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The canonical ideal and syzygies of curves with automorphisms

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The canonical ideal and syzygies of curves with automorphisms - Ph.D. Thesis.

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Contents

Introduction

This thesis studies canonical ideals of flat families of curves defined over discrete valuation rings. Such families arise naturally in the context of the problem of lifting curves with automorphisms. We focus our attention on the flat family that unifies Kummer theory and Artin-Schreier theory, also known as Oort-Sekiguchi-Suwa (OSS) theory, defined over a discrete valuation ring that dominates the ring of Witt vectors. In particular, we obtain explicit generators for the relative canonical ideal of the Artin-Schreier-Kummer-Witt family and study its syzygies.

Motivation

Let $\mathscr{X}_0 \to \mathrm{Spec}(k)$ be a smooth, projective curve, defined over an algebraically closed field k of prime characteristic $p > 0$, and let G be a finite subgroup of its automorphism group $\mathrm{Aut}_k(\mathscr{X}_0)$. The pair $(\mathscr{X}_0, G)$ is said to *lift to characteristic* 0 if there exist a characteristic 0 discrete valuation ring R with residue field k and a smooth, flat family of curves $\mathscr{X} \to \mathrm{Spec}(R)$ whose special fiber is G-equivariantly isomorphic to $\mathscr{X}_0 \to \mathrm{Spec}(k)$.

By a result of Grothendieck [28, XIII, Corollaire 2.12], if p does not divide the order of G, then the curve $\mathscr{X}_0$ with the action of G can always be lifted to characteristic 0; this is known as the *tame ramification case*. In particular, by taking G to be trivial, Grothendieck's result implies that lifting a curve to characteristic 0 without a group action is always possible.

The case in which the characteristic of k does divide the order of G, known as *wild ramification*, proved to be more challenging and remains open to this day. In [51], Frans Oort conjectured that if G is cyclic then the curve can always be lifted to characteristic 0 with its G-action. Following a series of results and developments in the field, see for example [59],[8], [26], Obus-Wewers in [49] and Pop in [53] proved Oort's conjecture and thus paved the road to two big research questions in the field:

1. For which groups G can the pair $(\mathscr{X}_0, G)$ be lifted to characteristic 0?

2. Which characteristic p open problems can be solved by lifting to characteristic 0?

A series of lifting obstructions, such as the Bertin obstruction of [7], the KGB obstruction of [19] and the Hurwitz tree obstruction of [14], reduced the candidates for the group G to the following shortlist: G must be either cyclic, dihedral of order $2p^n$ or the alternating group A_4. As mentioned above, the cyclic case is due to Obus-Wewers and Pop, while A_4 is due to Obus [47] (independently known to Bouw and Pop). This leaves the case for dihedral groups; Bouw-Wewers [13] solved the problem for dihedral groups of order $2p$ for p odd, while the case for $p = 2$ is due to Pagot [45]. D_4 is due to Weaver [74], D_9 is due to Obus [48], while D_{25} and D_{27} are due to our joint work with H. Dang, S. Das, A. Obus and V. Thatte [20]. These are the only dihedral groups actions known to satisfy the lifting problem; we believe that this thesis sheds some light on the general case by providing a new obstruction.

Regarding the second question, we will focus our attention on open problems related to $\Omega_{\mathscr{X}_0/k}$, the sheaf of holomorphic differentials on $\mathscr{X}_0$ and its n-th tensor power $\Omega_{\mathscr{X}_0/k}^{\otimes n}$, the sheaf of holomorphic n-polydifferentials. Using a classical theorem of M. Noether, F. Enriques and K. Petri we identify the direct sum of global sections

$$S_{\mathscr{X}_0} = \bigoplus_{n=1}^{\infty} H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/k}^{\otimes n})$$

with the curve's homogeneous coordinate ring in its canonical embedding, called the canonical ring. Here are a few open problems related to $S_{\mathscr{X}_0}$:

- The action of G on $\mathscr{X}_0$ can be naturally extended to the global sections $H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/k}^{\otimes n})$ and a classic problem, posed by Hecke in [33] and remaining still open, is to describe their $k[G]$-module structure.

- The problem of finding the dimension of the tangent space to the deformation functor of curves with automorphisms remains still open; Kontogeorgis proved in [39] that the coinvariants $H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/k}^{\otimes n})_G$ of the action can be used to facilitate the computation.

- Mark Green conjectured in [27] that the Betti numbers of $S_{\mathscr{X}_0}$ are related to the curve's Clifford index - a birational invariant which is a refinement of the gonality.

The above problems remain open in their full generality, even though there exist some partial results; see for example [69], [41], [55], [36]. However, the respective questions on characteristic 0 curves have been answered: the $k[G]$-module structure problem is due to Chevalley, Weil and Hecke [14] while Green's conjecture has been shown to hold for characteristic 0 general curves by Voisin [71], [70].

Our proposed strategy to tackle these problems in characteristic p is as follows: assuming that the pair $(\mathscr{X}_0, G)$ does lift to characteristic 0, let $\mathscr{X}_\eta \to \mathrm{Spec}\,(\mathrm{Quot}(R))$ be the generic fiber of its lift to characteristic 0. We may then consider the characteristic 0 canonical ring

$$S_{\mathscr{X}_\eta} = \bigoplus_{n=1}^{\infty} H^0(\mathscr{X}_\eta, \Omega_{\mathscr{X}_\eta}^{\otimes n})$$

for which the above questions have concrete answers, and combine them with reduction techniques in order to address the respective questions for $S_{\mathscr{X}_0}$. We would like to stress that this thesis does not give explicit answers to the above problems, but rather lays the foundations and develops techniques that will allow us to do so in future work.

We demonstrate our methods explicitly by applying them to the *Artin-Schreier-Kummer-Witt* family: let $\mathscr{X}_0 \to \mathrm{Spec}(k)$ be an Artin-Schreier curve, i.e. a degree p cyclic Galois cover of the projective line over the algebraically closed field $k = \overline{\mathbb{F}}_p$. Oort-Sekiguchi-Suwa in [59] used the language of group schemes to show that the pair $(\mathscr{X}_0, \mathbb{Z}/p\mathbb{Z})$ always lifts to characteristic 0; the respective relative curve is defined over $R = W(k)[\zeta_p]$, where $W(k)$ is the ring of Witt vectors over k, ζ_p is a p-th root of unity, and the family's generic fiber $\mathscr{X}_\eta \to \mathrm{Spec}\,(\mathrm{Quot}\,(R))$ is a Kummer curve.

In 2000, Bertin-Mézard [8] came up with an explicit model for the relative curve $\mathscr{X} \to \mathrm{Spec}(R)$ and were able to generalize the results of Oort-Sekiguchi-Suwa by considering equicharacteristic deformations of the Artin-Schreier curve $\mathscr{X}_0 \to \mathrm{Spec}(k)$. Using this model, Karanikolopoulos-Kontogeorgis [37] solved the Galois module structure problem for the relative 1-differentials $M_1 = H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})$ and obtained as a byproduct an explicit basis for the free R-module M_1 based on H. Boseck's work [12] on differentials of degree p cyclic covers of the projective line. Our study of the canonical ideal of such covers and its syzygies relies heavily on the work of Bertin-Mézard and Karanikolopoulos-Kontogeorgis.

This thesis is organized as follows: in Chapter 1 we study the canonical embedding of an arbitrary curve defined over a field of any characteristic and the canonical embedding of an arbitrary flat family of curves defined over a discrete valuation ring of mixed characteristic. In Chapter 2 we study minimal graded free resolutions, syzygies and Betti numbers of graded $R[w_1, \ldots, w_g]$-modules where R is a discrete valuation ring. In Chapter 3, we give a detailed, explicit construction of the generating set for the canonical ideal of the Artin-Schreier-Kummer-Witt flat family of curves. Finally, in Appendix A we recall some facts on Artin-Schreier and Kummer curves, and give some details both on the construction of the ring of Witt vectors and on the Artin-Schreier-Kummer-Witt flat family of curves.

Chapter-by-chapter summary and walkthrough

Chapter 1

Section 1.1 starts with the classical theorem published in 1927 by K. Petri bearing his name. Petri's Theorem combines his work with that of M. Noether, F. Enriques and C. Babbage: let X be a smooth, projective, non-hyperelliptic curve of genus $g \geq 4$ defined over an algebraically closed field F of arbitrary characteristic. If Ω_X denotes the sheaf of holomorphic differentials on X, the canonical map

$$\phi : F[w_1, \ldots, w_g] \to \bigoplus_{n \geq 0} H^0(X, \Omega_X^{\otimes n})$$

is the F-algebra homomorphism that assigns each variable w_i to a basis element $f_i dx$ of the global sections $H^0(X, \Omega_X)$.

Max Noether proved in 1880 that the canonical map is surjective and thus the curve's homogeneous coordinate ring in its canonical embedding is given by

$$F[w_1, \ldots, w_g]/I_X \cong \bigoplus_{n \geq 0} H^0(X, \Omega_X^{\otimes n})$$

where $I_X = \ker \phi$ is called the *canonical ideal.* Federigo Enriques showed in 1919 that the canonical ideal is generated in degree 2, unless the curve is trigonal or a degree 5 plane quintic. In 1927, Karl Petri found an explicit generating set in degree 2, while in 1939 Charles Babbage proved that in the above mentioned exceptional cases, the canonical ideal is generated in degree 2 and 3. The standard reference for a complete treatment in modern language is the paper by Bernard Saint-Donat [56].

Section 1.2 contains our first main result, a combinatorial criterion for a subset of the canonical ideal I_X to be a generating set of I_X. Let $S = F[w_1, \dots, w_g]$ and let G be a subset of I_X. In Proposition 1.2.2 we show that if the number of monomials of S that do not appear as initial terms of polynomials in G is less than or equal to $3(g-1)$ then G is a generating set of I_X.

The construction of the explicit generating set that we will give depends on an assumption on the format of the basis of $H^0(X, \Omega_X)$, which is motivated by Boseck's bases for Artin-Schreier and Kummer curves. In Section 1.3 we define such bases, call them *Boseck-type bases* and use them to define a particular multigrading on the polynomial ring which will be essential in what follows.

In Section 1.4 and Section 1.5 we proceed to construct an explicit generating set for I_X using the criterion of Proposition 1.2.2. The generating set G of I_X is built on a set of binomials (Proposition 1.4.1), consisting of differences of monomials of the same multidegree. We supplement the binomials with a set of polynomials (Proposition 1.5.1) uniquely determined by the curve's defining equation as an extension of the rational function field $F(x)$. Our main tool comes from the study of an index set A that determines the basis of 1-differentials and of the Minkowski sum of A with itself.

The reader should note that we do not prove in full generality that the union of binomials in Proposition 1.4.1 and polynomials in Proposition 1.5.1 generate I_X: this requires replacing Definition 1.3.1 with an explicit basis and will be done in Chapter 3. However, since our method has been used on other types of curves (an example involving Fermat curves can be found in [40]) it felt more appropriate to collect all relevant results in Chapter 1.

Section 1.6 studies the canonical embedding of an arbitrary flat relative curve $\mathscr{X} \to \mathrm{Spec}(R)$ defined over a discrete valuation ring R of mixed characteristic. The existence of the relative canonical embedding is established in Theorem 1.6.1, while Lemma 1.6.2 is a Nakayama-type result which reduces the problem of finding a generating set for the relative canonical ideal to finding compatible generating sets for the fibers of the relative curve $\mathscr{X} \to \mathrm{Spec}(R)$.

Chapter 2

This chapter studies syzygies and Betti numbers of graded $R[w_1, \dots, w_g]$-modules; the standard setting in the bibliography [23], [52] is for R to be a field, while our treatment makes the more general assumption that the base is a discrete valuation ring. Our results are applicable to finitely generated graded modules over polynomial rings over discrete valuation rings, but the reader should keep in mind as a motivating example always the relative canonical ring introduced in Section 1.6.

We begin by fixing our notation: let R be a discrete valuation ring with residue field k and quotient field L. We write S, S_k and S_L for the polynomial rings in g variables over R, k and L respectively, and consider a finitely generated graded S-module M which is flat as an R-module. In Proposition 2.0.1 we observe that flatness over R does imply that the generic fiber $M \otimes S_L$ and the special fiber $M \otimes S_k$ have the same Hilbert polynomial, however the Betti tables need not be the same. This is made explicit in Example 2.0.2, which serves as our motivation to explain the reasons behind the different Betti tables.

In Section 2.1 we give an overview of the construction of a graded free resolution of M as an S-module. We briefly discuss syzygies, minimality of the resolution and Betti numbers, and conclude the Section with Remark 2.1.1, where we explain that to compute the Betti numbers of M one needs to tensor the minimal graded free resolution with R.

Section 2.2 addresses the problem of measuring the difference between the Betti numbers of the generic and the special fiber: if $F_\bullet$ is a minimal graded free resolution of M as an S-module, then it is known that $F_\bullet \otimes S_k$ and $F_\bullet \otimes S_L$ are graded free resolutions of the S_k-module $M \otimes S_k$ and the S_L-module $M \otimes S_L$ respectively - see Proposition 2.2.1. The former is always minimal, and thus the Betti numbers of M and $M \otimes S_k$ coincide; however, the latter need not be minimal and our main result, Theorem 2.2.2, provides a formula for computing the Betti numbers of $M \otimes S_L$ in terms of the Betti numbers of M and the number of non-trivial cyclic summands of its syzygies, regarded as R-modules. In Remark 2.2.3 we sketch an argument that could lead to a new obstruction to the lifting problem of curves with automorphisms.

In Section 2.3 we deal with the issue of explicitly computing the difference in the Betti numbers; in Corollary 2.3.2 we show that the number of cyclic summands of the syzygies is given by the rank of the Smith normal form of the matrix of differentials. We apply in detail our results to the motivating Example 2.0.2 and explain why the Betti numbers differ only if the characteristic of the base field is 2. Chapter 2 concludes with Algorithm 1 which gives the set of primes p for which the Betti numbers of a homogeneous ideal in characteristic p differ from characteristic 0.

Chapter 3

The main result of this chapter is the explicit description of a generating set for the relative canonical ideal of the Artin-Schreier-Kummer-Witt flat family of curves whose existence is discussed in Appendix A. The generators are given in Proposition 3.3.1 and Proposition 3.4.2 and the proof relies heavily on the results of Chapter 1.

The starting point is the explicit bases for 1-differentials on Artin-Schreier and Kummer curves introduced by H. Boseck in [12] and the respective relative basis introduced by Karanikolopoulos-Kontogeorgis in [37]: in Section 3.1 we study the finite index set A which uniquely determines all three bases and the respective Minkowski sum $A+A$ introduced in Chapter 1. We prove a series of intermediate results to identify in Proposition 3.1.6 a subset $C(0) \subseteq A+A$ whose complement in $A+A$ has cardinality bounded by $3(g-1)$, as dictated by the combinatorial criterion of of Proposition 1.2.2.

In Section 3.2 we match the subset $(A+A) \setminus C(0)$ with a subset of the canonical ideal $I_{\mathscr{X}_\eta}$ of the generic fiber. This subset, consisting of the binomials of Proposition 3.2.1 and the polynomials of Proposition 3.2.2 is proven to be a generating set for $I_{\mathscr{X}_\eta}$ in Theorem 3.2.3. We work similarly in Section 3.3 to obtain the generators of the canonical ideal $I_{\mathscr{X}_0}$ of the special fiber: in Theorem 3.3.3 we prove that the binomials of Proposition 3.3.1 and the polynomials of Proposition 3.3.2 generate $I_{\mathscr{X}_0}$ by setting up a correspondence of the same type with the subset $(A+A) \setminus C(0)$.

The chapter concludes with Section 3.4 where we obtain the generators for the relative canonical ideal $I_{\mathscr{X}}$. Their explicit description is given in Proposition 3.3.1 and Proposition 3.4.2 and the generating property is established in Theorem 3.4.3, the main result of this section. We give two equivalent proofs of Theorem 3.4.3; both proofs make essential use of our Nakayama-type Lemma 1.6.2.

Appendix A

In the appendix we give an overview of the construction of the flat family of curves which unifies Kummer theory with Artin-Schreier theory. In particular, we explain that given a degree p cyclic cover of the projective line over an algebraically closed field of characteristic p, there exists a flat family of curves, defined over a discrete valuation ring that dominates the ring of Witt vectors whose special fiber is $\mathbb{Z}/p\mathbb{Z}$-equivariantly isomorphic to the original curve.

In Section A.1 we recall some basic facts on Artin-Schreier curves, i.e. degree p cyclic Galois covers in characteristic p. In Theorem A.1.2 we review the classic result by E. Artin and O. Schreier which provides an explicit affine model for Artin-Scheier curves and proceed with proving some fundamental properties of such curves in Corollary A.1.3.

Section A.2 is the characteristic 0 analog of Section A.1; degree p cyclic Galois covers in characteristic 0 were studied by E. Kummer who provided the explicit model for such curves - see Theorem A.2.2. The respective fundamental properties are presented in Corollary A.2.3.

The proof that Artin-Schreier curves with the action of their automorphism group do lift to Kummer curves requires first lifting the algebraically closed field of characteristic p to characteristic 0, i.e. finding a discrete valuation ring R of characteristic 0 whose residue field is isomorphic to $\overline{\mathbb{F}}_p$. We briefly discuss aspects of this construction in Section A.3 which is due to E. Witt and generalizes Hensel's construction of the p-adic integers $\mathbb{Z}_p$. The existence of the ring of Witt vectors and its properties are presented in Theorem A.3.1.

In Section A.4 we give an overview of deformation theory of curves with automorphisms. In particular, following a short overview of Schlessinger's approach to infinitesimal deformations, we explain that Artin-Schreier curves with their automorphism groups deform to Kummer curves following the approach of Oort-Sekiguchi-Suwa [59]. We conclude this section by obtaining Bertin-Mézard's model [8] for the flat family of curves whose special fiber is an Artin-Schreier curve and whose generic fiber is a Kummer curve.

Εισαγωγή

Η παρούσα διατριβή μελετά κανονικά ιδεώδη flat οικογενειών καμπυλών οι οποίες είναι ορισμένες υπεράνω δακτυλίων διακριτής εκτίμησης. Οι οικογένειες αυτές παίζουν κεντρικό ρόλο στο πρόβλημα της ανύψωσης καμπυλών με αυτομορφισμούς. Επικεντρωνόμαστε στην flat οικογένεια που ενοποιεί τις θεωρίες των Kummer και Artin-Schreier, επίσης γνωστή ως Oort-Sekiguchi-Suwa (OSS) θεωρία, ορισμένη υπεράνω ενός δακτυλίου διακριτής εκτίμησης που κυριαρχεί πάνω στον δακτύλιο των διανυσμάτων του Witt. Συγκεκριμένα, βρίσκουμε γεννήτορες για το σχετικό κανονικό ιδεώδες της Artin-Schreier-Kummer-Witt flat οικογένειας και μελετάμε τις συζυγίες του.

Κίνητρο

Έστω $\mathscr{X}_0 \to \mathrm{Spec}(k)$ μία ομαλή, προβολική καμπύλη, ορισμένη υπέρανω ενός αλγεβρικώς κλειστού σώματος k πρώτης χαρακτηριστικής $p > 0$, και έστω G μία πεπερασμένη υποομάδα της ομάδας αυτομορφισμών $\mathrm{Aut}_k(\mathscr{X}_0)$. Θα λέμε ότι το ζεύγος $(\mathscr{X}_0, G)$ *ανυψώνεται στη χαρακτηριστική* 0 εάν υπάρχουν ένας δακτύλιος διακριτής εκτίμησης R με σώμα υπολοίπων k και μία ομαλή, flat οικογένεια καμπυλών $\mathscr{X} \to \mathrm{Spec}(R)$ της οποίας η ειδική ίνα είναι G-ισόμορφη με την αρχική καμπύλη $\mathscr{X}_0 \to \mathrm{Spec}(k)$.

Ο Grothendieck απέδειξε [28, XIII, Corollaire 2.12] ότι εάν το p δεν διαιρεί την τάξη της G, τότε η καμπύλη $\mathscr{X}_0$ μαζί με η δράση της G μπορούν πάντοτε να ανυψωθούν στη χαρακτηριστική 0: πρόκειται για την περίπτωση της *ήμερης διακλάδωσης*. Ως πόρισμα, το αποτέλεσμα του Grothendieck δίνει πως η ανύψωση μίας καμπύλης στη χαρακτηριστική 0 χωρίς να λαμβάνεται υπόψη η δράση της ομάδας είναι πάντοτε εφικτή.

Η περίπτωση στην οποία η χαρακτηριστική του k διαιρεί την τάξη της G, γνωστή ως *άγρια διακλάδωση*, αποδείχθηκε πιο δύσκολη και παραμένει ανοιχτή ως σήμερα. Ο Frans Oort στο [51] διατύπωσε την εικασία ότι εάν η G είναι κυκλική, τότε η καμπύλη μπορεί πάντοτε να ανυψωθεί στη χαρακτηριστική 0 μαζί με τη δράση της G. Βασιζόμενοι σε μία σειρά αποτελεσμάτων, ενδεικτικά αναφέρουμε τα [59], [8], [26], οι Obus-Wewers [49] και Pop [53] απέδειξαν την εικασία του Oort και άνοιξαν το δρόμο για δύο μεγάλα ερευνητικά ερωτήματα:

1. Για ποιές ομάδες G είναι δυνατή η ανύψωση του ζεύγους $(\mathscr{X}_0, G)$ στη χαρακτηριστική 0;

2. Ποιά ανοικτά προβλήματα στη χαρακτηριστική p μπορούν να λυθούν ανυψώνοντάς τα στη χαρακτηριστική 0;

Μία σειρά εμποδίων στην ανύψωση, όπως για παράδειγμα το εμπόδιο του Bertin [7], το KGB εμπόδιο [19], ή το εμπόδιο του δένδρου του Hurwitz [14], έχουν περιορίσει τις υποψήφιες ομάδες στην ακόλουθη λίστα: η G θα πρέπει να είναι είτε κυκλική, είτε διεδρική τάξης $2p^n$ είτε η εναλάσσουσα ομάδα A_4. Όπως αναφέραμε παραπάνω, η κυκλική περίπτωση λύθηκε από τους Obus-Wewers [49] και Pop [53], και η A_4 από τον Obus [47], ενώ ήταν ανεξάρτητα γνωστή στην Bouw και στον Pop. Αναφορικά με τις διεδρικές ομάδες, οι Bouw-Wewers [13] έλυσαν το πρόβλημα για διεδρικές ομάδες τάξης $2p$ όπου ο p είναι περιττός, ενώ η περίπτωση $p = 2$ οφείλεται στον Pagot [45]. Η D_4 οφείλεται στον Weaver [74], η D_9 στον Obus [48], ενώ η D_{25} και η D_{27} οφείλονται σε κοινή μας εργασία με τον H. Dang, τον S. Das, τον A. Obus και τη V.Thatte [20]. Αυτές είναι οι μόνες διεδρικές ομάδες για τις οποίες έχει λυθεί το πρόβλημα της ανύψωσης στη χαρακτηριστική 0 και πιστεύουμε πως η παρούσα διατριβή μπορεί να ρίξει φως στη γενική περίπτωση δίνοντας ένα νέο εμπόδιο.

Αναφορικά με τη δεύτερη ερώτηση, επικεντρώνουμε την προσοχή μας σε ανοιχτά ερωτήματα που αφορούν το $\Omega_{\mathscr{X}_0/k}$, τη δέσμη των ολόμορφων διαφορικών πάνω στη $\mathscr{X}_0$ και τη n-οστή τανυστική της δύναμη $\Omega_{\mathscr{X}_0/k}^{\otimes n}$, τη δέσμη των ολόμορφων n-πολυδιαφορικών. Χρησιμοποιώντας ένα κλασσικό θεώρημα των M. Noether, F. Enriques και K. Petri, μπορούμε να ταυτίσουμε το ευθύ άθροισμα των ολικών τομών

$$S_{\mathscr{X}_0} = \bigoplus_{n=1}^{\infty} H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/k}^{\otimes n})$$

με τον ομογενή δακτύλιο συντεταγμένων της καμπύλης μέσα στην κανονική της εμβάπτιση, ο οποίος καλείται κανονικός δακτύλιος. Παραθέτουμε μερικά ανοιχτά προβλήματα που αφορούν τον $S_{\mathscr{X}_0}$:

- Η δράση της G πάνω στη $\mathscr{X}_0$ μπορεί να επεκταθεί φυσικά στις ολικές τομές $H^0(\mathscr{X}_0, \Omega^{\otimes n}_{\mathscr{X}_0/k})$ και ένα κλασσικό πρόβλημα, το οποίο διατυπώθηκε από τον Hecke στο [33] και παραμένει ανοιχτό, είναι να περιγραφεί η δομή τους ως $k[G]$-modules.

- Το πρόβλημα της εύρεσης της διάστασης του εφαπτόμενου χώρου του συναρτητή παραμόρφωσης καμπυλών με αυτομορφισμούς παραμένει ανοιχτό και ο Κοντογεώργης απέδειξε στο [39] ότι οι συν-αναλλοίωτες $H^0(\mathscr{X}_0, \Omega^{\otimes n}_{\mathscr{X}_0/k})_G$ της δράσης μπορούν να χρησιμοποιηθούν για να διευκολύνουν τους υπολογισμούς.

- Ο Mark Green διατύπωσε μία εικασία στο [27] που συνδέει τους Betti αριθμούς του $S_{\mathscr{X}_0}$ με τον δείκτη Clifford της καμπύλης - μία αμφίρρητη αναλλοίωτη η οποία αποτελεί εκλέπτυνση της έννοιας της gonality.

Τα παραπάνω προβλήματα παραμένουν ανοιχτά στη γενική περίπτωση, παρόλο που υπάρχουν μερικά αποτέλεσματα [69], [41], [55], [36]. Παρολαυτά, τα αντίστοιχα προβλήματα στη χαρακτηριστική 0 έχουν λυθεί: το πρόβλημα της $k[G]$-module δομής λύθηκε από τους Chevalley, Weil και Hecke [14], ενώ η εικασία του Green έχει αποδειχθεί σωστή για γενικές καμπύλες στη χαρακτηριστική 0 από την Voisin [71], [70].

Η στρατηγική που προτείνουμε για την επίλυση των προβλημάτων αυτών στη χαρακτηριστική 0 έχει ως εξής: υποθέτοντας ότι το ζεύγος $(\mathscr{X}_0, G)$ ανυψώνεται στη χαρακτηριστική 0, μπορούμε να θεωρήσουμε τη γενική ίνα της ανύψωσης $\mathscr{X}_\eta \to \mathrm{Spec}(\mathrm{Quot}(R))$ και τον αντίστοιχο κανονικό δακτύλιο χαρακτηριστικής 0

$$S_{\mathscr{X}_\eta} = \bigoplus_{n=1}^{\infty} H^0(\mathscr{X}_\eta, \Omega^{\otimes n}_{\mathscr{X}_\eta}).$$

Εφόσον λοιπόν για τον $S_{\mathscr{X}_\eta}$ τα παραπάνω ερωτήματα έχουν συγκεκριμένες απαντήσεις, μπορούμε να επιχειρήσουμε να συνδυάσουμε τις απαντήσεις αυτές με τεχνικές αναγωγής ώστε να απαντήσουμε τα ερωτήματα για τον δακτύλιο $S_{\mathscr{X}_0}$. Θα θέλαμε να τονίσουμε ότι η παρούσα διατριβή δε δίνει ακριβείς απαντήσεις στα παραπάνω ερωτήματα, αλλά θέτει τα θεμέλια και αναπτύσσει τεχνικές ώστε να απαντηθούν σε μελλοντικές εργασίες.

Η μεθοδολογία μας παρουσιάζεται αναλυτικά εφαρμόζοντάς την στη λεγόμενη Artin-Schreier-Kummer-Witt οικογένεια καμπυλών: έστω $\mathscr{X}_0 \to \mathrm{Spec}(k)$ μία Artin-Schreier καμπύλη, δηλαδή ένα βαθμού p, κυκλικό, Galois κάλυμμα της προβολικής ευθείας υπεράνω του αλγεβρικώς κλειστού σώματος $\overline{\mathbb{F}}_p$. Οι Oort-Sekiguchi-Suwa [59] χρησιμοποίησαν τη γλώσσα των group schemes για να δείξουν ότι το ζεύγος $(\mathscr{X}_0, \mathbb{Z}/p\mathbb{Z})$ ανυψώνεται στη χαρακτηριστική 0, με την αντίστοιχη σχετική καμπύλη να ορίζεται υπεράνω του δακτυλίου $R = W(k)[\zeta_p]$, όπου $W(k)$ είναι ο δακτύλιος των διανυσμάτων του Witt υπεράνω του k, ζ_p είναι μία p-οστή ρίζα της μονάδας και η γενική ίνα της οικογένειας είναι μία καμπύλη Kummer.

Το 2000, οι Bertin-Mézard στο [8] κατασκέυασαν ένα μοντέλο που περιγράφει τη σχετική καμπύλη $\mathscr{X} \to \mathrm{Spec}(R)$ και γενίκευσαν τα αποτελέσματα των Oort-Sekiguchi-Suwa θεωρώντας ισοχαρακτηριστικές παραμορφώσεις της καμπύλης Artin-Schreier $\mathscr{X}_0 \to \mathrm{Spec}(k)$. Χρησιμοποιώντας αυτό το μοντέλο, οι Καρανικολόπουλος - Κοντογεώργης στο [37] έλυσαν το πρόβλημα της Galois module δομής για τα σχετικά 1-διαφορικά $M_1 = H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})$ βρίσκοντας παράλληλα μία βάση για το ελεύθερο R-module M_1 βασιζόμενοι στην εργασία του H. Boseck [12]. Η μελέτη μας του κανονικού ιδεώδους και των συζυγιών του βασίζεται πάνω στις εργασίες των Bertin-Mézard και Καρανικολόπουλου-Κοντογεώργη.

Η παρούσα διατριβή είναι οργανωμένη ως εξής: στο Κεφάλαιο 1 μελετάμε την κανονική εμβάπτιση μίας τυχαίας καμπύλης ορισμένης υπεράνω ενός σώματος οποιασδήποτε χαρακτηριστικής. Επίσης μελετάμε την κανονική εμβάπτιση μίας τυχαίας flat οικογένειας καμπυλών ορισμένης υπεράνω ενός δακτυλίου διακριτής εκτίμησης μικτής χαρακτηριστικής. Στο Κεφάλαιο 2 μελετάμε ελάχιστες ελεύθερες επιλύσεις, συζυγίες και αριθμούς Betti βαθμωτών $R[w_1, \ldots, w_g]$-modules, όπου ο R είναι ένας δακτυλίος διακριτής εκτίμησης. Στο Κεφάλαιο 3 δίνουμε μία λεπτομερή κατασκευή ενός συνόλου γεννητόρων του κανονικού ιδεώδους της flat οικογένειας καμπυλών των Artin-Schreier-Kummer-Witt. Τέλος, στο Παράρτημα A υπενθυμίζουμε κάποιες ιδιότητες των καμπυλών Artin-Schreier και των καμπυλών Kummer, ενώ συζητάμε περιληπτικά την κατασκευή τόσο του δακτυλίου των διανυσμάτων του Witt όσο και της Artin-Schreier-Kummer-Witt οικογένειας καμπυλών.

Αναλυτική Περίληψη ανά Κεφάλαιο

Κεφάλαιο 1

Η Ενότητα 1.1 ξεκινά με ένα κλασσικό θεωρημα το οποίο δημοσιεύθηκε το 1927 από τον Karl Petri και φέρει το όνομα του. Το Θεωρημα του Petri συνδυάζει την εργασία του με εκείνες των M. Noether, F. Enriques και C. Babbage: έστω X μία ομαλή, προβολική καμπύλη γένους $g \geq 4$ ορισμένη υπεράνω ενός αλγεβρικώς κλειστού σώματος F οποιασδήποτε χαρακτηριστικής. Συμβολίζοντας με Ω_X τη δέσμη των ολόμορφων διαφορικών πανω στην X, η κανονική απεικόνιση

$$\phi : F[w_1, \dots, w_g] \to \bigoplus_{n \geq 0} H^0(X, \Omega_X^{\otimes n})$$

είναι ο ομομορφισμός F-αλγεβρών που στέλνει καθε μεταβλητή w_i σε ένα στοιχείο $f_i dx$ της βασης των ολικών τομών $H^0(X, \Omega_X)$.

