A NEW OBSTRUCTION TO THE LOCAL LIFTING PROBLEM

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Abstract. We study the local lifting problem of actions of semidirect products of a cyclic p-group by a cyclic prime to p group, where p is the characteristic of the special fibre. We give a criterion based on Harbater-Katz-Gabber compactification of local actions, which allows us to decide whether a local action lifts or not. In particular for the case of dihedral group we give an example of dihedral local action that can not lift and in this way we give a stronger obstruction than the KGB-obstruction.

1. Introduction

Let G be a finite group, k and algebraically closed field of characteristic $p > 0$ and consider the homomorphism

$$\rho : G \hookrightarrow \mathrm{Aut}(k[[t]]),$$

which will be called *a local G-action*. Let $W(k)$ denote the ring of Witt vectors of k. The local lifting problem considers the following question: Does there exist an extension $\Lambda/W(k)$, and a representation

$$\tilde{\rho} : G \hookrightarrow \mathrm{Aut}(\Lambda[[T]]),$$

such that if t is the reduction of T, then the action of G on $\Lambda[[T]]$ reduces to the action of G on $k[[t]]$? If the answer to the above question is positive, then we say that the G-action lifts to characteristic zero. A group G for which every local G-action on $k[[t]]$ lifts to characteristic zero is called *a local Oort group for k*.

After studying certain obstructions (the Bertin-obstruction, the KGB-obstruction, the Hurwitz tree obstruction etc) it is known that the only possible local Oort groups are known to be

(1) Cyclic groups
(2) Dihedral groups D_{p^h} of order $2p^h$
(3) The alternating group A_4

The Oort conjecture states that every cyclic group C_q of order $q = p^h$ lifts locally. This conjecture was proved recently by F. Pop [26] using the work of A. Obus and S. Wewers [24]. A. Obus proved that A_4 is local Oort group in [21] and this was also known to F. Pop and I. Bouw and S. Wewers [6]. The case of dihedral groups D_p are known to be local Oort by I. Bouw and S. Wewers for p odd [6] and by G. Pagot [25]. Several cases of dihedral groups D_{p^h} for small p^h have been studied by A. Obus [22] and H. Dang, S. Das, K. Karagiannis, A. Obus, V. Thatte [11], while the D_4 was studied by B. Weaver [30] For more details on the lifting problem we refer to [8], [9], [10], [20].

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Probably, the most important of the known so far obstructions is the KGB obstruction [9]. It was conjectured that this is the only obstruction for the local lifting problem, see [20], [22]. In particular, the KGB-obstruction for the dihedral group D_q is known to vanish, so the conjecture asserts that the local action of D_q always lifts. We will provide in section 6.1 a counterexample to this conjecture by proving that the HKG-cover corresponding to D_{125}, with a selection of lower jumps $9, 189, 4689$, which does not lift.

In this article, we will give a necessary and sufficient condition for a $C_q \rtimes C_m$-action and in particular for a D_q to lift. In order to do so, we will employ the Harbater-Katz-Gabber-compactification (HKG for short), which can be used in order to construct complete curves out of local actions. In this way, we have a variety of tools at our disposal and we can transform the local action and its deformations into representations of lineal groups acting on spaces of differentials of the HKG-curve. We have laid the necessary tools in our article [17], where we have collected several facts about the relation of liftings of local actions, liftings of curves and liftings of linear representations.

More precisely let us consider a local action $\rho : G \to \mathrm{Aut}k[[t]]$ of the group $G = C_q \rtimes C_m$. The Harbater-Katz-Gabber compactification theorem asserts that there is a Galois cover $X \to \mathbb{P}^1$ ramified wildly and completely only at one point P of X with Galois group $G = \mathrm{Gal}(X/\mathbb{P}^1)$ and tamely on a different point P' with ramification group C_m, so that the action of G on the completed local ring $\mathscr{O}_{X,P}$ coincides with the original action of G on $k[[t]]$. Moreover, it is known that the local action lifts if and only if the corresponding HKG-cover lifts.

In particular, we have proved that in order to lift a subgroup $G \subset \mathrm{Aut}(X)$, the representation $\rho : G \to \mathrm{GL}H^0(X, \Omega_X)$ should be lifted to characteristic zero and also the lifting should be compatible with the deformation of the curve. More precisely, in [17] we have proved the following relative version of Petri's theorem

Proposition 1. *Let $f_1, \ldots, f_r \in S := \mathrm{Sym}H^0(X, \Omega_X) = k[\omega_1, \ldots, \omega_g]$ be quadratic polynomials which generate the canonical ideal I_X of a curve X defined over an algebraic closed field k. Any deformation $\mathscr{X}_A$ is given by quadratic polynomials $\tilde{f}_1, \ldots, \tilde{f}_r \in \mathrm{Sym}H^0(\mathscr{X}_A, \Omega_{\mathscr{X}_A/A}) = A[W_1, \ldots, W_g]$, which reduce to $f_1, \ldots, f_r$ modulo the maximal ideal $\mathfrak{m}_A$ of A.*

And we also gave the following liftability criterion:

Theorem 2. *Consider an epimorphism $R \to k \to 0$ of local Artin rings. Let X be a curve which is is canonically embedded in $\mathbb{P}^g_k$ and the canonical ideal is generated by quadratic polynomials, and acted on by the group G. The curve $X \to \mathrm{Spec}(k)$ can be lifted to a family $\mathscr{X} \to \mathrm{Spec}(R) \in D_{\mathrm{gl}}(R)$ if and only if the representation $\rho_k : G \to \mathrm{GL}_g(k) = \mathrm{GL}(H^0(X, \Omega_X))$ lifts to a representation $\rho_R : G \to \mathrm{GL}_g(R) = \mathrm{GL}(H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}))$ and moreover the lift of the canonical ideal is left invariant by the action of $\rho_R(G)$.*

In section 3 we collect results concerning deformations of HKG covers, Artin representations and orbit actions and also provide a geometric explanation of the KGB-obstruction in remark 10. In section 4 we prove that the HKG-cover is canonically generated by quadratic polynomials, therefore theorem 2 can be applied.

In order to decide whether a linear representation of $G = C_q \rtimes C_m$ can be lifted we will employ the following

Theorem 3. *Consider a $k[G]$-module M which is decomposed as a direct sum*

$$M = V_\alpha(\epsilon_1, \kappa_1) \oplus \cdots \oplus V_\alpha(\epsilon_s, \kappa_s).$$

The module lifts to an $R[G]$-module if and only if the set $\{1, \ldots, s\}$ can be written as a disjoint union of sets I_ν, $1 \leq \nu \leq t$ so that

- *a.* $\sum_{\mu \in I_\nu} \kappa_\mu \leq q$, *for all* $1 \leq \nu \leq t$.
- *b.* $\sum_{\mu \in I_\nu} \kappa_\mu \equiv a \bmod m$ *for all* $1 \leq \nu \leq t$, *where* $a \in \{0, 1\}$.
- *c. For each ν, $1 \leq \nu \leq t$ there is an enumeration $\sigma : \{1, \ldots, \#I_\nu\} \to I_\nu \subset \{1, .., s\}$, such that*

$$\epsilon_{\sigma(2)} = \epsilon_{\sigma(1)} \alpha^{\kappa_{\sigma(1)}}, \epsilon_{\sigma(3)} = \epsilon_{\sigma(3)} \alpha^{\kappa_{\sigma(3)}}, \ldots, \epsilon_{\sigma(s)} = \epsilon_{\sigma(s-1)} \alpha^{\kappa_{\sigma(s-1)}}.$$

Condition b., with $a = 1$ happens only if the lifted C_q-action in the generic fibre has an eigenvalue equal to 1 for the generator τ of C_q.

Proof. See [18]. □

The idea of the above theorem is that indecomposable $k[G]$-modules in the decomposition of $H^0(X, \Omega_X)$ of the special fibre, should be combined together in order to give indecomposable modules in the decomposition of holomorphic differentials of the relative curve.

We will have the following strategy. We will consider a HKG-cover

$$\begin{tikzcd} X \arrow[r, "C_q"'] \arrow[rr, bend left, "G"] & \mathbb{P}^1 \arrow[r, "C_m"'] & \mathbb{P}^1 \end{tikzcd}$$

of the G-action. This has a cyclic subcover $X \to \mathbb{P}^1$ with Galois group C_q. We lift this cover using Oort's conjecture for C_q-groups to a cover $\mathscr{X} \to Spec\Lambda$. This gives rise to a representation

$$\rho : G \longrightarrow \mathrm{GL} H^0(X, \Omega_X), \tag{1}$$

together with a lifting

$$\begin{tikzcd} & \mathrm{GL} H^0(\mathscr{X}, \Omega_{\mathscr{X}/\Lambda}) = \mathrm{GL}_g(\Lambda) \arrow[d, "\bmod \mathfrak{m}_\Lambda"] \\ C_q \arrow[ur] \arrow[r] & \mathrm{GL} H^0(X, \Omega_X) = \mathrm{GL}_g(k) \end{tikzcd} \tag{2}$$

of the representation of the cyclic part C_q of G. We then lift, checking the conditions of theorem 3 the linear action of eq. (1) in characteristic zero in a such a way that the restriction to the C_q group is our initial lifting of the representation of the C_q subgroup coming from the lifting assured by Oort's conjecture given in eq. (2). Notice that the lifting of the cyclic group acting on a curve of characteristic zero in the generic fibre has the additional property that every eigenvalue of a generator of C_q is different than one, see eq. 13. Then using theorem 2 we will modify the initial lifting $\mathscr{X}$ to a lifting $\mathscr{X}'$ so that $\mathscr{X}'$ is acted on by G.

