euler's totient function.

Appendix B. Statements and Declarations

The authors have no conflict of interest to declare that are relevant to this article. Data availability: Not Applicable.

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