Ο Max Noether απέδειξε το 1880 ότι η κανονική απεικόνιση είναι επιμορφισμός και αρα ο ομογενής δακτύλιος συντεταγμένων της καμπύλης μέσα στην κανονική της εμβαπτιση δίνεται από

$$F[w_1, \dots, w_g]/I_X \cong \bigoplus_{n \geq 0} H^0(X, \Omega_X^{\otimes n})$$

όπου το $I_X = \ker \phi$ ονομάζεται *το κανονικό ιδεωδες*. Ο Federigo Enriques έδειξε το 1919 ότι το κανονικό ιδεώδες παράγεται από πολυώνυμα βαθμού 2, εκτός εάν η καμπύλη είναι τριγωνική ή επίπεδη βαθμού 5. Ο Karl Petri βρήκε ένα σύνολο γεννητόρων βαθμού 2, ενω ο Charles Babbage απέδειξε ότι για τις παραπάνω εξαιρέσεις το κανονικό ιδεώδες παράγεται σε βαθμό 2 και 3. Η κλασσική αναφορα για μία πλήρη παρουσίαση σε μοντέρνα γλωσσα είναι η εργασία του Bernard Saint-Donat [56].

Η Ενότητα 1.2 περιλαμβάνει το πρώτο από τα βασικά μας αποτελέσματα, ένα συνδυαστικό κριτήριο ωστε ένα υποσύνολο του κανονικού ιδεωδους I_X να αποτελεί σύνολο γεννητόρων του I_X. Έστω $S = F[w_1, \dots, w_g]$ και G ένα υποσύνολο του I_X. Στην Πρόταση 1.2.2 δείχνουμε ότι εάν το πλήθος των μονωνύμων του S τα οποία δεν εμφανίζονται ως αρχικοί όροι πολυωνύμων στο G είναι μικρότερο ή ίσο του $3(g-1)$, τότε το G παράγει το I_X.

Η κατασκευή του συνόλου γεννητόρων που θα δώσουμε εξαρτάται από μία υπόθεση για τη μορφή της βάσης του $H^0(X, \Omega_X)$ η οποία προέρχεται από τις βάσεις του Boseck για καμπύλες Artin-Schreier και Kummer. Στην Ενότητα 1.3 ορίζουμε αυτές τις βάσεις, τις οποίες καλούμε *βάσεις τύπου Boseck* και τις χρησιμοποιούμε για να ορίσουμε μία πολυδιαβάθμιση στον πολυωνυμικό δακτύλιο η οποία θα παίξει καθοριστικό ρόλο στα επόμενα.

Στην Ενότητα 1.4 και στην Ενότητα 1.5 προχωρούμε με την κατασκευή ενός συνόλου γεννητόρων για το I_X χρησιμοποιώντας το παραπάνω κριτήριο. Το σύνολο γεννητόρων G του I_X χτίζεται πανω σε ένα σύνολο διωνύμων (Πρόταση 1.4.1), το οποίο αποτελείται από διαφορές μονωνύμων του ίδιου πολυβαθμού. Για την κατασκευή του συνόλου γεννητόρων, συμπληρώνουμε τα διώνυμα με ένα σύνολο πολυωνύμων (Πρόταση 1.5.1) τα οποία καθορίζονται πλήρως από την εξίσωση ορισμού της καμπύλης ως επέκταση του ρητού σωματος συναρτήσεων $F(x)$. Το βασικό μας εργαλείο είναι η μελέτη ενός συνόλου δεικτών A που καθορίζει τη βάση των 1-διαφορικών και του αθροίσματος Minkowski του A με τον εαυτό του.

Θα θέλαμε να τονίσουμε ότι δεν αποδεικνύουμε στη γενική περίπτωση ότι η ένωση των διωνύμων της Πρότασης 1.4.1 και των πολυωνύμων της Πρότασης 1.5.1 παράγουν το I_X: αυτό απαιτεί την αντικατάσταση του Ορισμού 1.3.1 με μία συγκεκριμένη βάση και θα γίνει στο Κεφαλαιο 3. Ωστόσο, εφόσον η μέθοδός μας έχει χρησιμοποιηθεί και σε καμπύλες άλλου τύπου (για ένα παράδειγμα που περιλαμβανει καμπύλες Fermat βλ. [40]) θεωρήσαμε καταλληλότερο να συγκεντρώσουμε όλα τα σχετικά αποτελέσματα στο Κεφάλαιο 1.

Στην Ενότητα 1.6 μελετάμε την κανονική εμβάπτιση μίας τυχαίας flat σχετικής καμπύλης $\mathscr{X} \to \mathrm{Spec}(R)$ ορισμένης υπεράνω ενός δακτυλίου διακριτής εκτίμησης R μικτής χαρακτηριστικής. Η ύπαρξη της σχετικής κανονικής εμβάπτισης αποδεικνύεται στο Θεωρημα 1.6.1, ενω το Λήμμα 1.6.2 είναι ένα αποτέλεσμα τύπου Nakayama το οποίο ανάγει την εύρεση ενός συνόλου γεννητόρων του σχετικού κανονικού ιδεώδους στην εύρεση συμβατών συνόλων γεννητόρων για τα κανονικά ιδεώδη στις ίνες της σχετικής καμπύλης $\mathscr{X} \to \mathrm{Spec}(R)$.

Κεφάλαιο 2

Στο Κεφάλαιο αυτό μελετάμε συζυγίες και αριθμούς Betti βαθμωτών $R[w_1, \ldots, w_g]$-modules. Η συνήθης υπόθεση στη βιβλιογραφία [23], [52] είναι πως το R είναι σώμα, ενώ η δική μας ανάλυση κάνει τη γενικότερη υπόθεση πως οι συντελεστές προέρχονται από έναν δακτύλιο διακριτής εκτίμησης. Τα αποτελέσματα μας αφορούν πεπερασμένα παραγόμενα βαθμωτά modules πάνω από έναν πολυωνυμικό δακτύλιο με συντελεστές σε ένα δακτύλιο διακριτής εκτίμησης και το πρωταρχικό μας παραδειγμα είναι ο σχετικός κανονικός δακτύλιος που ορίστηκε στην Ενότητα 1.6.

Το Κεφάλαιο ξεκινά εισάγοντας το συμβολισμό: έστω R ένας δακτύλιος διακριτής εκτίμησης με σώμα υπολοίπων k και σώμα κλασμάτων L. Γραφουμε S, S_k και S_L για τους πολυωνυμικούς δακτυλίους σε g μεταβλητές υπεράνω των R, k και L αντίστοιχα, και θεωρούμε ένα πεπερασμένα παραγόμενο βαθμωτό S-module M το οποίο είναι flat ως R-module. Στην Πρόταση 2.0.1 παρατηρούμε ότι παρόλο που η γενική ίνα $M \otimes S_L$ και η ειδική ίνα $M \otimes S_k$ έχουν το ίδιο Hilbert πολυώνυμο, οι πίνακες Betti δεν είναι απαραίτητα ίδιοι. Αυτό γίνεται σαφές στο Παράδειγμα 2.0.2, το οποίο και αποτέλεσε κίνητρο για να εξηγήσουμε αυτή τη συμπεριφορά.

Στην Ενότητα 2.1 δίνουμε μία περίληψη της κατασκευής της βαθμωτής ελεύθερης επίλυσης του M ως S-module. Αναφερόμαστε περιληπτικά στις συζυγίες, στους αριθμούς Betti και στο πότε μία επίλυση είναι ελάχιστη, κλείνοντας την Ενότητα με την Παρατήρηση 2.1.1, όπου εξηγούμε ότι για να υπολογιστούν οι αριθμοί Betti του M χρειάζεται να πάρουμε το τανυστικό γινόμενο της ελάχιστης ελεύθερης επίλυσης με το R.

Η Ενότητα 2.2 συζητά το πρόβλημα του υπολογισμού της διαφοράς μεταξύ των Betti αριθμών της γενικής και της ειδικής ίνας: εάν $F_\bullet$ είναι μία ελάχιστη βαθμωτή ελεύθερη επίλυση του M ως S-module, τότε είναι γνωστό ότι η $F_\bullet \otimes S_k$ και η $F_\bullet \otimes S_L$ είναι βαθμωτές ελεύθερες επιλύσεις του S_k-module $M \otimes S_k$ και του S_L-module $M \otimes S_L$ αντίστοιχα -βλ. Πρόταση 2.2.1. Η πρώτη είναι πάντοτε ελάχιστη και άρα οι αριθμοί Betti του M και του $M \otimes S_k$ ταυτίζονται. Ωστόσο, η δεύτερη δεν είναι απαραίτητα ελάχιστη και το βασικό αποτέλεσμα του Κεφαλαίου, το Θεώρημα 2.2.2, δίνει έναν τύπο για τον υπολογισμό των αριθμών Betti του $M \otimes S_L$ συναρτήσει των αριθμών Betti του M και του πλήθους των μη-τετριμμένων κυκλικών προσθεταίων των συζυγιών, ιδωμένων ως R-modules. Στην Παρατήρηση 2.2.3 παρουσιάζουμε ένα επιχείρημα που μπορεί να οδηγήσει σε ένα νέο εμπόδιο για το πρόβλημα της ανύψωσης καμπυλών με αυτομορφισμούς.

Στην Ενότητα 2.3 αντιμετωπίζουμε το ζήτημα του ακριβούς υπολογισμού της διαφοράς μεταξύ των αριθμών Betti και στο Πόρισμα 2.3.2 δείχνουμε ότι το πλήθος των κυκλικών προσθεταίων των συζυγιών δίνεται από την τάξη της κανονικής μορφής Smith του πίνακα των διαφορικών. Εφαρμόζουμε αναλυτικά τα αποτελέσματα μας στο Παράδειγμα 2.0.2 και εξηγούμε γιατί οι αριθμοί Betti διαφέρουν μόνον όταν η χαρακτηριστική του σώματος συντελεστών είναι ίση με 2. Το Κεφάλαιο 2 ολοκληρώνεται με τον Αλγόριθμο 1, ο οποίος δίνει το σύνολο των πρώτων αριθμών p για τους οποίους οι αριθμοί Betti ενός ομογενούς ιδεώδους στη χαρακτηριστική p διαφέρουν από τη χαρακτηριστική 0.

Κεφάλαιο 3

Το βασικό αποτέλεσμα αυτού του Κεφαλαίου είναι η αναλυτική περιγραφή ενός συνόλου γεννητόρων του σχετικού κανονικού ιδεώδους της flat οικογένειας καμπυλών των Artin-Schreier-Kummer-Witt, η ύπαρξη της οποίας συζητείται στο Παραρτημα A. Οι γεννήτορες δίνονται στην Πρόταση 3.3.1 και την Πρόταση 3.4.2, ενώ η απόδειξη βασίζεται στα αποτελέσματα του Κεφαλαίου 1.

Το σημείο εκκίνησης είναι οι βάσεις που έδωσε ο H. Boseck στην εργασία του [12] για τα 1-διαφορικά σε καμπύλες Artin-Schreier και Kummer και η αντίστοιχη σχετική βάση που έδωσαν οι Καρανικολόπουλος-Κοντογεώργης στην εργασία τους [37]: η Ενότητα 3.1 μελετά το πεπερασμένο σύνολο δεικτών A που καθορίζει μοναδικά και τις τρεις βάσεις, καθώς και το άθροισμα Minkowski $A+A$ που ορίσαμε στο Κεφάλαιο 1. Αποδεικνύουμε μία σειρά ενδιάμεσων αποτελεσμάτων που οδηγούν στην Πρόταση 3.1.6, όπου ορίζουμε ένα υποσύνολο $C(0) \subseteq A+A$ του οποίου το συμπλήρωμα μέσα στο $A+A$ έχει πληθικότητα φραγμένη από το $3(g-1)$, όπως απαιτείται από το συνδυαστικό κριτήριο της Πρότασης 1.2.2.

Στην Ενότητα 3.2 αντιστοιχίζουμε το υποσύνολο $(A+A) \setminus C(0)$ με ένα υποσύνολο του κανονικού ιδεώδους $I_{\mathscr{X}_\eta}$ στη γενική ίνα. Το εν λόγω υποσύνολο, το οποίο αποτελείται από τα διώνυμα της Πρότασης 3.2.1 και τα πολυώνυμα της Πρότασης 3.2.2, αποδεικνύεται πως είναι σύνολο γεννητόρων του $I_{\mathscr{X}_\eta}$ στο Θεώρημα 3.2.3. Εργαζόμαστε παρόμοια στην Ενότητα 3.3, όπου παίρνουμε τους γεννήτορες του κανονικού ιδεώδους $I_{\mathscr{X}_0}$ της ειδικής ίνας: στο Θεώρημα 3.3.3 αποδεικνύουμε ότι τα διώνυμα της Πρότασης 3.3.1 και τα πολυώνυμα της Πρότασης 3.3.2 παράγουν το $I_{\mathscr{X}_0}$, χρησιμοποιώντας μία παρόμοια αντιστοιχία με το υποσύνολο $(A+A)\setminus C(0)$.

Το Κεφάλαιο ολοκληρώνεται με την Ενότητα 3.4, όπου βρίσκουμε γεννήτορες για το σχετικό κανονικό ιδεώδες $I_{\mathscr{X}}$. Η ακριβής περιγραφή τους δίνεται στην Πρόταση 3.3.1 και την Πρόταση 3.4.2, ενω η ιδιότητα τους ως σύνολο γεννητόρων αποδεικνύεται στο Θεωρημα 3.4.3 το οποίο είναι το βασικό αποτέλεσμα αυτής της Ενότητας. Δίνουμε δύο ισοδύναμες αποδείξεις του Θεωρήματος 3.4.3, οι οποίες βασίζονται στο Λήμμα 1.6.2.

Παράρτημα A

Στο Παράρτημα δίνουμε μία περίληψη της κατασκευής της flat οικογένειας καμπυλών που ενοποιούν τη θεωρία του Kummer και τη θεωρία των Artin-Schreier. Συγκεκριμένα, εξηγούμε ότι δοσμένου ενός βαθμού p κυκλικού καλύμματος της προβολικής ευθείας υπεράνω ενός αλγεβρικώς κλειστού σώματος χαρακτηριστικής p, υπαρχει μία flat οικογένεια καμπυλών, ορισμένη υπεράνω ενός δακτυλίου διακριτής εκτίμησης που κυριαρχεί πάνω στον δακτύλιο των διανυσμάτων του Witt και η οποία έχει ειδική ίνα $\mathbb{Z}/p\mathbb{Z}$-ισόμορφη με την αρχική καμπύλη.

Στην Ενότητα A.1 υπενθυμίζουμε κάποιες βασικές ιδιότητες των καμπυλών Artin-Schreier, δηλαδή των βαθμού p κυκλικών Galois καλυμμάτων στη χαρακτηριστική p. Στο Θεώρημα A.1.2 παρουσιάζουμε το κλασσικό αποτέλεσμα των E. Artin και O. Schreier το οποίο δίνει ένα αφφινικό μοντέλο για τις καμπύλες Artin-Schreier και προχωρούμε στην απόδειξη μερικών θεμελιωδών ιδιοτήτων αυτών των καμπυλών στο Πόρισμα A.1.3.

Η Ενότητα A.2 είναι ανάλογη με την Ενότητα A.1 μόνο που τωρα δουλεύουμε στη χαρακτηριστική 0. Τα βαθμού p κυκλικά Galois καλύμματα στη χαρακτηριστική 0 μελετήθηκαν από τον E. Kummer, ο οποίος έδωσε το αφφινικό μοντέλο για αυτές τις καμπύλες -βλ. Θεώρημα A.2.2. Οι αντίστοιχες θεμελιώδεις ιδιότητες παρουσιάζονται στο Πόρισμα A.2.3.

Η απόδειξη ότι οι καμπύλες Artin-Schreier μαζί με τη δράση της ομάδας αυτομορφισμών τους ανυψώνονται σε καμπύλες Kummer απαιτεί πρώτα την ανύψωση του αλγεβρικώς κλειστού σώματος χαρακτηριστικής p στη χαρακτηριστική 0, δηλαδή την εύρεση ενός δακτυλίου διακριτής εκτίμησης R χαρακτηριστικής 0 του οποίου το σώμα υπολοίπων είναι ισόμορφο με το $\overline{\mathbb{F}}_p$. Συζητάμε περιληπτικά κάποια βήματα της κατασκευής στην Ενότητα A.3, η οποία δόθηκε από τον E. Kummer και γενικεύει την κατασκευή των p-αδικών ακεραίων $\mathbb{Z}_p$ από τον Hensel. Η ύπαρξη του δακτυλίου των διανυσμάτων του Witt και οι βασικές του ιδιότητες συνοψίζονται στο Θεωρημα A.3.1.

Η Ενότητα A.4 αφορά τη θεωρία παραμορφώσεων καμπυλών με αυτομορφισμούς. Συγκεκριμένα, μετά από μία σύντομη περίληψη της προσέγγισης του Schlessinger στις απειροστές παραμορφώσεις, εξηγούμε πώς οι καμπύλες Artin-Schreier μαζί με την ομαδα αυτομορφισμών τους παραμορφώνονται σε καμπύλες Kummer, ακολουθώντας τους Oort-Sekiguchi-Suwa [59]. Ολοκληρωνουμε την Ενότητα επαληθεύοντας το μοντέλο των Bertin-Mézard [8] για την flat οικογένεια καμπυλών της οποίας η ειδική ίνα είναι καμπύλη Artin-Schreier και η γενική ίνα είναι καμπύλη Kummer.

Chapter 1

The Canonical Ideal

The main results of this chapter can be found in [17] which has been submitted for publication.

1.1 Petri's Theorem

Throughout this chapter, X is a complete, non-singular, non-hyperelliptic curve of genus $g \geq 4$ over an algebraically closed field F of arbitrary characteristic. Let $\Omega_{X/F}$ denote the sheaf of holomorphic differentials on X and, for $n \geq 1$, let $\Omega_{X/F}^{\otimes n}$ be the n-th tensor power of $\Omega_{X/F}$; the global sections $H^0(X, \Omega_{X/F}^{\otimes n})$ form an F-vector space of dimension d_n. By the Riemann Roch Theorem [29, Theorem IV.1.3] it follows that

$$d_n = \begin{cases} g, & \text{if } n = 1 \\ (2n-1)(g-1), & \text{if } n > 1. \end{cases} \tag{1.1}$$

The multiplication

$$\begin{array}{rcl} H^0(X, \Omega_{X/F}^{\otimes n}) \times H^0(X, \Omega_{X/F}^{\otimes m}) & \to & H^0(X, \Omega_{X/F}^{\otimes (n+m)}) \\ (f_1 dx^{\otimes n}, f_2 dx^{\otimes m}) & \mapsto & f_1 f_2 dx^{\otimes (n+m)}. \end{array}$$

endows the direct sum $\bigoplus_{n \geq 0} H^0(X, \Omega_{X/F}^{\otimes n})$, called **the canonical ring**, with the structure of a graded F-algebra.

The polynomial ring $S = F[\omega_1, \ldots, \omega_g]$ is also equipped with a graded ring structure:

its n-th graded piece consists of homogeneous polynomials of degree n, i.e.

$$S = \bigoplus_{n\geq 0} S_n \text{ where } S_n = \{f \in S : f \text{ is homogeneous with } \deg f = n\}. \tag{1.2}$$

We note that for $n \geq 1$, S_n is an F-vector space of dimension $\binom{g-1+n}{n}$, generated by

$$\{\omega_1^{a_1} \cdots \omega_g^{a_g} \in S : a_1 + \ldots + a_g = n\}.$$

Let $\{f_1 dx, \ldots, f_g dx\}$ be a basis for $H^0(X, \Omega_{X/F})$. For $n \geq 0$, we define the F-linear maps

$$\phi_n : S_n \longrightarrow H^0(X, \Omega_{X/F}^{\otimes n}),\ \omega_1^{a_1} \cdots \omega_g^{a_g} \mapsto f_1^{a_1} \cdots f_g^{a_g} dx^{\otimes(a_1+\cdots+a_g)},$$

which give rise to a graded F-algebra homomomorphism

$$\phi = \bigoplus_{n\geq 0} \phi_n : F[\omega_1, \ldots, \omega_g] \longrightarrow \bigoplus_{n\geq 0} H^0(X, \Omega_{X/F}^{\otimes n})$$

called **the canonical map**. The kernel of ϕ, called **the canonical ideal** and denoted by I_X, is a graded ideal with

$$I_X = \bigoplus_{n\geq 0} (I_X)_n \text{ where } (I_X)_n = \{f \in I_X : f \text{ is homogeneous with } \deg f = n\}. \tag{1.3}$$

The main tool for studying the canonical ideal is the following theorem:

Theorem 1.1.1 (M. Noether 1880, F. Enriques 1919, K. Petri 1927, C. Babbage 1939)**.** *Let X be a complete, non-singular, non-hyperelliptic curve of genus $g \geq 4$ over an algebraically closed field F of arbitrary characteristic and let*

$$\phi : F[\omega_1, \ldots, \omega_g] \longrightarrow \bigoplus_{n\geq 0} H^0(X, \Omega_{X/F}^{\otimes n})$$

denote the canonical map. Then:

1. *ϕ is surjective.*

2. *The kernel I_X of ϕ is generated by elements of degree 2 and 3.*

3. *I_X is generated by elements of degree 2 except in the following cases:*

(a) X is a non-singular plane quintic (in this case $g = 6$).

(b) X is trigonal, i.e. a triple covering of $\mathbb{P}^1_F$.

For a a complete proof and a modern treatment over a field of arbitrary characteristic we refer to the article of B. Saint-Donat [56].

1.2 A combinatorial criterion for generators

Let $S = F[\omega_1, \ldots, \omega_g]$ be the polynomial ring over F in g indeterminates. We write $\mathbb{T}$ for the set of monomials of S and $\mathbb{T}^n$ for the set of monomials of total degree n, i.e.

$$\mathbb{T}^n = \{\omega_1^{a_1} \cdots \omega_g^{a_g} \in S : a_1 + \ldots + a_g = n\}.$$

We remark that there is a one-to-one correspondence between the monomials $\omega_1^{a_1} \cdots \omega_g^{a_g} \in \mathbb{T}$ and the lattice points $\mathbf{a} = (a_1, \ldots, a_g) \in \mathbb{N}^g$. A **term order** on S is a total order on $\mathbb{N}^g$ with the following two properties:

- The zero vector $(0, \ldots, 0)$ is the unique minimal element.
- For all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{N}^g$ we have that $\mathbf{a} \prec \mathbf{b}$ implies $\mathbf{a} + \mathbf{c} \prec \mathbf{b} + \mathbf{c}$.

For more details and examples see [67, Chapter 1]. Let I be a graded ideal of S and let $\prec$ be a term order on S. We note that each $f \in S$ has a unique leading term with respect to $\prec$, denoted by $\mathrm{in}_\prec(f)$. The initial ideal of I with respect to $\prec$ is defined as $\mathrm{in}_\prec(I) = \langle \mathrm{in}_\prec(f) : f \in I \rangle$; it is a monomial ideal with

$$\mathrm{in}_\prec(I) = \bigoplus_{n \geq 0} \mathrm{in}_\prec(I)_n \text{ where } \mathrm{in}_\prec(I)_n = \{\mathrm{in}_\prec(f) : f \in (I)_n\}. \tag{1.4}$$

The grading of the polynomial ring and its ideals is naturally inherited to quotients, i.e.

$$S/I = \bigoplus_{n \geq 0} (S/I)_n \text{ and } S/\mathrm{in}_\prec(I) = \bigoplus_{n \geq 0} (S/\mathrm{in}_\prec(I))_n$$

and since quotients commute with direct sums, we have that

$$(S/I)_n \cong S_n/(I)_n \text{ and } (S/\mathrm{in}_\prec(I))_n \cong S_n/\mathrm{in}_\prec(I)_n. \tag{1.5}$$

The following proposition gives the dimension of $(I_X)_n$ in terms of the dimension of the global sections $H^0(X, \Omega_X^{\otimes n})$.

Proposition 1.2.1. *Let X be a complete, non-singular, non-hyperelliptic curve of genus $g \geq 4$ over F, let $S = F[\omega_1, \ldots, \omega_g]$ and let $\prec$ be a term order on S. The following holds for all $n \in \mathbb{N}$:*

$$\dim_F (I_X)_n = \dim_F (\mathrm{in}_\prec(I_X))_n \quad = \quad \binom{g-1+n}{n} - d_n.$$

Proof. By Macaulay's Theorem [22, Theorem 15.3] we have $\dim_F (I_X)_n = \dim_F (\mathrm{in}_\prec(I_X))_n$. Petri's Theorem 1.1.1 implies that $\dim_F (S/I_X)_n = \dim_F H^0(X, \Omega_X) = d_n$ and thus eq. (1.5) gives $\dim_F (I_X)_n = \dim_F S_n - d_n$. For the dimension of S_n as an F-vector space, it suffices to count the cardinality of $\mathbb{T}^n$, i.e. the number of monomials of degree n in g variables. The proof is a simple application of the so-called *stars and bars trick* from combinatorics, see [65, page 26]. □

We proceed with a combinatorial criterion for a subset of the canonical ideal to be a generating set:

Proposition 1.2.2. *Let $G \subseteq I_X$ be a set of homogeneous polynomials of degree 2 in I_X. Write $\mathbb{T}^2$ for the set of monomials of degree 2 in S and $\mathrm{in}_\prec(G)$ for the set of initial terms of elements in G. If*

$$|\mathbb{T}^2 \setminus \mathrm{in}_\prec(G)| \leq 3(g-1),$$

then $I_X = \langle G \rangle$.

Proof. We first note that by [67, Prop. 1.1]

$$\dim_F (S/\mathrm{in}_\prec(I_X))_2 = \dim_F (S/I_X)_2 . \tag{1.6}$$

By Proposition 1.2.1 we have that

$$\dim_F (S/\mathrm{in}_\prec(I_X))_2 = d_2 = 3(g-1).$$

Further, since $G \subseteq I_X$, $\langle \mathrm{in}_\prec(G) \rangle_2$ is an F-subspace of $\mathrm{in}_\prec(I_X)_2$. Therefore

$$\dim_F (S_2/\mathrm{in}_\prec(I_X)_2) \leq \dim_F (S_2/\langle \mathrm{in}_\prec(G) \rangle_2)$$

which, by eq. (1.5), is equivalent to

$$\dim_F (S/\mathrm{in}_\prec(I_X))_2 \leq \dim_F (S/\langle \mathrm{in}_\prec(G)\rangle)_2$$

so that

$$\dim_F (S/\langle \mathrm{in}_\prec(G)\rangle)_2 \geq 3(g-1). \tag{1.7}$$

Now $\langle \mathrm{in}_\prec(G)\rangle$ is a monomial ideal and thus

$$|\mathbb{T}^2 \setminus \mathrm{in}_\prec(G)| = \dim_F (S/\langle \mathrm{in}_\prec(G)\rangle)_2 . \tag{1.8}$$

Combining the assumption

$$|\mathbb{T}^2 \setminus \mathrm{in}_\prec(G)| \leq 3(g-1)$$

with eq. (1.7) and eq. (1.8) gives

$$\dim_F (S/\langle \mathrm{in}_\prec(G)\rangle)_2 = 3(g-1).$$

Finally, as in eq. (1.6), we have

$$\dim_F (S/\langle \mathrm{in}_\prec(G)\rangle)_2 = \dim_F (S/\langle G\ \rangle)_2 .$$

and we conclude that

$$\dim_F \big(S/I_X\big)_2 = \dim_F \big(S/\langle G\rangle\big)_2 \quad \Rightarrow \quad (I_X)_2 = \langle G\rangle_2 \Rightarrow I_X = \langle (I_X)_2\rangle = \langle G\rangle$$

completing the proof. □

1.3 The multigrading of the polynomial ring

Let $A \subseteq \mathbb{Z}^2$ be a finite index set of cardinality g with elements denoted by (N,μ). We take $S = F[\{z_{N,\mu} : (N,\mu) \in A\}]$ and assign to each variable $z_{N,\mu}$ the multidegree $\mathrm{mdeg}(z_{N,\mu}) = (1, N, \mu)$ and the multidegree $(0,0,0)$ to all elements of F. This gives a multigrading on S via

$$\mathrm{mdeg}(z_{N_1,\mu_1} z_{N_2,\mu_2} \cdots z_{N_d,\mu_d}) = (d, N_1 + N_2 + \cdots N_d, \mu_1 + \mu_2 + \cdots + \mu_d). \tag{1.9}$$

Note that d is the *standard degree* of the monomial $z_{N_1,\mu_1} z_{N_2,\mu_2} \cdots z_{N_d,\mu_d}$. We remark that the multidegree of an arbitrary monomial $z_{N,\mu} z_{N',\mu'} \in \mathbb{T}^2$ of standard degree 2 can be written as

$$\mathrm{mdeg}\left(z_{N,\mu} z_{N',\mu'}\right) = (2, N+N', \mu+\mu') = (1, N, \mu) + (1, N', \mu')$$

so we consider the Minkowski sum of the index set A with itself, defined as

$$A + A = \{(N+N', \mu+\mu') \ : \ (N,\mu), (N',\mu') \in A\} \subseteq \mathbb{Z}^2. \qquad (1.10)$$

and observe that

$$z_{N,\mu} z_{N',\mu'} \in \mathbb{T}^2 \Leftrightarrow \mathrm{mdeg}\left(z_{N,\mu} z_{N',\mu'}\right) \in A + A. \qquad (1.11)$$

The explicit description of a generating set G for I_X requires the choice of an explicit basis of the g-dimensional F-vector space $H^0(X, \Omega_{X/F})$. Let $g(x,y) = 0$ be an affine model of the curve X over F, so that the curve's function field $F(X)$ is given by the extension $F(x)(g)$ over $F(x)$. Since $g(x,y) \in F(x,y)$ we may divide it by an appropriate monomial and assume without loss of generality that

$$g(x,y) = 1 + \sum_{(i,j) \in D} b_{ij} x^i y^j, \text{ for some } D \subseteq \mathbb{Z} \times \mathbb{Z}_- \text{ and some } b_{ij} \in F. \qquad (1.12)$$

Definition 1.3.1. *Let D be as above and let $A \subset \mathbb{Z}^2$ be a finite index set of cardinality g with elements denoted by (N,μ). An F-basis $\mathbf{b}_A = \{f_{N,\mu} dx : (N,\mu) \in A, \ f_{N,\mu} \in F(X)\}$ of $H^0(X, \Omega_{X/F})$ will be called a **Boseck-type basis** if the following two properties hold:*

1. *If $(N_1 + N_2, \mu_1 + \mu_2) = (N_3 + N_4, \mu_3 + \mu_4)$, for $(N_\kappa, \mu_\kappa) \in A$, $1 \leq \kappa \leq 4$, then $f_{N_1,\mu_1} \cdot f_{N_2,\mu_2} = f_{N_3,\mu_3} \cdot f_{N_4,\mu_4}$.*
2. *If $(N_1 + N_2 + i, \mu_1 + \mu_2 - j) = (N_3 + N_4, \mu_3 + \mu_4)$, for $(N_\kappa, \mu_\kappa) \in A$, $1 \leq \kappa \leq 4$ and $(i,j) \in D$, then $x^i y^j f_{N_1,\mu_1} \cdot f_{N_2,\mu_2} = f_{N_3,\mu_3} \cdot f_{N_4,\mu_4}$.*

Remark 1.3.2. *The above definition is motivated by H. Boseck's study of cyclic ramified coverings of the projective line in [12].*

Example 1.3.3. *Let $X \to \mathrm{Spec}(\overline{\mathbb{F}}_5)$ be the curve with affine model*

$$X : \ y^5 - y = x^{-6}.$$

We note that X is an Artin-Schreier curve of genus $g = 10$, see also Section A.1. Rewrite the affine model of X in the form of eq. (1.12) as follows:

$$g(x, y) = 1 - y^{-4} - x^{-6}y^{-5}, \tag{1.13}$$

so the set of exponents D is $\{(0, -4), (-6, -5)\} \subseteq \mathbb{Z} \times \mathbb{Z}_-$. A Boseck-type basis $\mathbf{b}_A$ for $H^0(X, \Omega_{X/F})$, see Section 3.3, is given by $\mathbf{b}_A = \{x^N y^\mu : (N, \mu) \in A\}$, where

$$A \;=\; \{(-2, 4), (-3, 3), (-2, 3), (-4, 3), (-3, 3), (-2, 2), (-5, 1), (-4, 1), (-3, 1), (-2, 1)\}.$$

To demonstrate property (1) of Def. 1.3.1, the points $(-2, 4), (-2, 3), (-2, 2) \in A$ satisfy $(-2, 3) + (-2, 3) = (-4, 6) = (-2, 4) + (-2, 2)$ and the respective rational functions satisfy

$$f_{-2,3} \cdot f_{-2,3} = x^{-2}y^3 \cdot x^{-2}y^3 = x^{-4}y^6 = x^{-2}y^4 \cdot x^{-2}y^2 = f_{-2,4} \cdot f_{-2,2}.$$

To demonstrate property (2) of Def. 1.3.1, the points $(-2, 4), (-2, 3), (-5, 1), (-2, 2)(-2, 1) \in A$ and the points $(0, -4), (-6, -5) \in D$ satisfy

- $(-6, -5) + (-2, 4) + (-2, 3) = (-10, 2) = (-5, 1) + (-5, 1)$ *and*
- $(0, -4) + (-2, 4) + (-2, 3) = (-4, 3) = (-2, 2) + (-2, 1)$.

and the respective rational functions satisfy

- $x^{-6}y^{-5} \cdot f_{-2,4} \cdot f_{-2,3} = f_{-5,2} \cdot f_{-5,2}$ *and*
- $y^{-4} \cdot f_{-2,4} \cdot f_{-2,3} = f_{-2,2} \cdot f_{-2,1}$.