Notice that $m = 2$, that is for the case of dihedral groups D_q of order $2q$, there is no need to pair two indecomposable $k[D_q]$-modules togehterh in order to lift them into an indecomposable $R[D_q]$-module. The sets I_ν can be singletons and the conditions of theorem 3 are trivially satisfied. For example, condition 3.b. does not give any information since every integer is either odd or even. This means that the linear representations always lift.

In our geometric setting on the other hand, we know that in the generic fibre cyclic actions do not have identity eigenvalues, see proposition 13. This means that we have to consider lifts that satisfy 3.b. with $a = 0$. Therefore, indecomposable modules for $G = C_q \rtimes C_2 = D_q$ of odd dimmension d_1 should find an other indecomposable module of odd dimension d_2 in order lift to an $R[G]$-indecomposable module of even dimension $d_1 + d_2$. Moreover this dimension should satisfy $d_1 + d_2 \leq q$. If we also take care of the condition 3.c. we arrive at the following

Criterion 4. The HKG-curve with action of D_q lifts in characteristic zero if and only if all indecomposable summands $U(\epsilon, d)$, where $\epsilon \in \{0, 1\}$ and $1 \leq d \leq q^h$ with d odd have a pair $U(\epsilon', d')$, with $\epsilon' \in \{0, 1\} - \{\epsilon\}$ and $d + d' \leq q^h$.

In section 5 we will show that given a lifting $\mathscr{X}$ of the C_q action using Oort conjecture, and a lifting of the linear representation satisfying criterion 4 the lift $\mathscr{X}$ can be modified to a lift $\mathscr{X}'$, which lifts the action of D_q. In order to apply this idea we need a detailed study of the direct $k[G]$-summands of $H^0(X, \Omega_X)$, for $G = C_q \rtimes C_m$. This is considered in section 6, where we employ the joint work of the first author with F. Bleher and T. Chinburg [4], in order to compute the decomposition of $H^0(X, \Omega_X)$ into indecomposable kG-modules, in terms of the ramification filtration of the local action.

Then the lifting criterion of theorem 3 is applied. Our method gives rise to an algorithm which takes as input a group $C_q \rtimes C_m$, with a given sequence of lower jumps and decides whether the action lifts to characteristic zero.

In section 6.1 we give an example of an $C_{125} \rtimes C_4$ HKG-curve which does not lift and then we restrict ourselves to the case of dihedral groups. The possible ramification filtrations for local actions of the group $C_q \rtimes C_m$ were computed in the work of A. Obus and R. Pries in [23]. We focus on the case of dihedral groups D_q with lower jumps

$$b_\ell = w_0 \frac{p^{2\ell} + 1}{p + 1}, 0 \leq \ell \leq h - 1. \tag{3}$$

For the values $w_0 = 9$ we show in this section that the local action does not lift, providing a counterexample to the conjecture that the KGB-obstruction is the only obstruction to the local lifting problem.

Finally, in section 6.2 we prove that the jumps of eq. (3) for the value $w_0 = 1$ lift in characteristic zero.

We also have developed a programm in sage [28] in order to compute the decomposition of $H^0(X, \Omega_X)$ into intecomposable summands, which is freely available[1].

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[1] `https://www.dropbox.com/sh/uo0dg91l0vuqulr/AACarhRxsru_zuIp5ogLvy6va?dl=0`

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2. Notation

In this article we will study *metacyclic* groups $G = C_q \rtimes C_m$, where $q = p^h$ is a power of the characteristic and $m \in \mathbb{N}, (m,p) = 1$. Let τ be a generator of the cyclic group C_q and σ be a generator of the cyclic group C_m.

The group G is given in terms of generators and relations as follows:
(4)
$$G = \langle \sigma, \tau | \tau^q = 1, \sigma^m = 1, \sigma\tau\sigma^{-1} = \tau^{\alpha} \text{ for some } \alpha \in \mathbb{N}, 1 \leq \alpha \leq p^h - 1, (\alpha, p) = 1 \rangle.$$

The integer α satisfies the following congruence:

$$\alpha^m \equiv 1 \,\mathrm{mod} q \tag{5}$$

as one sees by computing $\tau = \sigma^m \tau \sigma^{-m} = \tau^{\alpha^m}$. Also the α can be seen as an element in the finite field $\mathbb{F}_p$, and it is a $(p-1)$-th root of unity, not necessarily primitive. In particular the following holds:

Lemma 5. *Let ζ_m be a fixed primitive m-th root of unity. There is a natural number a_0, $0 \leq a_0 < m - 1$ such that $\alpha = \zeta_m^{a_0}$.*

Proof. The integer α if we see it as an element in k is an element in the finite field $\mathbb{F}_p \subset k$, therefore $\alpha^{p-1} = 1$ as an element in $\mathbb{F}_p$. Let $\mathrm{ord}_p(\alpha)$ be the order of α in $\mathbb{F}_p^*$. By eq. 5 we have that $\mathrm{ord}_p(\alpha) \mid p-1$ and $\mathrm{ord}_p(\alpha) \mid m$, that is $\mathrm{ord}_p(\alpha) \mid (p-1, m)$.

The primitive m-th root of unity ζ_m generates a finite field $\mathbb{F}_p(\zeta_m) = \mathbb{F}_{p^\nu}$ for some integer ν, which has cyclic multiplicative group $\mathbb{F}_{p^\nu} \backslash \{0\}$ containing both the cyclic groups $\langle \zeta_m \rangle$ and $\langle \alpha \rangle$. Since for every divisor δ of the order of a cyclic group C there is a unique subgroup $C' < C$ of order δ we have that $\alpha \in \langle \zeta_m \rangle$, and the result follows. $\square$

Remark 6. For the case $C_q \rtimes C_m$ the KGB-obstruction vanishes if and only if the first lower jump h satisfies $h \equiv -1 \,\mathrm{mod} m$. For this to happen the conjugation action of C_m on C_q has to be faithful, see [20, prop. 5.9]. Also notice that by [23, th. 1.1], that if $u_0, u_1, \ldots, u_{h-1}$ is the sequence upper ramification jumps for the C_q subgroup, then the condition $h \equiv -1 \,\mathrm{mod} m$, then all upper jumps $u_i \equiv -1 \,\mathrm{mod} m$. In remark 10 we will explain the necessity of the KGB-obstruction in terms of the action of C_m, on the fixed horizontal divisor of the C_q group.

3. Deformation of covers

3.1. **Splitting the branch locus.** Consider a deformation $\mathscr{X} \to \mathrm{S}pec A$ of the curve X together with the action of G. Denote by $\tilde{\tau} = \tilde{\rho}(\tau)$ a lift of the action of the element $\tau \in \mathrm{Aut}(X)$. Weierstrass preparation theorem [5, prop. VII.6] implies that:

$$\tilde{\tau}(T) - T = g_{\tilde{\tau}}(T) u_{\tilde{\tau}}(T),$$

where $g_{\tilde{\tau}}(T)$ is a distinguished Weierstrass polynomial of degree $m+1$ and $u_{\tilde{\tau}}(T)$ is a unit in $R[[T]]$.

The polynomial $g_{\tilde{\tau}}(T)$ gives rise to a horizontal divisor that corresponds to the fixed points of $\tilde{\tau}$. This horizontal divisor might not be irreducible. The branch divisor corresponds to the union of the fixed points of any element in $G_1(P)$. Next lemma gives an alternative definition of a horizontal branch divisor for the relative curves $\mathscr{X} \rightarrow \mathscr{X}^G$, that works even when G is not a cyclic group.

Lemma 7. *Let $\mathscr{X} \rightarrow \mathrm{Spec}A$ be an A-curve, admitting a fibrewise action of the finite group G, where A is a Noetherian local ring. Let $S = \mathrm{Spec}A$, and $\Omega_{\mathscr{X}/S}$, $\Omega_{\mathscr{Y}/S}$ be the sheaves of relative differentials of $\mathscr{X}$ over S and $\mathscr{Y}$ over S, respectively. Let $\pi : \mathscr{X} \rightarrow \mathscr{Y}$ be the quotient map. The sheaf*

$$\mathscr{L}(-D_{\mathscr{X}/\mathscr{Y}}) = \Omega^{-1}_{\mathscr{X}/S} \otimes_S \pi^* \Omega_{\mathscr{Y}/S}$$

is the ideal sheaf of the horizontal Cartier divisor $D_{\mathscr{X}/\mathscr{Y}}$. The intersection of $D_{\mathscr{X}/\mathscr{Y}}$ with the special and generic fibre of $\mathscr{X}$ gives the ordinary branch divisors for curves.

