THE CANONICAL IDEAL AND THE DEFORMATION THEORY OF CURVES WITH AUTOMORPHISMS

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ABSTRACT. The deformation theory of curves is studied by using the canonical ideal. The deformation problem of curves with automorphisms is reduced to a deformation problem of linear representations.

1. INTRODUCTION

The deformation theory of curves with automorphisms is an important generalization of the classical deformation theory of curves. This theory is related to the lifting problem of curves with automorphisms, since one can consider liftings from characteristic $p > 0$ to characteristic zero in terms of a sequence of local Artin-rings.

J. Bertin and A. Mézard in [4], following Schlessinger's [33] approach introduced a deformation functor D_{gl} and studied it in terms of Grothendieck's equivariant cohomology theory [12]. In Schlessinger's approach to deformation theory, we want to know the tangent space to the deformation functor $D_{\mathrm{gl}}(k[\epsilon])$ and the possible obstructions to lift a deformation over an Artin local ring Γ to a small extension $\Gamma' \rightarrow \Gamma$. The reader who is not familiar with deformation theory is referred to section 2.1 for terminology and references to the literature. The tangent space of the global deformation functor $D_{\mathrm{gl}}(k[\epsilon])$ can be identified as Grothendieck's equivariant cohomology group $H^1(G, X, \mathscr{T}_X)$, which is known to be equal to the invariant space $H^1(X, \mathscr{T}_X)^G$. Moreover, a local local-global theorem is known, which can be expressed in terms of the short exact sequence:

(1)
$$0 \longrightarrow H^1(X/G, \pi_*^G(\mathscr{T}_X)) \longrightarrow H^1(G, X, \mathscr{T}_X) \longrightarrow H^0(X/G, R^1\pi_*^G(\mathscr{T}_X)) \longrightarrow 0$$
$$\Big\downarrow \wr \|$$
$$\bigoplus_{i=1}^{r} H^1\left(G_{x_i}, \widehat{\mathscr{T}_{X,x_i}}\right)$$

The lifting obstruction can be seen as an element in

$$H^2(G, X, \mathscr{T}_X) \cong \bigoplus_{i=1}^{r} H^2\left(G_{x_i}, \widehat{\mathscr{T}_{X,x_i}}\right).$$

In the above equations $x_1, \ldots, x_r \in X$ are the ramified points, G_{x_i} are the corresponding isotropy groups and $\widehat{\mathscr{T}_{X,x_i}}$ are the completed local tangent spaces, that is $\widehat{\mathscr{T}_{X,x_i}} = k[[t_i]]\frac{d}{dt_i}$, where t_i is a local uniformizer at x_i. The space $k[[t_i]]\frac{d}{dt_i}$ is

seen as G_{x_i}-module by the adjoint action, see [7, 2.1], [22, 1.5]. Bertin and Mézard reduced the computation of obstruction to the infinitesimal lifting problem of representations of the isotropy group G_{x_i} to the difficult group $\mathrm{Aut}k[[t]]$. In this article for a ring Γ, $\mathrm{Aut}\Gamma[[t]]$ denotes the group of continous automorphisms of $\Gamma[[t]]$.

This article aims to give a new approach to the deformation theory of curves with automorphisms, which is not based on the deformation theory of representations on the subtle object $\mathrm{Aut}k[[t]]$, but on the deformation theory of the better understood general linear group. In order to do so, we will restrict ourselves to curves that satisfy the mild assumptions of Petri's theorem

Theorem 1 (Petri's theorem)**.** *For a non-singular non-hyperelliptic curve X of genus $g \geq 3$ defined over an algebraically closed field with sheaf of differentials Ω_X there is the following short exact sequence:*

$$0 \to I_X \to \mathrm{Sym}H^0(X, \Omega_X) \to \bigoplus_{n=0}^{\infty} H^0(X, \Omega_X^{\otimes n}) \to 0,$$

where I_X is generated by elements of degree 2 and 3. Also if X is not a non-singular quintic of genus 6 or X is not a trigonal curve, then I_X is generated by elements of degree 2.

For a proof of this theorem we refer to [11], [32]. The ideal I_X is called *the canonical ideal* and it is the homogeneous ideal of the embedded curve $X \to \mathbb{P}^{g-1}$.

For curves that satisfy the assumptions of Petri's theorem and their canonical ideal is generated by quadrics, we prove in section 3 the following relative version of Petri's theorem

Proposition 2. *Let $f_1, \ldots, f_r \in S := \mathrm{Sym}H^0(X, \Omega_X) = k[\omega_1, \ldots, \omega_g]$ be quadratic polynomials which generate the canonical ideal I_X of a curve X defined over an algebraic closed field k. Any deformation $\mathscr{X}_A$ is given by quadratic polynomials $\tilde{f}_1, \ldots, \tilde{f}_r \in \mathrm{Sym}H^0(\mathscr{X}_A, \Omega_{\mathscr{X}_A/A}) = A[W_1, \ldots, W_g]$, which reduce to $f_1, \ldots, f_r$ modulo the maximal ideal $\mathfrak{m}_A$ of A.*

This approach allows us to replace several of Grothendieck's equivariant cohomology constructions in terms of linear algebra. Let us mention that in general, it is not so easy to perform explicit computations with equivariant Grothendieck cohomology groups and usually, spectral sequences or a complicated equivariant Chech cohomology is used, see [3], [23, sec.3].

Let $i : X \to \mathbb{P}^{g-1}$ be the canonical embedding and let $M_g(k)/\langle\mathbb{I}_g\rangle$ be the space of $g \times g$ matrices with coefficients in k, modulo the vector subspace of scalar multiples of the identity matrix. In proposition 28 we prove that elements $[f] \in H^1(X, \mathscr{T}_X)^G = D_{\mathrm{gl}}k[\epsilon]$ correspond to cohomology classes in $H^1(G, M_g(k)/\langle\mathbb{I}_g\rangle)$.

Furthermore, in our setting, the obstruction to liftings is reduced to an obstruction to the lifting of the linear canonical representation

$$(2) \qquad\qquad\qquad \rho : G \to \mathrm{GL}\big(H^0(X, \Omega_X)\big).$$

Also we will give a compatibility criterion involving the defining quadratic equations of our canonically embedded curve, namely in section 4 we will prove the following:

Theorem 3. *Consider an epimorphism $\Gamma' \to \Gamma \to 0$ of local Artin rings. A deformation $x \in D_{\mathrm{gl}}(\Gamma)$ can be lifted to a deformation $x' \in D_{\mathrm{gl}}(\Gamma')$ if and only if the representation $\rho_\Gamma : G \to \mathrm{GL}_g(\Gamma)$ lifts to a representation $\rho_{\Gamma'} : G \to \mathrm{GL}_g(\Gamma')$*

and moreover there is a lifting $X_{\Gamma'}$ of the embedded deformation of X_Γ which is invariant under the lifted action of $\rho_{\Gamma'}$.

Remark 4. The liftability of the representation ρ is a strong condition. In proposition 31 we give an example of a representation $\rho : G \to \mathrm{GL}_2(k)$, for a field k of positive characteristic p, which can not be lifted to a representation $\tilde{\rho} : G \to \mathrm{GL}_2(R)$ for $R = W(k)[\zeta_{p^h}]$, meaning that a lifting in some small extension $R/\mathfrak{m}_R^{i+1} \to R/\mathfrak{m}_R^i$ is obstructed. Here R denotes the Witt ring of k with a primitive p^h root of unity added, which has characteristic zero. In our counterexample $G = C_q \rtimes C_m$, $q = p^h$, $(m, p) = 1$.

Remark 5. One can always pass from the local lifting problem of $\rho : G \to \mathrm{Aut}\Gamma[[t]]$ to a global lifting problem, by considering the Harbater-Katz-Gabber (HKG for short) compactification X of the local action. Then one can consider the criterion involving the linear representation $\rho : G \to \mathrm{Gl}(H^0(X, \Omega_X))$. Notice that in [27] the canonical ideal for HGK-curves is explicitly described.

Remark 6. In 4.1 we will use the tools developed in this article to show that certain automorphisms of the Hermitian curve do not lift, even in possible characteristic. This is expected since the Hermitian curve is the unique curve with an extreme size of its automorphism group, see [36].

Remark 7. The invariance of the canonical ideal I_{X_Γ} under the action of G can be checked using Gauss elimination and echelon normal forms, see [25, sec 2.2].

Remark 8. The canonical ideal I_{X_Γ} is determined by r quadratic polynomials which form a $\Gamma[G]$-invariant Γ-submodule V_Γ in the free Γ-module of symmetric $g \times g$ matrices with entries in Γ. When we pass from a deformation $x \in D_{\mathrm{gl}}(\Gamma)$ to a deformation $x' \in D_{\mathrm{gl}}(\Gamma')$ we ask that the canonical ideal $I_{X_{\Gamma'}}$ is invariant under the lifted action given by the representation $\rho_{G'} : G \to \mathrm{GL}_g(\Gamma')$. In definition 11.1 we introduce an action $T(g)$ on the vector space of symmetric $g \times g$ matrices, and the invariance of the canonical ideal is equivalent to the invariance under the T-action of the Γ'-submodule $V_{\Gamma'}$ generated by the quadratic polynomials generating the ideal $I_{X'}$. Therefore, we can write one more representation

(3) $$\rho^{(1)} : G \to \mathrm{GL}\big(\mathrm{Tor}_1^S(k, I_X)\big).$$

Set $r = \binom{g-2}{2}$. Liftings of the representations $\rho, \rho^{(1)}$ defined by eq. (2), (3) in $\mathrm{GL}_g(\Gamma)$ resp. $\mathrm{GL}_r(\Gamma)$ will be denoted by ρ_Γ resp. $\rho_\Gamma^{(1)}$.

Notice that if the representation ρ_Γ lifts to a representation $\rho_{\Gamma'}$ and moreover there is a lifting $X_{\Gamma'}$ of the relative curve X_Γ so that $X_{\Gamma'}$ has an ideal $I_{X_{\Gamma'}}$ which is $\rho_{\Gamma'}$ invariant, then the representation $\rho_\Gamma^{(1)}$ also lifts to a representation $\rho_{\Gamma'}^{(1)}$, see also [25, prop. 5]

The deformation theory of linear representations $\rho, \rho^{(1)}$ gives rise to cocycles D_σ, $D_{\sigma^{-1}}^{(1)}$ in $H^1(G, M_g(k))$, $H^1(G, M_{\binom{g-2}{2}}(k))$, while the deformation theory of curves with automorphisms introduces a cocycle $B_\sigma[f]$ corresponding to $[f] \in H^1(X, \mathscr{T}_X)^G$. We will introduce a compatibility condition in the section 4.2 among these cocycles, using the isomorphism

$$\psi : M_g(k)/\langle \mathbb{I}_g \rangle \xrightarrow{\cong} H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \hookrightarrow \mathrm{Hom}_S(I_X, S/I_X) = H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$$
$$B \longmapsto \psi_B$$

defined in proposition 25.

Proposition 9. *The following compatibility condition is satisfied*

$$(4) \qquad \qquad \psi_{D_\sigma} - \psi_{B_\sigma[f]} = D^{(1)}_{\sigma^{-1}}.$$

The structure of the article is as follows. In section 2.2 we will present side by side the deformation theory of linear representations $\rho : G \to \mathrm{GL}(V)$ and the deformation theory of representations of the form $\rho : G \to \mathrm{Aut}k[[t]]$. The deformation theory of linear representations is a better-understood object of study, see [29], which played an important role in topology [20] and also in the proof of Fermat's last theorem, see [30]. The deformation theory of representations in $\mathrm{Aut}k[[t]]$ comes out from the study of local fields and it is related to the deformation problem of curves with automorphisms after the local global theory of Bertin Mézard. There is also an increased interest related to the study of Nottingham groups and $\mathrm{Aut}k[[t]]$, see [5], [9],[26].

It seems that the similarities between these two deformation problems are known to the expert, see for example [31, prop. 3.13]. For the convenience of the reader and in order to fix the notation, we also give a detailed explanation and comparison of these two deformation problems.

In section 3 we revise the theory of relative canonical ideals and the work of the first author together with H. Charalambous and K. Karagiannis [6] aiming at the deformation problem of curves with automorphisms. More precisely a relative version of Petri's theorem is proved, which implies that the relative canonical ideal is generated by quadratic polynomials.

In section 4 we study both the obstruction and the tangent space problem of the deformation theory of curves with automorphisms using the relative canonical ideal point of view. In this section theorem 3 is proved.

2. DEFORMATION THEORY OF CURVES WITH AUTOMORPHISMS

2.1. **Global deformation functor.** Let Λ be a complete local Noetherian ring with residue field k, where k is an algebraically closed field of characteristic $p \geq 0$. Let $\mathscr{C}$ be the category of local Artin Λ-algebras with residue field k and homomorphisms the local Λ-algebra homomorphisms $\phi : \Gamma' \to \Gamma$ between them, that is homomorphisms ϕ that statisfy $\phi^{-1}(\mathfrak{m}_\Gamma) = \mathfrak{m}_{\Gamma'}$. The deformation functor of curves with automorphisms is a functor D_{gl} from the category $\mathscr{C}$ to the category of sets

$$D_{\mathrm{gl}} : \mathscr{C} \to \mathrm{Sets}, \Gamma \mapsto \left\{ \begin{array}{c} \text{Equivalence classes} \\ \text{of deformations of} \\ \text{couples } (X, G) \text{ over } \Gamma \end{array} \right\}$$

defined as follows. For a subgroup G of the group $\mathrm{Aut}(X)$, a deformation of the couple (X, G) over the local Artin ring Γ is a proper, smooth family of curves

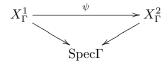$$X_\Gamma \to \mathrm{Spec}(\Gamma)$$

parametrized by the base scheme $\mathrm{Spec}(\Gamma)$, together with a group homomorphism $G \to \mathrm{Aut}_\Gamma(X_\Gamma)$, such that there is a G-equivariant isomorphism ϕ from the fibre

over the closed point of Γ to the original curve X:

$$\phi : X_\Gamma \otimes_{\mathrm{Spec}(\Gamma)} \mathrm{Spec}(k) \to \mathrm{X}.$$

Two deformations X_Γ^1, X_Γ^2 are considered to be equivalent if there is a G-equivariant isomorphism ψ that reduces to the identity in the special fibre and making the following diagram commutative:

$$
\begin{array}{ccc}
X_\Gamma^1 & \xrightarrow{\;\;\psi\;\;} & X_\Gamma^2 \\
& \searrow \qquad \swarrow & \\
& \mathrm{Spec}\Gamma &
\end{array}
$$

Given a small extension of Artin local rings

$$(5) \qquad\qquad 0 \to E \cdot k \to \Gamma' \to \Gamma \to 0$$

and an element $x \in D_{\mathrm{gl}}(\Gamma)$ we have that the set of lifts $x' \in D_{\mathrm{gl}}(\Gamma')$ extending x is a principal homogeneous space under the action of $D_{\mathrm{gl}}(k[\epsilon])$ and such an extension x' exists if certain obstruction vanishes. It is well known, see section 2.2, that similar behavior have the deformation functors of representations.