Example 1.3.4. *[40, Section 2.2.1] Let $X \to \mathrm{Spec}(\mathbb{C})$ be the curve with affine model*

$$X : \; x^6 + y^6 + 1 = 0.$$

We note that X is a Fermat curve of genus $g = 10$. Rewrite the affine model of X in the form of eq. (1.12) as follows:

$$g(x, y) = 1 + y^{-6} + x^6 y^{-6}, \tag{1.14}$$

so the set of exponents D is $\{(0, -6), (6, -6)\} \subseteq \mathbb{Z} \times \mathbb{Z}_-$. A Boseck-type basis $\mathbf{b}_A$ for

$H^0(X, \Omega_{X/F})$, see [38], is given by $\mathbf{b}_A = \{x^N y^\mu dx : (N, \mu) \in A\}$, where

$$A = \{(0,-5),(0,-6),(0,-7),(0,-8),(1,-5),(1,-6),(1,-7),(2,-5),(2,-6),(3,-5)\}.$$

To demonstrate property (1) of Def. 1.3.1, the points $(0,-8),(1,-7),(2,-6) \in A$ satisfy $(1,-7)+(1,-7) = (2,-14) = (0,-8)+(2,-6)$ and the respective rational functions satify

$$f_{1,-7} \cdot f_{1,-7} = xy^{-7} \cdot xy^{-7} = x^2y^{-14} = y^{-8} \cdot x^2y^{-6} = f_{0,-8} \cdot f_{2,-6}.$$

To demonstrate property (2) of Def. 1.3.1, the points $(0,-8),(0,-5),(3,-5) \in A$ and the points $(0,-6),(6,-6) \in D$ satisfy

- $(0,6)+(0,-8)+(0,-8) = (0,-10) = (0,-5)+(0,-5)$ *and*
- $(6,6)+(0,-8)+(0,-8) = (6,-10) = (3,-5)+(3,-5)$.

and the respective rational functions satisfy

- $y^{-6} \cdot f_{0,-8} \cdot f_{0,-8} = f_{0,-5} \cdot f_{0,-5}$ *and*
- $x^6y^{-6} \cdot f_{0,-8} \cdot f_{0,-8} = f_{3,-5} \cdot f_{3,-5}$.

1.4 The binomial part of the canonical ideal

The correspondence between monomials in $\mathbb{T}^2$ and points of $A + A$ is not one-to-one: consider two monomials $z_{N_1,\mu_1}z_{N_1',\mu_1'}$, $z_{N_2,\mu_2}z_{N_2',\mu_2'} \in \mathbb{T}^2$ of standard degree 2 that satisfy

$$\mathrm{mdeg}(z_{N_1,\mu_1}z_{N_1',\mu_1'}) = \mathrm{mdeg}(z_{N_2,\mu_2}z_{N_2',\mu_2'}).$$

Then $N_1 + N_1' = N_2 + N_2'$ and $\mu_1 + \mu_1' = \mu_2 + \mu_2'$ and thus by Definition 1.3.1, the binomial $z_{N_1,\mu_1}z_{N_1',\mu_1'} - z_{N_2,\mu_2}z_{N_2',\mu_2'}$ is contained in the canonical ideal, since

$$\phi(z_{N_1,\mu_1}z_{N_1',\mu_1'} - z_{N_2,\mu_2}z_{N_2',\mu_2'}) = f_{N_1,\mu_1}f_{N_1',\mu_1'}dx^{\otimes 2} - f_{N_2,\mu_2}f_{N_2',\mu_2'}dx^{\otimes 2} = 0.$$

where ϕ is the canonical map defined in Theorem 1.1.1.

We have thus arrived at the following:

Proposition 1.4.1. *Let $\mathbf{b}_A$ be a Boseck-type basis of $H^0(X, \Omega_{X/F})$ and let G_1 be the following set of binomials:*

$$G_1 = \{z_{N_1,\mu_1} z_{N_1',\mu_1'} - z_{N_2,\mu_2} z_{N_2',\mu_2'} \in S \ : \ z_{N_1,\mu_1} z_{N_1',\mu_1'}, z_{N_2,\mu_2} z_{N_2',\mu_2'} \in \mathbb{T}^2 \\ \textit{and } \operatorname{mdeg}(z_{N_1,\mu_1} z_{N_1',\mu_1'}) = \operatorname{mdeg}(z_{N_2,\mu_2} z_{N_2',\mu_2'})\}.$$

Then $G_1 \subseteq I_X$.

By Proposition 1.2.2, to test whether G_1 generates I_X, we need to count the monomials that do not appear as initial terms in G_1 with respect to some term order. We shall use the following term order in our analysis:

Definition 1.4.2. *Let $\prec_t$ be the lexicographic order on the variables $\{z_{N,\mu} : (N, \mu) \in A\}$ where A is as defined in Definition 1.3.1. We define a binary relation $\prec$ on the monomials of S as follows:*

$$z_{N_1,\mu_1} z_{N_2,\mu_2} \cdots z_{N_d,\mu_d} \prec z_{N_1',\mu_1'} z_{N_2',\mu_2'} \cdots z_{N_s',\mu_s'} \textit{ if and only if}$$

(i) $d < s$ *or*

(ii) $d = s$ *and* $\sum \mu_i > \sum \mu_i'$ *or*

(iii) $d = s$ *and* $\sum \mu_i = \sum \mu_i'$ *and* $\sum N_i < \sum N_i'$

(iv) $d = s$ *and* $\sum \mu_i = \sum \mu_i'$ *and* $\sum N_i = \sum N_i'$ *and*

$$z_{N_1,\mu_1} z_{N_2,\mu_2} \cdots z_{N_d,\mu_d} \prec_t z_{N_1',\mu_1'} z_{N_2',\mu_2'} \cdots z_{N_s',\mu_s'}.$$

Proposition 1.4.3. *The relation $\prec$ is a monomial term order.*

Proof. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{T}$ be three monomials of standard degree d, s, r respectively.

- **Transitivity:** If $\mathbf{a} \prec \mathbf{b}$ and $\mathbf{b} \prec \mathbf{c}$ then we must have $d \leq s \leq r$. If either inequality is strict then $d < r$ so $\mathbf{a} \prec \mathbf{c}$. If we have $d = s = r$ then we must have

$\sum \mu_{i,\mathbf{a}} \geq \sum \mu_{i,\mathbf{b}} \geq \sum \mu_{i,\mathbf{c}}$ and by the same argument, either at least one inequality is strict, so $\mathbf{a} \prec \mathbf{c}$, or all three sums are equal. The same argument applies for the sum of the N_i and thus transitivity follows from transitivity of the lexicographic term order $\prec_t$.

- **Connexity:** Assume that the statment $\mathbf{a} \prec \mathbf{b}$ is false. Then either $d > s$, so $\mathbf{b} \prec \mathbf{a}$, or $d = s$. This implies that $\sum \mu_{i,\mathbf{a}} \leq \sum \mu_{i,\mathbf{b}}$ and again if the inequality is strict, we have $\mathbf{b} \prec \mathbf{a}$. If the sums of the μ_i are equal, we can use the same argument on the sum of the N_i to conclude that either $\mathbf{b} \prec \mathbf{a}$ or the comparison is made using the lexicographic term order $\prec_t$. Thus either $\mathbf{b} \prec \mathbf{a}$ or $\mathbf{b} = \mathbf{a}$, as required.

- **Antisymmetry:** Let $\mathbf{a} \prec \mathbf{b}$ and $\mathbf{b} \prec \mathbf{a}$. Then we must necessarily have $d = s$, $\sum \mu_{i,\mathbf{a}} = \sum \mu_{i,\mathbf{b}}$, $\sum N_{i,\mathbf{a}} = \sum N_{i,\mathbf{b}}$. Antisymmetry follows from antisymmetry of the lexicographic term order $\prec_t$.

- **Multiplicativity:** Let $\mathbf{a} \prec \mathbf{b}$. The standard degree of $\mathbf{a} \cdot \mathbf{c}$ is then $\leq$ than the standard degree of $\mathbf{b} \cdot \mathbf{c}$ so either $\mathbf{a} \cdot \mathbf{c} \prec \mathbf{b} \cdot \mathbf{c}$ or, as above, we iterate the argument for the sum of the μ_i, the sum of the N_i and end up comparing $\mathbf{a} \cdot \mathbf{c}$ to $\mathbf{b} \cdot \mathbf{c}$ using the lexicographic term order $\prec_t$, which is multiplicative.

□

Recall that for any binomial $z_{N_1,\mu_1} z_{N_1',\mu_1'} - z_{N_2,\mu_2} z_{N_2',\mu_2'} \in G_1$, the monomials $z_{N_1,\mu_1} z_{N_1',\mu_1'}$ and $z_{N_2,\mu_2} z_{N_2',\mu_2'}$ have the same multidegree and thus correspond to the same point of $A+A$. In fact, points of $A+A$ are in bijection with monomials of S that do not appear as initial terms of binomials in G_1:

Proposition 1.4.4. *Let A be as in Definition 1.3.1. Then $|A+A| = |\mathbb{T}^2 \setminus \mathrm{in}_{\prec}(G_1)|$.*

Proof. Write (ρ, T) for the arbitrary point of $A+A$ and set

$$B_{\rho,T} := \{ z_{N,\mu} z_{N',\mu'} \in \mathbb{T}^2 \ : \ \mathrm{mdeg}\left(z_{N,\mu} z_{N',\mu'} \right) = (\rho, T) \}$$

so that

$$\mathbb{T}^2 = \bigcup_{(\rho,T) \in A+A} B_{\rho,T} \text{ with } B_{(\rho,T)} \cap B_{(\rho',T')} = \emptyset \text{ for } (\rho,T) \neq (\rho',T').$$

Observe that the differences of elements of $B_{\rho,T}$ are in G_1. We define the map of sets:

$$\begin{aligned} \sigma : A+A &\rightarrow \mathbb{T}^2 \\ (\rho,T) &\mapsto \min_{\prec} B_{\rho,T}. \end{aligned} \tag{1.15}$$

Since $B_{\rho,T}$ is non-empty and $\prec$ is a total order, it has a unique minimal element. Hence, the map σ is well-defined, $1-1$ and $\sigma(A+A) \subseteq \mathbb{T}^2 \setminus \mathrm{in}_{\prec}(G_1)$. For the inverse inclusion let $z_{N_1,\mu_1} z_{N_1',\mu_1'}$ be a monomial in $\mathbb{T}^2 \setminus \mathrm{in}_{\prec}(G_1)$. Then, either $z_{N_1,\mu_1} z_{N_1',\mu_1'}$ is the unique monomial of multidegree (ρ,T) or $z_{N_1,\mu_1} z_{N_1',\mu_1'} \prec z_{N_2,\mu_2} z_{N_2',\mu_2'}$ for all other monomials $z_{N_2,\mu_2} z_{N_2',\mu_2'}$ with the same multidegree. In both cases, $z_{N_1,\mu_1} z_{N_1',\mu_1'} = \min_{\prec} B_{\rho,T}$. Hence $\sigma(A+A) = \mathbb{T}^2 \setminus \mathrm{in}_{\prec}(G_1)$ and thus $|A+A| = |\sigma(A+A)| = |\mathbb{T}^2 \setminus \mathrm{in}_{\prec}(G_1)|$. □

1.5 The non-binomial part of the canonical ideal

Recall that by eq. (1.12) there exists an affine model for the curve of X over F of the form

$$g(x,y) = 1 + \sum_{(i,j)\in D} b_{ij} x^i y^j, \text{ for some } D \subseteq \mathbb{Z} \times \mathbb{Z}_- \text{ and some } b_{ij} \in F.$$

Motivated by the second property of Definition 1.3.1, we define $M_D \subseteq \mathbb{T}^2$ to be the set of monomials $\mathbf{a} \in \mathbb{T}^2$ such that for all $(i,j) \in D$ there exists at least one monomial $\mathbf{b} \in \mathbb{T}^2$ of multidegree $\mathrm{mdeg}(\mathbf{b}) = \mathrm{mdeg}(\mathbf{a}) + (0,i,-j)$, i.e.

$$M_D = \{\mathbf{a} \in \mathbb{T}^2 : \forall\ (i,j) \in D,\ \exists\ \mathbf{b} \in \mathbb{T}^2 \text{ with } \mathrm{mdeg}(\mathbf{b}) = \mathrm{mdeg}(\mathbf{a}) + (0,i,-j)\}. \tag{1.16}$$

Given a monomial $\mathbf{a} = z_{N,\mu} z_{N',\mu'} \in M_D$ and a point $(i,j) \in D$, denote by $z_{N_{ij},\mu_{ij}} z_{N_{ij}',\mu_{ij}'}$ the minimal monomial of $\mathbb{T}^2$ with respect to $\prec$ that satisfies $\mathrm{mdeg}(z_{N_{ij},\mu_{ij}} z_{N_{ij}',\mu_{ij}'}) = \mathrm{mdeg}(z_{N,\mu} z_{N',\mu'}) + (0,i,-j)$. This allows us to obtain a set of relations in I_X:

Proposition 1.5.1. *Let D be as in eq. (1.12), M_D be as in eq. (1.16) and let*

$$G_2 = \{z_{N,\mu} z_{N',\mu'} + \sum_{(i,j)\in D} b_{ij} z_{N_{ij},\mu_{ij}} z_{N_{ij}',\mu_{ij}'} \in S \ :\ z_{N,\mu} z_{N',\mu'} \in M_D\}.$$

Then $G_2 \subseteq I_X$.

Proof. Let

$$h := z_{N,\mu}z_{N',\mu'} + \sum_{(i,j)\in D} b_{ij} z_{N_{ij},\mu_{ij}} z_{N'_{ij},\mu'_{ij}} \in G_2$$

so that $\mathrm{mdeg}(z_{N_{ij},\mu_{ij}} z_{N'_{ij},\mu'_{ij}}) = \mathrm{mdeg}(z_{N,\mu}z_{N',\mu'}) + (0,i,-j)$. This implies that $N_{ij}+N'_{ij} = N+N'+i$ and $\mu_{ij}+\mu'_{ij} = \mu+\mu'-j$, so applying the canonical map and using the second property of Boseck-type bases of Definition 1.3.1 gives

$$\begin{aligned}
\phi(h) &= f_{N'\mu} f_{N',\mu'} dx^{\otimes 2} + \sum_{(i,j)\in D} b_{ij} f_{N_{ij},\mu_{ij}} f_{N'_{ij},\mu'_{ij}} dx^{\otimes 2} \\
&= f_{N'\mu} f_{N',\mu'} dx^{\otimes 2} + \sum_{(i,j)\in D} b_{ij} x^i y^j f_{N'\mu} f_{N',\mu'} dx^{\otimes 2} \\
&= f_{N'\mu} f_{N',\mu'} dx^{\otimes 2} \left(1 + \sum_{(i,j)\in D} b_{ij} x^i y^j \right) = 0.
\end{aligned}$$

□

Recall that, by Proposition 1.2.2, to show that $I_X = \langle G_1 \cup G_2 \rangle$, it suffices to show that $|\mathbb{T}^2 \setminus \mathrm{in}_{\prec}(G)| \le 3(g-1)$. In Proposition 1.4.4 we saw that the monomials which do not appear as initial terms of the binomials in G_1 are in bijection with $A+A$. We now prove that monomials that do not appear as initial terms of the polynomials in $G_1 \cup G_2$ are in bijection with an appropriate subset of $A+A$:

Proposition 1.5.2. *Let A be as in Definition 1.3.1 and let*

$$C_D = \{(\rho, T) \in A+A : (\rho+i, T-j) \in A+A \text{ for all } (i,j) \in D\}.$$

Then

$$|\mathbb{T}^2 \setminus \mathrm{in}_{\prec}(G_1 \cup G_2)| \le |(A+A) \setminus C_D|.$$

Proof. As in the proof of Proposition 1.4.4 let

$$\begin{aligned}
\sigma : A+A &\to \mathbb{T}^2 \\
(\rho, T) &\mapsto \min_{\prec} B_{\rho,T}.
\end{aligned}$$

where

$$B_{\rho,T} := \{z_{N,\mu}z_{N',\mu'} \in \mathbb{T}^2 \ : \ \mathrm{mdeg}\left(z_{N,\mu}z_{N',\mu'}\right) = (\rho, T)\}.$$

If $(\rho, T) \in C_D$ then by definition $(\rho + i, T - j) \in A + A$ for all $(i, j) \in D$. We define the monomials $z_{N,\mu} z_{N',\mu'} := \sigma(\rho, T)$ and $z_{N_{i,j},\mu_{i,j}} z_{N'_{i,j},\mu'_{i,j}} := \sigma(\rho + i, T - j)$ for $(i, j) \in D$, so that the polynomial

$$f = z_{N,\mu} z_{N',\mu'} + \sum_{(i,j) \in D} b_{ij} z_{N_{ij},\mu_{ij}} z_{N'_{ij},\mu'_{ij}}$$

is in G_2. Since j is negative, we have that $\mu + \mu' = T < T - j = \mu_{ij} + \mu'_{ij}$. The term order in Definition 1.4.2 then gives that $z_{N_{ij},\mu_{ij}} z_{N',\mu'} \prec z_{N,\mu} z_{N',\mu'}$ for all $(i, j) \in D$ so that $\text{in}_\prec(f) = z_{N,\mu} z_{N',\mu'}$. This implies that $\sigma(C_D) \subseteq \text{in}_\prec(G_2)$ and, since σ is $1 - 1$, we get

$$|C_D| \leq |\text{in}_\prec(G_2)| \Rightarrow |\mathbb{T}^2 \setminus \text{in}_\prec(G_1 \cup G_2)| \leq |(A + A) \setminus C_D|$$

completing the proof. □

We conclude this section by remarking that by Proposition 1.4.1 and Proposition 1.5.1 we have that $\langle G_1 \cup G_2 \rangle \subseteq I_X$. By Proposition 1.5.2 and Lemma 1.2.2, to show that $G_1 \cup G_2$ is a generating set, it remains to show that $|(A + A) \setminus C_D| \leq 3(g - 1)$. In Section 3, we will see that this holds for Artin-Schreier and Kummer curves and we will use this result to find a generating set for the canonical ideal of the respective relative curve.

1.6 The relative version of Petri's theorem

Let $(R, \mathfrak{m}_R)$ is a discrete valuation ring with residue field k and fraction field L and let $\mathscr{X} \to \text{Spec}(R)$ be a flat family of smooth, projective curves. We write $\mathscr{X}_\eta \to \text{Spec}(L)$ for the family's generic fiber and $\mathscr{X}_0 \to \text{Spec}(k)$ for the special fiber. We thus obtain a commutative diagram

$$\begin{array}{ccccc} \text{Spec}(k) \times_{\text{Spec}(R)} \mathscr{X} = \mathscr{X}_0 & \hookrightarrow & \mathscr{X} & \hookleftarrow & \mathscr{X}_\eta = \text{Spec}(L) \times_{\text{Spec}(R)} \mathscr{X} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(k) & \hookrightarrow & \text{Spec}(R) & \hookleftarrow & \text{Spec}(L) \end{array} \tag{1.17}$$

Note that, by [3, Prop. 2.2], flatness of $\mathscr{X} \to \text{Spec}(R)$ implies that the two fibers have the same Hilbert polynomial

$$h_{\mathscr{X}_\eta}(t) = h_{\mathscr{X}_0}(t) = at + (1 - g) \tag{1.18}$$

and thus both fibers have degree a and genus g. In what follows, we assume that $g \geq 4$.

Let $\Omega_{\mathscr{X}/R}$ denote the sheaf of relative holomorphic differentials on $\mathscr{X}$ and, for $n \geq 1$, let $\Omega_{\mathscr{X}/R}^{\otimes n}$ be the n-th tensor power of $\Omega_{X/F}$; by [29, lemma II.8.9] the global sections $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n})$ form a free $R-$module of rank d_n for all $n \geq 1$, with d_n given by eq. (1.1). The multiplication

$$\begin{array}{rcl} H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n}) \times H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes m}) & \to & H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes (n+m)}) \\ (f_1 dx^{\otimes n}, f_2 dx^{\otimes m}) & \mapsto & f_1 f_2 dx^{\otimes (n+m)} \end{array}$$

endows the direct sum $\bigoplus_{n\geq 0} H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n})$, called **the relative canonical ring**, with the structure of a graded R-algebra. The polynomial ring $S = R[W_1, \ldots, W_g]$ is equipped with a graded ring structure: its n-th graded piece consists of homogeneous polynomials of degree n, i.e.

$$S = \bigoplus_{n\geq 0} S_n \text{ where } S_n = \{f \in S : f \text{ is homogeneous with } \deg f = n\}. \tag{1.19}$$

We note that for $n \geq 1$, S_n is a free R-module of rank $\binom{g-1+n}{n}$, generated by $\{W_1^{a_1} \cdots W_g^{a_g} \in S : a_1 + \ldots a_g = n\}$. Let $\{f_1 dx, \ldots, f_g dx\}$ be a basis for $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})$. For $n \geq 1$, we define the R-linear maps

$$\phi_n : \; S_n \longrightarrow H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n}), \; W_1^{a_1} \cdots W_g^{a_g} \mapsto f_1^{a_1} \cdots f_g^{a_g} dx^{\otimes (a_1 + \cdots a_g)},$$

which give rise to a graded R-algebra homomomorphism

$$\phi = \bigoplus_{n\geq 0} \phi_n : R[W_1, \ldots, W_g] \longrightarrow \bigoplus_{n\geq 0} H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n})$$

called **the relative canonical map**. The kernel of ϕ, called **the relative canonical ideal** and denoted by $I_{\mathscr{X}}$, is a graded ideal with

$$I_{\mathscr{X}} = \bigoplus_{n\geq 0} (I_{\mathscr{X}})_n \text{ where } (I_{\mathscr{X}})_n = \{f \in I_{\mathscr{X}} : f \text{ is homogeneous with } \deg f = n\}. \tag{1.20}$$

Our next result concerns the canonical embedding of the family $\mathscr{X} \to \operatorname{Spec}(R)$:

Theorem 1.6.1. *Diagram (1.17) induces a deformation-theoretic diagram of canonical embeddings*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_{\mathscr{X}_\eta} & \hookrightarrow & S_L := L[\omega_1,\dots,\omega_g] & \xrightarrow{\phi_\eta} & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}_\eta, \Omega^{\otimes n}_{\mathscr{X}_\eta/L}) & \longrightarrow & 0 \\
 & & \uparrow \otimes_R L & & \uparrow \otimes_R L & & \uparrow \otimes_R L & & \\
0 & \longrightarrow & I_{\mathscr{X}} & \hookrightarrow & S_R := R[W_1,\dots,W_g] & \xrightarrow{\phi} & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}, \Omega^{\otimes n}_{\mathscr{X}/R}) & \longrightarrow & 0 \\
 & & \downarrow \otimes_R R/\mathfrak{m} & & \downarrow \otimes_R R/\mathfrak{m} & & \downarrow \otimes_R R/\mathfrak{m} & & \\
0 & \longrightarrow & I_{\mathscr{X}_0} & \hookrightarrow & S_k := k[w_1,\dots,w_g] & \xrightarrow{\phi_0} & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}_0, \Omega^{\otimes n}_{\mathscr{X}_0/k}) & \longrightarrow & 0
\end{array} \tag{1.21}$$

where $I_{\mathscr{X}_\eta} = \ker \phi_\eta$, $I_{\mathscr{X}} = \ker \phi$, $I_{\mathscr{X}_0} = \ker \phi_0$, *each row is exact and each square is commutative.*

Proof. Exactness of the top and bottom row of diagram (1.21) are due to Theorem 1.1.1, the classical result of Enriques, Petri and M. Noether. For the middle row, we select generators $f_1 dx, \dots, f_g dx$ for $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})$ and note that the assignment $W_i \mapsto f_i dx$ gives rise to a homogeneous homomorphism of graded rings

$$\phi : R[W_1, \dots, W_g] \longrightarrow \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}, \Omega^{\otimes n}_{\mathscr{X}/R}).$$

We prove surjectivity of ϕ by diagram chasing: let $r \in \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}, \Omega^{\otimes n}_{\mathscr{X}/R})$ and write $\overline{r} = r \otimes_R 1_{R/\mathfrak{m}} \in \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}, \Omega^{\otimes n}_{\mathscr{X}/k})$. Since ϕ_0 is onto, there exists $\overline{s} \in \mathrm{Sym}(H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/k}))$ with $\phi_0(\overline{s}) = \overline{r}$. Similarly, since $\mathrm{Sym}(H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})) \to \mathrm{Sym}(H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/k}))$ is onto, there exists $s \in \mathrm{Sym}(H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}))$ with $s \otimes_R 1_{R/\mathfrak{m}} = \overline{s}$. By construction, $\phi(s) = r$, proving that ϕ is onto as well. □

We proceed with establishing a Nakayama-type criterion for a subset of the kernel $I_{\mathscr{X}}$ to generate the relative canonical ideal:

Lemma 1.6.2. *Let G be a set of homogeneous polynomials in $I_{\mathscr{X}}$ such that $G \otimes_R L$ generates $I_{\mathscr{X}_\eta}$ and $G \otimes_R k$ generates $I_{\mathscr{X}_0}$. Then:*

(i) For any $n \in \mathbb{N}$, the R-modules $(S_R/\langle G \rangle)_n$ are free of rank d_n.

(ii) $I_{\mathscr{X}} = \langle G \rangle$.

Proof. For (i): Let $n \in \mathbb{N}$. Since by assumption $G \otimes_R L$ and $G \otimes_R k$ generate $I_{\mathscr{X}_\eta}$ and $I_{\mathscr{X}_0}$ respectively, we have that

$$(S_R/\langle G \rangle)_n \otimes_R L \cong \left(S_L/I_{\mathscr{X}_\eta}\right)_n \text{ and } (S_R/\langle G \rangle)_n \otimes_R k \cong \left(S_k/I_{\mathscr{X}_0}\right)_n.$$

By Petri's Theorem 1.1.1 we get that

$$\left(S_L/I_{\mathscr{X}_\eta}\right)_n \cong H^0(\mathscr{X}_\eta, \Omega^{\otimes n}_{\mathscr{X}_\eta/L}) \text{ and } \left(S_k/I_{\mathscr{X}_0}\right)_n \cong H^0(\mathscr{X}_0, \Omega^{\otimes n}_{\mathscr{X}_0/k})$$

and by eq. (1.1)

$$\dim_L H^0(\mathscr{X}_\eta, \Omega^{\otimes n}_{\mathscr{X}_\eta/L}) = \dim_k H^0(\mathscr{X}_0, \Omega^{\otimes n}_{\mathscr{X}_0/k}) = d_n.$$

The result follows from [29, lemma II.8.9].

For (ii): let $s \in I_{\mathscr{X}}$ and assume for contradiction that $s \notin \langle G \rangle$. Since $s \otimes 1_L \in I_{\mathscr{X}_\eta}$ and $G \otimes_R L$ generates $I_{\mathscr{X}_\eta}$, there exist $g_i \in G$ and $s_i \in S_L$ such that $s \otimes 1_L = \sum g_i s_i \otimes 1_L$. Choosing $d \in R$ to be the lcm of the denominators of the coefficients of the s_i, we may clear denominators to obtain $ds \otimes 1_L = \sum g_i ds_i \otimes 1_L$, with $ds_i \in S_R$ or equivalently $ds = \sum g_i ds_i$ with $ds_i \in S_R$, implying that $ds \in \langle G \rangle$. If $s \notin \langle G \rangle$, then s is a torsion element of $S_R/\langle G \rangle$, with its homogeneous components being torsion elements of the free $R-$modules $(S_R/\langle G \rangle)_n$ for some $n \in \mathbb{N}$. By (i), the latter are free R-modules, so we conclude that if $s \notin \langle G \rangle$ then s must be zero, completing the proof. □

Lemma 1.6.2 reduces the problem of determining the generating set of the relative canonical ideal to determining compatible generating sets for the canonical ideals of the two fibers. We will demonstrate this technique via an explicit example in Section 3.

Chapter 2

Syzygies of Ideals over Principal Ideal Domains

The main results of this chapter have been published in [16].

Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}_R$, field of fractions L and residue field k. We write $S = R[W_1, \ldots, W_g]$, $S_L = S \otimes_R L$ and $S_k = S \otimes_R k$ for the respective homogeneous polynomial rings and denote by M a finitely generated S-module. Further, we assume that M satisfies the following equivalent properties:

Proposition 2.0.1. *The following are equivalent:*

1. *The generator x of $\mathfrak{m}_R$ is not a zero-divisor on M.*

2. *M is flat as an R-module.*

3. *The S_L-module $\widehat{M} := M \otimes_R S_L$ and the S_k-module $\overline{M} := M \otimes_R S_k$ have the same Hilbert polynomial.*

Proof. For (1) $\Leftrightarrow$ (2) see [22, Cor. 6.3]. For (2) $\Leftrightarrow$ (3) see [4, Proposition 2.2]. □

The reader should always keep in mind as a motivating example the relative canonical ring $M = \bigoplus H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n})$ with the S-module structure inherited by the canonical map $\phi : S \twoheadrightarrow M$. The common Hilbert polynomial in this case $h(t) = dt + 1 - g$ where d is the degree of the curve $\mathscr{X}_0 \to \operatorname{Spec}(k)$ and g is its genus.

The goal of this chapter is to examine how the syzygies and the Betti numbers of the S_L-module $\widehat{M}$ and the S_k-module $\overline{M}$ are related. The study of syzygies becomes automatically more challenging over S since the non-zero elements of the base PID R may not be invertible and modules might have torsion, see [1, chap. 4] for a more comprehensive account and also [57]. On the other hand simplicial homology over $\mathbb{Z}$ has been extensively studied and techniques have been developed to account for that case and the different behavior over $\mathbb{Q}$, [21]. It is well known that, even in the case of monomial ideals, the minimal free resolution depends on the characteristic of the ground field, the classical example being the triangulation of the projective plane.