Proof. We will first prove that the above defined divisor $D_{\mathscr{X}/\mathscr{Y}}$ is indeed an effective Cartier divisor. According to [16, Cor. 1.1.5.2] it is enough to prove that

- $D_{\mathscr{X}/\mathscr{Y}}$ is a closed subscheme which is flat over S.
- for all geometric points $\mathrm{Spec}k \rightarrow S$ of S, the closed subscheme $D_{\mathscr{X}/\mathscr{Y}} \otimes_S k$ of $\mathscr{X} \otimes_S k$ is a Cartier divisor in $\mathscr{X} \otimes_S k/k$.

In our case the special fibre is a nonsingular curve. Since the base is a local ring and the special fibre is nonsingular, the deformation $\mathscr{X} \rightarrow \mathrm{Spec}A$ is smooth. (See the remark after the definition 3.35 p.142 in [19]). The smoothness of the curves $\mathscr{X} \rightarrow S$, and $\mathscr{Y} \rightarrow S$, implies that the sheaves $\Omega_{\mathscr{X}/S}$ and $\Omega_{\mathscr{X}/S}$ are S-flat, [19, cor. 2.6 p.222].

On the other hand the sheaf $\Omega_{\mathscr{Y},\mathrm{Spec}A}$ is by [16, Prop. 1.1.5.1] $\mathscr{O}_{\mathscr{Y}}$-flat. Therefore, $\pi^*(\Omega_{\mathscr{Y},\mathrm{Spec}A})$ is $\mathscr{O}_{\mathscr{X}}$-flat and $\mathrm{Spec}A$-flat [14, Prop. 9.2]. Finally, observe that the intersection with the special and generic fibre is the ordinary branch divisor for curves according to [14, IV p.301]. $\square$

For a curve X and a branch point P of X we will denote by $i_{G,P}$ the order function of the filtration of G at P. The Artin representation of the group G is defined by $\mathrm{ar}_P(\sigma) = -f_P i_{G,P}(\sigma)$ for $\sigma \neq 1$ and $\mathrm{ar}_P(1) = f_P \sum_{\sigma \neq 1} i_{G,P}(\sigma)$ [27, VI.2]. We are going to use the Artin representation at both the special and generic fibre. In the special fibre we always have $f_P = 1$ since the field k is algebraically closed. The field of quotients of A should not be algebraically closed therefore a fixed point there might have $f_P \geq 1$. The integer $i_{G,P}(\sigma)$ is equal to the multiplicity of $P \times P$ in the intersection of $\Delta.\Gamma_\sigma$ in the relative A-surface $\mathscr{X} \times_{\mathrm{Spec}A} \mathscr{X}$, where Δ is the diagonal and Γ_σ is the graph of σ [27, p. 105].

Since the diagonals Δ_0, Δ_η and the graphs of σ in the special and generic fibres respectively of $\mathscr{X} \times_{\mathrm{Spec}A} \mathscr{X}$ are algebraically equivalent divisors we have:

Proposition 8. *Assume that A is an integral domain, and let $\mathscr{X} \rightarrow \mathrm{Spec}A$ be a deformation of X. Let $\bar{P}_i$, $i = 1, \cdots, s$ be the horizontal branch divisors that intersect at the special fibre, at point P, and let P_i be the corresponding points on the generic fibre. For the Artin representations attached to the points P, P_i we have:*

$$\mathrm{ar}_P(\sigma) = \sum_{i=1}^{s} \mathrm{ar}_{P_i}(\sigma). \tag{6}$$

This generalizes a result of J. Bertin [3]. Moreover if we set $\sigma = 1$ to the above formula we obtain a relation for the valuations of the differents in the special and the generic fibre, since the value of the Artin's representation at 1 is the valuation of the different [27, prop. 4.IV,prop. 4.VI]. This observetion is equivalent to claim 3.2 in [13] and is one direction of a local criterion for good reduction theorem proved in [13, 3.4], [15, sec. 5].

3.2. **The Artin representation on the generic fibre.** We can assume that after a base change of the family $\mathscr{X} \rightarrow \mathrm{S}pec(A)$ the points P_i at the generic fibre have degree 1. Observe also that at the generic fibre the Artin representation can be computed as follows:

$$\mathrm{ar}_Q(\sigma) = \begin{cases} 1 \text{ if } \sigma(Q) = Q, \\ 0 \text{ if } \sigma(Q) \neq Q. \end{cases}$$

The set of points $S := \{P_1, \ldots, P_s\}$ that are the intersections of the ramification divisor and the generic fibre are acted on by the group G.

We will now restrict our attention to the case of a cyclic group $H = C_q$ of order q. Let S_k be the subset of S fixed by $C_{p^{h-k}}$, i.e.

$$P \in S_k \text{ if and only if } H(P) = C_{p^{h-k}}.$$

Let s_k be the order of S_k. Observe that since for a point Q in the generic fibre $\sigma(Q)$ and Q have the same stabilizers (in general they are conjugate, but here H is abelian) the sets S_k are acted on by H. Therefore $\#S_k =: s_k = p^k i_k$ where i_k is the number of orbits of the action of H on S_k.

Let $b_0, b_1, \ldots, b_{h-1}$ be the jumps in the lower ramification filtration. Observe that

$$H_{j_k} = \begin{cases} C_{p^{h-k}} & \text{for } 0 \leq k \leq h-1 \\ \{1\} & \text{for } k \geq h. \end{cases}$$

An element in H_{b_k} fixes only elements in S with stabilizers that contain H_{b_k}. So H_{b_0} fixes only S_0, H_{b_1} fixes both S_0 and S_1 and H_{b_k} fixes all elements in $S_0, S_1, \ldots, S_k$. By definition of the Artin representation an element σ in $H_{b_k} - G_{b_{k+1}}$ satisfies $\mathrm{ar}_P(\sigma) = b_k + 1$ and by using equation (6) we arive at

$$b_k + 1 = i_0 + pi_1 + \cdots + p^k i_k.$$

Remark 9. This gives us a geometric interpretation of the Hasse-Arf theorem, which states that for the cyclic p-group of order $q = p^h$, the lower ramification filtration is given by

$$H_0 = H_1 = \cdots = H_{b_0} \gneqq H_{b_0+1} = \cdots = H_{b_1} \gneqq H_{b_1+1} = \cdots = H_{b_{h-1}} \gneqq \{1\},$$

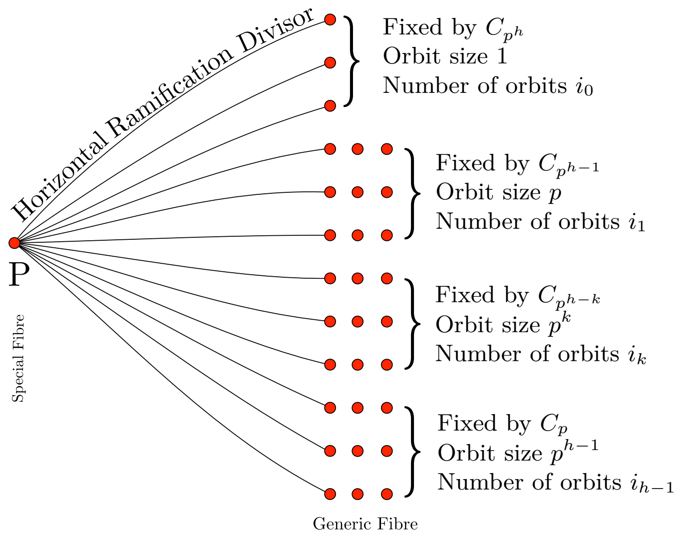
i.e. the jumps of the ramification filtration appear at the integers $b_0, \ldots, b_{h-1}$. Then

$$b_k + 1 = i_0 + i_1 p + i_2 p^2 + \cdots + i_k p^k. \tag{7}$$

The set of horizontal branch divisors is illustrated in figure 1. Notice that the group C_m acts on the set of ramification points of $H = C_q$ on the special fibre but it can't fix any of them since there are already fixed by a subgroup of C_q and if a branch point P of C_q was also fixed by an element of C_m, then the isotropy subgroup of P could not be cyclic. This proves that m divides the numbers of all orbits $i_0, \ldots, i_{n-1}$.

Remark 10. In this way we can recover the necessity of the KGB-obstruction since by eq. (7) the upper ramification jumps are $i_0 - 1, i_0 + i_1 - 1, \ldots, i_0 + \cdots + i_{n-1} - 1$.

Figure 1. The horizontal Ramification divisor



The Galois cover $X \to X/G$ breaks into two covers $X \to X^{C_q}$ and $X^{C_q} \to C^G$. The genus of C^G is zero by assumption and in the cover $X^{C_q} \to C^G$ there are exactly two ramified points with ramification indices m. An application of the Riemann-Hurwitz formula shows that the genus of X^{C_q} is zero as well.

The genus of the curve X can be computed either by the Riemann-Hurwitz formula in the special fibre

$$
\begin{aligned}
g &= 1 - p^n + \frac{1}{2}\sum_{i=0}^{\infty}(|G_i| - 1) \\
&= 1 - p^n + \frac{1}{2}\big((b_0+1)(p^n-1) + (b_1-b_0)(p^{n-1}-1) + (b_2-b_1)(p^{n-2}-1) + \cdots \\
&\cdots + (b_n - b_{n-1})(p-1)\big)
\end{aligned}
$$

or by the Riemann-Hurwitz formula on the generic fibre:

$$
g = 1 - p^n + \frac{1}{2}\left(i_0(p^n-1) + i_1 p(p^{n-1}-1) + \cdots i_{n-1}p^{n-1}(p-1)\right). \tag{8}
$$

Using eq. (7) we see that the two formulas for g give the same result as expected.