2.2. **Lifting of representations.** Let $\mathscr{G} : \mathscr{C} \to$ Groups be a group functor, see [8, ch. 2]. In this article, we will be mainly interested in two group functors. The first one, GL_g, will be represented by the by the group scheme $G_g = \Lambda[x_{11}, \ldots, x_{gg}, \det(x_{ij})^{-1}]$, that is $\mathrm{GL}_g(\Gamma) = \mathrm{Hom}_\Lambda(G_g, \Gamma)$. The second one is the group functor from the category of rings to the category of groups $\mathscr{N} : \Gamma \mapsto \mathrm{Aut}\Gamma[[t]]$.

We also assume that each group $\mathscr{G}(\Gamma)$ is embedded in the group of units of some ring $\mathscr{R}(\Gamma)$ depending functorially on Γ. This condition is asked since our argument requires us to be able to add certain group elements. We also assume that the additive group of the ring $\mathscr{R}(\Gamma)$ has the structure of direct product Γ^I, while $\mathscr{R}(\Gamma) = \mathscr{R}(\Lambda) \otimes_\Lambda \Gamma$. Notice, that I might be an infinite set, but since all rings involved are Noetherian Γ^I is flat, see [28, 4F].

A representation of the finite group G in $\mathscr{G}(\Gamma)$ is a group homomorphism

$$\rho : G \to \mathscr{G}(\Gamma),$$

where Γ is a commutative ring.

Remark 10. Consider two sets X, Y acted on by the group G. Then, every function $f : X \to Y$ is acted on by G, by defining the function $^\sigma f : X \to Y$, sending $x \mapsto \sigma f \sigma^{-1}(x)$. This construction will be used throughout this article.

More precisely we will use the following actions

Definition 11. (1) Let $M_g(\Gamma)$ denote the set of $g \times g$ matrices with entries in ring Γ. An element $A \in M_g(\Gamma)$ will be acted on by $g \in G$ in terms of the action

$$T(g)A = \rho(g^{-1})^t A \rho(g^{-1}).$$

This is the natural action coming from the action of G on $H^0(X, \Omega_{X/k})$ and on the quadratic forms $\omega^t A \omega$. We raise the group element in -1 in order to have a left action, that is $T(gh)A = T(g)T(h)A$. Notice also that $T(g)$ restricts to an action on the space $\mathscr{S}_g(\Gamma)$ of symmetric $g \times g$ matrices with entries in Γ.

(2) The adjoint action on elements $A \in M_g(\Gamma)$, comes from the action to the tangent space of the general linear group.

$$\mathrm{Ad}(g)A = \rho(g)A\rho(g^{-1}).$$

(3) Actions on elements which can be seen as functions between G-spaces as in remark 10. This action will be denoted as $f \mapsto^\sigma f$.

Examples
1. Consider the groups $\mathrm{GL}_g(\Gamma)$ consisted of all invertible $g \times g$ matrices with coefficients in Γ. The group functor

$$\Gamma \mapsto \mathrm{GL}_g(\Gamma) = \mathrm{Hom}(R, \Gamma),$$

is representable by the affine Λ-algebra $R = k[x_{11}, \ldots, x_{gg}, \det\left((x_{ij})\right)^{-1}]$, see [37, 2.5]. In this case the ring $\mathscr{R}(\Gamma)$ is equal to $\mathrm{End}(\Gamma^g)$, while $I = \{i, j \in \mathbb{N} : 1 \le i, j, \le g\}$.

We can consider the subfunctor $\mathrm{GL}_{g,\mathbb{I}_g}$ consisted of all elements $f \in \mathrm{GL}_g(\Gamma)$, which reduce to the identity modulo the maximal ideal $\mathfrak{m}_\Gamma$. The tangent space $T_{\mathbb{I}_g}\mathrm{GL}_g$ of GL_g at the identity element $\mathbb{I}_g$, that is the space $\mathrm{Hom}(\mathrm{Spec}k[\epsilon], \mathrm{Spec}R)$ or equivalently the set $\mathrm{GL}_{g,\mathbb{I}_g}(k[\epsilon])$ consisted of $f \in \mathrm{Hom}(R, k[\epsilon])$, so that $f \equiv \mathbb{I}_g \mod\langle\epsilon\rangle$. This set is a vector space according to the functorial construction given in [30, p. b 272] and can be identified to the space of $\mathrm{End}(k^g) = M_g(k)$, by identifying

$$\mathrm{Hom}(R, k[\epsilon]) \ni f \mapsto \mathbb{I}_g + \epsilon M, M \in M_g(k).$$

The later space is usually considered as the tangent space of the algebraic group $\mathrm{GL}_g(k)$ at the identity element or equivalently as the Lie algebra corresponding to $\mathrm{GL}_g(k)$.

The representation $\rho : G \to \mathrm{GL}_g(\Gamma)$ equips the space $T_{\mathbb{I}_g}\mathrm{GL}_g = M_g(k)$ with the adjoint action, which is the action described in remark 10, when the endomorphism M is seen as an operator $V \to V$, where V is a G-module in terms of the representation ρ:

$$G \times M_g(k) \longrightarrow M_g(k)$$
$$(g, M) \longmapsto \mathrm{Ad}(g)(M) = gMg^{-1}.$$

In order to make clear the relation with the local case below, where the main object of study is the automorphism group of a completely local ring we might consider the completion $\hat{R}_\mathbb{I}$ of the localization of $R = k[x_{11}, \ldots, x_{gg}, \det\left((x_{ij})\right)^{-1}]$ at the identity element. We can now form the group $\mathrm{Aut}\hat{R}_\mathbb{I}$ of automorphisms of the ring $\hat{R}_\mathbb{I}$ which reduce to the identity modulo $\mathfrak{m}_{\hat{R}_\mathbb{I}}$. The later automorphism group is huge but it certainly contains the group G acting on $\hat{R}_\mathbb{I}$ in terms of the adjoint representation. We have that elements $\sigma \in \mathrm{Aut}\hat{R}_\mathbb{I} \otimes k[\epsilon]$ are of the form

$$\sigma(x_{ij}) = x_{ij} + \epsilon\beta(x_{ij}), \text{ where } \beta(x_{ij}) \in \hat{R}_\mathbb{I}.$$

Moreover, the relation

$$\sigma(f \cdot g) = fg + \epsilon\beta(fg) = (f + \epsilon\beta(f))(g + \epsilon\beta(f)),$$

implies that the map β is a derivation and

$$\beta(fg) = f\beta(g) + \beta(f)g.$$

Therefore, β is a linear combination of $\frac{\partial}{\partial x_{ij}}$, with coefficients in $\hat{R}_{\mathbb{I}}$, that is

$$\beta = \sum_{0 \leqq i,j \leqq g} a_{i,j} \frac{\partial}{\partial x_{ij}}.$$

Remark 12. In the literature of Lie groups and algebras, the matrix notation $M_g(k)$ for the tangent space is frequently used for the Lie algebra-tangent space at identity, instead of the later vector field-differential operator approach, while in the next example, the differential operator notation for the tangent space is usually used.

2. Consider now the group functor $\Gamma \mapsto \mathscr{N}(\Gamma) = \mathrm{Aut}\Gamma[[t]]$. An element $\sigma \in \mathrm{Aut}\Gamma[[t]]$ is fully described by its action on t, which can be expressed as an element in $\Gamma[[t]]$. When Γ is an Artin local algebra then an automorphism is given by

$$\sigma(t) = \sum_{\nu=0}^{\infty} a_\nu t^\nu, \text{ where } a_i \in \Gamma, a_0 \in \mathfrak{m}_\Gamma \text{ and } a_1 \text{ is a unit in } \Gamma.$$

If a_1 is not a unit in Γ or $a_0 \notin \mathfrak{m}_\Gamma$, then σ is an endomorphism of $\Gamma[[t]]$. In this way $\mathrm{Aut}\Gamma[[t]]$ can be seen as the group of invertible elements in $\Gamma[[t]] = \mathrm{End}\Gamma[[t]] = \mathscr{R}(\Gamma)$. In this case, the set I is equal to the set of natural numbers, where Γ^I can be identified as the set of coefficients of each powerseries.

$$\mathrm{Aut}(k[\epsilon][[t]]) = \left\{ t \mapsto \sigma(t) = \sum_{\nu=1}^{\infty} a_i t^\nu : a_i = \alpha_i + \epsilon\beta_i, \ \alpha_i, \beta_i \in k, \alpha_1 \neq 0 \right\}.$$

Exactly as we did in the general linear group case, let us consider the subfunctor $\Gamma \mapsto \mathscr{N}_{\mathbb{I}}(\Gamma)$, where $\mathscr{N}_{\mathbb{I}}(\Gamma)$ consists of all elements in $\mathrm{Aut}\Gamma[[t]]$, which reduce to the identity mod $\mathfrak{m}_\Gamma$.

Such an element $\sigma \in \mathscr{N}_{\mathbb{I}}(k[\epsilon])$ transforms $f \in k[[t]]$ to a formal powerseries of the form

$$\sigma(f) = f + \epsilon F_\sigma(f),$$

where $F_\sigma(f)$ is fully determined by the value of $\sigma(t)$. The multiplication condition $\sigma(f_1 f_2) = \sigma(f_1)\sigma(f_2)$ implies that

$$F_\sigma(f_1 f_2) = f_1 F_\sigma(f_2) + F_\sigma(f_1) f_2,$$

that is F_σ is a $k[[t]]$-derivation, hence an element in $k[[t]]\frac{d}{dt}$.

The local tangent space of $\Gamma[[t]]$ is defined to be the space of differential operators $f(t)\frac{d}{dt}$, see [4], [7], [22]. The G action on the element $\frac{d}{dt}$ is given by the adjoint action, which is given as a composition of operators, and is again compatible with the action given in remark 10:

$$\Gamma[[t]] \xrightarrow{\rho(\sigma^{-1})} \Gamma[[t]] \xrightarrow{\frac{d}{dt}} \Gamma[[t]] \xrightarrow{\rho(\sigma)} \Gamma[[t]]$$

$$t \longmapsto \rho(\sigma^{-1})(t) \longmapsto \frac{d\rho(\sigma^{-1})(t)}{dt} \longmapsto \rho(\sigma)\left(\frac{d\rho(\sigma^{-1})(t)}{dt}\right)$$

So the G-action on the local tangent space $k[[t]]\frac{d}{dt}$ is given by

$$f(t)\frac{d}{dt} \longmapsto \mathrm{Ad}(\sigma)\left(f(t)\frac{d}{dt}\right) = \rho(\sigma)(f(t)) \cdot \rho(\sigma)\left(\frac{d\rho(\sigma^{-1})(t)}{dt}\right)\frac{d}{dt},$$

see also [22, lemma 1.10], for a special case.

$\mathscr{G}(\Gamma)$	$\mathscr{R}(\Gamma)$	tangent space	action
$\mathrm{GL}_g(\Gamma)$	$\mathrm{End}_g(\Gamma)$	$\mathrm{End}_g(k) = M_g(k)$	$M \mapsto \mathrm{Ad}(\sigma)(M)$
$\mathrm{Aut}\Gamma[[t]]$	$\mathrm{End}(\Gamma[[t]])$	$k[[t]]\frac{d}{dt}$	$f(t)\frac{d}{dt} \longmapsto \mathrm{Ad}(\sigma)\left(f(t)\frac{d}{dt}\right)$

TABLE 1. Comparing the two group functors

Motivated by the above two examples we can define

Definition 13. Let $\mathscr{G}_{\mathbb{I}}$ be the subfunctor of $\mathscr{G}$, defined by

$$\mathscr{G}_{\mathbb{I}}(\Gamma) = \{f \in \mathscr{G}(\Gamma) : f = \mathbb{I} \bmod \mathfrak{m}_\Gamma\}.$$

The tangent space to the functor $\mathscr{G}$ at the identity element is defined as $\mathscr{G}_{\mathbb{I}}(k[\epsilon])$, see [30]. Notice, that $\mathscr{G}_{\mathbb{I}}(k[\epsilon]) \cong \mathscr{R}(k)$, is k-vector space, acted on in terms of the adjoint representation, given by

$$G \times \mathscr{G}_{\mathbb{I}}(\Gamma) \longrightarrow \mathscr{G}_{\mathbb{I}}(\Gamma)$$
$$(\sigma, f) \longmapsto \rho(\sigma) \cdot f \cdot \rho(\sigma)^{-1}.$$

If $\mathscr{R}(\Gamma)$ can be interpreted as an endomorphism ring, then the above action can be interpreted in terms of the action on functions as described in remark 10.

We will define the tangent space in our setting as $\mathscr{T} = \mathscr{R}(k)$, which is equipped with the adjoint action.

2.3. **Deforming representations.** We can now define the deformation functor F_ρ for any local Artin algebra Γ with maximal ideal $\mathfrak{m}_\Gamma$ in $\mathscr{C}$ to the category of sets:

(6)
$$F_\rho : \Gamma \in \mathrm{Ob}(\mathscr{C}) \mapsto \left\{ \begin{array}{l} \text{liftings of } \rho : G \to \mathscr{G}(k) \\ \text{to } \rho_\Gamma : G \to \mathscr{G}(\Gamma) \text{ modulo} \\ \text{conjugation by an element} \\ \text{of } \ker(\mathscr{G}(\Gamma) \to \mathscr{G}(k)) \end{array} \right\}$$

Let

(7)
$$0 \longrightarrow \langle E \rangle = E \cdot \Gamma' = E \cdot k \xrightarrow{\phi'} \Gamma' \underset{\phi}{\overset{i}{\rightrightarrows}} \Gamma \longrightarrow 0$$

be a small extension in $\mathscr{C}$, that is the kernel of the natural onto map ϕ is a principal ideal, generated by E and $E \cdot \mathfrak{m}_{\Gamma'} = 0$. In the above diagram $i : \Gamma \to \Gamma'$ is a section, which is not necessarily a homomorphism. Since the kernel of ϕ is a principal ideal $E \cdot \Gamma'$ annihilated by $\mathfrak{m}_{\Gamma'}$ it is naturally a $k = \Gamma'/\mathfrak{m}_{\Gamma'}$-vector space, which is one dimensional.