Example 2.0.2. *The Betti numbers of*

$$B = \langle abc, abf, ace, ahe, ahf, bch, bhe, bef, chf, cef \rangle \ \triangleleft \ F[a, b, c, e, f, h]$$

differ when $\mathrm{char}(F) = 0$ *(table on the left) and* $\mathrm{char}(F) = 2$ *(table on the right).*

	0	*1*	*2*	*3*
0	*1*	*0*	*0*	*0*
1	*0*	*0*	*0*	*0*
2	*0*	*10*	*15*	*6*
3	*0*	*0*	*0*	*0*

	0	*1*	*2*	*3*	*4*
0	*1*	*0*	*0*	*0*	*0*
1	*0*	*0*	*0*	*0*	*0*
2	*0*	*10*	*15*	*6*	*1*
3	*0*	*0*	*0*	*1*	*0*

Thus when $\mathrm{char}(F) = 2$*, the ideal* B *has a third and a fourth graded syzygy of degree* 6 *which do not appear over characteristic zero (or any other characteristic* $p \neq 2$ *for that matter), see also [52, Ex. 12.4].*

For more examples, see Exercise 5.5.4 and the Notes after Section 7.3 of [15].

Returning to the general case, recall by eq. (1.19) that the polynomial ring S is graded by assigning the degree 1 to each variable W_i for $i = 1, \ldots, g$. We write $\mathfrak{m} = \langle W_1, \ldots, W_g \rangle$ for the prime irrelevant ideal of S and $\mathfrak{m}_S = \mathfrak{m} + \mathfrak{m}_R S$ for the maximal ideal of S and observe that $k = S/\mathfrak{m}_S = R/\mathfrak{m}_R$. Further, we assume that the finitely generated S-module M is positively graded, i.e.

$$M = \bigoplus_{i \geq a} M_i \text{ where } S_j M_i \subset M_{i+j} \text{ and in particular } M_i \text{ is an } R\text{-module for } i \geq a.$$

Let $\{m_1, \ldots, m_n\}$ form a generating set of M. It is clear that $\{\overline{m}_i = m_i + \mathfrak{m}_S M,\ i = 1, \ldots, n\}$ is a generating set of $M/\mathfrak{m}_S M$. The converse, i.e. Nakayama's lemma, holds when the elements m_i are homogeneous. The proof follows the same lines as the standard proof for the graded case, [23, lemma 1.4]. We include it here for completeness of the exposition.

Lemma 2.0.3 (Nakayama)**.** *Let M be a finitely generated, positively graded S-module, $m_1, \ldots, m_n \in M$ homogeneous so that $\overline{m}_i,\ i = 1, \ldots, n$ generate $M/\mathfrak{m}_S M$. Then $m_1, \ldots, m_n$ generate M.*

Proof. Let M' be the S-module generated by $\{m_1, \ldots, m_n\}$ and consider the finitely generated graded S-module $N = M/M'$. By our assumption on the m_i, $M' + \mathfrak{m}_S M = M$, thus $N/\mathfrak{m}_S N = 0$ and $\mathfrak{m}_S N = N$. If $N \neq 0$, there is a non-zero homogeneous element of least degree in N. Since $\mathfrak{m}_S N = N$, this element must have degree zero. It follows that $N_0 = \mathfrak{m}_R N_0$. Since R is a local PID, Nakayama's lemma in the local case gives that $N_0 = 0$. It follows that $N = 0$ as desired. □

2.1 Graded free resolutions and Betti numbers

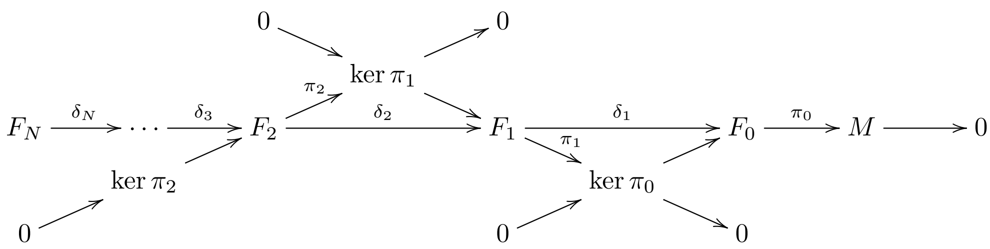
It follows from Lemma 2.0.3 that the least number of homogeneous elements needed to generate M is the dimension of the $S/\mathfrak{m}_S$-vector space $M/\mathfrak{m}_S M$. We proceed to construct a **minimal graded free resolution** of M. Let $m_1, \ldots, m_n$ be a minimal set of homogeneous generators of M. We let F_0 be the free S-module $F_0 = \bigoplus_i Se_i$ on the generators e_i with $\deg(e_i) = \deg(m_i)$ for $i = 1, \ldots n$, and let $\pi_0 : F_0 \longrightarrow M$ be the epimorphism determined by $\pi_0(e_i) = m_i$. This gives the short exact sequence

$$0 \to \ker \pi_0 \xrightarrow{\iota_0} F_0 \xrightarrow{\pi_0} M \to 0,$$

where $\ker \pi_0 \subset \mathfrak{m}_S F_0$. Since $\ker \pi_0$ is a finitely generated graded S-module we repeat this procedure to obtain an epimorphim $\pi_1 : F_1 \longrightarrow \ker \pi_0$ where F_1 is the free S-module on the number of generators of $\ker \pi_0$. We define $\delta_1 : F_1 \longrightarrow F_0$ as the composition $F_1 \xrightarrow{\pi_1} \ker \pi_0 \xrightarrow{\iota_0} F_0$ and note that, by construction, $\delta_1(F_1) = \ker \pi_0$ and we have specified the basis of F_1 that maps to a minimal homogeneous generating set of $\ker \pi_0$. Iterating this procedure, we obtain at the i-th step an epimorphism $\pi_i : F_i \longrightarrow \ker \pi_{i-1}$ and define

$\delta_i : F_i \longrightarrow F_{i-1}$ as the composition $F_i \xrightarrow{\pi_i} \ker \pi_{i-1} \xrightarrow{\iota_{i-1}} F_{i-1}$. We thus construct an exact complex

$(F_\bullet, \delta_\bullet)$:



called **a free graded resolution** of M which is minimal - and thus unique up to isomorphism - since by construction $\ker \pi_i \subset \mathfrak{m}_S F_i$ for all $i \geq 0$. The fact that the above process terminates in a finite number of steps is known as **Hilbert's Syzygy Theorem** [22, Corollary 19.7] and the number N is called **the projective dimension** of M. Note that for each $i \geq 1$, the resolution above breaks into a short exact sequence

$$0 \to \ker \pi_i \xrightarrow{\iota_i} F_i \xrightarrow{\pi_i} \ker\ \pi_{i-1} \to 0, \tag{2.1}$$

and $\delta_i : F_i \longrightarrow F_{i-1}$ is given by the composition $\iota_{i-1} \circ \pi_i$ for $i \geq 1$. We remark that the differentials δ_i are of degree zero, $\delta_i(F_i) \subset \mathfrak{m}_S F_{i-1}$ and that δ_i maps a basis of F_i to a minimal set of homogeneous generators of $\delta_i(F_i)$, as in [23, Corollary 1.5]. We write each F_i as a direct sum, indexed by $\mathbb{Z}$, of copies of S shifted by the degrees of the generators

$$F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}},$$

where finitely many of the $\beta_{i,j}$ are nonzero. The exponent $\beta_{i,j} \in \mathbb{N}$ that counts the number of minimal generators of degree j in F_i is called the (i, j)-**graded Betti number** of M and equals $\dim_k \operatorname{Tor}_i^S(k, M)_j$ as in [23, Corollary 1.7]. We write $\beta_{i,j}(M)$ when needed to emphasize the module M. The S-modules

$$\Pi_i = \ker \pi_{i-1} = \ker \delta_{i-1}, \tag{2.2}$$

for $i \geq 1$, are known as the i-th **syzygies** of M, and we set $\Pi_0 = M$, so that Π_i is a graded S-module for all i. By successively taking homology of the short exact sequences in (2.1) we get that

$$\mathrm{Tor}_1^S(\Pi_{i-1}, R) = \mathrm{Tor}_i^S(M, R), \ i \geq 1. \tag{2.3}$$

We let $F_{i,j}$ be the direct summand of F_i at degree j and denote by $\delta_{i,j}$ the restriction

$$\delta_{i,j} = \delta_i|_{F_{i,j}} : F_{i,j} \longrightarrow F_{i-1,j}.$$

Remark 2.1.1. *Let $F_\bullet$ be a minimal graded free resolution of the graded S-module M. By tensoring $F_\bullet$ with $R = S/\mathfrak{m}$ over S we obtain a graded complex of R-modules:*

$$F_\bullet \otimes R : F_N \otimes R \stackrel{\delta_N \otimes 1_R}{\longrightarrow} \cdots \longrightarrow F_1 \otimes R \stackrel{\delta_1 \otimes 1_R}{\longrightarrow} F_0 \otimes R \longrightarrow M \otimes R \longrightarrow 0$$

whose homology at the i-th position is $\mathrm{Tor}_i^S(R, M)$, a graded S-module. Note that, for $\alpha > 0$, $S(-\alpha) \otimes R$ only lives in degree α. Thus, to compute $\mathrm{Tor}_i^S(R, M)_j = \mathrm{Tor}_1^S(R, \Pi_{i-1})_j$ one needs to consider the following s.e.s. derived from (2.1)*:*

$$0 \to \mathrm{Tor}_1^S(\Pi_i, R)_j \to (\Pi_{i+1})_j \otimes R \longrightarrow (S(-j)^{\beta_{i,j}} \otimes R)_j \longrightarrow (\Pi_i)_j \otimes R \to 0. \tag{2.4}$$

2.2 The Betti numbers of the two fibers

Recall that since R is a discrete valuation ring every finitely generated R-module N must be a direct sum of the form

$$N = \bigoplus^{\mathrm{rk}(N)} R \oplus \mathrm{tor}(N),$$

where $\mathrm{rk}(N)$ is the rank of N as an R-module, while $\mathrm{tor}(N)$, the torsion part of N, is a direct sum of the form

$$\mathrm{tor}(N) = \bigoplus_{\nu=1}^{t(N)} R/Rx^{a(\nu,N)}, \text{ where } a(\nu, N) \in \mathbb{N}, \text{ for } \nu = 1, \ldots, t(N) \ . \tag{2.5}$$

Observe that $\mathrm{tor}(N)$ is still visible when tensoring with k (special fiber), since $N \otimes_R k = k^{\mathrm{rk}(N)+t(N)}$, while it disappears when tensoring with L (generic fiber), since $N \otimes_R L = L^{\mathrm{rk}(N)}$. It is known that under our assumptions, reduction of the graded minimal free resolution of M over S to the special fiber preserves exactness, see for example [52, Thm 20.3]. Further, flatness of L over R implies flatness of S_L over S. We collect these results in the proposition below for completeness of the exposition:

Proposition 2.2.1. *If $F_\bullet$ is a free resolution of M as an S-module then*

1. $\overline{F}_\bullet = F_\bullet \otimes S_k$ *is a free resolution of* $\overline{M} = M \otimes S_k$ *as an* S_k*-module.*

2. $\widehat{F}_\bullet = F_\bullet \otimes S_L$ *is a free resolution of* $\widehat{M} = M \otimes S_L$ *as an* S_L*-module.*

Note that if the resolution $F_\bullet$ above is minimal, then $\overline{F}_\bullet$ is also minimal but $\widehat{F}_\bullet$ might fail to be so. We write $\Pi_{i,j}$ for the j-th graded piece of $\Pi_i = \ker(\delta_{i-1})$ and decompose the R-module $\Pi_{i,j} \otimes R$ into its cyclic R-components. We will see that the quantities $f_{i,j} := \mathrm{rk}(\Pi_{i,j} \otimes R)$, $t_{i,j} := t(\Pi_{i,j} \otimes R)$ - see eq. (2.5) - and $s_{i,j} = \mathrm{rk}(\mathrm{Tor}_1^S(R, \Pi_i)_j)$ are critical when we measure the difference between the graded Betti numbers of the generic and the special fiber.

Theorem 2.2.2. *Let M be a finitely generated graded S-module which is flat as an R-module, Π_i be the i-th syzygy of M and $t_{i,j}$ be the number of nonzero cyclic summands of $\Pi_{i,j} \otimes R$, for $i \geq 0$. Then*

1. $\beta_{i,j}(M) = \beta_{i,j}(\overline{M})$*, for* $i \geq 0$.

2. $\beta_{i,j}(M) = \beta_{i,j}(\widehat{M}) + t_{i,j} + t_{i-1,j}$ *for* $i \geq 1$.

Proof. Let $F_\bullet$ be a minimal graded free resolution of the graded S-module M. By part (1) of Proposition 2.2.1, it follows that $\overline{F}_\bullet = F_\bullet \otimes S/xS$ is a free resolution of $\overline{M}$. Moreover, since $\delta_i(F_i) \subset \mathfrak{m}_S F_{i-1}$, it follows that $\overline{\delta_i}(\overline{F_i}) \subset \overline{\mathfrak{m}} \overline{F}_{i-1}$ and $\overline{F}_\bullet$ is a minimal free resolution of $\overline{M}$. Thus $\beta_{i,j}(M) = \beta_{i,j}(\overline{M})$.

For the generic fiber, by part 2 of Proposition 2.2.1, $\widehat{F}_\bullet = F_\bullet \otimes S_L$ is a free resolution of $\widehat{M}$ and thus $\beta_{i,j}(\widehat{M}) = \dim_L \mathrm{Tor}_i^{S_L}(\widehat{M}, L)_j$. By the Künneth formula [75, Th. 3.6.1], $\mathrm{Tor}_i^{S_L}(\widehat{M}, L)$ is the localization of $\mathrm{Tor}_i^S(M, R)$ at R^* and

$$\mathrm{Tor}_i^{S_L}(\widehat{M}, L)_j \cong \mathrm{Tor}_i^S(M, R)_j \otimes L.$$

Thus by eq. (2.3) it suffices to examine the structure of $\mathrm{Tor}_1^S(\Pi_{i-1}, R)_j$ as an R-module. We consider the tensor product $F_\bullet \otimes R$. By (2.4) we have the s.e.s.

$$0 \to \mathrm{Tor}_1^S(\Pi_{i-1}, R)_j \to \Pi_{i,j} \otimes R \longrightarrow R^{\beta_{i-1,j}} \longrightarrow \Pi_{i-1,j} \otimes R \to 0.$$

Since $(\Pi_{i,j} \otimes R) / \mathrm{Tor}_1^S(\Pi_{i-1}, R)_j \hookrightarrow R^{\beta_{i-1,j}}$, it follows that $(\Pi_{i,j} \otimes R) / \mathrm{Tor}_1^S(\Pi_{i-1}, R)_j$ is free and

$$\operatorname{tor}(\operatorname{Tor}_1^S(\Pi_{i-1}, R)_j) = \operatorname{tor}(\Pi_{i,j} \otimes R).$$

Thus

$$(\Pi_{i,j} \otimes R) / \operatorname{Tor}_1^S(\Pi_{i-1}, R)_j = R^{f_{i,j} - s_{i-1,j}}.$$

By the short exact sequence

$$\begin{array}{ccccccc} 0 \to (\Pi_{i,j} \otimes R) / \operatorname{Tor}_1^S(\Pi_{i-1}, R)_j & \to & R^{\beta_{i-1,j}} & \longrightarrow & \Pi_{i-1,j} \otimes R & \longrightarrow & 0 \\ \| & & & & \| & & \\ R^{f_{i,j} - s_{i-1,j}} & & & & R^{f_{i-1,j}} \oplus \operatorname{tor}(\Pi_{i-1,j} \otimes R) & & \end{array}$$

we have that

- $\beta_{i-1,j} = f_{i-1,j} + t_{i-1,j}$, from the epimorphism and
- $\beta_{i-1,j} = (f_{i,j} - s_{i-1,j}) + f_{i-1,j}$, from the additivity of ranks.

It follows that the rank $s_{i-1,j}$ of $\operatorname{Tor}_1^S(\Pi_{i-1}, R)_j$ is equal to

$$s_{i-1,j} = (f_{i,j} + f_{i-1,j}) - \beta_{i-1,j} = f_{i,j} + (f_{i-1,j} - \beta_{i-1,j}) = (\beta_{i,j} - t_{i,j}) - t_{i-1,j}.$$

We tensor $\operatorname{Tor}_i^S(R, M)_j$ with L to obtain that

$$\beta_{i,j}(\widehat{M}) = s_{i-1,j} = \beta_{i,j}(M) - t_{i,j} - t_{i-1,j}.$$

□

We believe that Theorem 2.2.2 could give a new obstruction to the problem of lifting curves with automorphisms. We give a sketch of the argument in the remark below which can be further developed in future work:

Remark 2.2.3. *With the notation of Section 1.6, let $\mathscr{X}_0 \to \operatorname{Spec}(k)$ be a smooth projective curve over k and let G be a finite subgroup of its automorphism group. The following three assumptions yield a contradiction:*

1. *The graded pieces of the syzygy modules of the canonical ring $\widehat{M} = S_{\mathscr{X}_0}$ are indecomposable as $k[G]$-modules.*
2. *The pair $(\mathscr{X}_0, G)$ lifts to characteristic 0.*

3. *The graded pieces* $\Pi_{i,j} \otimes R$ *of the syzygy modules of the relative canonical ring* $M = S_{\mathscr{X}}$ *have torsion, i.e.* $t_{i,j} \neq 0$.

Indeed, the decomposition of $\Pi_{i,j} \otimes R$ *into its free part and its torsion part should be respected by the action of* G *and this situtation should be carried out in the special fiber. This implies that the syzygy modules on the special fiber* $\Pi_{i,j} \otimes k$ *cannot be indecomposable.*

2.3 Smith normal forms and Betti numbers

How does one compute $t_{i,j}$? We claim that this can be done by computing the Smith normal form of the matrix of differentials $\delta_{i,j} \otimes 1_R$. We first recall the definition:

Definition 2.3.1. *Let* B *be an* $n \times m$ *matrix with entries in* R. *There exist invertible matrices* S, T *with entries in* R *such that the product* $A = SBT$ *is of the form*

$$A = \begin{pmatrix} b_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & b_t & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \qquad \textit{with } b_1 \mid b_2 \mid \ldots \mid b_t \ .$$

The matrix A *is called the* ***Smith normal form*** *of* B.

We proceed as in [21]: note that $R^{\beta_{i,j}} = F_{i,j} \otimes R$, while $R^{\beta_{i-1,j}} = F_{i-1,j} \otimes R$ and $\delta_{i,j} \otimes 1_R : R^{\beta_{i,j}} \longrightarrow R^{\beta_{i-1,j}}$. Let $B_{i,j}$ be the matrix of $\delta_{i,j} \otimes 1_R$ with respect to the canonical bases of $R^{\beta_{i,j}}$ and $R^{\beta_{i-1,j}}$. There is a change of basis for $R^{\beta_{i,j}}$ and $R^{\beta_{i-1,j}}$ so that the matrix of $\delta_{i,j} \otimes 1_R$ with respect to these new bases is the *Smith normal form* of $B_{i,j}$, say $A_{i,j}$. The Smith normal form $A_{i,j}$ contains an upper left diagonal block

$$\operatorname{diag}\left(b_1, b_2, \ldots, b_{t(i-1,j)}\right),$$

with $b_1 | b_2 | \cdots | b_{t(i-1,j)} \neq 0$, while the rest of the blocks of $A_{i,j}$ are zero. We note that since $F_\bullet$ is a minimal resolution, all $b_a \in \mathfrak{m}_R$, for $a = 1, \ldots, t(i-1,j)$ and thus $b_a = x^{e(a)}$, for some positive integer $e(a)$. It is clear that $t(i-1,j)$ is the rank of $\operatorname{Im}(\delta_{i,j} \otimes 1_R)$ and thus the rank of $\ker(\delta_{i,j} \otimes 1_R)$ equals $\beta_{i,j} - t(i-1,j)$. Let us now consider the Smith normal form of $\delta_{i+1,j} \otimes 1_R$. Suppose that its nonzero block is

$$\operatorname{diag}\left(c_1, c_2, \ldots, c_{t(i,j)}\right)$$

and let $\epsilon_1, \ldots, \epsilon_{\beta_{i,j}}$ be the basis of $R^{\beta_{i,j}}$ relative to this normal form. Thus $c_a \epsilon_a \in \operatorname{Im}\left(\delta_{i+1,j} \otimes 1_R\right)$. Since $(\delta_{i,j} \otimes 1_R) \circ (\delta_{i+1,j} \otimes 1_R) = 0$, we have that $(\delta_{i,j} \otimes 1_R)(c_a \epsilon_a) = c_a (\delta_{i,j} \otimes 1_R)(\epsilon_a) = 0$, and we conclude that $\epsilon_1, \ldots, \epsilon_{t(i,j)}$ are in $\ker(\delta_{i,j} \otimes 1_R)$, for $a = 1, \ldots, t(i,j)$. Thus,

$$\operatorname{Tor}_i^S(M,R)_j \cong R^{\beta_{i,j} - t(i-1,j) - t(i,j)} \oplus R/c_1 R \oplus \cdots \oplus R/c_{t(i,j)} R.$$

By the uniqueness of the decomposition of $\operatorname{Tor}_i^S(M,R)_j$ and induction on i, it follows that $t(i,j) = t_{i,j}$ for all i. We have shown the following:

Corollary 2.3.2. *If $(F_\bullet, \delta_\bullet)$ is a minimal graded free resolution of M over S and the Smith normal form of the matrix of $\delta_{a,j} \otimes 1_R$ has rank $t(a-1,j)$, $a \geq 1$, then $t_{i,j} = t(i,j)$ for $i \geq 0$ and $\beta_{i,j}(\widehat{M}) = \beta_{i,j}(M) - t_{i,j} - t_{i-1,j}$.*

Let us now return to Example 2.0.2. Let $\mathbb{Z}_2$ be the ring of 2-adic integers with fraction field $\mathbb{Q}_2$ and residue field $\mathbb{F}_2$. Let $S = \mathbb{Z}_2[a, \ldots, h]$, $\mathfrak{m} = \langle a, \ldots, h \rangle$ and $M = S/B$ where $B = \langle abc, abf, ace, ahe, ahf, bch, bhe, bef, chf, cef \rangle$. We will show that

$$\beta_{0,0}(M) = 1, \beta_{1,3}(M) = 10, \beta_{2,4}(M) = 15, \beta_{3,5}(M) = 6, \beta_{3,6}(M) = 1, \beta_{4,6}(M) = 1,$$

and that M has a minimal graded free resolution over S of the form

$$\begin{array}{ccccccccccccc} & F_4 & & F_3 & & F_2 & & F_1 & & F_0 & & & \\ & \| & & \| & & \| & & \| & & \| & & & \\ 0 \longrightarrow & S(-6) & \xrightarrow{\delta_4} & S(-5)^6 \oplus S(-6) & \xrightarrow{\delta_3} & S(-4)^{15} & \xrightarrow{\delta_2} & S(-3)^{10} & \xrightarrow{\delta_1} & S & \longrightarrow & M & \longrightarrow 0 \end{array} \tag{2.6}$$

We will show that Π_4, the kernel of $\delta_3 : F_3 \to F_2$, has a minimal generating set of two elements, with the generator of degree 6 becoming torsion in $F_\bullet \otimes_S R \cong F_\bullet \otimes S/\mathfrak{m}$, implying that $t_{4,6} = 1$. This means that for $\widehat{S} = \mathbb{Q}_2[a, \ldots, h]$, we get the following exact diagram:

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & \widehat{S}(-6) & \xrightarrow{\delta_4} & \widehat{S}(-5)^6 \oplus \widehat{S}(-6) & \xrightarrow{\delta_3} & \widehat{S}(-4)^{15} & \xrightarrow{\delta_2} & \widehat{S}(-3)^{10} & \xrightarrow{\delta_1} & \widehat{S} & \longrightarrow & \widehat{M} & \longrightarrow 0 \\ & \uparrow & & \uparrow & & & & & & & & & \\ 0 \longrightarrow & \widehat{S}(-6) & \xrightarrow{\cong} & \widehat{S}(-6) & \longrightarrow & 0 & & & & & & & \end{array}$$

The degree 6 elements in both F_4 and F_3 have to be removed in order to obtain a minimal free resolution in the generic fiber. We used Macaulay2 [25] in order to compute the above resolution. The following code

```
T = ZZ[a,b,c,e,f,h]

J = ideal(a*b*c,a*b*f,a*c*e,a*h*e,a*h*f,b*c*h,b*h*e,b*e*f,c*h*f,c*e*f)

rs = res J

rs.dd
```

produces the free resolution $G_\bullet$ of T/J over T

$$0 \longrightarrow T^2 \xrightarrow{\theta_4} T^{10} \xrightarrow{\theta_3} T^{17} \xrightarrow{\theta_2} T^{10} \xrightarrow{\theta_1} T, \tag{2.7}$$

where the differentials θ_3, θ_4 correspond to the matrices (also denoted for simplicity by θ_3, θ_4)

$$\theta_4 = \begin{pmatrix} 0 & f \\ e & 0 \\ -b & 0 \\ -h & 0 \\ 0 & -c \\ -c & 0 \\ 0 & a \\ a & 0 \\ \boxed{-1} & \boxed{1} \\ \boxed{-1} & \boxed{-1} \end{pmatrix} \qquad \theta_3 = \begin{pmatrix} 0 & -h & 0 & e & 0 & 0 & 0 & 0 & -eh & -eh \\ -h & 0 & 0 & -f & 0 & 0 & 0 & 0 & fh & 0 \\ -b & 0 & -f & 0 & 0 & 0 & 0 & 0 & bf & 0 \\ 0 & 0 & -c & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & 0 & -e & 0 & 0 & 0 & 0 \\ e & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ef \\ a & 0 & 0 & 0 & 0 & 0 & -f & 0 & 0 & 0 \\ -c & 0 & 0 & 0 & -f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & h & 0 & 0 & 0 & 0 & -ch \\ 0 & a & 0 & 0 & 0 & 0 & 0 & -e & 0 & 0 \\ 0 & -b & -e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & h & 0 & 0 \\ 0 & 0 & h & -b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{-1} & \boxed{1} & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{-1} & \boxed{-1} & 0 & -a \end{pmatrix}$$

so for bases $\{r_1,\ r_2\}$ of T^2 and $\{s_1, \ldots, s_{10}\}$ of T^{10}, the map $T^2 \xrightarrow{\theta_4} T^{10}$ is given by

$$\begin{aligned} r_1 &\mapsto es_2 - bs_3 - hs_4 - cs_6 + as_8 - s_9 - s_{10}, \\ r_2 &\mapsto fs_1 - cs_5 + as_7 + s_9 - s_{10} \end{aligned}$$

Notice that the matrix of θ_4 is reduced modulo $\langle a, \ldots, h\rangle$ to a 10×2 matrix, which is zero in all entries except for the lower 2×2 submatrix, i.e.

$$\theta_4 \otimes 1_R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \boxed{-1} & \boxed{1} \\ \boxed{-1} & \boxed{-1} \end{pmatrix}$$

We see that

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

and so the Smith Normal Form of $\theta_4 \otimes 1_R$ is given by

$$\mathrm{SNF}(\theta_4 \otimes 1_R) = \begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The only prime that divides a diagonal non-zero entry of $\mathrm{SNF}(\theta_4 \otimes 1_R)$ is 2. A similar computation shows that the reduction of the matrix of θ_3 modulo $\langle a, \ldots, h \rangle$ has a Smith normal form whose nonzero diagonal block is the 2×2 identity matrix. We conclude that the Betti numbers differ only in characteristic 2.

Let $S = \mathbb{Z}_2$. After a change of basis we get that $G_\bullet \otimes 1_S$ breaks into

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & S & \xrightarrow{\cong} & S & \longrightarrow & 0 & & & & \\
 & & \oplus & & \oplus & & & & & & \\
0 & \longrightarrow & S & \xrightarrow{\delta_4} & S^7 & \xrightarrow{\delta_3} & S^{15} & \xrightarrow{\delta_2} & S^{10} & \xrightarrow{\delta_1} & S \\
 & & & & \oplus & & \oplus & & & & \\
 & & 0 & \longrightarrow & S^2 & \xrightarrow{\cong} & S^2 & \longrightarrow & 0 & &
\end{array}$$

The middle row above gives the minimal graded free resolution $(F_\bullet, \delta_\bullet)$ of B in S. In particular, with respect to the appropriate basis of S^7, the differential δ_4 is

$$\delta_4 = \begin{pmatrix} -f & e & -b & h & c & a & 2 \end{pmatrix}^T$$

and we can see that the kernel of $\delta_3 \otimes_S 1_{S/\mathfrak{m}}$ is isomorphic to $\mathbb{Z}_2^6 \oplus \mathbb{Z}_2/2\mathbb{Z}_2$. The minimal graded free resolution of B in $\overline{S} = \mathbb{F}_2[a, \ldots, h]$ is

$$\overline{F}_\bullet : 0 \longrightarrow \overline{S} \xrightarrow{\overline{\delta}_4} \overline{S}^7 \xrightarrow{\overline{\delta}_3} \overline{S}^{15} \xrightarrow{\overline{\delta}_2} \overline{S}^{10} \xrightarrow{\overline{\delta}_1} \overline{S}$$

where

$$\overline{\delta}_4 = \begin{pmatrix} -f & e & -b & h & c & a & 0 \end{pmatrix}^T,$$

while the minimal graded free resolution of B in $\widehat{S} = \mathbb{Q}_2[a, \ldots, h]$ is given by the second row of

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \widehat{S} & \longrightarrow & \widehat{S} & \longrightarrow & 0 & & & & \\
 & & \oplus & & \oplus & & & & & & \\
 & & 0 & \longrightarrow & \widehat{S}^6 & \longrightarrow & \widehat{S}^{15} & \longrightarrow & \widehat{S}^{10} & \longrightarrow & \widehat{S}
\end{array}$$

Let us now consider B in $S = \mathbb{Z}_p[a, \ldots, h]$, where p is a prime, $p \neq 2$. We note that 2 is now a unit and through a series of base changes $G_\bullet \otimes S$ breaks into

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & S^2 & \xrightarrow{\cong} & S^2 & \longrightarrow & 0 & & & & \\
 & & & & \oplus & & & & & & \\
 & & 0 & \longrightarrow & S^6 & \xrightarrow{\delta_3} & S^{15} & \xrightarrow{\delta_2} & S^{10} & \xrightarrow{\delta_1} & S \\
 & & & & \oplus & & \oplus & & & & \\
 & & 0 & \longrightarrow & S^2 & \xrightarrow{\cong} & S^2 & \longrightarrow & 0 & &
\end{array}$$

where the middle row above gives the minimal graded free resolution S/B. In this case the Betti numbers of $M = S/B$ in the special and generic fiber coincide. The above example leads us to the following algorithm.

Algorithm 2.1: Testing whether the minimal free resolution depends on the characteristic of the base field.

Input: Homogeneous elements $f_1, \ldots, f_s \in T = \mathbb{Z}[W_1, \ldots, W_g]$.

Output: The set of primes p for which the Betti numbers of $I = \langle f_1, \ldots, f_s \rangle$ in characteristic p differ from characteristic 0.

Method:

1. Compute a graded free resolution $(G_\bullet, \theta_\bullet)$ of T/I.
2. Let A_i be the corresponding matrices of the differentials, for $i \geq 1$. Set $W_1, \ldots, W_g = 0$ for all entries of A_i to obtain the matrices B_i, for $i \geq 1$.
3. Compute the Smith normal form of B_i, for $i \geq 1$.
4. Collect all primes p that divide some nonzero entry of the Smith normal form of B_i, for $i \geq 1$.

We note that given a graded ideal I of $\mathbb{Z}[W_1, \ldots, W_g]$, the above algorithm indicates the primes for which the Betti numbers of $I \cdot \mathbb{Q}_p[W_1, \ldots, W_g]$ differ from the Betti numbers of $I \cdot \mathbb{F}_p[W_1, \ldots, W_g]$ and provides a possible obstruction to the lifting problem as discussed in Remark 2.2.3.