4. HKG-covers and their canonical ideal

Lemma 11. *Consider the Harbater-Katz-Gabber curve corresponding to the local group action $C_m \rtimes C_q$, where $q = p^h$ that is a power of the characteristic p. If one of the following conditions holds:*

- $h \geq 2$
- $h = 1$ *and the first jump* i_0 *in the ramification filtration for the cyclic group satisfies* $i_0 \neq 1$ *and* $q \geq \frac{12}{i_0 - 1} + 1$,

then the curve X has canonical ideal generated by quadratic polynomials.

Proof. We will prove that the curve X has genus $g \geq 6$ provided that p or h is big enough. We will also prove that the curve X is not hyperelliptic nor trigonal.

Remark 12. Let us first recall that a cyclic group of order $q = p^h$ for $h \geq 2$ can not act on the rational curve, see [29, thm 1]. Also let us recall that a cyclic group of order p can act on a rational curve and in this case the first and only break in the ramification filtration is $i_0 = 1$. This latter case is excluded.

Consider first the case $p^h = p$ and $i_0 \neq 1$. In this case we compute the genus g of the HKG-curve X using Riemann-Hurwitz formula:

$$2g = 2 - 2mq + q(m-1) + qm - 1 + i_0(q-1),$$

where the contribution $q(m-1)$ is from the q-points above the unique tame ramified point, while $qm - 1 + i_0(q-1)$ is the contribution of the wild ramified point. This implies that,

$$2g = (i_0 - 1)(q-1),$$

therefore if $i_0 \geq 2$, it suffices to have $q = p^h \geq 13$ and more generally it is enough to have $q \geq \frac{12}{i_0 - 1} + 1$ in order to ensure that $g \geq 6$.

For the case $h \geq 2$, we can write a stronger inequality based on Riemann-Hurwitz theorem as (recall that $i_0 \equiv i_1 \bmod p$ so $i_0 - i_1 \geq p$)

$$2g \geq (i_0 - 1)(p^h - 1) + (i_0 - i_1)(p^{h-1} - 1) \geq p^h - p, \tag{9}$$

which implies that $g \geq 6$ for $p > 3$ or $h > 3$.

In order to prove that the curve is not hyperelliptic we observe that hyperelliptic curves have a normal subgroup generated by the hyperelliptic involution j, so that $X \to X/\langle j \rangle = \mathbb{P}^1$. It is known that the automorphism group of a hyperelliptic curve fits in the short exact sequence

$$1 \to \langle j \rangle \to \mathrm{Aut}(X) \to H \to 1, \tag{10}$$

where H is a subgroup of $\mathrm{PGL}(2, k)$, see [7]. If m is odd then the hyperelliptic involution is not an element in C_m. If m is even, let σ be a generator of the cyclic group of order m and τ a generator of the group C_q. The involution $\sigma^{m/2}$ again can't be the hyperelliptic involution. Indeed, the hyperelliptic involution is central, while the conjugation action of σ on τ is faithful that is $\sigma^{m/2}\tau\sigma^{-m/2} \neq \tau$. In this case $G = C_m \rtimes C_q$ is a subgroup of H which should act on the rational function field. By the classification of such groups in [29, Th. 1] this is not possible. Thus X can't be hyperelliptic.

We will prove now that the curve is not trigonal. Using Clifford's theorem we can show [2, B-3 p.137] that a non hyperelliptic curve of genus $g \geq 5$ cannot have two distinct g_3^1. So if there is a g_3^1, then this is unique. Moreover, the g_3^1 gives rise to a map $\pi : X \to \mathbb{P}^1$ and every automorphism of the curve X fixes this map. Therefore, we obtain a morphism $\phi : C_m \rtimes C_q \to \mathrm{PGL}_2(k)$ and we arrive at the short exact sequence

$$1 \to \ker\phi \to C_m \rtimes C_q \to H \to 1,$$

for some finite subgroup H of $\mathrm{PGL}(2, k)$. If $\ker\phi = \{1\}$, then we have the tower of curves $X \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\pi'} \mathbb{P}^1$, where π' is a Galois cover with group $C_m \rtimes C_q$. This implies that X is a rational curve contradicting remark 12. If $\ker\phi$ is a cyclic group of order 3, then we have that $3 \mid m$ and the tower $X \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\pi'} \mathbb{P}^1$, where π is a cyclic Galois cover of order 3 and π' is a Galois cover with group $C_{m/3} \rtimes C_q$. As before this contradicts remark 12 and is not possible. $\square$

5. Invariant subspaces of vector spaces

The $g\times g$ symmetric matrices $A_1,\ldots,A_r$ defining the quadratic canonical ideal of the curve X, define a vector subspace of the vector space V of $g\times g$ symmetric matrices. By Oort conjecture, we know that there are symmetric matrices $\tilde{A}_1,\ldots,\tilde{A}_r$ with entries in a local principal ideal domain R, which reduce to the initial matrices $A_1,\ldots,A_r$. These matrices $\tilde{A}_1,\ldots,\tilde{A}_r$ correspond to the lifted relative curve $\tilde{X}$. Moreover, the submodule $\tilde{V}=\langle \tilde{A}_1,\ldots,\tilde{A}_r\rangle$ is left invariant under the action of a lifting $\tilde{\rho}$ of the representation $\rho: C_q \rightarrow \mathrm{GL}_g(k)$.

Proposition 13. *Let $\tilde{g}$ be the genus of the quotient curve X/H for a subgroup H of the automorphism group of a curve X in characteristic zero. We have*

$$\dim H^0(X,\Omega_X^{\otimes d})^H = \begin{cases} \tilde{g} & \text{if } d=1 \\ (2d-1)(\tilde{g}-1)+\sum_{P\in X/G}\left\lfloor d\left(1-\frac{1}{e(\tilde{P})}\right)\right\rfloor & \text{if } d>1 \end{cases}$$

Proof. See [12, eq. 2.2.3,2.2.4 p. 254]. □

Therefore, a generator of C_q acting on $H^0(X,\Omega_X)$ has no identity eigenvalues and m should divide g. This means that we have to consider liftings of indecomposable summands of C_q, which satisfy condition 3.b. with $a=0$. We now assume that condition 3.b. of theorem 3 can be fulfilled, so there is a lifting of the representation

$$\begin{tikzcd} & \mathrm{GL}_g(R) \arrow[d, "\mathrm{mod}\mathfrak{m}_R"] \\ C_q \rtimes C_m \arrow[ur, "\tilde{\rho}"] \arrow[r, "\rho"] & \mathrm{GL}_g(k) \end{tikzcd}$$

satisfying condition, see also the discussion in the introduction after the statement of this theorem after eq. (2).

We have to show that we can modify the space $\tilde{V}\subset \mathrm{Sym}_g(R)$ to a space $\tilde{V}'$ with the same reduction V modulo $\mathfrak{m}_R$ so that $\tilde{V}$ is $C_q\rtimes C_m$-invariant.

Consider the sum of the free modules

$$W=\tilde{V}+\tilde{\rho}(\sigma)\tilde{V}+\tilde{\rho}(\sigma^2)\tilde{V}+\cdots+\tilde{\rho}(\sigma^{m-1})\tilde{V}\subset R^N.$$

Observe that W is an $R[C_q\rtimes C_m]$-module and also it is a free submodule of R^N and by the theory of modules over local principal ideal domain there is a basis $E_1,\ldots,E_N$ of R^N such that

$$W=E_1\oplus\cdots\oplus E_r\oplus \pi^{a_{r+1}}E_{r+1}\oplus\cdots\oplus\pi^{a_N}E_N,$$

where $E_1,\ldots,E_r$ form a basis of $\tilde{V}$, while $\pi^{a_{r+1}}E_{r+1},\ldots,\pi^{a_N}E_N$ form a basis of the kernel W_1 of the reduction modulo $\mathfrak{m}_R$. Since the reduction is compatible with the actions of $\rho,\tilde{\rho}$ we have that W_1 is an $R[C_q\rtimes C_m]$-module, while $\tilde{V}$ is just a C_q-module.

Let π be the $R[C_q]$-equivariant projection map $W=\tilde{V}\oplus_{R[C_q]-\text{modules}}W_1\rightarrow W_1$. Since m is an invertible element of R, we can employ the proof of Mascke's theorem in order to construct a module $\tilde{V}'$, which is $R[C_q\rtimes C_m]$ stable and reduces to V modulo $\mathfrak{m}_R$, see also [1, I.3 p.12]. Indeed, consider the endomorphism $\bar{\pi}: W\rightarrow W$ defined by

$$\bar{\pi}=\frac{1}{m}\sum_{i=0}^{m-1}\tilde{\rho}(\sigma^i)\pi\tilde{\rho}(\sigma^{-i}).$$

We see that $\bar{\pi}$ is the identity on W_1 since π is the identity on W_1. Moreover $\tilde{V}' := \ker\bar{\pi}$ is both C_q and C_m invariant and reduces to V modulo $\mathfrak{m}_R$.