Lemma 14. *For a small extension as given in eq. (7) consider two liftings $\rho_{\Gamma'}^1, \rho_{\Gamma'}^2$ of the representation ρ_Γ. The map*

$$d : G \longrightarrow \mathscr{T} := \mathscr{R}(k)$$
$$\sigma \longmapsto d(\sigma) = \frac{\rho_{\Gamma'}^1(\sigma)\rho_{\Gamma'}^2(\sigma)^{-1} - \mathbb{I}_{\Gamma'}}{E}$$

is a cocycle.

Proof. We begin by observing that $\phi\left(\rho^1_{\Gamma'}(\sigma)\rho^2_{\Gamma'}(\sigma)^{-1} - \mathbb{I}_{\Gamma'}\right) = 0$, hence

$$\rho^1_{\Gamma'}(\sigma)\rho^2_{\Gamma'}(\sigma)^{-1} = \mathbb{I}_{\Gamma'} + E \cdot d(\sigma), \text{ where } d(\sigma) \in \mathscr{T}.$$

Also, we compute that

$$\begin{aligned}
\mathbb{I}_{\Gamma'} + E \cdot d(\sigma\tau) &= \rho^1_{\Gamma'}(\sigma\tau)\rho^2_{\Gamma'}(\sigma\tau)^{-1} \\
&= \rho^1_{\Gamma'}(\sigma)\rho^1_{\Gamma'}(\tau)\rho^2_{\Gamma'}(\tau)^{-1}\rho^2_{\Gamma'}(\sigma)^{-1} \\
&= \rho^1_{\Gamma'}(\tau)\left(\mathbb{I}_{\Gamma'} + Ed(\sigma)\right)\rho^2_{\Gamma'}\tau)\right)^{-1} \\
&= \rho^1_{\Gamma'}(\tau)\rho^2_{\Gamma'}(\tau)^{-1} + E \cdot \rho^1_{\Gamma'}(\tau)d(\sigma)\rho^2_{\Gamma'}(\tau)^{-1} \\
&= \mathbb{I}_{\Gamma'} + E \cdot d(\tau) + E \cdot \rho_k(\tau)d(\sigma)\rho_k(\tau)^{-1},
\end{aligned}$$

since E annihilates $\mathfrak{m}_{\Gamma'}$, so the values of both $\rho^1_{\Gamma'}(\tau))$ and $\rho^2_{\Gamma'}(\tau)$ when multiplied by E are reduced modulo the maximal ideal $\mathfrak{m}_{\Gamma'}$. Therefore, we conclude that

$$d(\sigma\tau) = d(\tau) + \rho_k(\tau)d(\sigma)\rho_k(\tau)^{-1} = d(\tau) + \mathrm{Ad}(\tau)d(\sigma).$$

$\square$

Similarly if $\rho^1_{\Gamma'}, \rho^2_{\Gamma'}$ are equivalent extensions of ρ_Γ, that is

$$\rho^1_{\Gamma'}(\sigma) = \left(\mathbb{I}_{\Gamma'} + EQ\right)\rho^2_{\Gamma'}(\sigma)\left(\mathbb{I}_{\Gamma'} + EQ\right)^{-1},$$

then

$$d(\sigma) = Q - \mathrm{Ad}(\sigma)Q,$$

that is $d(\sigma)$ is a coboundary. This proves that the set of liftings $\rho_{\Gamma'}$ of a representation $\rho_{\Gamma'}$ is a principal homogeneous space, provided it is non-empty.

The obstruction to the lifting can be computed by considering a naive lift $\rho_{\Gamma'}$ of ρ_Γ (that is we don't assume that $\rho_{\Gamma'}$ is a representation) and by considering the element

$$\phi(\sigma, \tau) = \rho_{\Gamma'}(\sigma) \circ \rho_{\Gamma'}(\tau) \circ \rho_{\Gamma'}(\sigma\tau)^{-1}, \quad \text{for } \sigma, \tau \in G$$

which defines a cohomology class as an element in $H^2(G, \mathscr{T})$. Two naive liftings $\rho^1_{\Gamma'}, \rho^2_{\Gamma'}$ give rise to cohomologous elements ϕ^1, ϕ^2 if their difference $\rho^1_{\Gamma'} - \rho^2_{\Gamma'}$ reduce to zero in Γ'. If this class is zero, then the representation ρ_Γ can be lifted to Γ'.
Examples Notice that in the theory of deformations of representations of the general linear group, this is a classical result, see [30, prop. 1], [29, p.30] while for deformations of representations in $\mathrm{Aut}\Gamma[[t]]$, this is in [7],[4].

The functors in these cases are given by

$$(8) \qquad F : \mathrm{Ob}(\mathscr{C}) \ni \Gamma \mapsto \left\{ \begin{array}{l} \text{liftings of } \rho : G \to \mathrm{GL}_n(k) \\ \text{to } \rho_\Gamma : G \to \mathrm{GL}_n(\Gamma) \text{ modulo} \\ \text{conjugation by an element} \\ \text{of } \ker(\mathrm{GL}_n(\Gamma) \to \mathrm{GL}_n(k)) \end{array} \right\}$$

$$(9) \qquad D_P : \mathrm{Ob}(\mathscr{C}) \ni \Gamma \mapsto \left\{ \begin{array}{l} \text{lifts } G \to \mathrm{Aut}(\Gamma[[t]]) \text{ of } \rho \text{ mod-} \\ \text{ulo conjugation with an element} \\ \text{of } \ker(\mathrm{Aut}\Gamma[[t]] \to \mathrm{Aut}k[[t]]) \end{array} \right\}$$

Let V be the n-dimensional vector space k, and let $\mathrm{End}_A(V)$ be the Lie algebra corresponding to the algebraic group $GL(V)$. The space $\mathrm{End}_A(V)$ is equipped with

the adjoint action of G given by:

$$\mathrm{End}_A(V) \to \mathrm{End}_A(V)$$

$$e \mapsto (g \cdot e)(v) = \rho(g)(e(\rho(g)^{-1})(v))$$

The tangent space of this deformation functor equals to

$$F(k[\epsilon]) = H^1(G, \mathrm{End}_A(V)),$$

where the later cohomology group is the group cohomology group and $\mathrm{End}_A(V)$ is considered as a G-module with the adjoint action.

More precisely, if

$$0 \to \langle E \rangle \to \Gamma' \xrightarrow{\phi} \Gamma \to 0$$

is a small extension of local Artin algebras then we consider the diagram of small extensions

$$\begin{array}{ccc}
 & & \mathrm{GL}_n(\Gamma') \\
 & \overset{\rho^1_{\Gamma'},\rho^2_{\Gamma'}}{\nearrow} & \downarrow \phi \\
G & \xrightarrow[\rho_\Gamma]{} & \mathrm{GL}_n(\Gamma)
\end{array}$$

where $\rho^1_{\Gamma'}, \rho^2_{\Gamma'}$ are two liftings of ρ_Γ in Γ'.

We have the element

$$d(\sigma) := \frac{1}{E} \left(\rho^1_{\Gamma'}(\sigma)\rho^2_{\Gamma'}(\sigma)^{-1} - \mathbb{I}_n \right) \in H^1(G, \mathrm{End}_n(k)).$$

To a naive lift $\rho_{\Gamma'}$ of ρ_Γ we can attach the 2-cocycle $\alpha(\sigma, \tau) = \rho_{\Gamma'}(\sigma)\rho_{\Gamma'}(\tau)\rho_{\Gamma'}(\sigma\tau)^{-1}$ defining a cohomology class in $H^2(G, \mathrm{End}_n(k))$.

The following proposition shows us that a lifting is not always possible.

Proposition 15. *Let k be an algebraically closed field of positive characteristic $p > 0$, end let $R = W(k)[\zeta_q]$ be the Witt ring of k with a primitive $q = p^h$ root adjoined. Consider the group $G = C_q \rtimes C_m$, where C_m and C_q are cyclic groups of orders m and q respectively and $(m, p) = 1$. Assume that σ and τ are generators for C_m and C_q respectively and moreover*

$$\sigma\tau\sigma^{-1} = \tau^a$$

for some integer a (which should satisfy $a^m \equiv 1 \bmod q$.) There is a linear representation $\rho : G \to \mathrm{GL}_2(k)$, which can not be lifted to a representation $\rho_R : G \to \mathrm{GL}_2(R)$.

Proof. Consider the field $\mathbb{F}_p \subset k$ and let λ be a generator of the cyclic group $\mathbb{F}_p^*$. The matrices

$$\sigma = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

satisfy

$$\sigma^{p-1} = 1, \tau^q = 1, \sigma\tau\sigma^{-1} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \sigma^a$$

and generate a subgroup of $\mathrm{GL}_2(k)$, isomorphic to $C_q \rtimes C_m$ for $m = p - 1$, giving a natural representation $\rho : G \to \mathrm{GL}_2(\bar{\mathbb{F}}_p) \subset \mathrm{GL}_2(k)$.

Suppose that there is a faithful representation $\tilde{\rho} : G \to \mathrm{GL}_n(R)$m, which gives a faithful representation of $\tilde{\rho} : G \to \mathrm{GL}_n(\mathrm{Quot}(R))$. Since $\tilde{\rho}(\tau)$ is of finite order after a $\mathrm{Quot}(R)$ linear change of basis we might assume that $\tilde{\rho}(\tau)$ is diagonal with q-roots

of unity in the diagonal (we have considered $R = W(k)[\zeta]$ so that the necessary diagonal elements exist in $\mathrm{Quot}(R)$). We have

$$\tilde{\rho}(\tau) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

At least one of the diagonal elements say $\lambda = \lambda_{i_0}$ in the above expression is a primitive q-th root of unity. Let E be an eigenvector, that is

$$\tilde{\rho}(\tau)E = \lambda E.$$

The equality $\tau\sigma = \sigma\tau^a$ implies that σE is an eigenvector of the eigenvalue λ^a. This means that n should be greater than the order of $a \bmod q$ since we have at least as many different (and linearly independent) eigenvectors as the different values $\lambda, \lambda^a, \lambda^{a^2}, \ldots$.

Since, for large prime $(p > 3)$ we have $2 = n < p - 1$ the representation ρ can not be lifted to R. $\qquad\qquad\square$

Remark 16. In [24] we give a necessary and sufficient condition for a modular representation of a group $C_{p^h} \rtimes C_m$ in a field of characteristic $p > 0$ to be lifted to a representation over a local principal ideal domain of characteristic zero containing the p^h roots of unity.

Local Actions By the local-global theorems of J.Bertin and A. Mézard [4] and the formal patching theorems of D. Harbater, K. Stevenson [14], [15], the study of the functor D_{gl} can be reduced to the study of the deformation functors D_P attached to each wild ramification point P of the cover $X \to X/G$, as defined in eq. (9). The theory of automorphisms of formal powerseries rings is not as well understood as is the theory of automorphisms of finite dimensional vector spaces, i.e. the theory of general linear groups.

As in the theory of liftings for the general linear group, we consider small extensions

$$1 \to \langle E \rangle \to \Gamma' \xrightarrow{\phi} \Gamma \to 1$$

An automorphism $\rho^\Gamma(\sigma) \in \mathrm{Aut}\Gamma[[t]]$ is completely described by a powerseries

$$\rho^\Gamma(\sigma)(t) = f_\sigma = \sum_{\nu=1}^{\infty} a_\nu^\Gamma(\sigma)t^\nu,$$

where $a_\nu^\Gamma(\sigma) \in \Gamma$. Given a naive lift

$$\rho^{\Gamma'}(\sigma)(t) = \sum_{\nu=1}^{\infty} a_\nu^{\Gamma'}(\sigma)t^\nu,$$

where $a_\nu^{\Gamma'}(\sigma) \in \Gamma'$ we can again form a two cocycle

$$\alpha(\sigma, \tau) = \rho^{\Gamma'}(\sigma) \circ \rho^{\Gamma'}(\tau) \circ \rho^{\Gamma'}(\sigma\tau)^{-1}(t),$$

defining a cohomology class in $H^2(G, \mathscr{T}_{k[[t]]})$. The naive lift $\rho^{\Gamma'}(\sigma)$ is an element of $\mathrm{Aut}\Gamma'[[t]]$ if and only if α is cohomologous to zero.

Suppose now that $\rho_1^{\Gamma'}, \rho_2^{\Gamma'}$ are two lifts in $\mathrm{Aut}\Gamma'[[t]]$. We can now define

$$d(\sigma) := \frac{1}{t}\left(\rho_1^{\Gamma'}(\sigma)\rho_2^{\Gamma'}(\sigma)^{-1} - \mathrm{Id}\right) \in H^1(G, \mathscr{T}_{k[[t]]}).$$

3. Relative Petri's theorem.

Recall that a functor $F : \mathscr{C} \to \text{Sets}$ can be extended to a functor $\hat{F} : \hat{\mathscr{C}} \to \text{Sets}$ by letting $\hat{F}(R) = \varprojlim F(R/\mathfrak{m}_R^{n+1})$ for every $R \in \text{Ob}(\hat{\mathscr{C}})$. An element $\hat{u} \in \hat{F}(R)$ is called a formal element, and by definition it can be represented as a system of elements $\{u_n \in F(R/\mathfrak{m}_R^{n+1})\}_{n \geq 0}$, such that for each $n \geq 1$, the map $F(R/\mathfrak{m}_R^{n+1}) \to F(R/\mathfrak{m}_R^n)$ induced by $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ sends $u_n \mapsto u_{n-1}$. For $R \in \text{Ob}(\hat{\mathscr{C}})$ and a formal element $\hat{u} \in \hat{F}(R)$, the couple $(R, \hat{u})$ is called a formal couple. It is known that there is a 1-1 correspondence between $\hat{F}(R)$ and the set of morphisms of functors $h_R := \text{Hom}_{\hat{\mathscr{C}}}(R, -) \to F$, see [35, lemma 2.2.2.]. The formal element $\hat{u} \in \hat{F}(R)$ will be called versal if the corresponding morphism $h_R \to F$ is smooth. For the definition of a smooth map between functors, see [35, def. 2.2.4]. The ring R will be called *versal deformation ring*.

Schlessinger [33, 3.7] proved that the deformation functor D for curves without automorphisms, admits a ring R as versal deformation ring. Schlessinger calls the versal deformation ring the hull of the deformation functor. Indeed, since there are no obstructions to liftings in small extensions for curves, see [33, rem. 2.10] the hull R of D_{gl} is a powerseries ring over Λ, which can be taken as an algebraic extension of $W(k)$. Moreover $R = \Lambda[[x_1, \ldots, x_{3g-3}]]$, as we can see by applying [3, cor. 3.3.5], when G is the trivial subgroup of the automorphism group. In this case the quotient map $f : X \to \Sigma = X/\{\text{Id}\} = X$ is the identity. Indeed, for the equivariant deformation functor, in the case of the trivial group, there are no ramified points and the short exact sequence in eq. (1) reduces to an isomorphism of the first two spaces. We have $\dim_k H^1(X/G, \pi_*^G(\mathscr{T}_X)) = \dim_k H^1(X, \mathscr{T}_X) = 3g-3$. The deformation $\mathscr{X} \to \text{Specf} R$ can be extended to a deformation $\mathscr{X} \to \text{Spec} R$ by Grothendieck's effectivity theorem, see [35, th. 2.5.13], [13].