Chapter 3

The Canonical Ideal of Relative Curves

The results of this chapter can be found in [17] which has been submitted for publication.

Let k be an algebraically closed field of prime characteristic $p \geq 5$. For $m \in \mathbb{N}$ prime to p, we consider the Artin-Schreier curve over k with affine model

$$X^p - X - x^{-m} \in k(x)[X].$$

We write $W(k)$ for the ring of Witt vectors over k, see also Section A.3, and ζ for a p-th root of unity. We define $q, \ell \in \mathbb{N}$ via the relation $m = pq - \ell,\ 1 \leq \ell \leq p-1$, and consider the Oort-Sekiquchi-Suwa ring, see Section A.4,

$$R = \begin{cases} W(k)[\zeta][[x_1, \ldots, x_q]] & \text{if } \ell = 1, \\ W(k)[\zeta][[x_1, \ldots, x_{q-1}]] & \text{if } \ell \neq 1. \end{cases}$$

A model for the equicharacteristic deformation of the Artin-Schreier curve over R is

$$X^p - X - \frac{x^\ell}{a(x)^p} \in K(x)[X].$$

where

$$K = \begin{cases} k((x_1, \ldots, x_q)) & \text{if } \ell = 1, \\ k((x_1, \ldots, x_{q-1})) & \text{if } \ell \neq 1. \end{cases}$$

and

$$a(x) = \begin{cases} x^q + x_1 x^{q-1} + \cdots + +x_{q-1}x + x_q, & \text{if } \ell = 1 \\ x^q + x_1 x^{q-1} + \cdots + x_{q-1}x, & \text{if } \ell \neq 1. \end{cases} \tag{3.1}$$

Let $\lambda = \zeta - 1$. By Theorem A.4.2, there exists a flat family of curves $\mathscr{X} \to \mathrm{Spec}(R)$ with generic fiber $\mathscr{X}_\eta \to \mathrm{Spec}(L)$ and special fiber $\mathscr{X}_0 \to \mathrm{Spec}(K)$ given by

$$\mathscr{X}_\eta : y^p = \lambda^p x^\ell + a(x)^p \text{ and } \mathscr{X}_0 : X^p - X = \frac{x^\ell}{a(x)^p} \tag{3.2}$$

where $y = a(x)(\lambda X + 1)$. In Section A.1 and Section A.2 we explain that the fibers given in eq. (3.2) are cyclic ramified coverings of the projective line over the respective field of genus $g = \frac{(p-1)(m-1)}{2}$. In [12], H. Boseck gives an explicit description of bases for the global sections of holomorphic differentials of such covers. In our context, Boseck's bases for $H^0(\mathscr{X}_\eta, \Omega_{\mathscr{X}_\eta/L})$ and $H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/K})$ respectively are

$$\begin{aligned} \mathbf{b}_\eta &= \left\{ x^N y^{-\mu} dx : \left\lfloor \frac{\mu\ell}{p} \right\rfloor \leq N \leq \mu q - 2,\ 1 \leq \mu \leq p-1 \right\} \\ \mathbf{b}_0 &= \left\{ x^N a(x)^{p-1-\mu} X^{p-1-\mu} dx : \left\lfloor \frac{\mu\ell}{p} \right\rfloor \leq N \leq \mu q - 2,\ 1 \leq \mu \leq p-1 \right\}. \end{aligned} \tag{3.3}$$

Using this analysis, the authors of [37] found an explicit basis for the global sections of holomorphic differentials on the relative curve $\mathscr{X} \to \mathrm{Spec}(R)$

$$\mathbf{b} = \left\{ x^N a(x)^{p-1-\mu} \frac{X^{p-1-\mu}}{a(x)^{p-1}(\lambda X + 1)^{p-1}} dx : \left\lfloor \frac{\mu\ell}{p} \right\rfloor \leq N \leq \mu q - 2,\ 1 \leq \mu \leq p-1 \right\}. \tag{3.4}$$

We remark that all three bases $\mathbf{b}_\eta, \mathbf{b}_0$ and $\mathbf{b}$ are Boseck-type bases, see Definition 1.3.1. The respective index set A is given by

$$A = \left\{ (N, \mu) : \left\lfloor \frac{\mu\ell}{p} \right\rfloor \leq N \leq \mu q - 2,\ 1 \leq \mu \leq p-1 \right\} \subseteq \mathbb{N}^2, \tag{3.5}$$

and property (1) of Definition 1.3.1 is satisfied trivially as multiplication of 1-differentials is additive on the exponents, which in turn are linear in N, μ. The same argument shows that property (2) of Definition 1.3.1 is satisfied, although the respective sets D are different in each case and will be defined and treated separately in the subsequent sections.

3.1 The index set A of the family

We start with the following simplified version of Boseck's proof in [12, eq. (34) p. 48]:

Proposition 3.1.1. *Let A be as in eq. (3.7). Then $|A| = g$.*

Proof. The definition of A in terms of bounding inequalities in eq. (3.7) implies that for each value of μ satisfying $1 \leq \mu \leq p-1$ there exist $(\mu q - 2) - \left\lfloor \frac{\mu \ell}{p} \right\rfloor + 1 = \mu q - \left\lfloor \frac{\mu \ell}{p} \right\rfloor - 1$ many points $(N, \mu) \in A$. This gives that

$$|A| = \sum_{\mu=1}^{p-1} \left(\mu q - \left\lfloor \frac{\mu \ell}{p} \right\rfloor - 1 \right) = \sum_{\mu=1}^{p-1} (\mu q - 1) - \sum_{\mu=1}^{p-1} \left\lfloor \frac{\mu \ell}{p} \right\rfloor . \tag{3.6}$$

By the definition of the floor function we have that $\mu \ell = \left\lfloor \frac{\mu \ell}{p} \right\rfloor p + r_\mu$ with $1 \leq r_\mu \leq p-1$, since $p \nmid \mu$. If $r_\mu = r_{\mu'}$ for $1 \leq \mu, \mu' \leq p-1$ then $\mu \ell \equiv \mu' \ell \mod p$ which in turn implies that $\mu = \mu'$ since $p \nmid \ell$. Hence, as μ varies through all integers between 1 an $p-1$, so does r_μ. This allows us to write

$$\sum_{\mu=1}^{p-1} \left\lfloor \frac{\mu \ell}{p} \right\rfloor = \sum_{\mu=1}^{p-1} \frac{\mu \ell - r_\mu}{p} = \frac{\ell}{p} \sum_{\mu=1}^{p=1} \mu - \frac{1}{p} \sum_{\mu=1}^{p-1} r_\mu = \frac{\ell - 1}{p} \sum_{\mu=1}^{p-1} \mu = \frac{(\ell-1)(p-1)}{2}.$$

We thus rewrite eq. (3.8)

$$\begin{aligned} |A| = \sum_{\mu=1}^{p-1} (\mu q - 1) - \sum_{\mu=1}^{p-1} \left\lfloor \frac{\mu \ell}{p} \right\rfloor &= \frac{p(p-1)q}{2} - (p-1) - \frac{(\ell-1)(p-1)}{2} \\ &= \frac{(p-1)\left(pq - 2 - (\ell - 1)\right)}{2} \\ &= \frac{(p-1)(m-1)}{2}. \end{aligned}$$

□

The combinatorial criterion of Proposition 1.2.2 and the analysis following Proposition 1.5.2 leads us to study the Minkowski sum $A + A$ defined in eq. (1.10).

Example 3.1.2. *Let $p = 5$, $q = 2$ and $\ell = 4$ so that $k = \overline{\mathbb{F}}_5$ and $L = \mathrm{Frac}\left(W(\overline{\mathbb{F}}_5)[\zeta_5][[x_1]]\right)$. We consider the Kummer extension of $L(x)$ given by $\mathscr{X}_\eta : y^5 = \lambda^5 x^4 + \left(x^2 + x_1 x\right)^5$ and the Artin-Schreier extension of $K(x)$ given by $\mathscr{X}_0 : X^5 - X = \frac{x^4}{(x^2+x_1 x)^5}$. Both curves are degree 5 cyclic covers of genus $g = 10$ and the respective bases of the global sections of* *holomorphic differentials are determined by the set*

$$\begin{aligned} A &= \left\{(N,\mu)\in\mathbb{Z}^2 : \left\lfloor \frac{4\mu}{5}\right\rfloor \le N \le 2\mu-2,\ 1\le\mu\le 4\right\} \\ &= \{(0,1),(1,2),(2,2),(2,3),(3,3),(4,3),(3,4),(4,4),(5,4),(6,4)\}. \end{aligned}$$

which is depicted in the figure below.

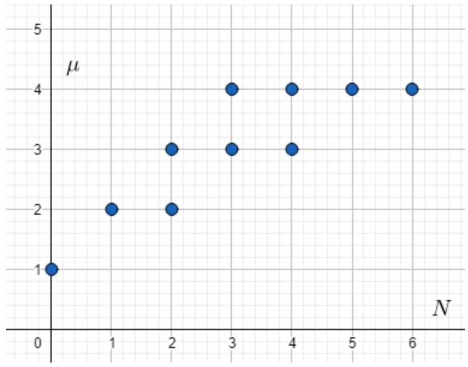


Figure 3.1: The set A for $p=5,\ q=2,\ \ell=4$.

The Minkowski sum $A+A$ in this case equals

$$\begin{aligned} A+A = \{&(0,2),(1,3),(2,3),(2,4),(3,4),(4,4),(3,5),(4,5),(5,5),(6,5),\\ &(4,6),(5,6),(6,6),(7,6),(8,6),(5,7),(6,7),(7,7),(8,7),(9,7),(10,7),\\ &(6,8),(7,8),(8,8),(9,8),(10,8),(11,8),(12,8)\} \end{aligned}$$

and is depicted in the figure below

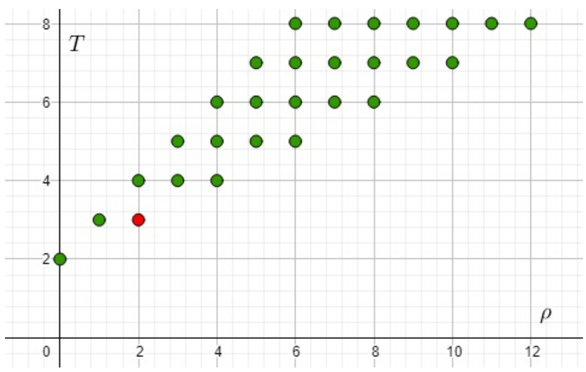


Figure 3.2: The Minkowski sum $A+A$ for $p=5,\ q=2,\ \ell=4$.

Observe that $|A+A|=28$ whereas the expected dimension of the degree 2 part of the *canonical ideal - for both fibers - is $3(g-1) = 27$. The difference is due to the red-colored point $(2,3)$ which corresponds to the subset C_D of $A + A$ introduced in Proposition 1.5.2. We will explain this behavior after we give some more details on $A + A$ and its subsets.*

We proceed with the description of $A + A$ in terms of bounding inequalities:

Lemma 3.1.3. *Let A be as in eq. (3.7). Then*

$$A + A = \{(\rho, T) : 2 \leq T \leq 2(p-1), b(T) \leq \rho \leq Tq - 4\} \subseteq \mathbb{N}^2$$

where

$$b(T) = \begin{cases} \left\lfloor \frac{T\ell}{p} \right\rfloor, & \text{if } \forall\, \mu, \mu' \geq 1 \text{ with } T = \mu + \mu' \text{ we have } \left\lfloor \frac{\mu\ell}{p} \right\rfloor + \left\lfloor \frac{\mu'\ell}{p} \right\rfloor = \left\lfloor \frac{T\ell}{p} \right\rfloor \\ \\ \left\lfloor \frac{T\ell}{p} \right\rfloor - 1, & \text{if } \exists\, \mu, \mu' \geq 1 \text{ with } T = \mu + \mu' \text{ and } \left\lfloor \frac{\mu\ell}{p} \right\rfloor + \left\lfloor \frac{\mu'\ell}{p} \right\rfloor = \left\lfloor \frac{T\ell}{p} \right\rfloor - 1. \end{cases}$$

Proof. By definition

$$(\rho, T) \in A + A \Leftrightarrow \exists\, (N, \mu), (N', \mu') \in A \times A \text{ with } (\rho, T) = (N, \mu) + (N', \mu').$$

Hence, both bounds for T as well as the upper bound of ρ are directly given by the respective bounds for N and μ in the description of A given in eq. (3.7). The formula for $b(T)$ is deduced by the property of the floor function $\lfloor x + y \rfloor - 1 \leq \lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$. □

We proceed with the study of the set C_D defined in Proposition 1.5.2. The equations (3.2) defining the two fibers can be rewritten to match the form of eq. (1.12) as follows

$$\mathscr{X}_\eta : 1 - \lambda^p x^\ell y^{-p} + a(x)^p y^{-p} = 0 \text{ and } \mathscr{X}_0 : 1 - X^{-(p-1)} - x^\ell a(x)^{-p} X^{-p} = 0. \tag{3.7}$$

This leads to the study of the following subsets of $A + A$, for $0 \leq i \leq p$:

$$C(i) = \{(\rho, T) \in A + A \;:\; (\rho + \ell, T + p) \text{ and } (\rho + j, T + p - i) \in A + A \text{ for } j_{\min}(i) \leq j \leq (p-i)q\}, \tag{3.8}$$

where

$$j_{\min}(i) = \begin{cases} 0, \text{ if } \ell = 1 \\ p - i, \text{ if } \ell \neq 1. \end{cases}$$

We will explain the relation between the set C_D of Proposition 1.5.2 and the sets $C(i)$ of eq. (3.10) in Proposition 3.2.4 and Proposition 3.3.4. First, we show that, for our purposes, it suffices to study the case for $i = 0$:

Lemma 3.1.4. *We have that $C(0) \subseteq C(i)$ for all $0 \leq i \leq p$.*

Proof. First show that for any $\alpha \in \mathbb{N}$,

$$b(T + \alpha) \leq b(T) + \alpha. \tag{3.9}$$

Indeed, the result follows trivially for $\alpha = 0$, whereas for $\alpha \geq 1$, since $\ell < p$, we have that

$$b(T+\alpha) \leq \left\lfloor \frac{(T+\alpha)\ell}{p} \right\rfloor = \left\lfloor \frac{T\ell}{p} + \frac{\alpha\ell}{p} \right\rfloor < \left\lfloor \frac{T\ell}{p} + \alpha \right\rfloor = \left\lfloor \frac{T\ell}{p} \right\rfloor + \alpha \leq b(T) + 1 + \alpha$$

and changing $<$ to $\leq$ gives $b(T + \alpha) \leq b(T) + \alpha$. Next, let $(\rho, T) \in C(0)$, so that, by Lemma 3.1.5, $b(T) \leq \rho \leq Tq - 4$ and $2 \leq T \leq p-2$. To show that $(\rho, T) \in C(i)$ it suffices to show that $(\rho + j, T + p - i) \in A + A$ for $j_{\min}(i) \leq j \leq (p-i)q$. First, we observe that

$$2 \leq T \leq T + p - i \leq p - 2 + p - i \leq 2(p-1)$$

and thus

$$\rho + j \leq \rho + (p-i)q \leq Tq - 4 + (p-i)q \leq (T + p - i)q - 4$$

For the lower bound of ρ, we distinguish the following cases:

- If $\ell = 1$, then $j_{\min}(i) = 0$ and

$$b(T + p - i) \leq b(T + p) \leq \rho \leq \rho + j.$$

- If $\ell > 1$ then $j_{\min}(i) = p - i$, and thus by eq. (3.11)

$$b(T + p - i) \leq b(T) + p - i \leq \rho + p - i \leq \rho + j.$$

We conclude that $2 \leq T \leq 2(p-1)$ and that $b(T+p-i) \leq \rho + j \leq (T+p-i)q - 4$ for $j_{\min}(i) \leq j \leq (p-i)q$. Lemma 3.1.3 implies that $(\rho + j, T + p - i) \in A + A$, completing the proof. □

Lemma 3.1.4 above implies that $|(A+A) \setminus C(i)| \leq |(A+A) \setminus C(0)|$ for all $0 \leq i \leq p$ and thus to show the desired $|(A+A) \setminus C(i)| \leq 3(g-1)$, it suffices to study the set $C(0)$. The result below gives a description of $C(0)$ in terms of bounding inequalities:

Lemma 3.1.5. *Let $b(T)$ be as in Lemma 3.1.3. Then*

$$C(0) = \{(\rho, T) \in A + A \ : \ b(T) \leq \rho \leq Tq - 4, \ 2 \leq T \leq p - 2\}$$

Proof. By definition, for all $j_{\min}(0) \leq j \leq pq$ we have that

$$(\rho, T) \in C(0) \Leftrightarrow (\rho, T) \in A + A, \ (\rho + \ell, T + p) \in A + A \text{ and } (\rho + j, T + p) \in A + A$$

Using Lemma 3.1.3 we rewrite

$$(\rho, T) \in C(0) \Leftrightarrow 2 \leq T \leq p-2 \text{ and } \max\{b(T), b(T+p)-\ell, b(T+p)-j_{\min}(0)\} \leq \rho \leq Tq-4.$$

We distinguish the following cases for $\max\{b(T), b(T+p) - \ell, b(T+p) - j_{\min}(0)\}$:

- If $\ell = 1$ then $j_{\min}(0) = 0$ and $b(T) = b(T+p) = 0$ since $\left\lfloor \frac{\mu\ell}{p} \right\rfloor = 0$ for all $1 \leq \mu \leq p-1$. Hence $\max\{b(T), b(T+p) - \ell, b(T+p) - j_{\min}(0)\} = b(T)$.

- If $\ell > 1$ then $j_{\min}(0) = p$, so $b(T+p) - j_{\min}(0) < b(T+p) - \ell$. Choosing an appropriate decomposition $T = \mu + \mu'$ we observe that

  $$b(T) = \left\lfloor \frac{T\ell}{p} \right\rfloor - 1 \Leftrightarrow b(T+p) = \left\lfloor \frac{(T+p)\ell}{p} \right\rfloor - 1.$$

  Finally, since $\left\lfloor \frac{(T+p)\ell}{p} \right\rfloor - \ell = \left\lfloor \frac{T\ell}{p} \right\rfloor$, we deduce that $b(T+p) - \ell = b(T)$, so that

  $$\max\{b(T), b(T+p) - \ell, b(T+p) - j_{\min}(0)\} = b(T).$$

We conclude that in both cases $\max\{b(T), b(T+p)-\ell, b(T+p)-j_{\min}(0)\} = b(T)$, meaning that $(\rho, T) \in C(0) \Leftrightarrow b(T) \leq \rho \leq Tq - 4, \ 2 \leq T \leq p - 2$. □

We conclude this section by proving that the cardinality of $(A+A)\setminus C(i)$ is bounded by $3(g-1)$:

Lemma 3.1.6. $|(A+A)\setminus C(i)| \leq 3(g-1)$, *for all* $0 \leq i \leq p$.

Proof. By Lemma 3.1.4, it suffices to show the result for $i=0$:

$$\begin{aligned}
|(A+A)\setminus C(0)| &= |A+A| - |C(0)| \\
&= \sum_{T=2}^{2(p-1)} (Tq - b(T) - 3) - \sum_{T=2}^{p-2} (Tq - b(T) - 3)\,, \text{ by Lem. 3.1.3 and 3.1.5} \\
&= \sum_{T=p-1}^{2(p-1)} (Tq - b(T) - 3) \\
&< \sum_{T=p-1}^{2(p-1)} \left(Tq - \left\lfloor \frac{T\ell}{p} \right\rfloor - 2\right), \text{ since by Lemma 3.1.3, } b(T) \geq \left\lfloor \frac{T\ell}{p} \right\rfloor - 1 \\
&= \sum_{T=p+1}^{2(p-1)} \left(Tq - \left\lfloor \frac{T\ell}{p} \right\rfloor - 2\right) + \left((p-1)q - \left\lfloor \frac{(p-1)\ell}{p} \right\rfloor - 2\right) \\
&\qquad + \left(pq - \left\lfloor \frac{p\ell}{p} \right\rfloor - 2\right). \qquad (3.10)
\end{aligned}$$

We wish to use the relation

$$\sum_{T=1}^{p-1} \left(Tq - \left\lfloor \frac{T\ell}{p} \right\rfloor - 1\right) = |A| = g \qquad (3.11)$$

so we change the index in the sum of eq. (3.12) by setting $T' = T - p$:

$$\begin{aligned}
\sum_{T=p+1}^{2(p-1)} \left(Tq - \left\lfloor \frac{T\ell}{p} \right\rfloor - 2\right) &= \sum_{T'=1}^{p-2} \left((T'+p)q - \left\lfloor \frac{(T'+p)\ell}{p} \right\rfloor - 2\right) \\
&= \sum_{T'=1}^{p-2} \left(T'q + pq - \left\lfloor \frac{T'\ell}{p} \right\rfloor - \ell - 2\right) \\
&= \sum_{T'=1}^{p-2} \left(T'q - \left\lfloor \frac{T'\ell}{p} \right\rfloor + m - 2\right), \text{ since } pq - \ell = m \\
&= \sum_{T'=1}^{p-2} \left(T'q - \left\lfloor \frac{T'\ell}{p} \right\rfloor - 1\right) + \sum_{T'=1}^{p-2} (m-1) \\
&= \sum_{T'=1}^{p-2} \left(T'q - \left\lfloor \frac{T'\ell}{p} \right\rfloor - 1\right) + (m-1)(p-2). \qquad (3.12)
\end{aligned}$$

Next, we observe that

$$\sum_{T'=1}^{p-2}\left(T'q-\left\lfloor\frac{T'\ell}{p}\right\rfloor-1\right)+\left((p-1)q-\left\lfloor\frac{(p-1)\ell}{p}\right\rfloor-2\right)=\sum_{T'=1}^{p-1}\left(T'q-\left\lfloor\frac{T'\ell}{p}\right\rfloor-1\right)-1. \tag{3.13}$$

Combining relations (3.12), (3.13), (3.14) and (3.15) gives:

$$\begin{aligned} |(A+A)\setminus C(0)| &< \sum_{T'=1}^{p-1}\left(T'q-\left\lfloor\frac{T'\ell}{p}\right\rfloor-1\right)-1+(m-1)(p-2)+\left(pq-\left\lfloor\frac{p\ell}{p}\right\rfloor-2\right) \\ &= g-1+mp-2m-p+2+m-2 \\ &= g+(m-1)(p-1)-2 \\ &= 3g-2 \end{aligned}$$

and changing $<$ to $\leq$ gives the desired

$$|(A+A)\setminus C(0)|\leq 3g-3.$$

□

3.2 The canonical ideal on the generic fiber

The generic fiber $\mathscr{X}_\eta \to \mathrm{Spec}(L)$ of the family $\mathscr{X} \to \mathrm{Spec}(R)$ has affine model

$$\mathscr{X}_\eta : y^p = \lambda^p x^\ell + a(x)^p \tag{3.14}$$

where $a(x)$ is given by eq. (3.1). Recall that by Theorem A.4.2, $\mathscr{X}_\eta \to \mathrm{Spec}(L)$ is a Kummer extension of $L(x)$ of genus $g = \frac{(p-1)(m-1)}{2}$, for $m = pq - \ell$, and that

$$\mathbf{b}_\eta = \left\{x^N y^{-\mu} dx : \left\lfloor\frac{\mu\ell}{p}\right\rfloor \leq N \leq \mu q - 2,\ 1 \leq \mu \leq p-1\right\}. \tag{3.15}$$

is a basis for the g-dimensional L-vector space $H^0(\mathscr{X}_\eta, \Omega_{\mathscr{X}_\eta/L})$ of global sections of holomorphic differentials on $\mathscr{X}_\eta \to \mathrm{Spec}(L)$. The basis $\mathbf{b}_\eta$ is a Boseck-type basis, see Definition 1.3.1, uniquely determined by the set of exponents

$$A = \left\{(N,\mu) \in \mathbb{Z}^2 : \left\lfloor\frac{\mu\ell}{p}\right\rfloor \leq N \leq \mu q - 2,\ 1 \leq \mu \leq p-1\right\}$$

so we let $\{\omega_{N,\mu} : (N,\mu) \in A\}$ be a set of variables indexed by A. The assignment $\omega_{N,\mu} \mapsto x^N y^{-\mu} dx$ gives rise to the canonical map

$$\begin{aligned} \phi_\eta : S_L = L[\{\omega_{N,\mu}\}] &\longrightarrow \bigoplus_{n\geq 0} H^0(\mathscr{X}_\eta, \Omega^{\otimes n}_{\mathscr{X}_\eta/L}), \\ \omega^{a_1}_{N_1,\mu_1}\cdots\omega^{a_d}_{N_d,\mu_d} &\longmapsto x^{(a_1N_1+\cdots+a_dN_d)}y^{-(a_1\mu_1+\cdots+a_d\mu_d)}dx^{\otimes(a_1+\cdots+a_d)} \end{aligned}$$

which is surjective by Petri's Theorem 1.1.1. Since $p \geq 5$, $\mathscr{X}_\eta$ is neither trigonal, nor a plane quintic, so the canonical ideal $I_{\mathscr{X}_\eta} = \ker\phi_\eta$ is generated in degree 2. Following the arguments of Section 1.4, to each variable $\omega_{N,\mu}$ we assign the multidegree $\mathrm{mdeg}(\omega_{N,\mu}) = (1, N, \mu) \in \mathbb{N}^3$ and obtain a multigrading on the polynomial ring S_L

$$\mathrm{mdeg}(\omega_{N_1,\mu_1}\omega_{N_2,\mu_2}\cdots\omega_{N_d,\mu_d}) = (d, N_1+N_2+\cdots N_d, \mu_1+\mu_2+\cdots+\mu_d). \tag{3.16}$$

For example, the canonical embedding of the Kummer curve

$$\mathscr{X}_\eta : \ y^5 = \lambda^5 x^4 + \left(x^2 + x_1 x\right)^5$$

of Example 3.1.2 with

$$A \ = \ \{(0,1),(1,2),(2,2),(2,3),(3,3),(4,3),(3,4),(4,4),(5,4),(6,4)\},$$

is determined by assigning

$$\begin{aligned} &\omega_{0,1} \mapsto y^{-1}dx, \omega_{1,2} \mapsto xy^{-2}dx, \omega_{2,2} \mapsto x^2y^{-2}dx, \omega_{2,3} \mapsto x^2y^{-3}dx, \omega_{3,3} \mapsto x^3y^{-3}dx, \\ &\quad \omega_{4,3} \mapsto x^4y^{-3}dx, \omega_{3,4} \mapsto x^3y^{-4}dx, \omega_{4,4} \mapsto x^4y^{-4}dx, \omega_{5,4} \mapsto x^5y^{-4}dx, \omega_{6,4} \mapsto x^6y^{-4}dx. \end{aligned}$$

Returning to the general case, for each $d \in \mathbb{N}$ we write $\mathbb{T}^d_L$ for the set of monomials of standard degree d in S_L. We remark that, as in Proposition 1.4.1, the following binomials are contained in $I_{\mathscr{X}_\eta}$.

Proposition 3.2.1. *We have that*

$$\begin{aligned} G^\eta_1 = \{&\omega_{N_1,\mu_1}\omega_{N'_1,\mu'_1} - \omega_{N_2,\mu_2}\omega_{N'_2,\mu'_2} \in S_L \ : \ \omega_{N_1,\mu_1}\omega_{N'_1,\mu'_1}, \omega_{N_2,\mu_2}\omega_{N'_2,\mu'_2} \in \mathbb{T}^2_L \\ &\textit{and } \mathrm{mdeg}(\omega_{N_1,\mu_1}\omega_{N'_1,\mu'_1}) = \mathrm{mdeg}(\omega_{N_2,\mu_2}\omega_{N'_2,\mu'_2})\} \subseteq I_{\mathscr{X}_\eta}. \end{aligned}$$

Proof. See Proposition 1.4.1. ☐

We proceed with expanding the p-th power of $a(x)$ to write

$$a(x)^p = \sum_{j=j_{\min}(0)}^{pq} c_{j,p} x^j \tag{3.17}$$

where by eq. (3.1)

$$j_{\min}(0) = \begin{cases} 0, \text{ if } \ell = 1 \\ p, \text{ if } \ell \neq 1 \end{cases} \tag{3.18}$$

and for any $j_{\min}(0) \leq j \leq pq$

$$c_{j,p} = \sum_{\substack{0 \leq t_i < p \\ t_1 + 2t_2 \cdots + q t_q = j}} \binom{p}{t_0, \ldots, t_q} \prod_{i=0}^{q} x_i^{t_i}.$$

Substituting eq. (3.19) into the equivalent form of the equation defining $\mathscr{X}_\eta$ given in eq. (3.9) gives that the set D_η of eq. (1.12) for $\mathscr{X}_\eta \to \operatorname{Spec}(L)$ is

$$D_\eta = \{(\ell, -p), (j, -p) : j_{\min}(0) \leq j \leq pq\} \tag{3.19}$$

Following the discussion that precedes Proposition 1.5.1 we define $M_{D_\eta} \subseteq \mathbb{T}_L^2$ to be the set of monomials $\mathbf{a} \in \mathbb{T}_L^2$ such that for all $(\iota, \kappa) \in D_\eta$ there exists at least one monomial $\mathbf{b} \in \mathbb{T}_L^2$ of multidegree $\operatorname{mdeg}(\mathbf{b}) = \operatorname{mdeg}(\mathbf{a}) + (0, \iota, -\kappa)$, i.e.

$$M_{D_\eta} = \{\mathbf{a} \in \mathbb{T}_L^2 : \forall\, (\iota, \kappa) \in D_\eta,\ \exists\, \mathbf{b} \in \mathbb{T}_L^2 \text{ with } \operatorname{mdeg}(\mathbf{b}) = \operatorname{mdeg}(\mathbf{a}) + (0, \iota, -\kappa)\}. \tag{3.20}$$

Let $\prec$ denote the term order of of Definition 1.4.2 on S_L. Given a monomial $\mathbf{a} = \omega_{N,\mu}\omega_{N',\mu'} \in M_{D_\eta}$, we denote by $\omega_{N'',\mu''}\omega_{N''',\mu'''}$ the minimal monomial of $\mathbb{T}_L^2$ with respect to $\prec$ that satisfies $\operatorname{mdeg}(\omega_{N'',\mu''}\omega_{N''',\mu'''}) = \operatorname{mdeg}(\omega_{N,\mu}\omega_{N',\mu'}) + (0, \ell, p)$. Similarly, for $j_{\min}(0) \leq j \leq pq$ we denote by $\omega_{N_j,\mu_j}\omega_{N'_j,\mu'_j}$ the minimal monomial of $\mathbb{T}_L^2$ with respect to $\prec$ that satisfies $\operatorname{mdeg}(\omega_{N_j,\mu_j}\omega_{N'_j,\mu'_j}) = \operatorname{mdeg}(\omega_{N,\mu}\omega_{N',\mu'}) + (0, j, p)$. This allows us to explicitly construct the following set of polynomials in $I_{\mathscr{X}_\eta}$:

Proposition 3.2.2. *Let D_η be as in eq. (3.21), M_{D_η} be as in eq. (3.22) and let*

$$G_2^\eta = \left\{ \omega_{N,\mu}\omega_{N',\mu'} - \lambda^p \omega_{N'',\mu''}\omega_{N''',\mu'''} - \sum_{j=j_{min}(0)}^{pq} c_{j,p} \cdot \omega_{N_j,\mu_j}\omega_{N'_j,\mu'_j} \in S_L \ : \omega_{N,\mu}\omega_{N',\mu'} \in M_{D_\eta} \right\}.$$

Then $G_2^\eta \subseteq I_X$.