6. Galois module structure of holomorphic differentials, special fibre

Consider the group $C_q \rtimes C_m$. Let τ be a generator of C_q and σ a generator of C_m. It is known that $\mathrm{Aut}(C_q) \cong \mathbb{F}_p^* \times Q$, for some abelian group Q. The representation $\psi : C_m \to \mathrm{Aut}(C_q)$ given by the action of C_m on C_q is known to factor through a character $\chi : C_m \to \mathbb{F}_p^*$. The order of χ divides $p-1$ and $\chi^{p-1} = \chi^{-(p-1)}$ is the trivial one dimensional character. In our setting, using the definition of G given in eq. (4) and lemma 5 we have that the character χ is defined by

$$\chi(\sigma) = \alpha = \zeta_m^{a_0} \in \mathbb{F}_p. \tag{11}$$

For all $i \in \mathbb{Z}$, χ^i defines a simple $k[C_m]$-module of k dimension one, which we will denote by S_{χ^i}. For $0 \leq \ell \leq m-1$ denote by S_ℓ the simple module on which σ acts as ζ_m^ℓ. Both S_{χ^i}, S_ℓ can be seen as $k[C_q \rtimes C_m]$-modules using inflation. Finally for $0 \leq \ell \leq m-1$ we define $\chi^i(\ell) \in \{0, 1, \ldots, m-1\}$ such that $S_{\chi^i(\ell)} \cong S_\ell \otimes_k S_{\chi^i}$. Using eq. (11) we arrive at

$$S_{\chi^i(\ell)} = S_{\ell + i a_0}. \tag{12}$$

There are $q \cdot m$ isomorphism classes of indecomposable $k[C_q \rtimes C_m]$-modules and are all uniserial. An indecomposable $k[C_q \rtimes C_m]$-module U is uniquely determined by its socle, which is the kernel of the action of $\tau - 1$ on U, and its k-dimension. For $0 \leq \ell \leq m-1$ and $1 \leq \mu \leq q$, let $U_{\ell,\mu}$ be the indecomposable $k[C_q \rtimes C_m]$ module with socle S_ℓ and k-dimension μ. Then $U_{\ell,\mu}$ is uniserial and its μ ascending composition factors are the first μ composition factors of the sequence

$$S_\ell, S_{\chi^{-1}(\ell)}, S_{\chi^{-2}(\ell)}, \ldots, S_{\chi^{-(p-2)}(\ell)}, S_\ell, S_{\chi^{-1}(\ell)}, S_{\chi^{-2}(\ell)}, \ldots, S_{\chi^{-(p-2)}(\ell)}$$

Notice that in our notation $V_\alpha(\lambda, k) = U_{\lambda+k,k}$.

Assume that $X \to \mathbb{P}^1$ is an HKG-cover with Galois group $C_q \rtimes C_m$. The subgroup I generated by the Sylow p-subgroups of the inertia groups of all closed points of X is equal to C_q.

Definition 14. For each $0 \leq j \leq q-1$ we define

$$D_j = \sum_{y \in \mathbb{P}^1} d_{y,j} y,$$

where the integers $d_{y,j}$ are defined as follows. Let x be a point of X above y and consider the i-th ramification group $I_{x,i}$ at x. The order of the inertia group at x is assumed to be $p^{n(x)}$ and we set $i(x) = h - n(x)$. Let $b_0, b_1, \ldots, b_{n(x)-1}$ be the jumps in the numbering of the lower ramification filtration subgroups of I_x. We define

$$d_{y,j} = \left\lfloor \frac{1}{p^{n(x)}} \sum_{l=1}^{n(x)} p^{n(x)-l} \big(p - 1 + (p-1-a_{l,t}) b_{l-1}\big) \right\rfloor$$

for all $t, j \geq 0$ satisfying

$$p^{i(x)} t \leq j < p^{i(x)}(t+1) \tag{13}$$

and

$$t = a_{1,t} + a_{2,t} p + \cdots + a_{n(x),t} p^{n(x)-1}$$

is the p-adic expansion of t. In particular $D_{q-1}=0$. Observe that $d_{y,j}\neq 0$ only for wildly ramified branch points.

Remark 15. For a divisor D on a curve Y define $\Omega_Y(D)=\Omega_Y\otimes \mathscr{O}_Y(D)$. In particular for $Y=\mathbb{P}^1$, and for $D=D_j=d_{P_\infty,j}P_\infty$, where D_j is a divisor supported at the infinity point P_∞ we have

$$H^0(\mathbb{P}^1,\Omega_{\mathbb{P}^1}(D_j))=\{f(x)dx : 0\leq \deg f(x)\leq d_{P_\infty,j}-2\}.$$

For the sake of simplicity, we will denote $d_{P_\infty,j}$ by d_j. The space $H^0(\mathbb{P}^1,\Omega_{\mathbb{P}^1}(D_j))$ has a basis given by $B=\{dx,xdx,\ldots,x^{d_j-2}dx\}$. Therefore, the number $n_{j,\ell}$ of simple modules appearing in the decomposition $\Omega_{\mathbb{P}^1}(D_j)$ isomorphic to S_ℓ for $0\leq \ell<m$, is equal to the number of monomials x^ν with

$$\nu\equiv \ell-1 \bmod m, 0\leq \nu\leq d_j-2.$$

If $d_j\leq 1$ then $B=\emptyset$ and $n_{j,\ell}=0$ for all $0\leq\ell<m$. If $d_j>1$, then we know that in the d_j-1 elements of the basis B, the first $m\left\lfloor \frac{d_j-1}{m}\right\rfloor$ elements contribute to every representative modulo m. Thus, we have at least $\left\lfloor \frac{d_j-1}{m}\right\rfloor$ elements in isomorphic to S_ℓ for every $0\leq\ell<m$. We will now count the rest elements, of the form $\{x^\nu dx\}$, where

$$m\left\lfloor \frac{d_j-1}{m}\right\rfloor\leq \nu\leq d_j-2 \text{ and } \nu\equiv \overline{\ell-1} \bmod m,$$

where $\overline{\ell-1}$ is the unique integer in $\{0,1,\ldots,m-1\}$ equivalent to $\ell-1$ modulo m. We observe that the number $y_j(\ell)$ of such elements ν is given by

$$y_j(\ell)=\begin{cases}1 & \text{if } \overline{\ell-1}\leq d_j-2-m\left\lfloor \frac{d_j-1}{m}\right\rfloor\\ 0 & \text{otherwise}\end{cases}$$

Therefore

$$n_{j,\ell}=\begin{cases}\left\lfloor \frac{d_j-1}{m}\right\rfloor+y_j(\ell) & \text{if } d_j\geq 2\\ 0 & \text{if } d_j\leq 1\end{cases}$$

For example if $d_j=9$ and $m=3$, then a basis for $H^0(\mathbb{P}^1,\Omega_{\mathbb{P}^1}(9P_\infty))$ is given by $\{dx,xdx,x^2dx,\ldots x^7dx\}$. This basis has 8 elements, and each triple $\{dx,xdx,x^2dx\}$, $\{x^3dx,x^4dx,x^5dx\}$ contributes one to each class S_0,S_1,S_2, while there are two remaining basis elements $\{x^6dx,x^7dx,\}$, which contribute one to S_1,S_2. Notice that $\left\lfloor\frac{8}{3}\right\rfloor=2$ and $y(\ell)=1$ for $\ell=1,2$.

In particular if $m=2$, then $n_{j,\ell}=0$ if $d_j\leq 1$ and for $d_j\geq 2$ we have

$$n_{j,\ell}=\begin{cases}\frac{d_j-1}{2} & \text{if } d_j\equiv 1 \bmod 2\\ \frac{d_j}{2}-1 & \text{if } \ell=0 \text{ and } d_j\equiv 0 \bmod 2\\ \frac{d_j}{2} & \text{if } \ell=1 \text{ and } d_j\equiv 0 \bmod 2\end{cases} \tag{14}$$

Lemma 16. *Assume that $d_{j-1}=d_j+1$. Then if $d_j\geq 2$*

$$n_{j-1,\ell}-n_{j,\ell}=\begin{cases}1 & \textit{if } d_{j-1}\equiv 1 \bmod 2 \textit{ and } \ell=0\\ & \textit{or } d_{j-1}\equiv 0 \bmod 2 \textit{ and } \ell=1\\ 0 & \textit{if } d_{j-1}\equiv 1 \bmod 2 \textit{ and } \ell=1\\ & \textit{or } d_{j-1}\equiv 0 \bmod 2 \textit{ and } \ell=0\end{cases}$$

If $d_j \leq 1$, then

$$n_{j-1,\ell} - n_{j,\ell} = \begin{cases} 0 & \text{if } d_j = 0 \text{ or } (d_j = 1 \text{ and } \ell = 0) \\ 1 & \text{if } d_j = 1 \text{ and } \ell = 1 \end{cases}$$

Proof. Assume that $d_j \geq 2$. We distinguish the following two cases, and we will use eq. (14)

- d_{j-1} is odd and d_j is even. Then, if $\ell = 0$
$$n_{j-1,\ell} - n_{j,\ell} = \frac{d_{j-1} - 1}{2} - \frac{d_j}{2} + 1 = 1$$
while $n_{j-1,\ell} - n_{j,\ell} = 0$ if $\ell = 1$.
- d_{j-1} is even and d_j is odd. Then, if $\ell = 0$
$$n_{j-1,\ell} - n_{j,\ell} = \frac{d_{j-1}}{2} - 1 - \frac{d_j - 1}{2} = 0,$$
while $n_{j-1,\ell} - n_{j,\ell} = 1$ if $\ell = 0$.