The versal element $\hat{u}$ corresponds to a deformation $\mathscr{X} \to \text{Spec} R$, with generic fibre $\mathscr{X}_\eta$ and special fibre $\mathscr{X}_0$. The couple $(R, \hat{u})$ is called the versal [35, def. 2.2.6] element of the deformation functor D of curves (without automorphisms). Moreover, the element u defines a map $h_{R/\Lambda} \to D$, which by definition of the hull is smooth, so every deformation $X_A \to \text{Spec} A$ defines a homomorphism $R \to A$, which allows us to see A as an R-algebra.

Indeed, for the Artin algebra $A \to A/\mathfrak{m}_A = k$ we consider the diagram

$$h_{R/\Lambda} = \text{Hom}_{\hat{\mathscr{C}}}(R, A) \to h_{R/\Lambda}(k) \times_{D(k)} D(A)$$

This section aims to prove the following

Proposition 17. *Let $f_1, \ldots, f_r \in k[\omega_1, \ldots, \omega_g]$ be quadratic polynomials which generate the canonical ideal of a curve X defined over an algebraic closed field k. Any deformation $\mathscr{X}_A$ is given by quadratic polynomials $\tilde{f}_1, \ldots, \tilde{f}_r \in A[W_1, \ldots, W_g]$, which reduce to $f_1, \ldots, f_r$ modulo the maximal ideal $\mathfrak{m}_A$ of A.*

For $n \geq 1$, we write $\Omega_{\mathscr{X}/R}^{\otimes n}$ for the sheaf of holomorphic polydifferentials on $\mathscr{X}$. By [17, lemma II.8.9] the $R-$modules $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n})$ are free of rank $d_{n,g}$ for all $n \geq 1$, with $d_{n,g}$ given by eq. (10)

$$(10) \qquad d_{n,g} = \begin{cases} g, & \text{if } n = 1 \\ (2n-1)(g-1), & \text{if } n > 1. \end{cases}$$

Indeed, by a standard argument using Nakayama's lemma, see [17, lemma II.8.9],[21] we have that the R-module $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n})$ is free. Notice that in order to use Nakayama's lemma we need the deformation over R to have both a special and generic fibre and this was the reason we needed to consider a deformation over the spectrum of R instead of the formal spectrum.

Lemma 18. *For every Artin algebra A the A-module $H^0(X_A, \Omega_{X_A/A}^{\otimes n})$ is free.*

Proof. This follows since $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})$ is a free R-module and [17, prop. II.8.10], which asserts that $\Omega_{X_A/A} \cong g'^*(\Omega_{\mathscr{X}/R})$, where g' is shown in the next commutative diagram:

$$
\begin{array}{ccc}
X_A = \mathscr{X} \times_{\mathrm{SpecR}} \mathrm{SpecA} & \xrightarrow{g'} & \mathscr{X} \\
\downarrow & & \downarrow \\
\mathrm{SpecA} & \longrightarrow & \mathrm{SpecR}
\end{array}
$$

We have by definition of the pullback

$$
(11) \qquad g'^*(\Omega_{\mathscr{X}/R})(X_A) = (g')^{-1}\Omega_{\mathscr{X}/R}(X_A) \otimes_{(g')^{-1}\mathscr{O}_{\mathscr{X}}(X_A)} \mathscr{O}_{X_A}(X_A)
$$

and by definition of the fibre product we obtain $\mathscr{O}_{X_A} = \mathscr{O}_{\mathscr{X}} \otimes_R A$. Observe also that since A is a local Artin algebra the schemes X_A and $\mathscr{X}$ share the same underlying topological space so

$$
g'^{-1}(\Omega_{\mathscr{X}/R}(X_A)) = \Omega_{\mathscr{X}/R}(\mathscr{X})
$$

and $g'^{-1}\mathscr{O}_{\mathscr{X}}(X_A) = \mathscr{O}_{\mathscr{X}}(\mathscr{X})$. So eq. (11) becomes

$$
\begin{aligned}
H^0(X_A, \Omega_{X_A/A}) = \Omega_{X_A/A}(X_A) = g'^*(\Omega_{\mathscr{X}/R})(X_A)) = \\
= \Omega_{\mathscr{X}/R}(\mathscr{X}) \otimes_{\mathscr{O}_{\mathscr{X}}(\mathscr{X})} \otimes \mathscr{O}_{\mathscr{X}}(\mathscr{X}) \otimes_{R_{gl}} A \\
= H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}) \otimes_R A.
\end{aligned}
$$

So $H^0(X_A, \Omega_{X_A/A})$ is a free A-module of the same rank as $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})$.

The proof for $H^0(X_A, \Omega_{X_A/A}^{\otimes n})$ follows in the same way. $\qquad\square$

We select generators $W_1, \ldots, W_g$ for the symmetric algebra

$$
\mathrm{Sym}(H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})) = R[W_1, \ldots, W_g].
$$

Similarly, we write

$$
\mathrm{Sym}(H^0(\mathscr{X}_\eta, \Omega_{\mathscr{X}_\eta/L})) = L[\omega_1, \ldots, \omega_g] \text{ and } \mathrm{Sym}(H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/k})) = k[w_1, \ldots, w_g],
$$

where

$$
\omega_i = W_i \otimes_R L \qquad w_i = W_i \otimes_R k \text{ for all } 1 \le i \le g.
$$

We have the following diagram relating special and generic fibres

(12)
$$
\begin{array}{ccccc}
\mathrm{Spec}(k) \times_{\mathrm{Spec}(R)} \mathscr{X} = \mathscr{X}_0 & \longhookrightarrow & \mathscr{X} & \longleftarrow\!\! & \mathscr{X}_\eta = \mathrm{Spec}(L) \times_{\mathrm{Spec}(R)} \mathscr{X} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(k) & \longhookrightarrow & \mathrm{Spec}(R) & \longleftarrow\!\! & \mathrm{Spec}(L)
\end{array}
$$

Our article is based on the following relative version of Petri's theorem.

Theorem 19. *Diagram (12) induces a deformation-theoretic diagram of canonical embeddings*

$$(13) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & I_{\mathscr{X}_\eta} & \hookrightarrow & S_L := L[\omega_1, \ldots, \omega_g] & \xrightarrow{\phi_\eta} & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}_\eta, \Omega_{\mathscr{X}_\eta/L}^{\otimes n}) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \otimes_R L} & & \uparrow{\scriptstyle \otimes_R L} & & \uparrow{\scriptstyle \otimes_R L} & & \\ 0 & \longrightarrow & I_{\mathscr{X}} & \hookrightarrow & S_R := R[W_1, \ldots, W_g] & \xrightarrow{\phi} & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \otimes_R R/\mathfrak{m}} & & \downarrow{\scriptstyle \otimes_R R/\mathfrak{m}} & & \downarrow{\scriptstyle \otimes_R R/\mathfrak{m}} & & \\ 0 & \longrightarrow & I_{\mathscr{X}_0} & \hookrightarrow & S_k := k[w_1, \ldots, w_g] & \xrightarrow{\phi_0} & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/k}^{\otimes n}) & \longrightarrow & 0 \end{array}$$

where $I_{\mathscr{X}_\eta} = \ker \phi_\eta$, $I_{\mathscr{X}} = \ker \phi$, $I_{\mathscr{X}_0} = \ker \phi_0$, each row is exact and each square is commutative. Moreover, the ideal $I_{\mathscr{X}}$ can be generated by elements of degree 2 as an ideal of S_R.

The commutativity of the above diagram was proved in [6] by H. Charalambous, K. Karagiannis and the first author. For proving that $I_{\mathscr{X}}$ is generated by elements of degree 2 as in the special and generic fibers we argue as follows: Since L is a field it follows by Petri's Theorem, that there are elements $\tilde{f}_1, \ldots, \tilde{f}_r \in S_L$ of degree 2 such that

$$I_{\mathscr{X}_\eta} = \langle \tilde{f}_1, \ldots, \tilde{f}_r \rangle.$$

Now we choose an element $c \in R$ such that $f_i := c\tilde{f}_i \in S_R$ for all i and notice that $\deg(f_i) = \deg(\tilde{f}_i) = 2$.

• Assume first that the element $c \in R$ is invertible in R. Consider the ideal $I = \langle f_1, \ldots, f_r \rangle$ of S_R. We will prove that $I = I_{\mathscr{X}}$. Consider the multiplicative system R^*. We will prove first $I \subset I_{\mathscr{X}} = \ker\phi$. Indeed, using the commuting upper square every element $a = \sum_{\nu=1}^{r} a_i f_i \in I$ maps to $\sum_{\nu=1}^{r} a_i f_i \otimes_R 1$ which in turn maps to 0 by ϕ_η. The same element maps to $\phi(a)$ and $\phi(a) \otimes_R 1$ should be zero. Since all modules $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n})$ are free $\phi(a) = 0$ and $a \in I_{\mathscr{X}}$.

Since the family $\mathscr{X} \to \mathrm{Spec} R$ is flat we have that $I_{\mathscr{X}} \otimes_R L = I_{\mathscr{X}_\eta}$, that is we apply the $\otimes_R L$ functor on the middle short exact sequence of eq. (13). The ideal $I = I_{\mathscr{X}_\eta} \cap S_R = (I_{\mathscr{X}} \otimes_R L) \cap S_R$. By [2, prop. 3.11ii] this gives that

$$I = \cup_{s \in R^*}(I_{\mathscr{X}} : s) \supset I_{\mathscr{X}},$$

so $I_{\mathscr{X}} = I$. In the above formula $(I_{\mathscr{X}} : s) = \{x \in S_R : xs \in I_{\mathscr{X}}\}$.

• From now on we don't assume that the element c is an invertible element of R.

Let $\bar{g}$ be an element of degree 2 in $I_{\mathscr{X}_0}$, we will prove that we can select an element $g \in I_{\mathscr{X}}$ such that $g \otimes 1_k = \bar{g}$, so that g has degree 2.

Let us choose a lift $\tilde{g} \in S_R$ of degree 2 by lifting each coefficient of g from k to R. This element is not necessarily in $I_{\mathscr{X}}$. We have $\phi(g) \otimes 1_k = \phi_0(g \otimes 1_k) = \phi_0(\bar{g}) = 0$. Let $\bar{e}_1, \ldots, \bar{e}_{3g-3}$ be generators of the free R-module $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes 2})$ and choose $e_1, \ldots, e_{3g-3} \in S_R$ such that $\phi(e_i) = \bar{e}_i$. Let us write $\phi(\tilde{g}) = \sum_{i=1}^{3g-3} \lambda_i \bar{e}_i$, with $\lambda_i \in R$. Since $\phi_0(\bar{g}) = 0$ we have that all $\lambda_i \in \mathfrak{m}_R$ for all $1 \le i \le 3g-3$. This

means that the element $g = \tilde{g} - \sum_{i=1}^{3g-3} \lambda_i e_i \in S_R$ reduces to $\bar{g}$ modulo $\mathfrak{m}_R$ and also $\phi(g) = \phi(\tilde{g}) - \sum_{i=1}^{3g-3} \lambda_i \bar{e}_i = 0$, so $g \in I_{\mathscr{X}}$.

Let $\bar{g}_1, \ldots, \bar{g}_s \in I_{\mathscr{X}_0}$ be elements of degree 2 such that

$$I_{\mathscr{X}_0} = \langle \bar{g}_1, \ldots, \bar{g}_s \rangle$$

and, using the previous construction, we take g_i lifts in $I_{\mathscr{X}} \lhd S_R$, i.e. such that $g_i \otimes 1_k = \bar{g}_i$ and also assume that the elements g_i have also degree 2.

We will now prove that the elements $g_1 \otimes_{S_R} 1_L, \ldots, g_s \otimes_{S_R} 1_L \in S_L$ generate the ideal $I_{\mathscr{X}_\eta}$. By the commutativity of the diagram in eq. (13) we have $\langle g_1 \otimes_{S_R} 1_L, \ldots, g_s \otimes_{S_R} 1_L \rangle \subset I_{\mathscr{X}_\eta} = \ker \phi_\eta$. Observe that any linear relation

$$\sum_{\nu=1}^{s} (a_\nu g_\nu \otimes_{S_R} 1_L) = 0, \text{ with } a_\nu \in L$$

gives rise to a relation for some $c \in R$

$$\sum_{\nu=1}^{s} c \cdot a_\nu g_\nu = 0, \qquad c \cdot a_\nu \in S_R,$$

which implies that $c \cdot a_\nu \in \mathfrak{m}_R$.

We will prove that the elements $g_i \otimes_{S_R} 1_L$ are linear independent.

Lemma 20. *Let $\bar{v}_1, \ldots, \bar{v}_n \in k^m$ be linear independent elements and $v_1, \ldots, v_n$ be lifts in R^m. Then*

$$\sum_{\nu=1}^{n} a_\nu v_\nu = 0 \qquad a_\nu \in R,$$

implies that $a_1 = \cdots = a_n = 0$.

Proof. We have $n \leq m$. We write the elements $v_1, \ldots, v_n$ (resp. $\bar{v}_1, \ldots, \bar{v}_n$) as columns and in this way we obtain an $m \times n$ matrix J (resp. $\bar{J}$). Since the elements are linear independent in k^m there is an $n \times n$ minor matrix with an invertible determinant. Without loss of generality, we assume that there is an $n \times n$ invertible matrix $\bar{Q}$ with coefficients in k such that $\bar{Q} \cdot \bar{J}^t = \left(\mathbb{I}_n \mid \bar{A} \right)$, where $\bar{A}$ is an $(m-n) \times n$ matrix. We now get lifts Q, J and A of $\bar{Q}, \bar{J}$ and $\bar{A}$ respectively, with coefficients in R, i.e.

$$Q \cdot J^t \equiv \left(\mathbb{I}_n \mid A \right) \mathrm{mod} \mathfrak{m}_R.$$

The columns of J are lifts of the elements $\bar{v}_1, \ldots, \bar{v}_n$. It follows that $Q \cdot J^t = \left(\mathbb{I}_n \mid A \right) + \left(C \mid D \right)$, where C, D are matrices with entries in $\mathfrak{m}_R$. The determinant of $\mathbb{I}_n + C$ is $1 + m$, for some element $m \in \mathfrak{m}_R$, and this is an invertible element in the local ring R. Similarly, the matrix Q is invertible. Therefore,

$$J^t = \left(Q^{-1}(\mathbb{I}_n + C) \mid Q^{-1}(A + D) \right)$$

has the first $n \times n$ block matrix invertible and the desired result follows.