Proof. We follow the arguments of the proof of Proposition 1.5.1. Let

$$f = \omega_{N,\mu}\omega_{N',\mu'} - \lambda^p \omega_{N'',\mu''}\omega_{N''',\mu'''} - \sum_{j=j_{\min}(0)}^{pq} c_{j,p} \cdot \omega_{N_j,\mu_j}\omega_{N'_j,\mu'_j} \in G_2^\eta$$

be a polynomial whose terms satisfy the multidegree relations

$$N'' + N''' = N + N' + \ell \quad , \quad \mu'' + \mu''' = \mu + \mu' + p \text{ and} \tag{3.21}$$
$$N_j + N'_j = N + N' + j \quad , \quad \mu_j + \mu'_j = \mu + \mu' + p.$$

Applying the canonical map ϕ_η to f gives

$$x^{N+N'}y^{-(\mu+\mu')}dx^{\otimes 2} - \lambda^p x^{N''+N'''}y^{-(\mu''+\mu''')}dx^{\otimes 2} - \sum_{j=j_{\min}(0)}^{pq} c_{j,p} \cdot x^{N_j+N'_j}y^{-(\mu_j+\mu'_j)}dx^{\otimes 2}, \tag{3.22}$$

and using the relations of eq. (3.23) we may rewrite eq. (3.24) as

$$x^{N+N'}y^{-(\mu+\mu')}dx^{\otimes 2} - \lambda^p x^{N+N'+\ell}y^{-(\mu+\mu'+p)}dx^{\otimes 2} - \sum_{j=j_{\min}(0)}^{pq} c_{j,p} \cdot x^{N+N'+j}y^{-(\mu+\mu'+p)}dx^{\otimes 2}. \tag{3.23}$$

Factoring out $x^{N+N'}y^{-(\mu+\mu')}dx^{\otimes 2}$ from eq. (3.25) gives

$$x^{N+N'}y^{-(\mu+\mu')}dx^{\otimes 2} \cdot \left(1 - \lambda^p x^\ell y^{-p} - \sum_{j=j_{\min}(0)}^{pq} c_{j,p} x^j y^{-p} \right)$$

and combining with the expansion of $a(x)^p$ in eq. (3.19) we get

$$x^{N+N'}y^{-(\mu+\mu')}dx^{\otimes 2} \cdot \left(1 - \lambda^p x^\ell y^{-p} - a(x)^p y^{-p} \right)$$

which is 0 by eq. (3.16), completing the proof. □

In the context of Example 3.1.2 with

$$A = \{(0,1),(1,2),(2,2),(2,3),(3,3),(4,3),(3,4),(4,4),(5,4),(6,4)\},$$

we give indicatively some binomials contained in G_1^η, since its elements are too many to be listed

$$\omega_{1,2}\omega_{2,2} - \omega_{0,1}\omega_{3,3},\ \omega_{3,3}^2 - \omega_{2,3}\omega_{4,3},\ \omega_{3,4}\omega_{6,4} - \omega_{4,4}\omega_{5,4}.$$

Regarding G_2^η, the expansion of $a(x)^p$ is given by

$$\left(x^2 + x_1 x\right)^5 = x^{10} + (5x_1)x^9 + (10x_1^2)x^8 + (10x_1^3)x^7 + (5x_1^4)x^6 + (x_1^5)x^5$$

and thus one element of G_2^η is

$$\omega_{0,1}\omega_{2,2} - \lambda^p\omega_{3,4}^2 - (x_1^5)\omega_{3,4}\omega_{4,4} - (5x_1^4)\omega_{4,4}^2 - (10x_1^3)\omega_{4,4}\omega_{5,4} - (10x_1^2)\omega_{5,4}^2 - (5x_1)\omega_{5,4}\omega_{6,4} - \omega_{6,4}^2.$$

The reader may verify that $\omega_{0,1}\omega_{2,2}$ is the only monomial that satisfies the multidegree conditions defining M_{D_η} in eq. (3.22), so the above relation is the only in G_2^η .

Returning to the general case, the main result of this section is the following:

Theorem 3.2.3. $I_{\mathscr{X}_\eta} = \langle G_1^\eta \cup G_2^\eta \rangle$.

To prove Theorem 3.2.3 we will use the dimension criterion of Proposition 1.2.2 which requires counting the monomials that do not appear as initial terms of polynomials in $G_1^\eta \cup G_2^\eta$ with respect to $\prec$. As in Section 1.3, the multidegree of an arbitrary monomial $\omega_{N,\mu}\omega_{N',\mu'} \in \mathbb{T}_L^2$ of standard degree 2 can be written as

$$\text{mdeg}\left(\omega_{N,\mu}\omega_{N',\mu'}\right) = (2, N+N', \mu+\mu') = (1,N,\mu) + (1,N',\mu')$$

so we consider the Minkowski sum of A with itself, which by Lemma 3.1.3 is given by

$$A + A = \{(\rho, T) : 2 \leq T \leq 2(p-1), b(T) \leq \rho \leq Tq - 4\}$$

In Example 3.1.2, we saw that the Minkowski sum $A + A$ is given by

$$A + A = \{(0,2),(1,3),(2,3),(2,4),(3,4),(4,4),(3,5),(4,5),(5,5),(6,5),$$
$$(4,6),(5,6),(6,6),(7,6),(8,6),(5,7),(6,7),(7,7),(8,7),(9,7),(10,7),$$
$$(6,8),(7,8),(8,8),(9,8),(10,8),(11,8),(12,8)\}$$

or equivalently

$$A + A = \{(\rho, T) : 2 \leq T \leq 8, b(T) \leq \rho \leq 2T - 4\} \subseteq \mathbb{N}^2,$$

where a direct computation verifies that in this case we always have $b(T) = \lfloor \frac{4T}{5} \rfloor - 1$.

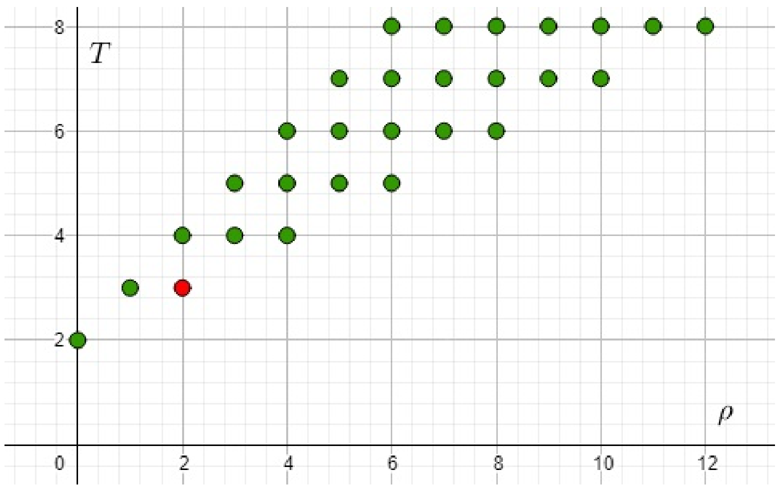


Figure 3.3: The Minkowski sum $A + A$ for $p = 5,\ q = 2,\ \ell = 4$.

By Proposition 1.4.4, we have that the points of $A+A$ are in bijectiion with the monomials that do not appear as initial terms of binomials in G_1^{η}. For example, in multidegree $(2, 7, 7)$ we have three monomials

$$\omega_{2,3}\omega_{5,4},\ \omega_{3,3}\omega_{4,4},\ \omega_{4,3}\omega_{3,4}$$

from which we construct the binomials

$$\omega_{2,3}\omega_{5,4} - \omega_{3,3}\omega_{4,4},\ \omega_{2,3}\omega_{5,4} - \omega_{4,3}\omega_{3,4},\ \omega_{3,3}\omega_{4,4} - \omega_{4,3}\omega_{3,4}.$$

The first binomial has initial term $\omega_{3,3}\omega_{4,4}$ while the second and third have the same initial term $\omega_{4,3}\omega_{3,4}$. Hence $\omega_{2,3}\omega_{5,4}$ is the unique monomial of multidegree $(2,7,7)$ that does not appear as an initial term in G_1^η. On the other hand, the red point $(2,3) \in A+A$ corresponds to the monomial $\omega_{0,1}\omega_{2,2}$, the unique monomial of multidegree $(2,2,3)$ which thus cannot appear in any binomial in G_1^η. However, $\omega_{0,1}\omega_{2,2}$ is the initial term of the relation in G_2^η

$$\omega_{0,1}\omega_{2,2}-\lambda^p\omega_{3,4}^2-(x_1^5)\omega_{3,4}\omega_{4,4}-(5x_1^4)\omega_{4,4}^2-(10x_1^3)\omega_{4,4}\omega_{5,4}-(10x_1^2)\omega_{5,4}^2-(5x_1)\omega_{5,4}\omega_{6,4}-\omega_{6,4}^2$$

which explains the difference between $|A+A|=28$ and $\dim_L\left(I_{\mathscr{X}_\eta}\right)_2 = 3(g-1)=27$.

Returning to the general case and motivated by the definition of G_2^η in Proposition 3.2.2, we consider the subset $C(0)$ of $A+A$ given by eq. (3.10)

$$\begin{aligned} C(0) = \{(\rho,T) \in A+A \mid (\rho+\ell, T+p) \in A+A \\ \text{and } (\rho+j, T+p) \in A+A \text{ for } j_{\min}(0) \leq j \leq pq\}, \end{aligned} \tag{3.24}$$

and observe that, in the context of Example 3.1.2, $C(0)=\{(2,3)\}$ and thus $|(A+A)\setminus C(0)| = 27 = \dim_L\left(I_{\mathscr{X}_\eta}\right)_2$. We prove this in full generality below, following the arguments of Proposition 1.5.2:

Proposition 3.2.4. $|\mathbb{T}_L^2 \setminus \mathrm{in}_\prec\left(G_1^\eta \cup G_2^\eta\right)| \leq |(A+A)\setminus C(0)|$.

Proof. As in the proof of Proposition 1.4.4, we write (ρ,T) for the arbitrary point of $A+A$ and consider the bijective map of sets

$$\begin{aligned} \sigma : A+A &\to \mathbb{T}_L^2 \setminus \mathrm{in}_\prec(G_1^\eta) \\ (\rho,T) &\mapsto \min_\prec\{\omega_{N,\mu}\omega_{N',\mu'} \in \mathbb{T}_L^2 \ : \ \mathrm{mdeg}\left(\omega_{N,\mu}\omega_{N',\mu'}\right) = (\rho,T)\}. \end{aligned}$$

If $(\rho,T)\in C(0)$ then by definition $(\rho,T)\in A+A$, $(\rho+\ell,T+p)\in A+A$ and $(\rho+j,T+p)\in A+A$ for all $j_{\min}(0)\leq j\leq pq$. Hence, the monomials

$$\omega_{N,\mu}\omega_{N',\mu'} := \sigma(\rho,T),\ \omega_{N'',\mu''}\omega_{N''',\mu'''} := \sigma(\rho+\ell,T+p),\ \omega_{N_j,\mu_j}\omega_{N_j',\mu_j'} := \sigma(\rho+j,T+p)$$

give rise to a polynomial

$$g = \omega_{N,\mu}\omega_{N',\mu'} - \lambda^p \omega_{N'',\mu''}\omega_{N''',\mu'''} - \sum_{j=j_{\min}(0)}^{pq} c_{j,p} \cdot \omega_{N_j,\mu_j}\omega_{N'_j,\mu'_j},$$

which, by construction, satisfies $g \in G_2^\eta$ and $\mathrm{in}_\prec(g) = \sigma(\rho, T)$. Hence $\sigma\left(C(0)\right) \subseteq \mathrm{in}_\prec(G_2^\eta)$ and so

$$\sigma\big((A+A) \setminus C(0)\big) \supseteq \mathbb{T}_L^2 \setminus \mathrm{in}_\prec(G_1^\eta \cup G_2^\eta). \tag{3.25}$$

Since σ is one-to-one, eq. (3.27) gives

$$|(A+A) \setminus C(0)| = |\sigma\big((A+A) \setminus C(0)\big)| \geq |\mathbb{T}_L^2 \setminus \mathrm{in}_\prec(G_1^\eta \cup G_2^\eta)|$$

completing the proof. □

We are now ready to prove that $G_1^\eta \cup G_2^\eta$ generates $I_{\mathscr{X}_\eta}$:

Proof of Theorem 3.2.3. By Proposition 3.2.1 and Proposition 3.2.2 we get that $\langle G_1^\eta \cup G_2^\eta \rangle \subseteq I_{\mathscr{X}_\eta}$. By Lemma 3.2.4 and Lemma 3.1.6 we get that $|\mathbb{T}_L^2 \setminus \mathrm{in}_\prec\left(G_1^\eta \cup G_2^\eta\right)| \leq 3(g-1)$. Proposition 1.2.2 implies that $I_{\mathscr{X}_\eta} = \langle G_1^\eta \cup G_2^\eta \rangle$. □

3.3 The canonical ideal on the special fiber

The special fiber $\mathscr{X}_0 \to \mathrm{Spec}(K)$ of the family $\mathscr{X} \to \mathrm{Spec}(R)$ has affine model

$$\mathscr{X}_0 : X^p - X = \frac{x^\ell}{a(x)^p} \tag{3.26}$$

where $a(x)$ is defined in eq. (3.1). Recall that by Theorem A.4.2, $\mathscr{X}_0 \to \mathrm{Spec}(K)$ is an Artin-Schreier extension of $K(x)$ of genus $g = \frac{(p-1)(m-1)}{2}$, for $m = pq - \ell$, and

$$\mathbf{b}_0 \quad = \quad \left\{ x^N a(x)^{p-1-\mu} X^{p-1-\mu} dx : \left\lfloor \frac{\mu\ell}{p} \right\rfloor \leq N \leq \mu q - 2,\ 1 \leq \mu \leq p-1 \right\} \tag{3.27}$$

is a basis for the g-dimensional K-vector space $H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/K})$ of global sections of holomorphic differentials on $\mathscr{X}_0 \to \mathrm{Spec}(K)$. The basis $\mathbf{b}_0$ is a Boseck-type basis, see Definition

1.3.1, uniquely determined by the set of exponents

$$A = \left\{ (N, \mu) \in \mathbb{Z}^2 : \left\lfloor \frac{\mu \ell}{p} \right\rfloor \leq N \leq \mu q - 2,\ 1 \leq \mu \leq p - 1 \right\}$$

so we let $\{w_{N,\mu} : (N, \mu) \in A\}$ be a set of variables indexed by A. The assignment $w_{N,\mu} \mapsto x^N a(x)^{p-1-\mu} X^{p-1-\mu} dx$ gives rise to the canonical map

$$\begin{aligned} \phi_0 : S_K = K[\{w_{N,\mu}\}] &\longrightarrow \bigoplus_{n \geq 0} H^0(\mathscr{X}_0, \Omega^{\otimes n}_{\mathscr{X}_0/K}), \\ w^{a_1}_{N_1,\mu_1} \cdots w^{a_d}_{N_d,\mu_d} &\longmapsto x^{(a_1 N_1 + \cdots + a_d N_d)} \left(a(x) X\right)^{-(a_1 \mu_1 + \cdots + a_d \mu_d)} dx^{\otimes (a_1 + \cdots + a_d)} \end{aligned}$$

which is surjective by Petri's Theorem 1.1.1. Since $p \geq 5$, $\mathscr{X}_0$ is neither trigonal, nor a plane quintic, so the canonical ideal $I_{\mathscr{X}_0} = \ker \phi_0$ is generated in degree 2. Following the arguments of Section 1.4, to each variable $w_{N,\mu}$ we assign the multidegree $\mathrm{mdeg}(w_{N,\mu}) = (1, N, \mu) \in \mathbb{N}^3$ and obtain a multigrading on the polynomial ring S_K

$$\mathrm{mdeg}(w_{N_1,\mu_1} w_{N_2,\mu_2} \cdots w_{N_d,\mu_d}) = (d, N_1 + N_2 + \cdots N_d, \mu_1 + \mu_2 + \cdots + \mu_d). \qquad (3.28)$$

For example, the canonical embedding of the Artin-Schreier curve

$$X^5 - X = \frac{x^4}{(x^2 + x_1 x)^5}$$

of Example 3.1.2 with

$$A = \{(0,1), (1,2), (2,2), (2,3), (3,3), (4,3), (3,4), (4,4), (5,4), (6,4)\},$$

is determined by assigning

$$w_{0,1} \mapsto (x^2 + x_1 x)^3 X^3 dx,\ w_{1,2} \mapsto x(x^2 + x_1 x)^2 X^2 dx,\ w_{2,2} \mapsto x^2 (x^2 + x_1 x)^2 X^2 dx,$$
$$w_{2,3} \mapsto x^2 (x^2 + x_1 x) X dx, w_{3,3} \mapsto x^3 (x^2 + x_1 x) X dx,\ w_{4,3} \mapsto x^4 (x^2 + x_1 x) X dx,$$
$$w_{3,4} \mapsto x^3 dx,\ w_{4,4} \mapsto x^4 dx,\ w_{5,4} \mapsto x^5 dx,\ w_{6,4} \mapsto x^6 dx.$$

Returning to the general case, for each $d \in \mathbb{N}$ we write $\mathbb{T}^d_K$ for the set of monomials of standard degree d in S_K. We remark that, as in Proposition 1.4.1, the following binomials are contained in $I_{\mathscr{X}_0}$.

Proposition 3.3.1. *We have that*

$$G_1^0 = \{w_{N_1,\mu_1} w_{N_1',\mu_1'} - w_{N_2,\mu_2} w_{N_2',\mu_2'} \in S_k \ : \ w_{N_1,\mu_1} w_{N_1',\mu_1'}, w_{N_2,\mu_2} w_{N_2',\mu_2'} \in \mathbb{T}_K^2$$
$$\textit{and } \mathrm{mdeg}(w_{N_1,\mu_1} w_{N_1',\mu_1'}) = \mathrm{mdeg}(w_{N_2,\mu_2} w_{N_2',\mu_2'})\} \subseteq I_{\mathscr{X}_0}.$$

Proof. See Proposition 1.4.1. □

Next, we expand the $(p-1)$-th power of $a(x)$ and write

$$a(x)^{p-1} = \sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1} x^j, \tag{3.29}$$

where by eq. (3.1)

$$j_{\min}(1) = \begin{cases} 0, \text{ if } \ell = 1 \\ p-1, \text{ if } \ell \neq 1 \end{cases} \tag{3.30}$$

and for any $j_{\min}(1) \leq j \leq (p-1)q$

$$c_{j,p-1} = \sum_{\substack{(t_0,\dots,t_q)\in\mathbb{N}^q \\ t_1+2t_2\cdots+qt_q=j}} \binom{p-1}{t_0,\dots,t_q} \prod_{i=0}^{q} x_i^{t_i}.$$

Substituting eq. (3.31) into the equivalent form of the equation defining $\mathscr{X}_0$ given in eq. (3.9) gives that the set D_0 of eq. (1.12) for $\mathscr{X}_0 \to \mathrm{Spec}(K)$ is

$$D_0 = \{(\ell, -p), (j, -(p-1)) : j_{\min}(1) \leq j \leq (p-1)q\} \tag{3.31}$$

Following the discussion that precedes Proposition 1.5.1 we define $M_{D_0} \subseteq \mathbb{T}_K^2$ to be the set of monomials $\mathbf{a} \in \mathbb{T}_K^2$ such that for all $(\iota, \kappa) \in D_0$ there exists at least one monomial $\mathbf{b} \in \mathbb{T}_K^2$ of multidegree $\mathrm{mdeg}(\mathbf{b}) = \mathrm{mdeg}(\mathbf{a}) + (0, \iota, -\kappa)$, i.e.

$$M_{D_0} = \{\mathbf{a} \in \mathbb{T}_K^2 : \forall\ (\iota, \kappa) \in D_0,\ \exists\ \mathbf{b} \in \mathbb{T}_K^2 \text{ with } \mathrm{mdeg}(\mathbf{b}) = \mathrm{mdeg}(\mathbf{a}) + (0, \iota, -\kappa)\}. \tag{3.32}$$

Let $\prec$ denote the term order of of Definition 1.4.2 on S_K. Given a monomial $\mathbf{a} = w_{N,\mu} w_{N',\mu'} \in M_{D_0}$, we denote by $w_{N'',\mu''} w_{N''',\mu'''}$ the minimal monomial of $\mathbb{T}_K^2$ with respect to $\prec$ that satisfies $\mathrm{mdeg}(w_{N'',\mu''}w_{N''',\mu'''}) = \mathrm{mdeg}(w_{N,\mu}w_{N',\mu'}) + (0,\ell,p)$. Similarly, for $j_{\min}(1) \leq j \leq (p-1)q$ we denote by $w_{N_j,\mu_j}w_{N'_j,\mu'_j}$ the minimal monomial of $\mathbb{T}^2_K$ with respect to $\prec$ that satisfies $\mathrm{mdeg}(w_{N_j,\mu_j}w_{N'_j,\mu'_j}) = \mathrm{mdeg}(w_{N,\mu}w_{N',\mu'}) + (0,j,p-1)$. This allows us to explicitly construct the following set of polynomials in $I_{\mathscr{X}_0}$:

Proposition 3.3.2. *Let D_0 be as in eq. (3.33), M_{D_0} be as in eq. (3.34) and let*

$$G_2^0 = \left\{ w_{N,\mu}w_{N',\mu'} - w_{N'',\mu''}w_{N''',\mu'''} - \sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1}w_{N_j,\mu_j}w_{N'_j,\mu'_j} \in S_K \ : \ w_{N,\mu}w_{N',\mu'} \in M_{D_0} \right\}.$$

Then $G_2^0 \subseteq I_{\mathscr{X}_0}$.

Proof. We follow the arguments of the proof of Proposition 1.5.1. Let

$$f := w_{N,\mu}w_{N',\mu'} - w_{N'',\mu''}w_{N''',\mu'''} - \sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1}w_{N_j,\mu_j}w_{N'_j,\mu'_j} \in G_2^0$$

be a polynomial whose terms satisfy the multidegree relations

$$\begin{aligned} N'' + N'''' &= N + N' + \ell \quad , \quad \mu'' + \mu''' = \mu + \mu' + p \\ N_j + N'_j &= N + N' + j \quad , \quad \mu_j + \mu'_j = \mu + \mu' + p - 1. \end{aligned} \tag{3.33}$$

Applying the canonical map ϕ_0 to f gives

$$\begin{aligned} &x^{N+N'}\,(a(x)X)^{2p-(\mu+\mu')}\,dx^{\otimes 2} - x^{N''+N'''}\,(a(x)X)^{2p-(\mu''+\mu''')}\,dx^{\otimes 2} \\ &\qquad - \sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1}x^{N_j+N'_j}\,(a(x)X)^{2p-(\mu_j+\mu'_j)}\,dx^{\otimes 2} \end{aligned} \tag{3.34}$$

and using the relations of eq. (3.35) we may rewrite eq. (3.36) as

$$\begin{aligned} &x^{N+N'}\,(a(x)X)^{2p-(\mu+\mu')}\,dx^{\otimes 2} - x^{N+N'+\ell}\,(a(x)X)^{2p-(\mu+\mu'+p)}\,dx^{\otimes 2} \\ &\qquad - \sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1}x^{N+N'+j}\,(a(x)X)^{2p-(\mu+\mu'+p-1)}\,dx^{\otimes 2} \end{aligned} \tag{3.35}$$

Factoring out ${}^{N+N'}\left(a(x)X\right)^{2p-(\mu+\mu')}\,dx^{\otimes 2}$ from eq. (3.37) gives

$$x^{N+N'}\left(a(x)X\right)^{2p-(\mu+\mu')}\,dx^{\otimes 2}\cdot\left(1-x^{\ell}\left(a(x)X\right)^{-p}-\sum_{j=j_{\min}(1)}^{(p-1)q}c_{j,p-1}x^{j}\left(a(x)X\right)^{-(p-1)}\right)$$

and combining with the expansion of $a(x)^{p-1}$ in eq. (3.31) we get

$$\begin{aligned}
&= x^{N+N'}\left(a(x)X\right)^{2p-(\mu+\mu')}\,dx^{\otimes 2}\cdot\left(1-x^{\ell}\left(a(x)X\right)^{-p}-a(x)^{p-1}\left(a(x)X\right)^{-(p-1)}\right)\\
&= x^{N+N'}\left(a(x)X\right)^{2p-(\mu+\mu')}\,dx^{\otimes 2}\cdot\left(1-x^{\ell}a(x)^{-p}X^{-p}-X^{-(p-1)}\right)
\end{aligned}$$

which is 0 by eq. (3.28), completing the proof. □

In the context of Example 3.1.2 with

$$A = \{(0,1),(1,2),(2,2),(2,3),(3,3),(4,3),(3,4),(4,4),(5,4),(6,4)\},$$

we give indicatively some binomials contained in G_1^0, since its elements are too many to be listed

$$w_{1,2}w_{2,2}-w_{0,1}w_{3,3},\; w_{3,3}^2-w_{2,3}w_{4,3},\; w_{3,4}w_{6,4}-w_{4,4}w_{5,4}.$$

Regarding G_2^0, the expansion of $a(x)^{p-1}$ is given by

$$\left(x^2+x_1x\right)^4 = x^8+(4x_1)x^7+(6x_1^2)x^6+(4x_1^3)x^5+(x_1^4)x^4$$

and thus one element of G_2^0 is

$$w_{0,1}w_{2,2}-w_{3,4}^2-(x_1^4)w_{2,3}w_{4,4}-(4x_1^3)w_{2,3}w_{5,4}-(6x_1^2)w_{3,3}w_{5,4}-(4x_1)w_{3,3}w_{6,4}-w_{4,3}w_{6,4}.$$

As was the case with the generic fiber, the reader may verify that $w_{0,1}w_{2,2}$ is the only monomial that satisfies the multidegree conditions defining M_{D_0} in eq. (3.34), so the only relation in G_2^0 is the one above.

Returning to the general case, the main result of this section is the following:

Theorem 3.3.3. $I_{\mathscr{X}_0} = \left\langle G_1^0 \cup G_2^0\right\rangle$.

To prove Theorem 3.3.3 we will use the dimension criterion of Proposition 1.2.2 which requires counting the monomials that do not appear as initial terms of polynomials in $G_1^0 \cup G_2^0$ with respect to $\prec$. As in Section 1.3, the multidegree of an arbitrary monomial $w_{N,\mu} w_{N',\mu'} \in \mathbb{T}_K^2$ of standard degree 2 can be written as

$$\operatorname{mdeg}\left(w_{N,\mu} w_{N',\mu'}\right) = (2, N+N', \mu+\mu') = (1, N, \mu) + (1, N', \mu')$$

so we consider once more the Minkowski sum of A with itself and its respective description in terms of bounding inequalities given in Lemma 3.1.3.

In Example 3.1.2, we saw that the Minkowski sum $A + A$ is given by

$$\begin{aligned} A + A = \{&(0,2),(1,3),(2,3),(2,4),(3,4),(4,4),(3,5),(4,5),(5,5),(6,5),\\ &(4,6),(5,6),(6,6),(7,6),(8,6),(5,7),(6,7),(7,7),(8,7),(9,7),(10,7),\\ &(6,8),(7,8),(8,8),(9,8),(10,8),(11,8),(12,8)\}\end{aligned}$$

or equivalently

$$A + A = \{(\rho, T) : 2 \le T \le 8, b(T) \le \rho \le 2T - 4\} \subseteq \mathbb{N}^2,$$

where a direct computation verifies that in this case we always have $b(T) = \left\lfloor \frac{4T}{5} \right\rfloor - 1$.

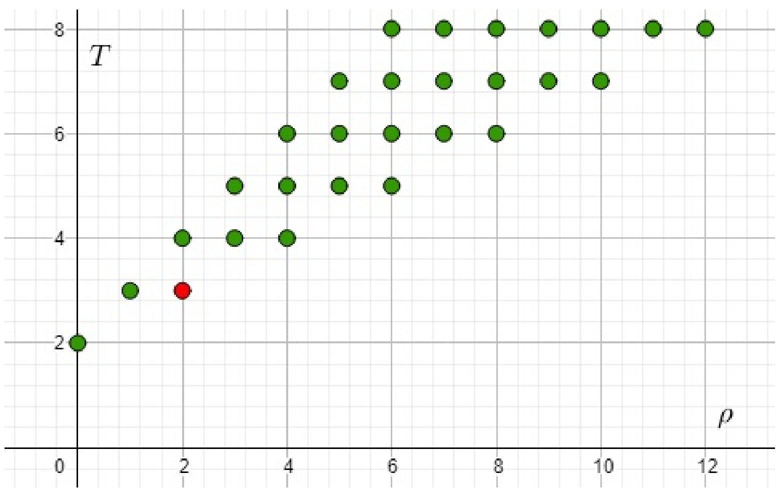


Figure 3.4: The Minkowski sum $A + A$ for $p = 5,\ q = 2,\ \ell = 4$.

The binomials in this case are the same as in the generic fiber and we thus have the examples of $w_{2,3} w_{5,4}$ being the unique monomial of multidegree $(2, 7, 7)$ that does not appear as an initial term in G_1^0 and $w_{0,1}w_{2,2}$ being the unique monomial of multidegree $(2,2,3)$ that does not appear as an initial term in G_1^0. Further, $w_{0,1}w_{2,2}$ is the initial term of the relation in G_2^0

$$w_{0,1}w_{2,2} - w_{3,4}^2 - (x_1^4)w_{2,3}w_{4,4} - (4x_1^3)w_{2,3}w_{5,4} - (6x_1^2)w_{3,3}w_{5,4} - (4x_1)w_{3,3}w_{6,4} - w_{4,3}w_{6,4}.$$

explaining the difference between $|A+A| = 28$ and $\dim_k (I_{\mathscr{X}_0})_2 = 3(g-1) = 27$.

The major difference between the generators of the generic and special fiber is that the initial terms of polynomials in G_1^0 are not in correspondence with the set $C(0)$ defined in 3.26; the definition of G_2^0 in Proposition 3.3.2 leads us to consider the set

$$\begin{aligned} C(1) = \{(\rho, T) \in A+A \mid (\rho+\ell, T+p) \in A+A \\ \text{and } (\rho+j, T+p-1) \in A+A \text{ for } j_{\min}(1) \leq j \leq (p-1)q\}, \end{aligned} \tag{3.36}$$

see also eq. (3.10). We prove the analogue of Proposition 3.2.4 in full generality in below, following once more the arguments of Proposition 1.5.2:

Proposition 3.3.4. $|\mathbb{T}_K^2 \setminus \mathrm{in}_\prec\left(G_1^0 \cup G_2^0\right)| \leq |(A+A) \setminus C(1)|$.