If now $d_j = 0$ and $d_{j-1} = 1$, then $n_{j-1,\ell} - n_{j,\ell} = 0$. If $d_j = 1$ and $d_{j-1} = 2$ then $n_{j,\ell} = 0$ while $n_{j-1,\ell} = 0$ if $\ell = 0$ and $n_{j-1,\ell} = 1$ if $\ell = 1$.

□

Theorem 17. *Let $M = H^0(X, \Omega_X)$, let τ be the generator of C_q, and for all $0 \leq j < q$ we define $M^{(j)}$ to be the kernel of the action of $k[C_q](\tau - 1)^j$. For $0 \leq a \leq m-1$ and $1 \leq b \leq q = p^h$, let $n(a,b)$ be the number of indecomposable direct $k[C_q \rtimes C_m]$-module summands of M that are isomorphic to $U_{a,b}$. Let $n_1(a,b)$ be the number of indecomposable direct $k[C_m]$-summands of $M^{(b)}/M^{(b-1)}$ with socle $S_{\chi^{-(b-1)}(a)}$ and dimension 1. Let $n_2(a,b)$ be the number of indecomposable direct $k[C_m]$-module summands of $M^{(b+1)}/M^{(b)}$ with socle $S_{\chi^{-b}(a)}$, where we set $n_2(a,b) = 0$ if $b = q$.*

$$n(a,b) = n_1(a,b) - n_2(a,b).$$

The numbers $n_1(a,b), n_2(a,b)$ can be computed using the isomorphism

$$M^{(j+1)}/M^{(j)} \cong S_{\chi^{-j}} \otimes_k H^0(Y, \Omega_Y(D_j)),$$

where $Y = X/C_q$ and D_j are the divisors on Y, given in definition 14.

For the case of HKG-covers, with ∞ the wild ramified point and 0 the tame ramified point the divisors D_j are supported only at the wild ramified point and are given by

$$D_j = \left\lfloor \frac{1}{p^h} \sum_{l=1}^{h} p^{h-l}(p - 1 + (p-1-a_{l,t})b_{l-1}) \right\rfloor P_\infty$$

$$t = a_{1,t} + a_{2,t}p + \cdots + a_{h,t}p^{h-1}$$

is the p-adic expansion of t. Notice that since $i(x) = 0$ eq. (13) reduces to $t = j$.

Corollary 18. *Set $d_j = \left\lfloor \frac{1}{p^h} \sum_{l=1}^{h} p^{h-l}(p-1+(p-1-a_{l,t})b_{l-1}) \right\rfloor$. The numbers $n(a,b), n_1(a,b)$ and $n_2(a,b)$ are given by*

$$n(a,b) = n_1(a,b) - n_2(a,b) = n_{b-1,a} - n_{b,a}.$$

Proof. We will treat the $n_1(a,b)$ case and the $n_2(a,b)$ follows similarly. By the equivariant isomorphism for $M = H^0(X, \Omega_X)$ we have that

$$M^{(b+1)}/M^{(b)} \cong S_{\chi^{-b}} \otimes_k H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(D_b)).$$

The number of idecomposable $k[C_m]$-summands of $M^{(b)}/M^{(b-1)}$ isomorphic to $S_{\chi^{-(b-1)(a)}} = S_{a-(b-1)a_0}$ equals to the number of idecomposable $k[C_m]$-summands of $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(D_j))$ isomorphic to S_a which is computed in remark 15. □

In [23, Th. 1.1] A. Obus and R. Pries described the upper jumps in the ramification filtration of $C_{p^h} \rtimes C_m$-covers.

Theorem 19. *Let $G = C_{p^h} \rtimes C_m$, where $p \nmid m$. Let $m' = |\mathrm{Cent}_G(\sigma)|/p^h$, where $\langle \tau \rangle = C_{p^h}$. A sequence $u_1 \leq \cdots u_n$ of rational numbers occurs as the set of positive breaks in the upper numbering of the ramification filtration of a G-Galois extension of $k((t))$ if and only if:*

(1) $u_i \in \frac{1}{m}\mathbb{N}$ *for* $1 \leq i \leq h$
(2) $\gcd(m, mu_1) = m'$
(3) $p \nmid mu_1$ *and for* $1 < i \leq h$, *either* $u_i = pu_{i-1}$ *or both* $u_i > pu_{i-1}$ *and* $p \nmid mu_i$.
(4) $mu_i \equiv mu_1 \bmod m$ *for* $1 \leq i \leq n$.

Notice that in our setting $\mathrm{Cent}_G(\tau) = \langle \tau \rangle$, therefore $m' = 1$. Also the set of upper jumps of C_{p^h} is given by $w_1 = mu_1, \ldots, w_h = mu_h, w_i \in \mathbb{N}$, see [23, lemma 3.5].

The theorem of Hasse-Arf [27, p. 77] applied for cyclic groups, implies that there are strictly positive integers $\iota_0, \iota_1, \ldots, \iota_{h-1}$ such that

$$b_s = \sum_{\nu=0}^{s-1} \iota_\nu p^\nu, \text{ for } 0 \leq s \leq h-1$$

Also, the upper jumps for the C_q extension are given by

$$w_0 = i_0 - 1, w_1 = i_0 + i_1 - 1, \ldots, w_h = i_0 + i_1 + \cdots + u_h - 1. \tag{15}$$

Assume that for all $0 < \nu \leq h-1$ we have $w_\nu = pw_{\nu-1}$. Equation (15) implies that

$$i_1 = (p-1)w_0, i_2 = (p-1)pw_0, i_3 = (p-1)p^2 w_0, \ldots, u_{h-1} = (p-1)p^{h-2}w_0.$$

Therefore,

$$\begin{aligned} b_\ell + 1 = \sum_{\nu=0}^{\ell} i_\nu p^\nu &= 1 + w_0 + (p-1)w_0 \cdot p + (p-1)pw_0 \cdot p^2 + \cdots + (p-1)p^{\ell-1}w_0 \cdot p^\ell \\ &= 1 + u_0 + p(p-1)u_0 \left(\sum_{\nu=0}^{\ell-1} p^{2\nu} \right) = 1 + w_0 + p(p-1)w_0 \frac{p^{2\ell}-1}{p^2-1} \\ &= 1 + w_0 + pw_0 \frac{p^{2\ell}-1}{p+1} = 1 + w_0 \frac{p^{2\ell+1}+1}{p+1} \end{aligned}$$

where we have used that $w_0 = b_0 = i_0 - 1$.

6.1. **Examples of local actions that don't lift.** Consider the curve with lower jumps $1, 21, 521$ and higer jumps $1, 5, 25$, acted on by $C_{125} \rtimes C_4$. According to eq. (5), the only possible values for α are $1, 57, 68, 124$. The value $\alpha = 1$ gives rise to a cyclic group G, while the value $\alpha = 124$ has order 2 modulo 125. The values $57, 68$ have order 4 modulo 125. The cyclic group $\mathbb{F}_5^*$ is generated by the primitive root 2 of order 4. We have that $57 \equiv 2 \bmod 5$, while $68 \equiv 3 \equiv 2^3 \bmod 5$.

Using corollary 18 together with remark 15 we have that $H^0(X, \Omega_X)$ is decomposed into the following indecomposable modules, each one appearing with multiplicity one:

$$\begin{gathered} U_{0,5}, U_{3,11}, U_{2,17}, U_{1,23}, U_{0,29}, U_{3,35}, U_{2,41}, U_{1,47}, U_{0,53}, U_{3,59}, \\ U_{2,65}, U_{1,71}, U_{0,77}, U_{3,83}, U_{2,89}, U_{1,95}, U_{0,101}, U_{3,107}, U_{2,113}, U_{1,119} \end{gathered}$$

We have that $119 \equiv 3 \bmod 4$ so the module $U_{1,119}$ can not be lifted by itself. Also it can't be paired with $U_{0,5}$ since $119 + 5 \equiv 4 \neq 1 \bmod 4$. All other modules have dimension d such that $d + 119 > 125$. Therefore, the representation of $H^0(G, \Omega_X)$ cannot be lifted.