$\square$

Remark 21. It is clear that over a ring where 2 is invertible, there is an 1-1 correspondence between symmetric $g \times g$ matrices and quadratic polynomials. Indeed, a quadratic polynomial can be written as

$$f(w_1, \ldots, w_g) = \sum_{1 \leq i,j \leq g} a_{ij} w_i w_j = w^t A w,$$

where $A = (a_{ij})$. Even if the matrix A is not symmetric, the matrix $(A + A^t)/2$ is and generates the same quadratic polynomial

$$w^t A w = w^t \left(\frac{A + A^t}{2} \right) w.$$

Notice that the map

$$A \mapsto \frac{A + A^t}{2}$$

is onto the space of symmetric matrices and has as kernel the space of antisymmetric matrices.

A minimal set of quadratic generators is given by a set of polynomials $f_1, \ldots, f_r$, with $f_i = w^t A_i w$, where the symmetric polynomials are linearly independent.

By the general theory of Betti tables we know that in the cases the canonical ideal is generated by quadratic polynomials, the dimension of this set of matrices equals $\binom{g-2}{2}$, see [10, prop. 9.5]. Therefore we begin on the special fibre with the $s = \binom{g-2}{2}$ generators $\bar{g}_1, \ldots, \bar{g}_s$ elements. As we have proved in theorem 19 we can lift them to elements $g_1, \ldots, g_s \in I_{\mathscr{X}}$ so that for $J := \langle g_1, \ldots, g_s \rangle$ we have

(i) $J \otimes_R L = I_{\mathscr{X}_\eta}$.
(ii) $J \otimes_R k = I_{\mathscr{X}_0}$.

In this way we obtain the linear independent elements $g_1 \otimes_{S_R} 1_L, \ldots, g_s \otimes_{S_R} 1_L$ in I_{X_η}. We have seen that the $s = \binom{g-2}{2}$ linear independent quadratic elements generate also $I_{\mathscr{X}_\eta}$.

By following Lemma 5 (ii) of [6] we have the next lemma.

Lemma 22. *Let G be a set of polynomials in S_R such that $\langle G \rangle \otimes_R L = I_{\mathscr{X}_\eta}$ and $\langle G \rangle \otimes_R k = I_{\mathscr{X}_0}$. Then $I_{\mathscr{X}} = \langle G \rangle$.*

Essential for the proof of lemma 22 was that the ring R has a generic fibre. The deformation theory is concerned with deformations over local Artin algebras which do not have generic fibres. But by tensoring with A in the middle sequence of eq. (13) we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I_{X_A} & \hookrightarrow & S_A := A[W_1, \ldots, W_g] & \overset{\phi}{\twoheadrightarrow} & \displaystyle\bigoplus_{n=0}^{\infty} H^0(X_A, \Omega_{X_A/A}^{\otimes n}) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \otimes_A A/\mathfrak{m}_A} & & \downarrow{\scriptstyle \otimes_A A/\mathfrak{m}_A} & & \downarrow{\scriptstyle \otimes_A A/\mathfrak{m}_A} & & \\
0 & \longrightarrow & I_{\mathscr{X}_0} & \hookrightarrow & S_k := k[w_1, \ldots, w_g] & \overset{\phi_0}{\twoheadrightarrow} & \displaystyle\bigoplus_{n=0}^{\infty} H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/k}^{\otimes n}) & \longrightarrow & 0
\end{array}
$$

Indeed, since $H^0(\mathscr{X}, \Omega_{\mathscr{X}/A}^{\otimes n})$ is free the left top arrow in the above diagram is injective. Moreover the relative canonical ideal I_{X_A} is still generated by quadratic polynomials in S_A.

3.1. **Embedded deformations.** Let Z be a scheme over k and let X be a closed subscheme of Z. An embedded deformation $X' \to \operatorname{Spec} k[\epsilon]$ of X over $\operatorname{Spec} k[\epsilon]$ is a

closed subscheme $X' \subset Z' = Z \times \operatorname{Spec} k[\epsilon]$ fitting in the diagram:

Let $\mathscr{I}$ be the ideal sheaf describing X as a closed subscheme of Z and

$$(14) \qquad \mathscr{N}_{X/Z} = \mathscr{H}om_Z(\mathscr{I}, \mathscr{O}_X) = \mathscr{H}om_X(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_X),$$

be the normal sheaf. In particular, for an affine open set U of X we set $B' = \mathscr{O}_{Z'}(U) = B \oplus \epsilon B$, where $B = \mathscr{O}_Z(U)$ and we observe that describing the sheaf of ideals $\mathscr{I}'(U) \subset \mathscr{B}'$ is equivalent to giving an element

$$\phi_U \in \operatorname{Hom}_{\mathscr{O}_Z(U)}\big(\mathscr{I}(U), \mathscr{O}_Z(U)/\mathscr{I}(U)\big),$$

see [18, prop. 2.3].

In this article, we will take $Z = \mathbb{P}^{g-1}$ and consider the canonical embedding $f : X \to \mathbb{P}^{g-1}$. We will denote by N_f the sheaf $\mathscr{N}_{X/\mathbb{P}^{g-1}}$. Let $\mathscr{I}_X$ be the sheaf of ideals of the curve X seen as a subscheme of $\mathbb{P}^{g-1}$. Since the curve X satisfies the conditions of Petri's theorem, it is fully described by certain quadratic polynomials $f_1 = \tilde{A}_1, \ldots, f_r = \tilde{A}_r$ which correspond to a set $g \times g$ matrices $A_1, \ldots, A_r$, see [25]. The elements $f_1, \ldots, f_r$ generate the ideal I_X corresponding to the projective cone $C(X)$ of X, $C(X) \subset \mathbb{A}^g$.

We have

$$H^0(X, N_f) = \operatorname{Hom}_S(I_X, \mathscr{O}_X).$$

Assume that X is deformed to a curve $X_\Gamma \to \operatorname{Spec}\Gamma$, where Γ is a local Artin algebra, $X_\Gamma \subset \mathbb{P}_\Gamma^{g-1} = \mathbb{P}^{g-1} \times \operatorname{Spec}\Gamma$. Our initial curve X is described in terms of the homogeneous canonical ideal I_X, generated by the elements $\{w^t A_1 w, \ldots, w^t A_r w\}$. For a local Artin algebra Γ let $\mathscr{S}_g(\Gamma)$ denote the space of symmetric $g \times g$ matrices with coefficients in Γ. The deformations X_Γ are expressed in terms of the ideals I_{X_Γ}, which by the relative Petri's theorem are also generated by elements $w^t A_1^\Gamma w, \ldots, w^t A_r^\Gamma w$, where A_i^Γ is in $\mathscr{S}_g(\Gamma)$. This essentially fits with Schlessinger's observation in [34], where the deformations of the projective variety are related to the deformations of the affine cone, notice that in our case all relative projective curves are smooth and the assumptions of [34, th. 2] are satisfied. We can thus replace the sheaf theoretic description of eq. (14) and work with the affine cone instead.

Remark 23. A set of quadratic generators $\{w^t A_1 w, \ldots, w^t A_r w\}$ is a minimal set of generators if and only if the elements $A_1, \ldots, A_r$ are linear independent in the free Γ-module $\mathscr{S}_g(\Gamma)$ of rank $(g+1)g/2$.

3.1.1. *Embedded deformations and small extensions.* Let

$$0 \to \langle E \rangle \to \Gamma' \xrightarrow{\pi} \Gamma \to 0$$

be a small extension and a curve $\mathbb{P}_{\Gamma'}^{g-1} \supset X_{\Gamma'} \to \operatorname{Spec}\Gamma'$ be a deformation of X_Γ and X. The curve $X_{\Gamma'}$ is described in terms of quadratic polynomials $w^t A_i^{\Gamma'} w$, where

$A_i^{\Gamma'} \in \mathscr{S}_g(\Gamma')$, which reduce to A_i^{Γ} modulo $\langle E \rangle$. This means that

$$\text{(15)} \qquad\qquad A_i^{\Gamma'} \equiv A_i^{\Gamma} \bmod \ker(\pi) \text{ for all } 1 \leq i \leq r$$

and if we select a naive lift $i(A_i^{\Gamma})$ of A_i^{Γ}, then we can write

$$A_i^{\Gamma'} = i(A_i^{\Gamma}) + E \cdot B_i, \text{ where } B_i \in \mathscr{S}_g(k).$$

The set of liftings of elements $A_i^{\Gamma'}$ of elements A_i^{Γ}, for $1 \leq i \leq r$ is a principal homogeneous space, under the action of $H^0(X, N_f)$, since two such liftings $\{A_i^{(1)}(\Gamma'), 1 \leq i \leq r\}$, $\{A_i^{(2)}(\Gamma'), 1 \leq i \leq r\}$ differ by a set of matrices in $\{B_i(\Gamma') = A_i^{(1)}(\Gamma') - A_i^{(2)}(\Gamma'), 1 \leq i \leq r\}$ with entries in $\langle E \rangle \cong k$, see also [18, thm. 6.2].

Define a map $\phi : \langle A_1, \ldots, A_r \rangle \to \mathscr{S}_g(k)$ by $\phi(A_i) = B_i(\Gamma')$ and we also define the corresponding map on polynomials $\tilde{\phi}(\tilde{A}_i) = w^t \phi(A_i) w$. we obtain a map $\tilde{\phi} \in \mathrm{Hom}_S(I_X, \mathscr{O}_X) = H^0(X, N_f)$, see also [18, th. 6.2], where $S = S_k$. Obstructions to such liftings are known to reside in $H^1(X, \mathscr{N}_{X/\mathbb{P}^{g-1}} \otimes_k \ker \pi)$, which we will prove it is zero, see remark 24.

3.1.2. *Embedded deformations and tangent spaces.* Let us consider the $k[\epsilon]/k$ case. Since $i : X \hookrightarrow \mathbb{P}^{g-1}$ is non-singular we have the following exact sequence

$$0 \to \mathscr{T}_X \to i^* \mathscr{T}_{\mathbb{P}^{g-1}} \to \mathscr{N}_{X/\mathbb{P}^{g-1}} \to 0$$

which gives rise to

$$0 \to H^0(X, \mathscr{T}_X) \to H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \to H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$$
$$\xrightarrow{\delta}$$
$$\to H^1(X, \mathscr{T}_X) \longrightarrow H^1(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \to H^1(X, \mathscr{N}_{X/\mathbb{P}^{g-1}}) \to 0$$

Remark 24. In the above diagram, the last entry in the bottom row is zero since it corresponds to a second cohomology group on a curve. By Riemann-Roch theorem we have that $H^0(X, \mathscr{T}_X) = 0$ for $g \geq 2$. Also, the relative Petri theorem implies that the map δ is onto. We will give an alternative proof that δ is onto by proving that $H^1(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) = 0$. This proves that $H^1(X, \mathscr{N}_{X/\mathbb{P}^{g-1}}) = 0$ as well, so there is no obstruction in lifting the embedded deformations.

Each of the above spaces has a deformation theoretic interpretation, see [16, p.96]:

- The space $H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})$ is the space of deformations of the map $i : X \hookrightarrow \mathbb{P}^{g-1}$, that is both $X, \mathbb{P}^{g-1}$ are trivially deformed, see [35, p. 158, prop. 3.4.2.(ii)]
- The space $H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$ is the space of embedded deformations, where $\mathbb{P}^{g-1}$ is trivially deformed see [18, p. 13, Th. 2.4)].
- The space $H^1(X, \mathscr{T}_X)$ is the space of all deformations of X.

The dimension of the space $H^1(X, \mathscr{T}_X)$ can be computed using Riemann-Roch theorem on the dual space $H^0(X, \Omega_X^{\otimes 2})$ and equals $3g - 3$. In next section we will give a linear algebra interpretation for the spaces $H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$, $H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})$ allowing us to compute its dimensions.

3.2. **Some matrix computations.** We begin with the Euler exact sequence (see. [17, II.8.13], [38, p. 581] and [19] MO)

$$0 \to \mathscr{O}_{\mathbb{P}^{g-1}} \to \mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g} \to \mathscr{T}_{\mathbb{P}^{g-1}} \to 0.$$

We restrict this sequence to the curve X:

$$0 \to \mathscr{O}_X \to i^* \mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g} = \omega_X^{\oplus g} \to i^* \mathscr{T}_{\mathbb{P}^{g-1}} \to 0.$$

We now take the long exact sequence in cohomology

(16)

$$0 \longrightarrow k = H^0(X, \mathscr{O}_X) \xrightarrow{\ f_1\ } H^0(X, i^* \mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g}) \xrightarrow{\ f_2\ } H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})$$

$$\xrightarrow{\quad\quad f_3 \quad\quad}$$

$$\hookrightarrow H^1(X, \mathscr{O}_X) \xrightarrow{\ f_4\ } H^1(X, i^* \mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g}) \xrightarrow{\ f_5\ } H^1(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \longrightarrow H^2(X, \mathscr{O}_X) = 0$$

The spaces involved above have the following dimensions:

- $i^* \mathscr{O}_{\mathbb{P}^{g-1}}(1) = \Omega_X$ (canonical bundle)
- $\dim H^0(X, i^* \mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g}) = g \cdot \dim H^0(X, \Omega_X) = g^2$
- $\dim H^1(X, \mathscr{O}_X) = \dim H^1(X, \Omega_X) = g$
- $\dim H^1(X, i^* \mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g}) = g \cdot \dim H^0(X, \mathscr{O}_X) = g$

We will return to the exact sequence given in eq. (16) and the above dimension computations in the next section.

3.2.1. *Study of $H^0(X, N_f)$.* By relative Petri theorem the elements $\phi(A_i)$ are quadratic polynomials not in I_X, that is elements in a vector space of dimension $(g+1)g/2 - \binom{g-2}{2} = 3g - 3$, where $(g+1)g/2$ is the dimension of the symmetric $g \times g$ matrices and $\binom{g-2}{2}$ is the dimension of the space generated by the generators of the canonical ideal, see [10, prop. 9.5].

The set of matrices $\{A_1, \ldots, A_r\}$ can be assumed to be linear independent but this does not mean that an arbitrary selection of quadratic elements $\omega^t B_i \omega \in \mathscr{O}_X$ will lead to a homomorphism of rings. Indeed, the linear independent elements A_i might satisfy some syzygies, see the following example, where the linear independent elements

$$x^2 = \begin{pmatrix} x & y \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad xy = \begin{pmatrix} x & y \end{pmatrix}^t \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

satisfy the syzygy

$$y \cdot x^2 - x \cdot xy = 0.$$

Therefore, a map of modules ϕ, should be compatible with the syzygy and satisfy the same syzygy. This is known as the fundamental Grothendieck flatness criterion, see [34, 1.1] and also [1, lem. 5.1, p. 28].