Proof. As in the proof of Proposition 1.4.4, we write (ρ, T) for the arbitrary point of $A+A$ and consider the bijective map of sets

$$\begin{aligned} \sigma : A+A &\to \mathbb{T}_K^2 \setminus \mathrm{in}_\prec(G_1^0) \\ (\rho, T) &\mapsto \min_\prec\{w_{N,\mu}w_{N',\mu'} \in \mathbb{T}_K^2 \ : \ \mathrm{mdeg}\left(w_{N,\mu}w_{N',\mu'}\right) = (\rho, T)\}. \end{aligned}$$

If $(\rho, T) \in C(1)$ then by definition $(\rho, T) \in A+A$, $(\rho+\ell, T+p) \in A+A$ and $(\rho+j, T+p-1) \in A+A$ for all $j_{\min}(1) \leq j \leq (p-1)q$. Hence the monomials

$$w_{N,\mu}w_{N',\mu'} := \sigma(\rho, T), w_{N'',\mu''}w_{N''',\mu'''} := \sigma(\rho+\ell, T+p),\ w_{N_j,\mu_j}w_{N_j',\mu_j'} := \sigma(\rho+j, T+p-1)$$

give rise to a polynomial

$$g = w_{N,\mu}w_{N',\mu'} - w_{N'',\mu''}w_{N''',\mu'''} - \sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1}w_{N_j,\mu_j}w_{N_j',\mu_j'},$$

which, by construction, satisfies $g \in G_2^0$ and $\mathrm{in}_{\prec}(g) = \sigma(\rho, T)$. Hence $\sigma(C(1)) \subseteq \mathrm{in}_{\prec}(G_2^0)$ and so

$$\sigma\big((A+A) \setminus C(1)\big) \supseteq \mathbb{T}_K^2 \setminus \mathrm{in}_{\prec}(G_1^0 \cup G_2^0). \tag{3.37}$$

Since σ is one-to-one, eq. (3.39) gives

$$|(A+A) \setminus C(1)| = |\sigma\big((A+A) \setminus C(1)\big)| \geq |\mathbb{T}_K^2 \setminus \mathrm{in}_{\prec}(G_1^0 \cup G_2^0)|$$

completing the proof. □

We are now ready to prove that $G_1^0 \cup G_2^0$ generates $I_{\mathscr{X}_0}$:

Proof of Theorem 3.3.3. By Proposition 3.3.1 and Proposition 3.3.2 we get that $\langle G_1^0 \cup G_2^0 \rangle \subseteq I_{\mathscr{X}_0}$. By Lemma 3.3.4 and Lemma 3.1.4 we get that $|\mathbb{T}_K^2 \setminus \mathrm{in}_{\prec}\left(G_1^0 \cup G_2^0\right)| \leq |(A+A) \setminus C(1)| \leq |(A+A) \setminus C(0)|$ so Lemma 3.1.6 gives that $|\mathbb{T}_K^2 \setminus \mathrm{in}_{\prec}\left(G_1^0 \cup G_2^0\right)| \leq 3(g-1)$. Proposition 1.2.2 implies that $I_{\mathscr{X}_0} = \langle G_1^0 \cup G_2^0 \rangle$. □

3.4 Thickening and reduction

Let $\mathscr{X} \to \mathrm{Spec}(R)$ denote the family of curves with generic fiber $\mathscr{X}_\eta : y^p = \lambda^p x^\ell + a(x)^p$ and special fiber $\mathscr{X}_0 : X^p - X = \frac{x^\ell}{a(x)^p}$, where $a(x)$ is defined in eq. (3.1). Recall that the global sections $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})$ of holomorphic differentials on $\mathscr{X} \to \mathrm{Spec}(R)$ is a free R-module of rank $g = \frac{(p-1)(m-1)}{2}$, for $m = pq - \ell$. The set

$$\mathbf{b} \;=\; \left\{ \frac{x^N a(x)^{p-1-\mu} X^{p-1-\mu}}{a(x)^{p-1}(\lambda X + 1)^{p-1}} dx : \left\lfloor \frac{\mu \ell}{p} \right\rfloor N \leq \mu q - 2,\ 1 \leq \mu \leq p-1 \right\} \tag{3.38}$$

is a Boseck-type basis, uniquely determined by the set of exponents

$$A = \left\{ (N, \mu) \in \mathbb{Z}^2 : \left\lfloor \frac{\mu \ell}{p} \right\rfloor \leq N \leq \mu q - 2,\ 1 \leq \mu \leq p-1 \right\}$$

so we let $\{W_{N,\mu} : (N,\mu) \in A\}$ be a set of variables indexed by A. The assignment

$$W_{N,\mu} \mapsto \frac{x^N a(x)^{p-1-\mu} X^{p-1-\mu}}{a(x)^{p-1}(\lambda X + 1)^{p-1}} dx$$

gives rise to the canonical map

$$\begin{aligned} \phi : S_R = R[\{W_{N,\mu}\}] &\longrightarrow \bigoplus_{n\geq 0} H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n}) \\ W_{N_1,\mu_1}^{a_1} \cdots W_{N_d,\mu_d}^{a_d} &\longmapsto \frac{x^{(a_1N_1+\cdots a_dN_d)}(a(x)X)^{a_1(p-1-\mu_1)+\cdots a_d(p-1-\mu_d)}}{a(x)^{(a_1+\ldots+a_d)(p-1)}(\lambda X+1)^{(a_1+\ldots+a_d)(p-1)}} dx^{\otimes(a_1+\ldots+a_d)} \end{aligned}$$

which is surjective by the relative version of Petri's Theorem. Since $p \geq 5$, the canonical ideal $I_{\mathscr{X}} = \ker\phi$ is generated in degree 2. Following the arguments of Section 1.4, to each variable $W_{N,\mu}$ we assign the multidegree $\mathrm{mdeg}(W_{N,\mu}) = (1, N, \mu) \in \mathbb{N}^3$ and obtain a multigrading on the polynomial ring S_R

$$\mathrm{mdeg}(W_{N_1,\mu_1}W_{N_2,\mu_2}\cdots W_{N_d,\mu_d}) = (d, N_1 + N_2 + \cdots N_d, \mu_1 + \mu_2 + \cdots + \mu_d). \tag{3.39}$$

We write $\mathbb{T}_R^d$ for the set of monomials of standard degree d in S_R. We remark that, as in Proposition 1.4.1, the following binomials are contained in $I_{\mathscr{X}}$.

Proposition 3.4.1. *We have that*

$$\begin{aligned} G_1 = \{W_{N_1,\mu_1}W_{N_1',\mu_1'} - W_{N_2,\mu_2}W_{N_2',\mu_2'} \in S_k \ : \ &W_{N_1,\mu_1}W_{N_1',\mu_1'}, W_{N_2,\mu_2}W_{N_2',\mu_2'} \in \mathbb{T}_R^2 \\ &\textit{and } \mathrm{mdeg}(W_{N_1,\mu_1}W_{N_1',\mu_1'}) = \mathrm{mdeg}(W_{N_2,\mu_2}W_{N_2',\mu_2'})\} \subseteq I_{\mathscr{X}}. \end{aligned}$$

Proof. See Proposition 1.4.1. □

Next, we expand the $(p-i)$-th power of $a(x)$ for $0 \leq i \leq p$ and write

$$a(x)^{p-i} = \sum_{j=j_{\min}(i)}^{(p-i)q} c_{j,p-i}x^j, \tag{3.40}$$

where by eq. (3.1)

$$j_{\min}(i) = \begin{cases} 0, & \text{if } \ell = 1 \\ p-i, & \text{if } \ell \neq 1 \end{cases} \tag{3.41}$$

and for any $j_{\min}(i) \leq j \leq (p-i)q$

$$c_{j,p-i} = \sum_{\substack{(t_0,\ldots,t_q)\in\mathbb{N}^q \\ t_1+2t_2\cdots+qt_q=j}} \binom{p-i}{t_0,\ldots,t_q} \prod_{s=0}^{q} x_s^{t_s}.$$

Arguing as we did when discussing the two fibers, eq. (3.42) implies that the set D of eq. (1.12) for $\mathscr{X} \to \mathrm{Spec}(R)$ is

$$D = \{(\ell, -p), (j, -(p-i)) : 0 \le i \le p,\ j_{\min}(i) \le j \le (p-i)q\} \tag{3.42}$$

Following the discussion that precedes Proposition 1.5.1 we define

$$M_D = \{\mathbf{a} \in \mathbb{T}_R^2 : \forall\ (\iota, \kappa) \in D,\ \exists\ \mathbf{b} \in \mathbb{T}_R^2 \text{ with } \mathrm{mdeg}(\mathbf{b}) = \mathrm{mdeg}(\mathbf{a}) + (0, \iota, -\kappa)\}. \tag{3.43}$$

Let $\prec$ denote the term order of of Definition 1.4.2 on S_R. Given a monomial $\mathbf{a} = W_{N,\mu}W_{N',\mu'} \in M_D$, We denote by $W_{N'',\mu''}W_{N''',\mu'''}$ the minimal monomial of $\mathbb{T}_R^2$ with respect to $\prec$ that satisfies $\mathrm{mdeg}(W_{N'',\mu''}W_{N''',\mu'''}) = \mathrm{mdeg}(W_{N,\mu}W_{N',\mu'}) + (0, \ell, p)$. Similarly, for $0 \le i \le p$ and $j_{\min}(i) \le j \le (p-i)q$ we denote by $W_{N_{i,j},\mu_{i,j}}W_{N'_{i,j},\mu'_{i,j}}$ the minimal monomial of $\mathbb{T}_R^2$ with respect to $\prec$ that satisfies $\mathrm{mdeg}(W_{N_{i,j},\mu_{i,j}}W_{N'_{i,j},\mu'_{i,j}}) = \mathrm{mdeg}(W_{N,\mu}W_{N',\mu'}) + (0, j, p-i)$. This allows us to explicitly construct the following set of polynomials in $I_{\mathscr{X}}$:

Proposition 3.4.2. *Let D be as in eq. (3.44), M_D be as in eq. (3.45) and let*

$$G_2 = \left\{ W_{N,\mu}W_{N',\mu'} - W_{N'',\mu}W_{N''',\mu'''} + \sum_{i=1}^{p-1} \sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p} \binom{p}{i} c_{j,p-i} W_{N_{i,j},\mu_{i,j}} W_{N'_{i,j},\mu'_{i,j}} \in S_R \ : \ W_{N,\mu}W_{N',\mu'} \in M_D \right\}.$$

Then $G_2 \subseteq I_{\mathscr{X}}$.

Proof. We follow the arguments of the proof of Proposition 1.5.1. Let

$$f := W_{N,\mu}W_{N',\mu'} - W_{N'',\mu}W_{N''',\mu'''} + \sum_{i=1}^{p-1} \sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p} \binom{p}{i} c_{j,p-i} W_{N_j,\mu_i} W_{N'_j,\mu'_i}$$

where

$$N'' + N''' = N + N' + \ell \quad , \quad \mu'' + \mu''' = \mu + \mu' + p \text{ and} \tag{3.44}$$

$$N_j + N'_j = N + N' + j \quad , \quad \mu_i + \mu'_i = \mu + \mu' + p - i.$$

We note that $f \in R[\{W_{N,\mu}\}]$, since by [9, sec. 4.3]

$$p \cdot \lambda^s \equiv \begin{cases} 0 \quad \bmod \mathfrak{m}, \text{ for } -(p-1) < s < 0 \\ -1 \quad \bmod \mathfrak{m}, \text{ for } s = -(p-1), \end{cases} \tag{3.45}$$

which implies that $\lambda^{i-p}\binom{p}{i} \in \mathfrak{m} \subseteq \mathfrak{m}_R \subseteq R$ for all $1 \leq i \leq p-1$. Applying the canonical map ϕ to f gives

$$\Bigg(\frac{x^{N+N'}\left(a(x)X\right)^{2(p-1)-(\mu+\mu')}}{(a(x)(\lambda X+1))^{2(p-1)}} dx^{\otimes 2} - \frac{x^{N''+N'''}\left(a(x)X\right)^{2(p-1)-(\mu''+\mu''')}}{(a(x)(\lambda X+1))^{2(p-1)}} dx^{\otimes 2} \\ + \sum_{i=1}^{p-1} \sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p}\binom{p}{i} c_{j,p-i} \frac{x^{N_j+N'_j}\left(a(x)X\right)^{2(p-1)-(\mu_i+\mu'_i)}}{(a(x)(\lambda X+1))^{2(p-1)}} dx^{\otimes 2} \Bigg), \tag{3.46}$$

and using the relations of eq. (3.46) we may rewrite eq. (3.48) as

$$\frac{x^{N+N'}\left(a(x)X\right)^{2(p-1)-(\mu+\mu')}}{(a(x)(\lambda X+1))^{2(p-1)}} dx^{\otimes 2} - \frac{x^{N+N'+\ell}\left(a(x)X\right)^{2(p-1)-(\mu+\mu'+p)}}{(a(x)(\lambda X+1))^{2(p-1)}} dx^{\otimes 2} \\ + \sum_{i=1}^{p-1} \sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p}\binom{p}{i} c_{j,p-i} \frac{x^{N+N'+j}\left(a(x)X\right)^{2(p-1)-(\mu+\mu'+p-i)}}{(a(x)(\lambda X+1))^{2(p-1)}} dx^{\otimes 2}.$$

If we write

$$h := \frac{x^{N+N'}\left(a(x)X\right)^{2(p-1)-(\mu+\mu')}}{(a(x)(\lambda X+1))^{2(p-1)}} dx^{\otimes 2},$$

then

$$\phi(f) = h\left(1 - x^{\ell}\left(a(x)X\right)^{-p} + \sum_{i=1}^{p-1} \sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p}\binom{p}{i} c_{j,p-i} x^{j}\left(a(x)X\right)^{i-p}\right).$$

and combining with the expansion of $a(x)^{p-i}$ in eq. (3.42) we get

$$\phi_{\mathbf{c}}(f) = h\left(1 - x^{\ell}\left(a(x)X\right)^{-p} + \sum_{i=1}^{p-1} \lambda^{i-p}\binom{p}{i} X^{i-p}\right).$$

We simplify the expression as follows:

$$\begin{aligned}
\phi(f) &= h\left(1 - x^{\ell}\,(a(x)X)^{-p} + \sum_{i=1}^{p-1} \lambda^{i-p}\binom{p}{i} X^{i-p}\right) \\
&= h\left(-x^{\ell}\,(a(x)X)^{-p} + \sum_{i=1}^{p} \lambda^{i-p}\binom{p}{i} X^{i-p}\right) \\
&= h\left(-x^{\ell}\,(a(x)X)^{-p} - \lambda^{-p}X^{-p} + \sum_{i=0}^{p} \lambda^{i-p}\binom{p}{i} X^{i-p}\right) \\
&= h\left(-x^{\ell}\,(a(x)X)^{-p} - \lambda^{-p}X^{-p} + \lambda^{-p}X^{-p}(\lambda X+1)^{p}\right). \qquad (3.47)
\end{aligned}$$

Finally, since $y = a(x)(\lambda X + 1)$, eq. (3.49) is equivalent to eq.(3.16), so $\phi(f) \otimes_R 1_L = 0$, completing the proof. □

The main result of this section is the following:

Theorem 3.4.3. $I_{\mathscr{X}} = \langle G_1 \cup G_2 \rangle$.

To prove Theorem 3.4.3, we will use the Nakayama-type criterion of Lemma 1.6.2 and a series of intermediate results. We first prove compatibility with the special fiber:

Proposition 3.4.4. $(G_1 \cup G_2) \otimes_R K$ *generates* $I_{\mathscr{X}_0}$.

Proof. It is immediate that $G_1 \otimes_R k = G_1^0$. For $G_2 \otimes_R K$, eq. (3.47) implies that in the expression

$$\sum_{i=1}^{p-1} \sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p}\binom{p}{i} c_{j,p-i} W_{N_j,\mu_i} W_{N'_j,\mu'_i}$$

only the term for $i = 1$ survives reduction, giving that

$$\left(\sum_{i=1}^{p-1} \sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p}\binom{p}{i} c_{j,p-i} W_{N_j,\mu_i} W_{N'_j,\mu'_i}\right) \otimes_R K = -\sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1} w_{N_j,\mu_j} w_{N'_j,\mu'_j},$$

and equivalently

$$\left(W_{N,\mu}W_{N',\mu'} - W_{N'',\mu}W_{N''',\mu'''} + \sum_{i=1}^{p-1} \sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p}\binom{p}{i} c_{j,p-i} W_{N_j,\mu_i} W_{N'_j,\mu'_i}\right) \otimes_R K =$$

$$w_{N,\mu}w_{N',\mu'} - w_{N'',\mu''}w_{N''',\mu'''} - \sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1} w_{N_j,\mu_j} w_{N'_j,\mu'_j},$$

so that $G_2 \otimes_R K = G_2^0$, completing the proof. □

Finally, we examine compatibility with the generic fiber and we shall give two different proofs:

Proposition 3.4.5. *$(G_1 \cup G_2) \otimes_R L$ generates $I_{\mathscr{X}_\eta}$.*

Proof. Again, it is immediate that $G_1 \otimes_R L = G_1^\eta$. For $G_2 \otimes_R L$, as in the proof of Proposition 1.4.4, we write (ρ, T) for the arbitrary point of $A+A$ and consider the bijective map of sets

$$\begin{aligned} \sigma : A + A \quad &\to \quad \mathbb{T}_R^2 \setminus \mathrm{in}_\prec(G_1^\eta) \\ (\rho, T) \quad &\mapsto \quad \min_\prec \{W_{N,\mu} W_{N',\mu'} \in \mathbb{T}_R^2 \; : \; \mathrm{mdeg}\left(W_{N,\mu} W_{N',\mu'}\right) = (\rho, T)\}. \end{aligned}$$

Next, for $0 \leq i \leq p$, we consider the subsets of $A + A$

$$C(i) = \{(\rho, T) \in A + A \; : \; (\rho + \ell, T + p) \text{ and } (\rho + j, T + p - i) \in A + A \qquad \text{for } j_{\min}(i) \leq j \leq (p - i)q\},$$

where

$$j_{\min}(i) = \begin{cases} 0, \text{ if } \ell = 1 \\ p - i, \text{ if } \ell \neq 1. \end{cases}$$

By Lemma 3.1.4, $C(0) \subseteq C(i)$, i.e. if $(\rho, T) \in C(0)$ then $(\rho, T) \in A + A$, $(\rho + \ell, T + p) \in A + A$ and $(\rho + j, T + p - i) \in A + A$ for all $j_{\min}(i) \leq j \leq (p - i)q$. Hence the monomials

$$W_{N,\mu} W_{N',\mu'} := \sigma(\rho, T), W_{N'',\mu} W_{N''',\mu'''} := \sigma(\rho+\ell, T+p), \; W_{N_j,\mu_i} W_{N'_j,\mu'_i} := \sigma(\rho+j, T+p-i)$$

give rise to the polynomial

$$g = W_{N,\mu} W_{N',\mu'} - W_{N'',\mu} W_{N''',\mu'''} + \sum_{i=1}^{p-1} \sum_{j=j_{\min}}^{(p-i)q} \lambda^{i-p} \binom{p}{i} c_{j,p-i} W_{N_j,\mu_i} W_{N'_j,\mu'_i} \in G_2.$$

which, by construction, satisfies $\mathrm{in}_\prec(g) = \sigma(\rho, T)$. Hence $\sigma(C(0)) \subseteq \mathrm{in}_\prec(G_2) \otimes_R L$ and so

$$\sigma\big((A + A) \setminus C(0)\big) \supseteq \mathbb{T}_L^2 \setminus \left(\mathrm{in}_\prec(G_1 \cup G_2) \otimes_R L\right). \tag{3.48}$$

Since σ is one-to-one, eq. (3.50) gives

$$|(A+A)\setminus C(0)| = |\sigma\big((A+A)\setminus C(0)\big)| \geq |\mathbb{T}_L^2 \setminus (\mathrm{in}_{\prec}(G_1\cup G_2)\otimes_R L)\,|.$$

Proposition 3.1.6 then implies that $|\mathbb{T}_L^2 \setminus (\mathrm{in}_{\prec}(G_1\cup G_2)\otimes_R L)\,| \leq 3(g-1)$ and the result follows from Proposition 1.2.2. □

We are now ready to prove that $G_1 \cup G_2$ generates $I_{\mathscr{X}}$.

Proof of Theorem 3.4.3. By Proposition 3.4.4, $(G_1\cup G_2)\otimes_R K$ generates $I_{\mathscr{X}_0}$ and by Proposition 3.4.5 $(G_1\cup G_2)\otimes_R L$ generates $I_{\mathscr{X}_\eta}$. The result follows from Lemma 1.6.2. □

We conclude this section by giving an alternative proof of Proposition 3.4.5. First, we fix the following notation: for a ring A and a set G we write A^G for the $A-$module generated by G.

Lemma 3.4.6.

$$\big(R^{G_2}/\left(R^{G_1}\cap R^{G_2}\right)\big)\otimes_R L \cong L^{G_2^\eta}/\left(L^{G_1^\eta}\cap L^{G_2^\eta}\right).$$

Proof. Let $\left\{\omega_{N,\mu}\omega_{N',\mu'}\in\mathbb{T}_L^2 : (N,\mu),\ (N',\mu')\in A\right\}$ be an L-basis for $(S_L)_2$. For $(\rho,T)\in A+A$ we write

$$\boldsymbol{\omega}_{\rho,T} = \left\{\omega_{N,\mu}\omega_{N',\mu'}\in\mathbb{T}_L^2 : N+N'=\rho,\ \mu+\mu'=T\right\}$$

and note that the quotient $(S_L)_2/L^{G_1^\eta}$ has an L-basis given by $\{\boldsymbol{\omega}_{\rho,T} : (\rho,T)\in A+A\}$. This implies that the quotient $L^{G_2^\eta}/\left(L^{G_1^\eta}\cap L^{G_2^\eta}\right)$ has an L-basis given by

$$\left\{\boldsymbol{\omega}_{\rho,T} - \lambda^p\boldsymbol{\omega}_{\rho+\ell,T+p} - \sum_{j=j_{\min}}^{pq} c_{j,p}\cdot\boldsymbol{\omega}_{\rho+j,T+p} : (\rho,T)\in C(0)\right\}. \tag{3.49}$$

Similarly, let $\left\{W_{N,\mu}W_{N',\mu'}\in\mathbb{T}_R^2 : (N,\mu),\ (N',\mu')\in A\right\}$ be an R-basis for $(S_R)_2$. For $(\rho,T)\in A+A$ we write

$$\boldsymbol{W}_{\rho,T} = \left\{W_{N,\mu}W_{N',\mu'}\in\mathbb{T}_R^2 : N+N'=\rho,\ \mu+\mu'=T\right\}$$

and note that the quotient $(S_R)_2/R^{G_1}$ has an R-basis given by $\{\boldsymbol{W}_{\rho,T} : (\rho,T)\in A+A\}$.

This implies that the quotient $R^{G_2}/\left(R^{G_1}\cap R^{G_2}\right)$ has an R-basis given by

$$\left\{\boldsymbol{W}_{\rho,T}-\boldsymbol{W}_{\rho+\ell,T+p}+\sum_{i=1}^{p-1}\sum_{j=j_{\min}(i)}^{(p-i)q}\lambda^{i-p}\binom{p}{i}c_{j,p-i}\boldsymbol{W}_{\rho+j,T+p-i}:(\rho,T)\in C(0)\right\}. \tag{3.50}$$

The assignments

$$\begin{array}{rcl}
\boldsymbol{\omega}_{\rho,T}-\lambda^{p}\boldsymbol{\omega}_{\rho+\ell,T+p}-\displaystyle\sum_{j=j_{\min}}^{pq}c_{j,p}\boldsymbol{\omega}_{\rho+j,T+p} & \mapsto & \boldsymbol{\omega}_{\rho,T}\\
\boldsymbol{W}_{\rho,T}-\boldsymbol{W}_{\rho+\ell,T+p}+\displaystyle\sum_{i=1}^{p-1}\sum_{j=j_{\min}(i)}^{(p-i)q}\lambda^{i-p}\binom{p}{i}c_{j,p-i}\boldsymbol{W}_{\rho+j,T+p-i} & \mapsto & \boldsymbol{W}_{\rho,T}
\end{array}$$

can be linearly extended to define linear maps

$$\begin{array}{rcl}
f^{\eta}:L^{G_2^{\eta}}/\left(L^{G_1^{\eta}}\cap L^{G_2^{\eta}}\right) & \longrightarrow & (S_L)_2/L^{G_1^{\eta}}\text{ and}\\
f:R^{G_2}/\left(R^{G_1}\cap R^{G_2}\right) & \longrightarrow & R[\{W_{N,\mu}\}]_2/R^{G_1}.
\end{array}$$

Note that $\mathrm{Im}f^{\eta}$ has basis $\{\boldsymbol{\omega}_{\rho,T}:(\rho,T)\in C(0)\}$ and $\mathrm{Im}f$ has basis $\{\boldsymbol{w}_{\rho,T}:(\rho,T)\in C(0)\}$, so that the L-vector spaces $\mathrm{Im}f^{\mathbf{b}}$ and $\mathrm{Im}f\otimes_R L$ are isomorphic over L as they both have dimension equal to $|C(0)|$.

The elements of $(S_L)_2/L^{G_1^{\mathbf{b}}}$ and $(S_R)_2/R^{G_1}$ come equipped with a natural ordering, induced by the term order on the respective polynomial rings. This implies that in the expressions for $f^{\mathbf{b}}$ and f the terms $\boldsymbol{\omega}_{\rho,T}$ and $\boldsymbol{W}_{\rho,T}$ precede all other terms, which in turn implies that the matrices defining $f^{\mathbf{b}}$ and f are upper triangular with diagonal entries equal to 1. Their determinants are thus the product of these diagonal entries, making the matrices invertible and the linear maps injective.

We thus obtain a commutative diagram

$$\begin{array}{ccc}
L^{G_2^{\eta}}/\left(L^{G_1^{\eta}}\cap L^{G_2^{\eta}}\right) & \longrightarrow & \mathrm{Im}f^{\eta}\\
\big\downarrow & & \big\downarrow\\
\left(R^{G_2}/\left(R^{G_1}\cap R^{G_2}\right)\right)\otimes_R L & \longrightarrow & \mathrm{Im}f\otimes_R L
\end{array}$$

and since the top, right and bottom maps are isomorphisms, so is the left. □

Alternative proof of Proposition 3.4.5. Once more, it is immediate that $G_1 \otimes_R L = G_1^\eta$. For $G_2 \otimes_R L$, we first observe that by Lemma 3.4.6 and the third isomorphism theorem, we have a commutative diagram

$$\begin{array}{ccc} \left(R^{G_2}/\left(R^{G_1} \cap R^{G_2}\right)\right) \otimes_R L & \longrightarrow & L^{G_2^\eta}/\left(L^{G_1^\eta} \cap L^{G_2^\eta}\right) \\ \downarrow & & \downarrow \\ \left(\left(R^{G_1} + R^{G_2}\right)/R^{G_1}\right) \otimes_R L & \longrightarrow & \left(L^{G_1^\eta} + L^{G_2^\eta}\right)/L^{G_1^\eta} \end{array} \tag{3.51}$$

and since the top, right and left maps are linear isomorphisms, so is the bottom. Tensoring commutes with taking quotients, so we obtain an isomorphism

$$\left(\left(R^{G_1} + R^{G_2}\right) \otimes_R L\right) / \left(R^{G_1} \otimes_R L\right) \longrightarrow \left(L^{G_1^\eta} + L^{G_2^\eta}\right)/L^{G_1^\eta}. \tag{3.52}$$

Since $R^{G_1} \otimes_R L \cong L^{G_1^\eta}$, so by the five lemma the isomorphism above lifts to an isomorphism

$$\left(R^{G_1} + R^{G_2}\right) \otimes_R L \longrightarrow L^{G_1^\eta} + L^{G_2^\eta}. \tag{3.53}$$

The result follows since $\langle G_1 \cup G_2\rangle \otimes_R L = \langle\left(R^{G_1} + R^{G_2}\right) \otimes_R L\rangle$ and $\langle G_1^\eta \cup G_2^\eta\rangle = \langle L^{G_1^\eta} + L^{G_2^\eta}\rangle$.

□

Appendix A

The Artin-Schreier-Kummer-Witt Family of Curves

A.1 Artin-Schreier curves

Let k be an algebraically closed field of prime characteristic $p > 0$. Let x be a transcedental element over k, so that the rational function field $k(x)$ is the function field of $\mathbb{P}^1_k$, the projective line over k.

Definition A.1.1. *An Artin-Schreier extension F/k is a degree p cyclic Galois extension of $k(x)$. An Artin-Schreier curve $\mathscr{X}_0/k$ is a non-singular, projective curve over k whose function field $k(\mathscr{X}_0)/k(x)$ is an Artin-Schreier extension.*

We note that Artin-Schreier curves are well-defined, as by [29, Cor. 4.5], two non-singular, projective curves are birational if and only if their function fields are isomorphic. The affine model for Artin-Schreier curves is given by the Fundamental Theorem of Artin-Schreier covers, whose proof we sketch below and refer the reader to [66, Prop. 3.7.8] for more detaills.

Theorem A.1.2 (E. Artin, O. Schreier, 1927)**.** *$F/k(x)$ is an Artin-Schreier extension if and only if it is the splitting field of a polynomial $X^p - X - f(x) \in k(x)[X]$, where X is a transcendental element over $k(x)$, $f(x) \in k(x)^\times$ and $f(x) \neq z^p - z$ for all $z \in k(x)$.*

Proof. Let α be a root of $X^p - X - f(x)$ in its splitting field $F/k(x)$ and let g be the minimal polynomial of α, so that $g \mid X^p - X - f(x)$ and $[F : k(x)] = \deg(g)$. By Fermat's

Little Theorem, for all $1 \leq i \leq p-1$, $\alpha + i$ is a root $X^p - X - f(x)$, since

$$(\alpha+i)^p - (\alpha+i) = (\alpha^p + i) - (\alpha+i) = \alpha^p - \alpha = f(x).$$

Thus the set of dinstict roots of $X^p - X - f(x)$ is $\{\alpha,\ \alpha+1, \ldots, \alpha+(p-1)\}$ and $X^p - X - f(x)$ is separable with splitting field $F = k(x)(\alpha) \cong k(x, \alpha)$. Next, let

$$X^p - X - f(x) = \prod_{i=0}^{p-1} h_i(X)$$

be its factorization in monic, irreducible polynomials. Thus, for $0 \leq i \leq p-1$, $h_i(x) \in k(x)[X]$ is the minimal polynomial of $\alpha + i$, so $\deg(h_i) \leq \deg(g)$. Then the polynomials $H_i(X) = h_i(X+i)$ have degree $\deg(H_i) = \deg(h_i) \leq \deg(g)$ and satisfy $H_i(\alpha) = 0$, so the minimality of g implies $H_i(X) = g(X)$ or equivalently $h_i(X) = g(X - i)$. Hence

$$X^p - X - f(x) = \prod_{i=0}^{p-1} g(X-i)$$

and thus $\deg(g) \mid \deg(X^p - X - f(x)) = p$. If $\alpha \in k(x)$, then $\alpha^p - \alpha = f(x)$ would imply $f(x) = 0$, contradicting the assumption $f(x) \neq 0$. Hence $deg(g) \neq 1$, which gives $\deg(g) = p = \deg(X^p - X - f(x))$ and thus $g = X^p - X - f(x)$. We conclude that $X^p - X - f(x)$ is separable and irreducible over $k(x)$ and its splitting field $F \cong K(x, \alpha)$ is a degree p cyclic Galois extension of $k(x)$.

For the opposite direction, let $F/k(x)$ be a degree p cyclic Galois cover of $k(x)$ with Galois group $G \cong \mathbb{Z}/p\mathbb{Z}$. Let σ be a generator of G and let $w \in F^\times$. We define two elements

$$\begin{aligned} z &= \sum_{i=0}^{p-1} \sigma^i(w). \\ y &= \prod_{i=0}^{p-1} (p-1-i)\sigma^i(w) \end{aligned}$$

and observe that by Dedekind's Independence of Characters Theorem, $z \neq 0$ and thus, since $\sigma(z) = z$, $z \in k(x)^\times$. Then $\sigma(y) = y + z$ and we may write $\sigma(\alpha) = \alpha + 1$ for $\alpha = y/z$. Hence $\alpha \notin k(x)$ and $F = k(x)(\alpha) \cong k(x, \alpha)$. This implies that the minimal polynomial of

α is given by

$$\prod_{i=0}^{p-1}\left(X-\sigma^i(\alpha)\right)=\prod_{i=0}^{p-1}(X-(\alpha+i))=X^p-X+(-1)^p\prod_{i=0}^{p-1}(\alpha+i)$$

completing the proof. □

We proceed with some properties of Artin-Schreier curves:

Corollary A.1.3. *Let $\mathscr{X}_0/k$ be an Artin-Schreier curve with function field $k(\mathscr{X}_0)$. Then:*

1. *The Galois group $G=\operatorname{Gal}(k(\mathscr{X}_0)/k(x))$ is generated by the automorphism*

$$\sigma:\mathscr{X}_0\to\mathscr{X}_0,\ (x,X)\mapsto(x,X+1).$$

2. *G acts on $\mathscr{X}_0$ and $\mathscr{X}_0/G=\mathbb{P}^1_k$.*

3. *If m denotes the conductor of the extension $k(\mathscr{X}_0)/k(x)$, then the genus of $\mathscr{X}_0/k$ is given by*

$$g=\frac{(p-1)(m-1)}{2}.$$

Proof. (1) follows directly from Theorem A.1.2 and (2) follows from (1). For (3) we observe that the Hurwitz genus formula gives

$$\begin{aligned}2g-2 &= -2[k(\mathscr{X}_0):k(x)]+\deg\operatorname{Diff}(k(\mathscr{X}_0)/k(x))\Leftrightarrow\\ g &= \frac{1}{2}\deg\operatorname{Diff}(k(\mathscr{X}_0)/k(x))-(p-1)\end{aligned}$$

By [66, Prop. 3.7.8], $\deg\operatorname{Diff}(k(\mathscr{X}_0)/k(x))=(m+1)(p-1)$ and substituting into the Hurwitz genus formula gives

$$g=\frac{(p-1)(m+1)}{2}-(p-1)=\frac{(p-1)(m-1)}{2}.$$

□

We conclude our brief overview of Artin-Schreier curves with a simple example.