The case of dihedral groups is more difficult to find an example that does not lift. We have the following

The HKB-curve with lower jumps $9, 9 \cdot 21 = 189, 9 \cdot 521 = 4689$ has genus 11656 and the following modules appear in its decomposition, each one appearing with multiplicity one:

$$\begin{aligned} &U_{0,1}, U_{1,1}, U_{0,2}, U_{1,2}, U_{1,3}, U_{0,4}, U_{1,4}, U_{0,5}, U_{1,6}, U_{0,7}, U_{1,7}, U_{0,8}, U_{1,8}, U_{0,9}, U_{1,9}, U_{0,11}, \\ &U_{1,11}, U_{0,12}, U_{1,12}, U_{0,13}, U_{1,13}, U_{0,14}, U_{1,15}, U_{0,16}, U_{0,17}, U_{1,17}, U_{0,18}, U_{1,18}, U_{0,19}, U_{1,19}, \\ &U_{0,21}, U_{1,21}, U_{0,22}, U_{1,22}, U_{0,23}, U_{1,23}, U_{1,24}, U_{0,25}, U_{1,26}, U_{0,27}, U_{1,27}, U_{0,28}, U_{1,28}, U_{0,29}, \\ &U_{1,29}, U_{0,31}, U_{1,31}, U_{0,32}, U_{1,32}, U_{0,33}, U_{0,34}, U_{1,34}, U_{1,35}, U_{0,36}, U_{0,37}, U_{1,37}, U_{0,38}, U_{1,38}, \\ &U_{0,39}, U_{1,39}, U_{0,41}, U_{1,41}, U_{0,42}, U_{1,42}, U_{0,43}, U_{1,43}, U_{1,44}, U_{0,45}, U_{0,46}, U_{1,46}, U_{1,47}, U_{0,48}, \\ &U_{1,48}, U_{0,49}, U_{1,49}, U_{0,51}, U_{1,51}, U_{0,52}, U_{1,52}, U_{0,53}, U_{0,54}, U_{1,54}, U_{1,55}, U_{0,56}, U_{0,57}, U_{1,57}, \\ &U_{0,58}, U_{1,58}, U_{0,59}, U_{1,59}, U_{0,61}, U_{1,61}, U_{0,62}, U_{1,62}, U_{0,63}, U_{1,63}, U_{1,64}, U_{0,65}, U_{0,66}, U_{1,66}, \\ &U_{1,67}, U_{0,68}, U_{1,68}, U_{0,69}, U_{1,69}, U_{0,71}, U_{1,71}, U_{0,72}, U_{1,72}, U_{0,73}, U_{1,73}, U_{0,74}, U_{1,75}, U_{0,76}, \\ &U_{0,77}, U_{1,77}, U_{0,78}, U_{1,78}, U_{0,79}, U_{1,79}, U_{0,81}, U_{1,81}, U_{0,82}, U_{1,82}, U_{0,83}, U_{1,83}, U_{1,84}, U_{0,85}, \\ &U_{1,86}, U_{0,87}, U_{1,87}, U_{0,88}, U_{1,88}, U_{0,89}, U_{1,89}, U_{0,91}, U_{1,91}, U_{0,92}, U_{1,92}, U_{0,93}, U_{1,93}, U_{0,94}, \\ &U_{1,95}, U_{0,96}, U_{1,96}, U_{0,97}, U_{0,98}, U_{1,98}, U_{0,99}, U_{1,99}, U_{0,101}, U_{1,101}, U_{0,102}, U_{1,102}, U_{1,103}, \\ &U_{0,104}, U_{1,104}, U_{0,105}, U_{1,106}, U_{0,107}, U_{1,107}, U_{0,108}, U_{1,108}, U_{0,109}, U_{1,109}, U_{0,111}, U_{1,111}, \\ &U_{0,112}, U_{1,112}, U_{0,113}, U_{1,113}, U_{0,114}, U_{1,115}, U_{0,116}, U_{1,116}, U_{0,117}, U_{0,118}, U_{1,118}, U_{0,119}, \\ &U_{1,119}, U_{0,121}, U_{1,121}, U_{0,122}, U_{1,122}, U_{0,123}, U_{1,123}, U_{1,124}, \end{aligned}$$

The above formulas were computed using Sage 9.8 [28]. In order to be completely sure that are correct we will compute the values we need by hand also. We have

$$\begin{aligned} d_j &= \left\lfloor \frac{1}{125} \left(5^2\big(4 + (4 - a_1)9\big) + 5\big(4 + (4 - a_2)189\big) + \big(4 + (4 - a_3)4689\big) \right) \right\rfloor \\ &= \left\lfloor \frac{1}{125} \left(23560 - 225a_1 - 945a_2 - 4689a_3 \right) \right\rfloor \end{aligned}$$

j	$p-$adic	d_j	$n_{j,0}$	$n_{j,1}$	$n_{j-1,0}-n_{j,0}$	$n_{j-1,1}-n_{j,1}$
0	$0,0,0$	$\lfloor \frac{23560}{125} \rfloor = 188$	93	94	$-$	$-$
1	$1,0,0$	$\lfloor \frac{23335}{125} \rfloor = 186$	92	93	1	1
2	$1,0,0$	$\lfloor \frac{23110}{125} \rfloor = 184$	91	92	1	1
3	$1,0,0$	$\lfloor \frac{22885}{125} \rfloor = 183$	91	91	0	1
4	$1,0,0$	$\lfloor \frac{22660}{125} \rfloor = 181$	90	90	1	1
5	$0,1,0$	$\lfloor \frac{22615}{125} \rfloor = 180$	89	90	1	0
6	$1,1,0$	$\lfloor \frac{22390}{125} \rfloor = 179$	89	89	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
120	$0,4,4$	$\lfloor \frac{1024}{125} \rfloor = 8$	3	4		
121	$1,4,4$	$\lfloor \frac{799}{125} \rfloor = 6$	2	3	1	1
122	$2,4,4$	$\lfloor \frac{574}{125} \rfloor = 4$	1	2	1	1
123	$3,4,4$	$\lfloor \frac{349}{125} \rfloor = 2$	0	1	1	1
124	$4,4,4$	$\lfloor \frac{124}{125} \rfloor = 0$	0	0	0	1

Notice that $U_{1,123}, U_{0,123}$ can be paired with $U_{1,0}, U_{1,1}$, and then for $U_{0,121}, U_{1,121}$ there is only one $U_{1,3}$ to be paired with. The lift is not possible.

6.2. **Examples of actions that lift.** Our aim now is to prove the following

Theorem 20. *Assume that the first lower jump equals $b_0 = w_0 = 1$ and each other lower jump is given by*

$$b_\ell = \frac{p^{2\ell+1}+1}{p+1}.$$

Then, the local action of the dihedral group D_{p^h} lifts.

Remark 21. Notice that in this case if $d_{j-1} > d_j$ then $d_{j-1} = d_j + 1$.

Lemma 22. *Write*

$$\begin{aligned} j-1 &= (p-1) + (p-1)p + \cdots + (p-1)p^{s-1} + a_s p^s + \cdots \\ j &= (a_s+1)p^s + \cdots \end{aligned}$$

where s is the first power in the p-adic expansion of $j-1$ such that the corresponding coefficient $0 \leq a_s < p-1$. Then

$$B(j) - B(j-1) = p^{h-s}.$$

Proof. By definition of the function $B(j)$ we have that

$$\begin{aligned} B(j) - B(j-1) &= b_{s-1}p^{h-s} - (p-1)(b_0 p^{h-1} + \cdots + b_{s-2}p^{h-s+1}) \\ &= \frac{p^{2s-1}+1}{p+1} p^{h-s} - (p-1)\sum_{\nu=1}^{s-1} p^{h-\nu}\frac{p^{2\nu-1}+1}{p+1} \\ &= p^{h-s}. \end{aligned}$$

□

Definition 23. We will call the element j of type s if all p-adic coefficients of j corresponding to p^ν for $\nu \leq s-1$ are $p-1$, while the coefficient corresponding to p^s is not $p-1$. For example $j-1$ in lemma 22 is of type s, while j is of type 1.

Proposition 24. *Write* $\pi_j = \left\lfloor \frac{B(j)}{p^h} \right\rfloor$. *Then,* $\pi_j = \pi_{j-1} + 1$ *if and only if* $j = k(p+1)$. *Also* $p^h \nmid B(j)$ *for all* $1 \leq j \leq p^h - 1$.

Proof. In the following table we present the change on $B(j)$ after increasing $j-1$ to j, where $j-1$ has type s, using lemma 22.

j	$B(j)$	$\left\lfloor \frac{B(j)}{p^h} \right\rfloor$
0	0	0
1	p^{h-1}	0
$a_1 = 2, \ldots, p-1$	$a_1 p^{h-1}$	0
p	$(p-1)p^{h-1} + p^{h-2}$	0
$\boxed{p+1}$	$p^h + p^{h-2}$	$\boxed{1}$
$p+2$	$p^h + p^{h-2} + p^{h-1}$	1
$p + a_1, a_1 = 3, \ldots, p-1$	$p^h + p^{h-2} + (a_1 - 1)p^{h-1}$	1
$2p$	$p^h + 2p^{h-2} + (p-2)p^{h-1}$	1
$2p+1$	$p^h + 2p^{h-2} + (p-1)p^{h-1}$	1
$\boxed{2p+2}$	$2p^h + 2p^{h-2}$	$\boxed{2}$
$2p+3$	$2p^h + 2p^{h-2} + p^{h-1}$	2
$2p + a_1$	$2p^h + 2p^{h-2} + (a_1 - 2)p^{h-1}$	2
$3p$	$2p^h + 3p^{h-2} + (p-3)p^{h-1}$	2
$\cdots$	$\cdots$	$\cdots$
$(p-1)p$	$(p-2)p^h + (p-1)p^{h-2} + p^{h-1}$	$p-2$
$\cdots$	$\cdots$	$\cdots$
$(p-1) + (p-1)p$	$(p-1)p^h + (p-1)p^{h-2}$	$p-1$
p^2	$(p-1)p^h + (p-1)p^{h-2} + p^{h-3}$	$p-1$
$\cdots$	$\cdots$	$\cdots$
$(p-1) + p^2$	$(p-1)p^h + (p-1)p^{h-2} + p^{h-3} + (p-1)p^{h-1}$	$p-1$
$\boxed{p+p^2}$	$p^{h+1} + p^{h-3}$	$\boxed{p}$
$1 + p + p^2$	$p^{h+1} + p^{h-1} + p^{h-3}$	p