Proposition 25. *The map*

$$\psi : M_g(k) \longrightarrow \mathrm{Hom}_S(I_X, S/I_X) = H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$$

$$B \longmapsto \psi_B : \omega^t A_i \omega \mapsto \omega^t (A_i B + B^t A_i) \omega \ \mathrm{mod} I_X$$

identifies the vector space $M_g(k)/\langle \mathbb{I}_g \rangle$ to $H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \subset H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$. The map ψ is equivariant, where $M_g(k)$ is equipped with the adjoint action

$$B \mapsto \rho(g) B \rho(g^{-1}) = \mathrm{Ad}(g) B,$$

that is

$$^g \psi_B = \psi_{\mathrm{Ad}(g)B}.$$

Proof. Recall that the space $H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})$ can be identified to the space of deformations of the map f, where X, $\mathbb{P}^{g-1}$ are both trivially deformed. By [34] a map $\phi \in \operatorname{Hom}_S(I_X, S/I_X) = \operatorname{Hom}_S(I_X, \mathscr{O}_X)$ gives rise to a trivial deformation if there is a map

$$w_j \mapsto w_j + \epsilon \delta_j(w),$$

where $\delta_j(w) = \sum_{\nu=1}^g b_{j,\nu} w_\nu$. The map can be defined in terms of the matrix $B = (b_{j,\nu})$,

$$w \mapsto w + \epsilon B w$$

so that for all $\tilde{A}_i$, $1 \le i \le r$

$$(17) \qquad\qquad \nabla \tilde{A}_i \cdot B w = \phi(\tilde{A}_i) = \phi(w^t A_i w) \bmod I_X.$$

But for $\tilde{A}_i = w^t A_i w$ we compute $\nabla \tilde{A}_i = w^t A_i$, therefore eq. (17) is transformed to

$$(18) \qquad\qquad w^t A_i B w = w^t B_i w \bmod I_X,$$

for a symmetric $g \times g$ matrix B_i in $\mathscr{S}_g(k[\epsilon])$. Therefore, if 2 is invertible according to remark 21 we replace the matrix $A_i B$ appearing in eq. (18) by the symmetric matrix $A_i B + B^t A_i$. Since we are interested in the projective algebraic set defined by homogeneous polynomials the $1/2$ factor of remark 21 can be omitted.

For every $B \in M_g(k)$ we define the map $\psi_B \in \operatorname{Hom}_S(I_X, S/I_X) = \operatorname{Hom}_S(I_X, \mathscr{O}_X)$ given by

$$\tilde{A}_i = \omega^t A_i \omega \mapsto \omega^t (A_i B + B^t A_i) \omega \bmod I_X,$$

and we have just proved that the functions ψ_B are all elements in $H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})$. The kernel of the map $\psi : B \mapsto \psi_B$ consists of all matrices B satisfying:

$$(19) \qquad\qquad A_i B = -B^t A_i \bmod I_X \text{ for all } 1 \le i \le \binom{g-2}{2}.$$

This kernel seems to depend on the selection of the elements A_i. This is not the case. We will prove that the kernel consists of all multiples of the identity matrix. Indeed,

$$\dim H^0(X, i^* \mathscr{T}_X) = g^2 - \ker \psi.$$

We now rewrite the spaces in eq. (16) by their dimensions we get

$$
\begin{array}{ccccccc}
(0) \longrightarrow (1) & \xrightarrow{f_1} & (g^2) & \xrightarrow{f_2} & (g^2 - \ker \psi) \\
& & & f_3 & \\
(g) \longrightarrow (g) & \longrightarrow & (?) & \longrightarrow & (0)
\end{array}
$$

So

- $\dim \ker f_2 = \dim \operatorname{Im} f_1 = 1$
- $\dim \ker f_3 = \dim \operatorname{Im} f_2 = g^2 - 1$
- $\dim \operatorname{Im} f_3 = (g^2 - \dim \ker \psi) - (g^2 - 1) = 1 - \dim \ker \psi$

It is immediate that $\dim \ker \psi = 0$ or 1. But obviously $\mathbb{I}_g \in \ker \psi$, and hence

$$\dim \ker \psi = 1.$$

Finally $\dim \operatorname{Im} f_3 = 0$, i.e. f_3 is the zero map and we get the small exact sequence,

$$0 \longrightarrow k = H^0(X, \mathscr{O}_X) \longrightarrow H^0(X, i^* \mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g}) \longrightarrow H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \longrightarrow 0$$

It follows that
$$\dim H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) = g^2 - 1.$$
We have proved that $\psi : M_g(k)/\langle \mathbb{I}_g \rangle \to H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})$ is an isomorphism of vector spaces. We will now prove it is equivariant.

Using remark 10 we have that the action of the group G on the function
$$\psi_B : A_i \mapsto A_i B + B^t A_i,$$
seen as an element in $H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})$ is given:
$$
\begin{aligned}
A_i \mapsto T(\sigma^{-1}) A_i &\xmapsto{\psi_B} T(\sigma) \left(\rho(\sigma)^t A_i \rho(\sigma) B + B^t \rho(\sigma)^t A_i \rho(\sigma) \right) \\
&= \left(A_i \rho(\sigma) B \rho(\sigma^{-1}) + (\rho(\sigma) B \rho(\sigma^{-1}))^t A_i \right)
\end{aligned}
$$

$\square$

Corollary 26. *The space $H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})^G$ is generated by the elements $B \neq \{\lambda \mathbb{I}_g : \lambda \in k\}$ such that*
$$\rho(\sigma) B \rho(\sigma^{-1}) B^{-1} = [\rho(\sigma), B] \in \langle A_1, \ldots, A_r \rangle \text{ for all } \sigma \in \operatorname{Aut}(X).$$

Remark 27. This construction allows us to compute the space $H^1(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})$. Indeed, we know that f_4 is isomorphism and hence f_5 is the zero map, on the other hand f_5 is surjective, it follows that $H^1(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) = 0$. This provides us with another proof of the exactness of the sequence

$$(20) \quad 0 \longrightarrow H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \longrightarrow H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}}) \xrightarrow{\delta} H^1(X, \mathscr{T}_X) \longrightarrow 0$$

3.3. **Invariant spaces.** Let
$$0 \to A \to B \to C \to 0$$
be a short exact sequence of G-modules. We have the following sequence of G-invariant spaces
$$0 \to A^G \to B^G \to C^G \xrightarrow{\delta_G} H^1(G, A) \to \cdots$$
where the map δ_G is computed as follows: an element c is given as a class $b \bmod A$ and it is invariant if and only if $gb - b = a_g \in A$. The map $G \ni g \mapsto a_g$ is the cocycle defining $\delta_G(c) \in H^1(G, A)$.

Using this construction on the short exact sequence of eq. (20) we arrive at

$$0 \longrightarrow H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})^G \longrightarrow H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})^G \xrightarrow{\delta} H^1(X, \mathscr{T}_X)^G$$
$$\xrightarrow{\delta_G}$$
$$\longrightarrow H^1\big(G, H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})\big) \longrightarrow \cdots$$

We will use eq. (20) in order to represent elements in $H^1(X, \mathscr{T}_X)$ as elements $[f] \in H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})/H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) = H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})/\operatorname{Im}\psi$.

Proposition 28. *Let $[f] \in H^1(X, \mathscr{T}_X)^G$ be a class of a map $f : I_X \to S/I_X$ modulo $\operatorname{Im}\psi$. For each element $\sigma \in G$ there is a matrix $B_\sigma[f]$, depending on f, which defines a class in $M_g(k)/\langle \mathbb{I}_g \rangle$ satisfying the cocycle condition in eq. (22), such that*
$$\delta_G(f)(\sigma) : A_i \mapsto A_i (B_\sigma[f]) + \big(B_\sigma^t[f]\big) A_i \bmod \langle A_1, \ldots, A_g \rangle.$$

Proof. Let $[f] \in H^1(X, \mathscr{T}_X)^G$, where $f : I_X \to S/I_X$ that is $f \in H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$. The $\delta_G(f)$ is represented by an 1-cocycle given by $\delta_G(f)(\sigma) = ^\sigma f - f$. Using the equivariant isomorphism of $\psi : M_g(k)/\langle \mathbb{I}_g \rangle \to H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})$ of proposition 25 we arrive at the diagram:

$$G \longrightarrow H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \xrightarrow{\psi^{-1}} M_g(k)/\langle \mathbb{I}_g \rangle$$

$$\sigma \longmapsto \delta_G(f)(\sigma) \longrightarrow B[f]_\sigma := \psi^{-1}(\delta_G(f)(\sigma))$$

We will now compute

$$^\sigma f : A_i \xrightarrow{T(\sigma^{-1})} T(\sigma^{-1})A_i \xrightarrow{f} f(T(\sigma^{-1})A_i) \xrightarrow{T(\sigma)} T(\sigma)f(T(\sigma^{-1})A_i).$$

We set

$$T(\sigma^{-1})(A_i) = \rho(\sigma)^t A_i \rho(\sigma) = \sum_{\nu=1}^{r} \lambda_{i,\nu}(\sigma)A_i$$

so

(21)
$$\delta_G(f)(\sigma)(A_i) = \sum_{\nu=1}^{r} \lambda_{i,\nu}(\sigma) \cdot \rho(\sigma^{-1})^t f(A_\nu)\rho(\sigma^{-1}) - f(A_i)$$

$$= A_i B_\sigma[f] + B_\sigma[f]^t A_i \bmod I_X$$

for some matrix $B_\sigma[f] \in M_g(k)$ such that for all $\sigma, \tau \in G$ we have

(22)
$$B_{\sigma\tau}[f] = B_\sigma[f] + \sigma B_\tau[f]\sigma^{-1} + \lambda(\sigma, \tau)\mathbb{I}_g$$

$$= B_\sigma[f] + \mathrm{Ad}(\sigma)B_\tau[f] + \lambda(\sigma, \tau)\mathbb{I}_g.$$

In the above equation we have used the fact that $\sigma \mapsto B_\sigma[f]$ is a 1-cocycle in the quotient space $M_g(k)/\mathbb{I}_g$, therefore the cocycle condition holds up to an element of the form $\lambda(\sigma, \tau)\mathbb{I}_g$. □

Remark 29. Let

$$\lambda(\sigma, \tau)\mathbb{I}_g = B_{\sigma\tau}[f] - B_\sigma[f] - \mathrm{Ad}(\sigma)\mathrm{B}_\tau[f].$$

The map $G \times G \to k$, $(\sigma, \tau) \mapsto \lambda(\sigma, \tau)$ is a normalized 2-cocycle (see [40, p. 184]), that is

$0 = \lambda(\sigma, 1) = \lambda(1, \sigma)$ for all $\sigma \in G$

$0 = \mathrm{Ad}(\sigma_1)\lambda(\sigma_2, \sigma_3) - \lambda(\sigma_1\sigma_2, \sigma_3) + \lambda(\sigma_1, \sigma_2\sigma_3) - \lambda(\sigma_1, \sigma_2)$ for all $\sigma_1, \sigma_2, \sigma_3 \in G$

$\quad = \lambda(\sigma_2, \sigma_3) - \lambda(\sigma_1\sigma_2, \sigma_3) + \lambda(\sigma_1, \sigma_2\sigma_3) - \lambda(\sigma_1, \sigma_2)$ for all $\sigma_1, \sigma_2, \sigma_3 \in G$

For the last equality notice that the Ad-action is trivial on scalar multiples of the identity.

Proof. The first equation is clear. For the second one,

$$\lambda(\sigma_1\sigma_2, \sigma_3)\mathbb{I}_g = B_{\sigma_1\sigma_2\sigma_3}[f] - B_{\sigma_1\sigma_2}[f] - \mathrm{Ad}(\sigma_1\sigma_2)\mathrm{B}_{\sigma_3}[f]$$

and

$$\lambda(\sigma_1, \sigma_2)\mathbb{I}_g = B_{\sigma_1\sigma_2}[f] - B_{\sigma_1}[f] - \mathrm{Ad}(\sigma_1)\mathrm{B}_{\sigma_2}[f].$$

Hence

$$
\begin{aligned}
\lambda(\sigma_1\sigma_2,\sigma_3)\mathbb{I}_g + \lambda(\sigma_1,\sigma_2)\mathbb{I}_g =& B_{\sigma_1\sigma_2\sigma_3}[f] - \mathrm{Ad}(\sigma_1\sigma_2)\mathrm{B}_{\sigma_3}[f] - \mathrm{B}_{\sigma_1}[f] - \mathrm{Ad}(\sigma_1)\mathrm{B}_{\sigma_2}[f] \\
=& B_{\sigma_1\sigma_2\sigma_3}[f] - B_{\sigma_1}[f] - \mathrm{Ad}(\sigma_1)\mathrm{B}_{\sigma_2\sigma_3}[f] + \\
& + \mathrm{Ad}(\sigma_1)\mathrm{B}_{\sigma_2,\sigma_3}[f] - \mathrm{Ad}(\sigma_1)\mathrm{B}_{\sigma_2}[f] - \mathrm{Ad}(\sigma_1\sigma_2)\mathrm{B}_{\sigma_3}[f] \\
=& \lambda(\sigma_1,\sigma_2\sigma_3)\mathbb{I}_g + \mathrm{Ad}(\sigma_1)\big(\mathrm{B}_{\sigma_2,\sigma_3}[f] - \mathrm{B}_{\sigma_2}[f] - \mathrm{Ad}(\sigma_1)\mathrm{B}_{\sigma_3}[f]\big) \\
=& \mathrm{Ad}(\sigma_1)\lambda(\sigma_2,\sigma_3)\mathbb{I}_g + \lambda(\sigma_1,\sigma_2\sigma_3)\mathbb{I}_g.
\end{aligned}
$$

$\square$

Corollary 30. *If $f(\omega^t A_i \omega) = \omega^t B_i \omega$, where $B_i \in M_g(k)$ are the images of the elements defining the canonical ideal in the small extension $\Gamma' \to \Gamma$, then the symmetric matrices defining the canonical ideal $I_X(\Gamma')$ are given by $A_i + E \cdot B_i$. Using proposition 28 we have*

$$
\begin{aligned}
(23) \qquad ({}^\sigma f - f)(A_i) &= \sum_{\nu=1}^{r} \lambda_{i,\nu}(\sigma)T(\sigma)(B_\nu) - B_i \\
&= \big(A_i B_\sigma[f] + B_\sigma^t[f]A_i\big) \bmod \langle A_1, \dots, A_r\rangle \\
&= \psi_{B_\sigma[f]}A_i.
\end{aligned}
$$

Therefore, using also eq. (21)

$$
(24) \qquad \sum_{\nu=1}^{r} \lambda_{i,\nu}(\sigma)(B_\nu) - T(\sigma^{-1})B_i = T(\sigma^{-1})\psi_{B_\sigma[f]}(A_i).
$$

4. On the deformation theory of curves with automorphisms

Let $1 \to \langle E \rangle \to \Gamma' \to \Gamma \to 0$ be a small extension of Artin local algebras and consider the diagram

$$
\begin{array}{ccccc}
X_\Gamma & \longrightarrow & X_{\Gamma'} & \longrightarrow & \mathscr{X} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(\Gamma) & \longrightarrow & \mathrm{Spec}(\Gamma') & \longrightarrow & \mathrm{Spec}(R)
\end{array}
$$

Suppose that G acts on X_Γ, that is every automorphism $\sigma \in G$ satisfies $\sigma(I_{X_\Gamma}) = I_{X_\Gamma}$. If the action of the group G is lifted to $X_{\Gamma'}$ then we should have a lift of the representations $\rho, \rho^{(1)}$ defined in eq. (2), (3) to Γ' as well. The set of all such liftings is a principal homogeneous space parametrized by the spaces $H^1(G, M_g(k)), H^1(G, M_r(k))$, provided that the corresponding lifting obstructions in $H^2(G, M_g(k)), H^2(G, M_r(k))$ both vanish.