Example A.1.4. *Let* $k = \overline{\mathbb{F}}_5$*. The splitting field of the polynomial*

$$X^5 - X - x^{-6} \in \overline{\mathbb{F}}_5(x)(X)$$

defines an Artin-Schreier curve $\mathscr{X}_0$ *of genus* $g = 10$ *and Galois group* $G = \mathbb{Z}/5\mathbb{Z}$*.*

A.2 Kummer curves

Let L be an algebraically closed field of characteristic 0. Let x be a transcedental element over L, so that the rational function field $L(x)$ is the function field of $\mathbb{P}^1_L$, the projective line over L.

Definition A.2.1. *A Kummer extension* F/L *is a degree* p *cyclic Galois extension of* $L(x)$ *that contains a primitive* p*-th root of unity. A Kummer curve* $\mathscr{X}_\eta/L$ *is a non-singular, projective curve over* L *whose function field* $L(\mathscr{X}_\eta)/L(x)$ *is a Kummer extension.*

We note that, as was the case with Artin-Schreier curves, Kummer curves are well-defined since two non-singular, projective curves are birational if and only if their function fields are isomorphic. We state the Fundamental Theorem of Kummer covers below as well as some basic properties of Kummer curves. The proofs are omitted as they are similar to the proofs of the respective results for Artin-Schreier curves.

Theorem A.2.2 (E. Kummer, 1840)**.** $F/L(x)$ *is a Kummer extension if and only if it is the splitting field of a polynomial* $y^p - g(x) \in L(x)[y]$*, where* y *is a transcendental element over* $L(x)$ *and* $g(x) \in L(x)^\times$*.*

Proof. Similar to the proof of Theorem A.1.2. □

We proceed with some properties of Kummer curves:

Corollary A.2.3. *Let* $\mathscr{X}_\eta/L$ *be a Kummer curve with function field* $L(\mathscr{X}_\eta)$*. Then:*

1. *The Galois group* $G = \mathrm{Gal}\,(L(\mathscr{X}_\eta)/L(x))$ *is generated by the automorphism*

$$\sigma : \mathscr{X}_\eta \to \mathscr{X}_\eta,\ (x, y) \mapsto (x, \zeta_p \cdot y).$$

2. G *acts on* $\mathscr{X}_\eta$ *and* $\mathscr{X}_\eta/G = \mathbb{P}^1_L$*.*

3. *If m denotes the conductor of the extension $k(\mathscr{X}_\eta)/L(x)$, then the genus of $\mathscr{X}_\eta/L$ is given by*

$$g = \frac{(p-1)(m-1)}{2}.$$

Proof. Similar to Corollary A.1.3. □

We conclude our brief overview of Kummer curves with an example.

Example A.2.4. *Let $W\left(\overline{\mathbb{F}}_5\right)[\zeta_5]$ denote the ring of Witt vectors over $\overline{\mathbb{F}}_5$ - see also Section A.3 - extended by a 5-th root of unity and let L be its field of fractions. If $\lambda = \zeta_5 - 1$ then the splitting field of the polynomial*

$$y^5 - x^4(\lambda^5 x^4 + x^6) \in L(x)(y)$$

defines a Kummer curve $\mathscr{X}_\eta$ of genus $g = 10$ and Galois group $G = \mathbb{Z}/5\mathbb{Z}$.

A.3 The ring of Witt vectors

Theorem A.3.1 (E. Witt, 1937)**.** *Let k be a perfect field of prime characteristic $p > 0$. There exists a discrete valuation ring of characteristic 0, called **the ring of Witt vectors** and denoted by $W(k)$, with local uniformizer $p \cdot \mathbf{1}_{W(k)} = \underbrace{\mathbf{1}_{W(k)} + \cdots + \mathbf{1}_{W(k)}}_{p\text{-times}}$ and residue field isomorphic to k.*

The theorem above first appeared in Witt's seminal paper [76]. We will refrain from reproducing Witt's construction and restrict ourselves to mentioning some properties; for a detailed treatment the reader may refer to Jacobson's [35, Section 8.10] and Hazewinkel's [32, Chapter III] books and Rabinoff's [54] survey notes.

The underlying set of $W(k)$ is $\prod_{n\in P} k = k^P$, for $P = \{1, p, p^2, \ldots\}$, and its elements are denoted by $\mathbf{x} = (x_0, \ldots, x_n, \ldots) = (x_n)_{n\in\mathbb{N}}$. To describe the ring structure on $W(k)$, Witt considered, for $n \in \mathbb{N}$, the n-th Witt polynomials

$$w_n = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n \in \mathbb{Z}[X_0, \ldots, X_n]$$

and was able to show that, for all $n \in \mathbb{N}$, the equations

$$\begin{aligned} w_n(X_0, \ldots, X_n) + w_n(Y_0, \ldots, Y_n) &= w_n(S_0, \ldots, S_n) \\ w_n(X_0, \ldots, X_n) w_n(Y_0, \ldots, Y_n) &= w_n(Z_0, \ldots, Z_n) \end{aligned} \tag{A.1}$$

can be solved for S_n and Z_n to define polynomials with integral coefficients in the variables $X_0, \ldots, X_n, Y_0, \ldots, Y_n$, i.e. $S_n, Z_n \in \mathbb{Z}[X_0, \ldots, X_n, Y_0, \ldots, Y_n]$. As an example, we remark that $S_0 = X_0 + Y_0$ and $Z_0 = X_0 Y_0$ and work out explicitly the case $n = 1$. For S_1 we have

$$\begin{aligned} w_1(S_0, S_1) &= w_1(X_0, X_1) + w_1(Y_0, Y_1) \Leftrightarrow \\ S_0^p + pS_1 &= X_0^p + pX_1 + Y_0^p + pY_1 \Leftrightarrow \\ pS_1 &= p(X_1 + Y_1) + X_0^p + Y_0^p - (X_0 + Y_0)^p \Leftrightarrow \\ pS_1 &= p(X_1 + Y_1) - \sum_{i=1}^{p-1} \binom{p}{i} X_0^i Y_0^{p-i} \Leftrightarrow \\ S_1 &= X_1 + Y_1 - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} X_0^i Y_0^{p-i} \end{aligned}$$

and since $\frac{1}{p}\binom{p}{i} \in \mathbb{Z}$, $S_1 \in \mathbb{Z}[X_0, X_1, Y_0, Y_1]$. Similarly, for Z_1 we have

$$\begin{aligned} w_1(Z_0, Z_1) &= w_1(X_0, X_1) w_1(Y_0, Y_1) \Leftrightarrow \\ Z_0^p + pZ_1 &= (X_0^p + pX_1)(Y_0^p + pY_1) \Leftrightarrow \\ pZ_1 &= -Z_0^p + (X_0 Y_0)^p + p(Y_0^p X_1 + X_0^p Y_1) + p^2 X_1 Y_1 \Leftrightarrow \\ Z_1 &= Y_0^p X_1 + X_0^p Y_1 + pX_1 Y_1 \in \mathbb{Z}[X_0, X_1, Y_0, Y_1]. \end{aligned}$$

Thus, given two Witt vectors $\mathbf{x} = (x_0, \ldots, x_n, \ldots)$ and $\mathbf{y} = (y_0, \ldots, y_n, \ldots)$ Witt defined their sum $\mathbf{s} = (s_0, \ldots, s_n, \ldots)$ and product $\mathbf{z} = (z_0, \ldots, z_n, \ldots)$ via the relations

$$s_n = S_n(x_0, \ldots, x_n, y_0, \ldots, y_n) \; z_n = Z_n(x_0, \ldots, x_n, y_0, \ldots, y_n).$$

where, for all $n \in \mathbb{N}$, S_n and Z_n are given by eq. (A.1). These operations endow $W(k)$ with the structure of a commutative ring with $\mathbf{0}_{W(k)} = (0, 0, 0, \ldots)$, $\mathbf{1}_{W(k)} = (1, 0, 0, \ldots)$.

It is important also to mention that the discrete valuation on $W(k)$ is the p-adic valuation and that $W(k)$ is complete and Hausdorff with respect to the induced topology. Further, the construction is functorial and unique up to canonical isomorphism. We conclude this section with the remark that when $k = \mathbb{Z}/p\mathbb{Z}$ then $W(k) = \mathbb{Z}_p$, the ring of p-adic integers.

A.4 Deformation theory of curves with automorphisms

A.4.1 Infinitesimal deformations

In this section we explain the infinitesimal point of view following M. Schlessinger [58], see also [8],[2]. Let X be a smooth, projective curve over an algebraically closed field k. Given a local ring A with maximal ideal $\mathfrak{m}_A$ and residue field $A/\mathfrak{m}_A \cong k$, a deformation of X over A is a proper, smooth, relative curve $\mathscr{X} \to \mathrm{Spec}(A)$ such that $\mathscr{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}\,(k) \cong X$, which fits in the following commutative diagram:

$$\begin{array}{ccc} X \cong \mathscr{X} \times_{\mathrm{Spec}A} \mathrm{Spec}\,(A/\mathfrak{m}_A) & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) \cong \mathrm{Spec}\,(A/\mathfrak{m}_A) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

Two deformations $\mathscr{X}_1$ $\mathscr{X}_2$ are considered to be equivalent if there is an isomorphism $\psi : \mathscr{X}_1 \to \mathscr{X}_2$ making the following diagram commute

$$\begin{array}{ccc} \mathscr{X}_1 & \xrightarrow{\psi} & \mathscr{X}_2 \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}A & \end{array} \tag{A.2}$$

and such that ψ gives the identity on the special fibers.

Definition A.4.1. *Let $\mathscr{C}$ be the category of local Artin algebras $(A, \mathfrak{m}_A)$ with $A/\mathfrak{m}_A \cong k$. We define the* ***deformation functor***

$$\begin{array}{rcl} D : \mathscr{C} & \to & \mathrm{Sets} \\ A & \mapsto & \left\{ \begin{array}{l} \textit{Equivalence classes of} \\ \textit{deformations of } X \textit{ over } A \end{array} \right\} \\ \left(A \xrightarrow{f} B\right) & \mapsto & \Big((\mathscr{X}_1 \to \mathrm{Spec}(A)) \mapsto \big(\mathscr{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \to \mathrm{Spec}(B)\big) \Big) \end{array}$$

Next, we fix a pair (X, G) where X is a curve as above and G is a subgroup of its automorphism group. A deformation of (X, G) over the local ring A is a deformation of the curve X over A together with a group isomorphism $G \to \mathrm{Aut}_A(\mathscr{X})$, and such that the isomorphism $\mathscr{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \cong X$ is G-equivariant. The notion of equivalence of (X, G) deformations is similar to the non equivariant case, but we now assume that the map ψ of diagram A.2 is also G-equivariant. A deformation functor is then defined

$$\begin{aligned} D_{(X,G)} : \mathscr{C} &\to \text{Sets}, \\ A &\mapsto \left\{ \begin{array}{l} \text{Equivalence classes} \\ \text{of deformations of} \\ \text{couples } (X, G) \text{ over } A \end{array} \right\} \end{aligned}$$

A deformation over an integral domain A is called **equicharacteristic** if the characteristics of the special and generic fiber coincide, that is if $\mathrm{char}(\mathrm{Quot}(A)) = \mathrm{char}(A/\mathfrak{m}_A)$. Otherwise the deformation is called **mixed characteristic**.

Versal elements. Let $W(k)$ be the ring of Witt vectors over k - see section A.3 - and let $\widehat{\mathscr{C}}$ be the category of $W(k)$-algebras which are local, noetherian and complete. Recall that a covariant functor admits a versal deformation $\mathscr{R}$ if there is an object $\mathscr{R} \in \hat{\mathscr{C}}$ and a smooth morphism of functors $\xi : D \to \mathrm{Hom}_{W(k)}(\mathscr{R}, \cdot) =: h_{\mathscr{R}}(\cdot)$, which induces an isomorphism $D(k[\epsilon]) \cong (\mathfrak{m}_{\mathscr{R}}/\mathfrak{m}_{\mathscr{R}}^2)^*$ - for more details see see [8], [58], [42]. Using Schlessinger's method, Bertin and Mézard proved that the deformation functor of curves with automorphisms $D_{(X,G)}$ posesses a versal element $\mathscr{R}_{\mathrm{gl}}$ which can be expressed as

$$\mathscr{R}_{\mathrm{gl}} = (R_1 \hat{\otimes} \cdots \hat{\otimes} R_r)[[U_1, \ldots, U_N]],$$

where $N = \dim_k H^1(\mathscr{X}_0/G, \pi_*^G(\mathscr{T}_{\mathscr{X}_0}))$, r is the number of wildly ramified points and $R_1, \ldots, R_r$ are complete local rings which are the versal elements to certain local deformation functors defined for each wildly ramified point, [8, cor. 3.3.5]. For the case of Artin-Schreier curves this construction can be made more explicit using the theory of Oort-Sekiguchi-Suwa, which we will briefly describe in the next section.

A.4.2 On the deformation of Artin-Schreier to Kummer

H. Hasse in [31] developed the theory of Artin-Schreier and Kummer extensions of curves side by side, showing that there are some fundamental similarities between the two theories.

In modern language - see [10], [35, p. 498] - Kummer theory studies abelian extensions of exponent n prime to the characteristic p; such extensions are related via the short exact sequence of multiplicative group schemes:

$$1 \longrightarrow \mu_{n,K} \longrightarrow \mathbb{G}_{m,K} \longrightarrow \mathbb{G}_{m,K} \longrightarrow 1$$
$$x \longmapsto x^n$$

where $\mathbb{G}_{m,K}$ is the multiplicative group scheme over the field K, which is assumed to contain the n-th roots of unity, while $\mu_{n,K}$ is the kernel of the map $x \mapsto x^n$. For definitions related to the theory of group schemes see [68]. In particular, Kummer theory can describe all cylic order n extensions of K.

A similar construction can be made for cyclic extensions of order p, which is summarized in the short exact sequence of additive group schemes:

$$1 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_{a,K} \longrightarrow \mathbb{G}_{a,K} \longrightarrow 1$$
$$x \longmapsto \wp(x) = x^p - x$$

where $\mathbb{G}_{a,K}$ is the additive group scheme of K. Notice, that the second sequence has a natural generalization to the ring of Witt vectors which can classify all abelian p-extensions, see [35, 8.11 p.509].

The theories of Artin-Schreier and Kummer extensions were first unified by W. Waterhouse [72] and then generalized by T. Sekiguchi, N. Suwa and F. Oort [59], [60], [61]. More precisely, for a discrete valuation ring $(R, \mathfrak{m}_R)$ with $K = \mathrm{Quot}(R)$, $R/\mathfrak{m}_R = k$ and $\lambda \in \mathfrak{m}_R \backslash \{0\}$, we define the group scheme $\mathscr{G}^{(\lambda)} := \mathrm{Spec} R[x, 1/(1 + \lambda x)]$ with multiplication law given by the ring homomorphism

$$R[x, 1/(\lambda x + 1)] \longrightarrow R[x, 1/(\lambda x + 1)] \otimes_R R[x, 1/(\lambda x + 1)]$$

$$x \longmapsto 1 \otimes x + x \otimes 1 + \lambda(x \otimes x),$$

inverse map given by

$$R[x, 1/(\lambda x + 1)] \longrightarrow R[x, 1/(\lambda x + 1)]$$

$$x \longmapsto -x/(\lambda x + 1)$$

and identity

$$R[x, 1/(\lambda x + 1)] \longrightarrow R$$

$$x \longmapsto 0.$$

Every flat group scheme over $\mathrm{Spec}(R)$ with special fibre $\mathbb{G}_a$ and generic fibre $\mathbb{G}_m$ is isomorphic to $\mathscr{G}^{(\lambda)}$ for some $\lambda \in \mathfrak{m}_R \backslash \{0\}$, [73, Theorem 2.5]. Moreover we have the following exact sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathscr{G}^{(\lambda)} \longrightarrow \mathscr{G}^{(\lambda^p)} \longrightarrow 0$$

which combines the Artin-Schreier and Kummer sequences.

A.4.3 Deforming Artin-Schreier curves

We now focus our attention on Artin-Schreier curves of the form

$$X^p - X - x^{-m} \in k(x)[X] \tag{A.3}$$

with Galois group $G = \mathbb{Z}/p\mathbb{Z}$. Assume that $p \neq 2$ and that $(m, p) = 1$ with $m = pq - \ell$, $1 \leq \ell \leq p - 1$. Define the Oort-Sekiguchi-Suwa factor

$$R = \begin{cases} W(k)[\zeta][[x_1, \ldots, x_q]] & \text{if } \ell = 1, \\ W(k)[\zeta][[x_1, \ldots, x_{q-1}]] & \text{if } \ell \neq 1 \end{cases} \tag{A.4}$$

The curve in eq. (A.3) has a unique ramified point and the versal ring R_σ corresponding to this point has a formally smooth quotient $R_\sigma \twoheadrightarrow R$. Set

$$a(x) = \begin{cases} x^q + x_1 x^{q-1} + \cdots + +x_{q-1}x + x_q, & \text{if } \ell = 1 \\ x^q + x_1 x^{q-1} + \cdots + x_{q-1}x, & \text{if } \ell \neq 1. \end{cases}$$

A model of a deformation over the ring R is given by

$$X^p - X - \frac{x^\ell}{a(x)^p} \in K(x)[X]. \tag{A.5}$$

where

$$K = \begin{cases} k((x_1, \ldots, x_q)) & \text{if } \ell = 1, \\ \\ k((x_1, \ldots, x_{q-1})) & \text{if } \ell \neq 1. \end{cases} \tag{A.6}$$

Theorem A.4.2 (Oort-Sekiguchi-Suwa, Bertin-Mézard)**.** *Let R be the ring defined in eq. (A.4), K be the field defined in eq. (A.6), let $L = \operatorname{Quot}(R)$ and $\lambda = \zeta - 1$. There exists a flat family of curves $\mathscr{X} \to \operatorname{Spec}(R)$ with special fiber G-isomorphic to the Artin-Schreier curve $\mathscr{X}_0 \to \operatorname{Spec}(K)$ given by $X^p - X - \frac{x^\ell}{a(x)^p}$ and generic fiber G-isomorphic to the Kummer curve $\mathscr{X}_\eta \to \operatorname{Spec}(L)$ given by $y^p - \lambda^p x^\ell - a(x)^p$ via the variable change $y = a(x)(\lambda X + 1)$.*

Proof. Substituting the variable change in the equation of the generic fiber gives:

$$\begin{aligned} y^p = \lambda^p x^\ell + a(x)^p \quad &\Leftrightarrow \quad a(x)^p(\lambda X + 1)^p = \lambda^p x^\ell + a(x)^p \\ &\Leftrightarrow \quad (\lambda X + 1)^p = \lambda^p \frac{x^\ell}{a(x)^p} + 1 \\ &\Leftrightarrow \quad \sum_{i=0}^{p} \binom{p}{i} \lambda^i X^i = \lambda^p \frac{x^\ell}{a(x)^p} + 1 \\ &\Leftrightarrow \quad \sum_{i=1}^{p} \binom{p}{i} \lambda^i X^i = \lambda^p \frac{x^\ell}{a(x)^p} \\ &\Leftrightarrow \quad \sum_{i=1}^{p} \binom{p}{i} \lambda^{i-p} X^i = \frac{x^\ell}{a(x)^p} \\ &\Leftrightarrow \quad X^p + \sum_{i=1}^{p-1} \binom{p}{i} \lambda^{i-p} X^i = \frac{x^\ell}{a(x)^p}. \end{aligned} \tag{A.7}$$

Let $\mathfrak{m}$ denote the maximal ideal of $W(k)[\zeta_p]$. By [9, sec. 4.3] we have that

$$p \cdot \lambda^s \equiv \begin{cases} 0 \mod \mathfrak{m}, \text{ for } -(p-1) < s < 0 \\ \\ -1 \mod \mathfrak{m}, \text{ for } s = -(p-1), \end{cases} \tag{A.8}$$

which implies that eq. (A.7) modulo $\mathfrak{m}$ becomes

$$X^p - X = \frac{x^\ell}{a(x)^p}.$$

□

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Bibliography

[1] William W. Adams and Philippe Loustaunau. *An introduction to Gröbner bases*, volume 3 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1994.

[2] Jannis A. Antoniadis and Aristides Kontogeorgis. Automorphisms of curves. In *Algebraic modeling of topological and computational structures and applications*, volume 219 of *Springer Proc. Math. Stat.*, pages 339–361. Springer, Cham, 2017.

[3] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[4] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[5] Dave Bayer and David Eisenbud. Ribbons and their canonical embeddings. *Trans. Amer. Math. Soc.*, 347(3):719–756, 1995.

[6] Christine Berkesch and Frank-Olaf Schreyer. Syzygies, finite length modules, and random curves. In *Commutative algebra and noncommutative algebraic geometry. Vol. I*, volume 67 of *Math. Sci. Res. Inst. Publ.*, pages 25–52. Cambridge Univ. Press, New York, 2015.

[7] José Bertin. Obstructions locales au relèvement de revêtements galoisiens de courbes lisses. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(1):55–58, 1998.

[8] José Bertin and Ariane Mézard. Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques. *Invent. Math.*, 141(1):195–238, 2000.

[9] José Bertin and Ariane Mézard. Problem of formation of quotients and base change. *Manuscripta Math.*, 115(4):467–487, 2004.

[10] B. J. Birch. Cyclotomic fields and Kummer extensions. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 85–93. Thompson, Washington, D.C., 1967.

[11] Christian Bopp and Frank-Olaf Schreyer. A version of Green's conjecture in positive characteristic. *Experimental Mathematics*, 0(0):1–6, 2019.

[12] Helmut Boseck. Zur Theorie der Weierstrasspunkte. *Math. Nachr.*, 19:29–63, 1958.

[13] Irene I. Bouw and Stefan Wewers. The local lifting problem for dihedral groups. *Duke Math. J.*, 134(3):421–452, 2006.

[14] Louis Hugo Brewis and Stefan Wewers. Artin characters, Hurwitz trees and the lifting problem. *Math. Ann.*, 345(3):711–730, 2009.

[15] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[16] Hara Charalambous, Kostas Karagiannis, Sotiris Karanikolopoulos, and Aristides Kontogeorgis. Syzygies of ideals of polynomial rings over principal ideal domains. In *Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation*, ISSAC '20, page 78-82, New York, NY, USA, 2020. Association for Computing Machinery.

[17] Hara Charalambous, Kostas Karagiannis, and Aristides Kontogeorgis. The relative canonical ideal of the Artin-Schreier-Kummer-Witt family of curves, 2019. `arXiv:arXiv:1905.05545`.

[18] C. Chevalley, A. Weil, and E. Hecke. Über das verhalten der integrale 1. gattung bei automorphismen des funktionenkörpers. *Abh. Math. Sem. Univ. Hamburg*, 10(1):358–361, 1934.

[19] Ted Chinburg, Robert Guralnick, and David Harbater. The local lifting problem for actions of finite groups on curves. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(4):537–605, 2011.

[20] Huy Dang, Soumyadip Das, Kostas Karagiannis, Andrew Obus, and Vaidehee Thatte. Local oort groups and the isolated differential data criterion, 2019. `arXiv:arXiv:1912.12797`.

[21] Jean-Guillaume Dumas, Frank Heckenbach, David Saunders, and Volkmar Welker. Computing simplicial homology based on efficient Smith normal form algorithms. In *Algebra, geometry, and software systems*, pages 177–206. Springer, Berlin, 2003.

[22] David Eisenbud. *Commutative algebra*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[23] David Eisenbud. *The geometry of syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry.

[24] Gerd Faltings. Calculus on arithmetic surfaces. *Ann. of Math. (2)*, 119(2):387–424, 1984.

[25] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at `http://www.math.uiuc.edu/Macaulay2/`.

[26] Barry Green and Michel Matignon. Liftings of Galois covers of smooth curves. *Compositio Math.*, 113(3):237–272, 1998.

[27] Mark L. Green. Koszul cohomology and the geometry of projective varieties. *J. Differential Geom.*, 19(1):125–171, 1984.

[28] Alexander Grothendieck and Michel Raynaud. *Revêtements étales et groupe fondamental (SGA 1)*, volume 3 of *Documents Mathématiques (Paris)*. Séminaire de géométrie algébrique du Bois Marie 1960–61. Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin]; Société Mathématique de France, Paris, 2003

[29] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[30] Robin Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, New York, 2010.

[31] Helmut Hasse. Theorie der relativ-zyklischen algebraischen Funktionenkörper, insbesondere bei endlichem Konstantenkörper. *J. Reine Angew. Math.*, 172:37–54, 1934.

[32] Michiel Hazewinkel. *Formal groups and applications*, volume 78 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[33] E. Hecke. Über ein fundamentalproblem aus der theorie der elliptischen modulfunktionen. *Abh. Math. Sem. Univ. Hamburg*, 6(1):235–257, 1928.

[34] Adolf Hurwitz. Über algebraische Gebilde mit eindeutigen Tranformationen in sich. *Math. Ann.*, 41:403–442, 1893.

[35] Nathan Jacobson. *Basic algebra. II*. W. H. Freeman and Company, New York, second edition, 1989.

[36] Sotiris Karanikolopoulos. On holomorphic polydifferentials in positive characteristic. *Math. Nachr.*, 285(7):852–877, 2012.

[37] Sotiris Karanikolopoulos and Aristides Kontogeorgis. Integral representations of cyclic groups acting on relative holomorphic differentials of deformations of curves with automorphisms. *Proc. Amer. Math. Soc.*, 142(7):2369–2383, 2014.

[38] Aristides Kontogeorgis. The group of automorphisms of the function fields of the curve $x^n + y^m + 1 = 0$. *J. Number Theory*, 72(1):110–136, 1998.

[39] Aristides Kontogeorgis. On the tangent space of the deformation functor of curves with automorphisms. *Algebra Number Theory*, 1(2):119–161, 2007.

[40] Aristides Kontogeorgis, Alexios Terezakis, and Ioannis Tsouknidas. Automorphisms and the canonical ideal, 2019.. `arXiv:arXiv:1909.10282`.

[41] Daniel J. Madden. Arithmetic in generalized Artin-Schreier extensions of $k(x)$. *J. Number Theory*, 10(3):303–323, 1978.

[42] B. Mazur. Deformation theory of Galois representations (in Modular forms and Fermat's last theorem). pages xx+582, 1997. Papers from the Instructional Conference

on Number Theory and Arithmetic Geometry held at Boston University, Boston, MA, August 9–18, 1995.

[43] Shōichi Nakajima. On abelian automorphism groups of algebraic curves. *J. London Math. Soc. (2)*, 36(1):23–32, 1987.

[44] Shōichi Nakajima. p-ranks and automorphism groups of algebraic curves. *Trans. Amer. Math. Soc.*, 303(2):595–607, 1987.

[45] Guillaume Pagot Relèvement en caractéristique zéro d'actions de groupes abéliens de type $(p, \ldots, p)$, *Thèse, Université Bordeaux I, available at //http://www.math.u-bordeaux1.fr/ mmatigno/Pagot-These.pdf*, 2002.

[46] Andrew Obus. The (local) lifting problem for curves. In *Galois-Teichmüller theory and arithmetic geometry*, volume 63 of *Adv. Stud. Pure Math.*, pages 359–412. Math. Soc. Japan, Tokyo, 2012.

[47] Andrew Obus. The local lifting problem for A_4. *Algebra Number Theory*, 10(8):1683–1693, 2016.

[48] Andrew Obus. A generalization of the Oort conjecture. *Comment. Math. Helv.*, 92(3):551–620, 2017.

[49] Andrew Obus and Stefan Wewers. Cyclic extensions and the local lifting problem. *Ann. of Math. (2)*, 180(1):233–284, 2014.

[50] Gilvan Oliveira and Karl-Otto Stöhr. Gorenstein curves with quasi-symmetric Weierstrass semigroups. *Geom. Dedicata*, 67(1):45–63, 1997.

[51] Frans Oort. Lifting algebraic curves, abelian varieties, and their endomorphisms to characteristic zero. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 165–195. Amer. Math. Soc., Providence, RI, 1987.

[52] Irena Peeva. *Graded syzygies*, volume 14 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011.

[53] Florian Pop. The Oort conjecture on lifting covers of curves. *Ann. of Math. (2)*, 180(1):285–322, 2014.

[54] Joseph Rabinoff. The theory of witt vectors. September 2014.

[55] Irving Reiner. Integral representations of cyclic p-groups. In *Topics in algebra (Proc. 18th Summer Res. Inst., Austral. Math. Soc., Austral. Nat. Univ., Canberra, 1978)*, volume 697 of *Lecture Notes in Math.*, pages 70–87. Springer, Berlin, 1978.

[56] B. Saint-Donat. On Petri's analysis of the linear system of quadrics through a canonical curve. *Math. Ann.*, 206:157–175, 1973.

[57] Mahrud Sayrafi. Computations over local rings in macaulay2, 2017.

[58] Michael Schlessinger. Functors of Artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.

[59] T. Sekiguchi, F. Oort, and N. Suwa. On the deformation of Artin-Schreier to Kummer. *Ann. Sci. École Norm. Sup. (4)*, 22(3):345–375, 1989.

[60] Tsutomu Sekiguchi and Noriyuki Suwa. Théories de Kummer-Artin-Schreier-Witt. *C. R. Acad. Sci. Paris Sér. I Math.*, 319(2):105–110, 1994.

[61] Tsutomu Sekiguchi and Noriyuki Suwa. Théorie de Kummer-Artin-Schreier et applications. *J. Théor. Nombres Bordeaux*, 7(1):177–189, 1995. Les Dix-huitièmes Journées Arithmétiques (Bordeaux, 1993).

[62] Edoardo Sernesi. *Deformations of algebraic schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[63] Jean-Pierre Serre. Sur la topologie des variétés algébriques en caractéristique p. In *Symposium internacional de topología algebraica International symposium on algebraic topology*, pages 24–53. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.

[64] Jean-Pierre Serre. Exemples de variétés projectives en caractéristique p non relevables en caractéristique zéro. *Proc. Nat. Acad. Sci. U.S.A.*, 47:108–109, 1961.

[65] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[66] Henning Stichtenoth. *Algebraic function fields and codes.* Springer-Verlag, Berlin, 1993.

[67] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series.* American Mathematical Society, Providence, RI, 1996.

[68] John Tate. Finite flat group schemes. In *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, pages 121–154. Springer, New York, 1997.

[69] Robert C. Valentini and Manohar L. Madan. Automorphisms and holomorphic differentials in characteristic p. *J. Number Theory*, 13(1):106–115, 1981.

[70] Claire Voisin. Green's generic syzygy conjecture for curves of even genus lying on a $K3$ surface. *J. Eur. Math. Soc. (JEMS)*, 4(4):363–404, 2002.

[71] Claire Voisin. Green's canonical syzygy conjecture for generic curves of odd genus. *Compos. Math.*, 141(5):1163–1190, 2005.

[72] William C. Waterhouse. A unified Kummer-Artin-Schreier sequence. *Math. Ann.*, 277(3):447–451, 1987.

[73] William C. Waterhouse and Boris Weisfeiler. One-dimensional affine group schemes. *J. Algebra*, 66(2):550–568, 1980.

[74] Bradley Weaver. The local lifting problem for D_4. *Israel J. Math.*, 228(2):587–626, 2018.

[75] Charles A. Weibel. *An introduction to homological algebra.* Cambridge University Press, Cambridge, 1994.

[76] Ernst Witt. Zyklische Körper und Algebren der Charakteristik p vom Grad p^n. Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik p. *J. Reine Angew. Math.*, 176:126–140, 1937.

[77] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.