Indeed, if the type of $j-1$ is $s = 1$ then $B(j) = B(j-1) + p^{h-1}$ and it is clear that we will get one more p^h at $kp + k$, for $1 \leq k \leq p$. We will prove the result in full generality by induction. Observe that if $j-1$ is of type s, and $\pi_j = \pi_{j-1} + 1$, then $B(j) = B(j-1) + p^{h-s}$ and moreover

$$\begin{aligned} B(j-1) &= (p-1)p^{h-1} + (p-1)p^{h-2} + \cdots + (p-1)p^{h-s} + \pi_{j-1}p^h + u \\ B(j) &= p^h + \pi_{j-1}p^h + u \end{aligned}$$

for some $u = \sum_{\nu=0}^{h-2} \gamma_\nu p^\nu$, $0 \leq \gamma_\nu < p$. Set $T = \pi_{j-1}p^h + u$. Assume by induction that this jump occurs at $j = k(p+1)$. Then the next jump will occur at $j =$

$k(p+1)+(p+1)$, since

$$\begin{aligned}
B(j+1) &= B(j)+p^{h-1}+T\\
B(j+2) &= B(j)+2p^{h-1}+T\\
&\cdots\\
B(j+(p-1)) &= B(j)+(p-1)p^{h-1}+T\\
B(j+p) &= B(j)+(p-1)p^{h-1}+p^{h-2}+T\\
B(j+p+1) &= B(j)+p^{h}+T+p^{h-2}.
\end{aligned}$$

□

Theorem 25. *Assume that $w_0=1$, and the jumps of the C_q action are as in theorem 20. Then each direct summand $U(\epsilon,j)$ of $H^0(X,\Omega_X)$ has a compatible pair according to criterion 4, which is given by*

$$\begin{aligned}
&U(\epsilon', p^h-1-j) \text{ if } h \text{ is odd}\\
&U(\epsilon', p^h-p-j) \text{ if } h \text{ is even}
\end{aligned}$$

Proof. For every $1\le j\le p^h-1$, set $\tilde{j}=p^h-1-j$. For every $1\le j\le p^h-1$ write $B(j)=\pi_j p^h+v_j$, $0\le v_j<p^h$. Recall that

$$d_j=\left\lfloor \frac{p^h-1+B(p^h-1)-B(j)}{p^h}\right\rfloor = \left\lfloor \frac{p^h-1+B(\tilde{j})}{p^h}\right\rfloor = 1+\pi_{\tilde{j}}+\left\lfloor \frac{-1+v_j}{p^h}\right\rfloor.$$

Since $v_j\neq 0$, we have that $\left\lfloor \frac{-1+v_j}{p^h}\right\rfloor=0$. Therefore, $d_{j-1}>d_j$ if and only if $\pi_{\tilde{j}+1}<\pi_{\tilde{j}}$ that is

$$\tilde{j}+1=k(p+1)\Rightarrow \tilde{j}=k(p+1)-1.$$

Observe now that if h is odd and $d_{j-1}=d_j+1$, that is $\tilde{j}=k(p+1)-1$. Then

$$j=p^h-1-\tilde{j}=p^h-k(p+1).$$

- If h is odd, then $\tilde{j}=p^h-(1+p^h)+k(p+1)=p^h-k'(p+1)$ for some integer $k'=\frac{p^h+1}{p+1}-k$, since in this case $p+1\mid p^h+1$. Since $\tilde{\tilde{j}}=j$ we can assume that $j<\tilde{j}$. Then $d_j-d_{\tilde{j}}$ is the number of jumps between $d_j,d_{\tilde{j}}$, that is the number of elements $x=p^h-l_x(p+1)\in\mathbb{N}$ of the form

$$j=p^h-k(p+1)<p^h-l_x(p+1)\le p^h-k'(p+1)$$

  that is $k'\le l_x<k$. This number equals $k-k'=2k-\frac{p^h+1}{p+1}$, which is odd since $\frac{p^h+1}{p+1}\sum_{\nu=0}^{h-1}(-p)^\nu$ is odd.
- If h is even, then $j'=p^h-p-j=p^h-(p+p^h)+k(p+1)=p^h-k'(p+1)$ for some integer k', since in this case $p+1\mid p^h+p$. Again since $j''=j$ we can assume that $j<j'$. Again $d_j-d_{j'}$ is the number of jumps between $d_j,d_{j'}$, which equals to $2k-\frac{p^h+p}{p+1}=p\frac{p^{h-1}+1}{p+1}$, which is odd.

Observe that we have proved in both cases that d_j is odd if and only if $d_{\tilde{j}}$ (resp. d'_j) is even. The change of ϵ to ϵ' follows by lemma 16. □

References

[1] J. L. Alperin. *Local representation theory*, volume 11 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Modular representations as an introduction to the local representation theory of finite groups.

[2] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[3] José Bertin. Obstructions locales au relèvement de revêtements galoisiens de courbes lisses. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(1):55–58, 1998.

[4] Frauke M. Bleher, Ted Chinburg, and Aristides Kontogeorgis. Galois structure of the holomorphic differentials of curves. *J. Number Theory*, 216:1–68, 2020.

[5] Nicolas Bourbaki. *Commutative Algebra. Chapters 1–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1989. Translated from the French, Reprint of the 1972 edition.

[6] Irene I. Bouw and Stefan Wewers. The local lifting problem for dihedral groups. *Duke Math. J.*, 134(3):421–452, 2006.

[7] Rolf Brandt and Henning Stichtenoth. Die Automorphismengruppen hyperelliptischer Kurven. *Manuscripta Math.*, 55(1):83–92, 1986.

[8] T. Chinburg, R. Guralnick, and D. Harbater. Oort groups and lifting problems. *Compos. Math.*, 144(4):849–866, 2008.

[9] Ted Chinburg, Robert Guralnick, and David Harbater. The local lifting problem for actions of finite groups on curves. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(4):537–605, 2011.

[10] Ted Chinburg, Robert Guralnick, and David Harbater. Global Oort groups. *J. Algebra*, 473:374–396, 2017.

[11] Huy Dang, Soumyadip Das, Kostas Karagiannis, Andrew Obus, and Vaidehee Thatte. Local oort groups and the isolated differential data criterion, 2019.

[12] Hershel M. Farkas and Irwin Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1980.

[13] Barry Green and Michel Matignon. Liftings of Galois covers of smooth curves. *Compositio Math.*, 113(3):237–272, 1998.

[14] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[15] Kazuya Kato. Vanishing cycles, ramification of valuations, and class field theory. *Duke Math. J.*, 55(3):629–659, 1987.

[16] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliptic curves*. Princeton University Press, Princeton, NJ, 1985.

[17] Aristides Kontogeorgis and Alexios Terezakis. The canonical ideal and the deformation theory of curves with automorphisms, 2021.

[18] Aristides Kontogeorgis and Alexios Terezakis. On the lifting problem of representations of a metacyclic group, 2023.

[19] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[20] Andrew Obus. The (local) lifting problem for curves. In *Galois-Teichmüller theory and arithmetic geometry*, volume 63 of *Adv. Stud. Pure Math.*, pages 359–412. Math. Soc. Japan, Tokyo, 2012.

[21] Andrew Obus. The local lifting problem for A_4. *Algebra Number Theory*, 10(8):1683–1693, 2016.

[22] Andrew Obus. A generalization of the Oort conjecture. *Comment. Math. Helv.*, 92(3):551–620, 2017.

[23] Andrew Obus and Rachel Pries. Wild tame-by-cyclic extensions. *J. Pure Appl. Algebra*, 214(5):565–573, 2010.

[24] Andrew Obus and Stefan Wewers. Cyclic extensions and the local lifting problem. *Ann. of Math. (2)*, 180(1):233–284, 2014.

[25] Guillaume Pagot. *Relèvement en caractéristique zéro d'actions de groupes abéliens de type* $(p, \ldots, p)$. PhD thesis, Bordeaux Univeristy, 2002.

[26] Florian Pop. The Oort conjecture on lifting covers of curves. *Ann. of Math. (2)*, 180(1):285–322, 2014.

[27] Jean-Pierre Serre. *Local fields*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.

[28] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.8)*, 2023. `https://www.sagemath.org`.

[29] Robert C. Valentini and Manohar L. Madan. A hauptsatz of L. E. Dickson and Artin-Schreier extensions. *J. Reine Angew. Math.*, 318:156–177, 1980.

[30] Bradley Weaver. The local lifting problem for D_4. *Israel J. Math.*, 228(2):587–626, 2018.

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