Assume that there is a lifting of the representation

$$
(25) \qquad
\begin{array}{c}
\mathrm{GL}_g(\Gamma') \\
{}^{\rho_{\Gamma'}}\nearrow \quad \Big\downarrow {\scriptstyle \bmod\langle E\rangle} \\
G \xrightarrow[\rho_\Gamma]{} \mathrm{GL}_g(\Gamma)
\end{array}
$$

This lift gives rise to a lifting of the corresponding automorphism group to the curve $X_{\Gamma'}$ if

$$
\rho_{\Gamma'}(\sigma) I_{X_{\Gamma'}} = I_{X_{\Gamma'}}, \quad \text{for all } \sigma \in G,
$$

that is if the relative canonical ideal is invariant under the action of the lifted representation $\rho_{\Gamma'}$. In this case the free Γ'-modules $V_{\Gamma'}$, defined in remark 8, are G-invariant and the T-action, as defined in definition 11.1 restricts to a lift of the representation

(26)
$$
\begin{array}{ccc}
 & & \mathrm{GL}_r(\Gamma') \\
 & {\scriptstyle \rho_{\Gamma'}^{(1)}} \nearrow & \downarrow {\scriptstyle \mathrm{mod}\langle E\rangle} \\
G & \xrightarrow[\;\rho_{\Gamma}^{(1)}\;]{} & \mathrm{GL}_r(\Gamma)
\end{array}
$$

In [25, sec. 2.2] we gave an efficient way to check this compatibility in terms of linear algebra:

Consider an ordered basis Σ of the free Γ-module $\mathscr{S}_g(\Gamma)$ generated by the matrices $\Sigma(ij) = (\sigma(ij))_{\nu,\mu}$, $1 \leq i \leq j \leq g$ ordered lexicographically, with elements

$$
\sigma(ij)_{\nu,\mu} = \begin{cases} \delta_{i,\nu}\delta_{j,\mu} + \delta_{i,\mu}\delta_{j,\nu}, & \text{if } i \neq j \\ \delta_{i,\nu}\delta_{i,\mu} & \text{if } i = j. \end{cases}
$$

For example, for $g = 2$ we have the elements

$$
\sigma(11) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \sigma(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma(22) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
$$

For every symmetric matrix A, let $F(A)$ be the column vector consisted of the coordinates of A in the basis Σ. Consider the symmetric matrices $A_1^{\Gamma'}, \ldots, A_r^{\Gamma'}$, which exist since at the level of curves there is no obstruction of the embedded deformation. For each $\sigma \in G$ the $(g+1)g/2 \times 2r$ matrix

(27)
$$
F_{\Gamma'}(\sigma) = \left[F\left(A_1^{\Gamma'}\right), \ldots, F\left(A_r^{\Gamma'}\right), F\left(\rho_{\Gamma'}(\sigma)^t A_1^{\Gamma'} \rho_{\Gamma'}(\sigma)\right), \ldots, F\left(\rho_{\Gamma'}(\sigma)^t A_r^{\Gamma'} \rho_{\Gamma'}(\sigma)\right) \right].
$$

The automorphism σ acting on the relative curve X_Γ is lifted to an automorphism σ of $X_{\Gamma'}$ if and only if the matrix given in eq. (27) has rank r.

Proposition 31. *The obstruction to lifting an automorphism of X_Γ to $X_{\Gamma'}$ has a global obstruction given by vanishing the class of*

$$
A(\sigma, \tau) = \rho_{\Gamma'}(\sigma)\rho_{\Gamma'}(\tau)\rho_{\Gamma'}(\sigma\tau)^{-1}
$$

in $H^2(G, M_g(k))$ and a compatibility rank condition is given by requiring that the rank of the matrix $F_{\Gamma'}(\sigma)$ equals r for all elements $\sigma \in G$.

4.1. **An example.** Let k be an algebraically closed field of positive characteristic $p > 0$. Consider the Hermitian curve, defined over k, given by the equation

(28)
$$
H : y^p - y = \frac{1}{x^{p+1}},
$$

which has the group $\mathrm{PGU}(3, p^2)$ as an automorphism group, [39, th. 7]. As an Artin-Schreier extension of the projective line, this curve fits within the Bertin-Mézard model of curves, and the deformation functor with respect to the subgroup $\mathbb{Z}/p\mathbb{Z} \cong \mathrm{Gal}(H/\mathbb{P}^1) = \{y \mapsto y + 1\}$ has versal deformation ring $W(k)[\zeta][[x_1]]$, where ζ is a primitive p root of unity which resides in an algebraic extension of $\mathrm{Quot}(W(k))$ [4], [21]. Indeed, $m = p + 1 = 2p - (p-1) = qp - l$, so in the notation of [4] $q = 2$ and $l = p - 1$.

The reduction of the universal curve in the Bertin-Mezard model modulo $\mathfrak{m}_{W(k)[\zeta]}$ is given by the Artin-Schrein equation:

$$(29) \qquad X^p - X = \frac{x^{p-1}}{(x^2 + x_1 x)^p}$$

which has special fibre at the specialization $x_1 = 0$ the original Hermitian curve given in eq. (28).

The initial Hermitian curve admits the automorphism $\sigma : y \mapsto y, x \mapsto \zeta_{p+1} x$, where ζ_{p+1} is a primitive $p + 1$ root of unity. We will use the tools developed in this article in order to show that the automorphism σ does not lift even in positive characteristic.

We set $a(x) = x^2 + x_1 x$ and $\lambda = \zeta - 1 \in W(k)[\zeta]$. In [21] the first author together with S. Karanikolopoulos proved that the free R-module $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})$ has basis

$$\mathbf{c} = \left\{ W_{N,\mu} = \frac{x^N a(x)^{p-1-\mu} X^{p-1-\mu}}{a(x)^{p-1}(\lambda X + 1)^{p-1}} dx : \left\lfloor \frac{\mu \ell}{p} \right\rfloor \leq N \leq \mu q - 2, \ 1 \leq \mu \leq p - 1 \right\}.$$

From the form of the holomorphic differentials, it is clear that the representation of $\langle \sigma \rangle$ on $H^0(H, \Omega_{H/k})$ is diagonal since $a(x) = x^2 + x_1 x$ reduces to x^2 for $x_1 = 0$. In our example, we have $q = \deg a(x) = 2$ so in the special fibre we have

$$w_{N,\mu} = x^{N-2\mu} X^{p-1-\mu} dx$$

$$\sigma(w_{N,\mu}) = \zeta_{p+1}^{N-2\mu+1} w_{N,\mu}$$

and

$$(30) \qquad \sigma(w_{N,\mu} w_{N',\mu'}) = \zeta_{p+1}^{N+N'-2(\mu+\mu')+2} w_{N,\mu} w_{N',\mu'}.$$

Thus, the action of σ on holomorphic differentials on the special fibre is given by a diagonal matrix.

To decide, using the tools developed in this article, whether the action lifts to the Artin local ring $k[\epsilon]$, we have to see first whether the diagonal representation can be lifted, that is whether we have the following commutative diagram:

$$\begin{array}{ccc} & & \mathrm{GL}_g(k[\epsilon]) \\ & \nearrow^{\tilde{\rho}} & \downarrow \\ \langle \sigma \rangle & \xrightarrow{\rho} & \mathrm{GL}_g(k) \end{array}$$

Since $\rho(\sigma) = \mathrm{diag}(\delta_1, \ldots, \delta_g) =: \Delta$ a possible lift will be given by $\tilde{\rho}(\sigma) = \Delta + \epsilon B$, for some $g \times g$ matrix B with entries in k. The later element should have order $p + 1$, that is

$$\mathbb{I}_g = (\Delta + \epsilon B)^{p+1} = \Delta^{p+1} + \epsilon \Delta^p B,$$

which in turn implies that $\Delta^p B = 0$ and since Δ is invertible $B = 0$. This means that the representation of the cyclic group generated by σ is trivially deformed to a representation into $\mathrm{GL}_g(k[\epsilon])$.

The next step is to investigate whether the canonical ideal is kept invariant under the action of σ for $x_1 \neq 0$. The canonical ideal for Bertin-Mézard curves was

recently studied by H. Haralampous K. Karagiannis and the first author, see [6].
Namely, using the notation of [6] we have

$$a(x)^{p-i} = (x^2 + x_1 x)^{p-i} = \sum_{j=j_{\min}}^{2(p-1)} c_{j,p-i} x^j$$

$$= \sum_{j=0}^{p-i} \binom{p-i}{j} x_1^{p-i-j} x^{j+p-i}$$

so by setting $J = j + p - i$, $p - i \leq J \leq 2(p - i)$ we have

$$c_{J,p-i} = \begin{cases} \binom{p-i}{J-(p-i)} x_1^{2(p-i)-J} & \text{if } J \geq p - i \\ 0 & \text{if } J < p - i \end{cases}$$

This means that $c_{2(p-i),p-i} = 1$, $c_{2(p-i)-1,p-i} = (p-i)x_1$ and for all other values
of J, the quantity $c_{J,p-i}$ is either zero or a monomial in x_1 of degree ≥ 2.

It is proved in [6] that the canonical ideal is generated by two sets of generators
G_1 and G_2 given by:

$$G_1^{\mathfrak{c}} = \{W_{N_1,\mu_1} W_{N_1',\mu_1'} - W_{N_2,\mu_2} W_{N_2',\mu_2'} \in S \ : \ W_{N_1,\mu_1} W_{N_1',\mu_1'}, W_{N_2,\mu_2} W_{N_2',\mu_2'} \in \mathbb{T}^2$$
$$\text{and } N_1 + N_1' = N_2 + N_2', \quad \mu_1 + \mu_1' = \mu_2 + \mu_2'\}.$$

$$G_2^{\mathfrak{c}} = \Big\{ W_{N,\mu} W_{N',\mu'} - W_{N'',\mu''} W_{N''',\mu'''}$$

$$+ \sum_{i=1}^{p-1} \sum_{j=j_{\min}(i)}^{(p-i)q} \lambda^{i-p} \binom{p}{i} c_{j,p-i} W_{N_j,\mu_i} W_{N_j',\mu_i'} \in S \ :$$

$$N'' + N''' = N + N' + p - 1, \quad \mu'' + \mu''' = \mu + \mu' + p,$$
$$N_j + N_j' = N + N' + j, \quad \mu_i + \mu_i' = \mu + \mu' + p - i$$

$$\text{for } 0 \leq i \leq p, \ j_{\min}(i) \leq j \leq (p-i)q \Big\}.$$

The reduction modulo $\mathfrak{m}_{W(k)[\zeta]}$, of the set $G_1^{\mathfrak{c}}$ is given by simply replacing each $W_{n,\mu}$
by $w_{N,\mu}$ and does not depend on x_1. Therefore it does not give us any condition
to deform σ.

The reduction of the set $G_2^{\mathfrak{c}}$ modulo $\mathfrak{m}_{W(k)[\zeta]}$ is given by

$$G_2^{\mathfrak{c}} \otimes_R k = \Big\{ w_{N,\mu} w_{N',\mu'} - w_{N'',\mu''} w_{N''',\mu'''} - \sum_{j=j_{\min}(1)}^{(p-1)q} c_{j,p-1} w_{N_j,\mu_j} w_{N_j',\mu_j'} \in S \ :$$

$$N'' + N''' = N + N' + p - 1, \quad \mu'' + \mu''' = \mu + \mu' + p,$$
$$N_j + N_j' = N + N' + j, \quad \mu_i + \mu_i' = \mu + \mu' + p - i$$

$$\text{for } j_{\min}(1) \leq j \leq (p-1)q \Big\}.$$

If we further consider this set modulo $\langle x_1^2 \rangle$, that is if we consider the canonical
curve as a family over first-order infinitesimals then, only the terms $c_{2(p-1),p-1} = 1$,
$c_{2(p-1)-1,p-1} = (p-1)x_1$ survive.

Using eq. (30) and the definition of $G_2^{\mathfrak{c}}$ we have that for

$$W = w_{N,\mu} w_{N',\mu'} - w_{N'',\mu''} w_{N''',\mu'''} - w_{N_{2(p-1)},\mu_{p-1}} w_{N'_{2(p-1)},\mu'_{p-1}}$$

$$\sigma(W) = \zeta_{p+1}^{N+N'-2(\mu+\mu')+2} W$$

Set

$$W'' = w_{N_{2(p-1)-1},\mu_{p-1}} w_{N'_{2(p-1)-1},\mu'_{p-1}}.$$

The automorphism lifts if and only if the element

$$W' = W + x_1 W''$$

we have

$$\sigma(W') = \chi(\sigma)\big(W'\big).$$

But this is not possible since for

$$\sigma(W'') = \zeta_{p+1}^{N_{2(p-1)-1}+N_{2(p-1)-1}-2(\mu_{p-1}+\mu'_{p-1})+2} W''$$

and

$$N_{2(p-1)-1} + N_{2(p-1)-1} - 2(\mu_{p-1} + \mu'_{p-1}) + 2 = N + N' - 2(\mu + \mu') + 2 - 1.$$

4.2. **A tangent space condition.** All lifts of X_Γ to $X_{\Gamma'}$ form a principal homogeneous space under the action of $H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$. This paragraph aims to provide the compatibility relation given in eq. (4) by selecting the deformations of the curve and the representations.

Let $\{A_1^\Gamma, \ldots, A_r^\Gamma\}$ be a basis of the canonical Ideal I_{X_Γ}, where X_Γ is a canonical curve. Assume also that the special fibre is acted on by the group G, and we assume that the action of the group G is lifted to the relative curve X_Γ. Since X_Γ is assumed to be acted on by G, we have the action

$$(31) \qquad T(\sigma^{-1})(A_i^\Gamma) = \rho_\Gamma(\sigma)^t A_i^\Gamma \rho_\Gamma(\sigma) = \sum_j \lambda_{i,j}^\Gamma(\sigma) A_j(\Gamma) \text{ for each } i = 1, \ldots, r,$$

where ρ_Γ is a lift of the representation ρ induced by the action of G on $H^0(X_\Gamma, \Omega_{X/\Gamma})$ and $\lambda_{i,j}^\Gamma(\sigma)$ are the entries of the matrix of the lifted representation $\rho_\Gamma^{(1)}$ induced by the action of G on $A_1^\Gamma, \ldots, A_r^\Gamma$. Notice that the matrix $\rho_\Gamma(\sigma) \in \mathrm{GL}_g(\Gamma)$. We will denote by $A_1^{\Gamma'}, \ldots, A_r^{\Gamma'} \in \mathscr{S}_g(\Gamma')$ a set of liftings of the matrices $A_1^\Gamma, \ldots, A_r^\Gamma$. Since the couple (X_Γ, G) is lifted to $(X_{\Gamma'}, G)$, there is an action

$$T(\sigma^{-1})(A_i^{\Gamma'}) = \rho_{\Gamma'}(\sigma)^t A_i^{\Gamma'} \rho_{\Gamma'}(\sigma) = \sum_j \lambda_{i,j}^{\Gamma'}(\sigma) A_j^{\Gamma'} \text{ for each } i = 1, \ldots, r,$$

where $\lambda_{ij}^{\Gamma'}(\sigma) \in \Gamma'$. All other liftings extending X_Γ form a principal homogeneous space under the action of $H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$ that is we can find matrices $B_1, \ldots, B_r \in \mathscr{S}_g(k)$, such that the set

$$\{A_1^{\Gamma'} + E \cdot B_1, \ldots, A_r^{\Gamma'} + E \cdot B_r\}$$

forms a basis for another lift $I_{X_{\Gamma'}^1}$ of the canonical ideal of I_{X_Γ}. That is all lifts of the canonical curve I_{X_Γ} differ by an element $f \in \mathrm{Hom}_S(I_X, S/I_X) = H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$ so that $f(A_i) = B_i$.

In the same manner, if $\rho_{\Gamma'}$ is a lift of the representation ρ_Γ every other lift is given by

$$\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma),$$

where $\tau(\sigma) \in M_g(k)$.

We have to find out when $\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma)$ is an automorphism of the relative curve $X_{\Gamma'}$, i.e. when

(32) $\ T(\rho_{\Gamma'}(\sigma^{-1}) + E \cdot \tau(\sigma^{-1}))(A_i^{\Gamma'} + E \cdot B_i) \in \text{span}_{\Gamma'}\{A_1^{\Gamma'} + E \cdot B_1, \dots, A_r^{\Gamma'} + E \cdot B_r\},$

that is

(33)

$$(\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma))^t \left(A_i^{\Gamma'} + E \cdot B_i \right) (\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma)) = \sum_{j=1}^r \tilde{\lambda}_{ij}^{\Gamma'}(\sigma) \left(A_j^{\Gamma'} + E \cdot B_j \right),$$

for some $\tilde{\lambda}_{ij}^{\Gamma'}(\sigma) \in \Gamma'$. Since

$$T_{\Gamma'}(\sigma^{-1})A_i^{\Gamma'} = \rho_{\Gamma}(\sigma)^t A_i^{\Gamma} \rho_{\Gamma}(\sigma) \,\text{mod}\langle E \rangle$$

we have that $\tilde{\lambda}_{ij}^{\Gamma'}(\sigma) = \lambda_{i,j}^{\Gamma}(\sigma) \,\text{mod}E$, therefore we can write

(34) $$\tilde{\lambda}_{ij}^{\Gamma'}(\sigma) = \lambda_{ij}^{\Gamma'}(\sigma) + E \cdot \mu_{ij}(\sigma),$$

for some $\mu_{ij}(\sigma) \in k$. We expand first the right-hand side of eq. (33) using eq. (34). We have

(35) $$\sum_{j=1}^r \tilde{\lambda}_{ij}^{\Gamma'}(\sigma) \left(A_j^{\Gamma'} + E \cdot B_j \right) = \sum_{j=1}^r \left(\lambda_{ij}^{\Gamma'}(\sigma) + E \cdot \mu_{ij}(\sigma) \right) \left(A_j^{\Gamma'} + E \cdot B_j \right)$$

(36) $$= \sum_{j=1}^r \lambda_{ij}^{\Gamma'}(\sigma) A_j^{\Gamma'} + E\big(\mu_{ij}(\sigma)A_j + \lambda_{ij}(\sigma)B_j\big).$$

Here we have used the fact that $E\mathfrak{m}_{\Gamma} = E\mathfrak{m}_{\Gamma'}$ so $E \cdot x = E \cdot (x \,\text{mod}\mathfrak{m}_{\Gamma'})$ for every $x \in \Gamma'$.

We now expand the left-hand side of eq. (33).

$$(\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma))^t \left(A_i^{\Gamma'} + E \cdot B_i \right) (\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma)) = \rho_{\Gamma'}(\sigma)^t A_i^{\Gamma'} \rho_{\Gamma'}(\sigma)$$
$$+ E \cdot \left(\rho(\sigma)^t B_i \rho(\sigma) + \tau^t(\sigma) A_i \rho(\sigma) + \rho(\sigma)^t A_i \tau(\sigma) \right).$$

Set $D_\sigma = \tau(\sigma)\rho(\sigma)^{-1} = d(\sigma)$ according to the notation of lemma 14, we can write

(37)
$$\tau(\sigma)^t A_i \rho(\sigma) + \rho(\sigma)^t A_i \tau(\sigma)$$
$$= \rho(\sigma)^t \rho(\sigma^{-1})^t \tau(\sigma)^t A_i \rho(\sigma) + \rho(\sigma)^t A_i \tau(\sigma)\rho(\sigma)^{-1}\rho(\sigma)$$
$$= \rho(\sigma)^t (D_\sigma^t A_i)\rho(\sigma) + \rho(\sigma)^t (A_i D_\sigma)\rho(\sigma)$$
$$= T(\sigma^{-1})\psi_{D_\sigma}(A_i).$$

while eq. (24) implies that

(38) $$\rho(\sigma)^t B_i \rho(\sigma) - \sum_{j=1}^r \lambda_{ij}(\sigma^{-1})B_j = -T(\sigma^{-1})\psi_{B_\sigma[f]}(A_i).$$

For the above computations recall that for a $g \times g$ matrix B, the map ψ_B is defined by

$$\psi_B(A_i) = A_i B + B^t A_i.$$

Combining now eq. (37) and (38) we have that eq. (33) is equivalent to

$$T(\sigma^{-1})\big(\psi_{D_\sigma}(A_i)\big) - T(\sigma^{-1})\psi_{B_\sigma[f]}(A_i) = \sum_{j=1}^{r} \mu_{ij}(\sigma)A_j$$

$$(39) \qquad \big(\psi_{D_\sigma}(A_i)\big) - \psi_{B_\sigma[f]}(A_i) = \sum_{j=1}^{r} T(\sigma)\mu_{ij}(\sigma)A_j.$$

$$= \sum_{j=1}^{r}\sum_{\nu=1}^{r} \mu_{ij}(\sigma)\lambda_{j\nu}(\sigma^{-1})A_\nu.$$

On the other hand the action T on $A_1, \ldots, A_r$ is given in terms of the matrix $(\lambda_{i,j})$ while the right-hand side of eq. (39) $\big(\mu_{i,j}(\sigma^{-1})\big)\big(\lambda_{ij}(\sigma)\big)$ corresponds to the derivation $D^{(1)}(\sigma^{-1})$ of the ρ_1-representation. Equation (4) is now proved.

References

[1] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris. `doi:10.1007/978-3-540-69392-5`.

[2] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[3] José Bertin and Ariane Mézard. Induction and Restriction In Formal Deformation of Coverings. *arXiv:math.AG/0205228*.

[4] José Bertin and Ariane Mézard. Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques. *Invent. Math.*, 141(1):195–238, 2000.

[5] Rachel Camina. The Nottingham group. In *New horizons in pro-p groups*, volume 184 of *Progr. Math.*, pages 205–221. Birkhäuser Boston, Boston, MA, 2000.

[6] Hara Charalambous, Kostas Karagiannis, and Aristides Kontogeorgis. The relative canonical ideal of the Artin-Schreier-Kummer-Witt family of curves. *Annales de l'institut Fourier*, 73(3):1085–1113, 2023. URL: `https://aif.centre-mersenne.org/articles/10.5802/aif.3543/`, `doi:10.5802/aif.3543`.

[7] Gunther Cornelissen and Fumiharu Kato. Equivariant deformation of Mumford curves and of ordinary curves in positive characteristic. *Duke Math. J.*, 116(3):431–470, 2003.

[8] Michel Demazure. *Lectures on p-divisible groups*, volume 302 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. Reprint of the 1972 original.

[9] Marcus du Sautoy and Ivan Fesenko. Where the wild things are: ramification groups and the Nottingham group. In *New horizons in pro-p groups*, volume 184 of *Progr. Math.*, pages 287–328. Birkhäuser Boston, Boston, MA, 2000.

[10] David Eisenbud. *The geometry of syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry.

[11] Mark Green and Robert Lazarsfeld. A simple proof of Petri's theorem on canonical curves. In *Geometry today (Rome, 1984)*, volume 60 of *Progr. Math.*, pages 129–142. Birkhäuser Boston, Boston, MA, 1985.

[12] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221, 1957.

[13] Alexander Grothendieck. Géométrie formelle et géométrie algébrique. In *Séminaire Bourbaki, Vol. 5*, pages Exp. No. 182, 193–220, errata p. 390. Soc. Math. France, Paris, 1995.

[14] David Harbater. Patching and Galois theory. In *Galois groups and fundamental groups*, volume 41 of *Math. Sci. Res. Inst. Publ.*, pages 313–424. Cambridge Univ. Press, Cambridge, 2003.

[15] David Harbater and Katherine F. Stevenson. Patching and thickening problems. *J. Algebra*, 212(1):272–304, 1999.

[16] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[17] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[18] Robin Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, New York, 2010. `doi:10.1007/978-1-4419-1596-2`.

[19] Enrique Acosta (https://mathoverflow.net/users/1724/enrique acosta). Geometric meaning of the Euler sequence on $\mathbb{P}^n$ (example 8.20.1 in ch ii of Hartshorne). MathOverflow. URL:https://mathoverflow.net/q/5211 (version: 2016-12-11). URL: `https://mathoverflow.net/q/5211`, `arXiv:https://mathoverflow.net/q/5211`.

[20] Michael Kapovich and John J. Millson. On the deformation theory of representations of fundamental groups of compact hyperbolic 3-manifolds. *Topology*, 35(4):1085–1106, 1996. `doi:10.1016/0040-9383(95)00060-7`.

[21] Sotiris Karanikolopoulos and Aristides Kontogeorgis. Integral representations of cyclic groups acting on relative holomorphic differentials of deformations of curves with automorphisms. *Proc. Amer. Math. Soc.*, 142(7):2369–2383, 2014. URL: `https://doi.org/10.1090/S0002-9939-2014-12010-7`.

[22] Aristides Kontogeorgis. On the tangent space of the deformation functor of curves with automorphisms. *Algebra Number Theory*, 1(2):119–161, 2007.

[23] Aristides Kontogeorgis. Polydifferentials and the deformation functor of curves with automorphisms. *Journal of Pure and Applied Algebra*, 210(2):551–558, 2007.

[24] Aristides Kontogeorgis and Alexios Terezakis. On the lifting problem of representations of a metacyclic group, 2023. URL: `https://arxiv.org/abs/2301.01032`, `doi:10.48550/ARXIV.2301.01032`.

[25] Aristides Kontogeorgis, Alexios Terezakis, and Ioannis Tsouknidas. Automorphisms and the canonical ideal. *Mediterr. J. Math.*, 18(6):Paper No. 261, 15, 2021. `doi:10.1007/s00009-021-01878-3`.

[26] Aristides Kontogeorgis and Ioannis Tsouknidas. A cohomological treatise of HKG-covers with applications to the Nottingham group. *J. Algebra*, 555:325–345, 2020. `doi:10.1016/j.jalgebra.2020.02.037`.

[27] Aristides Kontogeorgis and Ioannis Tsouknidas. A generating set for the canonical ideal of HKG-curves. *Res. Number Theory*, 7(1):Paper No. 4, 16, 2021. `doi:10.1007/s40993-020-00230-0`.

[28] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999. `doi:10.1007/978-1-4612-0525-8`.

[29] Alexander Lubotzky and Andy R. Magid. Varieties of representations of finitely generated groups. *Mem. Amer. Math. Soc.*, 58(336):xi+117, 1985. `doi:10.1090/memo/0336`.

[30] B. Mazur. Deformation theory of Galois representations (in Modular forms and Fermat's last theorem). pages xx+582, 1997. Papers from the Instructional Conference on Number Theory and Arithmetic Geometry held at Boston University, Boston, MA, August 9–18, 1995.

[31] Martin Olson. Tangent spaces and obstructed theories, 2019. URL: `https://math.berkeley.edu/~molsson/MSRISummer07.pdf`.

[32] B. Saint-Donat. On Petri's analysis of the linear system of quadrics through a canonical curve. *Math. Ann.*, 206:157–175, 1973. URL: `https://doi.org/10.1007/BF01430982`.

[33] Michael Schlessinger. Functors of Artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.

[34] Michael Schlessinger. On rigid singularities. *Rice Univ. Stud.*, 59(1):147–162, 1973.

[35] Edoardo Sernesi. *Deformations of algebraic schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[36] Henning Stichtenoth. Über die Automorphismengruppe eines algebraischen Funktionenkörpers von Primzahlcharakteristik. I. Eine Abschätzung der Ordnung der Automorphismengruppe. *Arch. Math. (Basel)*, 24:527–544, 1973.

[37] John Tate. Finite flat group schemes. In *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, pages 121–154. Springer, New York, 1997.

[38] Ravi Vakil. *The rising sea*. 2017.

[39] Robert C. Valentini and Manohar L. Madan. A Hauptsatz of L. E. Dickson and Artin-Schreier extensions. *J. Reine Angew. Math.*, 318:156–177, 1980.

[40] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, Cambridge, 1994.

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