Chapter 16
Automorphisms of Curves

Jannis A. Antoniadis and Aristides Kontogeorgis

Abstract This is a survey article concerning the groups of automorphisms of curves defined over algebraically closed fields of positive characteristic, their representations and applications to their deformation theory.

16.1 Introduction

By an (algebraic) curve we will mean a projective non-singular one-dimensional variety, defined over an algebraically closed field k of characteristic $p \geq 0$. Over the field $\mathbb{C}$ of complex numbers the notion of a projective algebraic curve coincides with the notion of compact Riemann surface. Every compact Riemann surface X is known to be an orientable two-dimensional real manifold and to any such surface we can attach a natural number $g_X \in \mathbb{N} \cup \{0\}$, called the genus, which topologically counts the number of holes of the surface X. Over an arbitrary field of positive characteristic we can still define the genus, by setting g_X to be the dimension of the space of global holomorphic differentials $H^0(X, \Omega_X)$, although a topological interpretation is less clear, see [72]. In Sect. 16.2.2 a topological interpretation can be given as the number of cycles of the graph of analytic reduction.

An automorphism of a curve X is an isomorphism $\sigma : X \to X$, and the set $\mathrm{Aut}_k(X)$ of all automorphisms form a group under composition. Since we assumed that the constant field is algebraically closed we will omit the index k from the notation, and we will denote the automorphism group by $\mathrm{Aut}(X)$.

If the genus is zero, then X is isomorphic to the projective line $\mathbb{P}^1$ and the automorphism group is the group of Möbius transformations $\mathrm{PGL}(2, k)$, which is infinite. If

J.A. Antoniadis
Department of Mathematics & Applied Mathematics, University of Crete,
Voutes Campus, 70013 Heraklion, Greece
e-mail: antoniad@math.uoc.gr

A. Kontogeorgis (✉)
Department of Mathematics, National and Kapodistrian University of Athens,
Panepistimioupolis, 15784 Athens, Greece
e-mail: kontogar@math.uoa.gr

© Springer International Publishing AG 2017

S. Lambropoulou et al. (eds.), *Algebraic Modeling of Topological and Computational Structures and Applications*, Springer Proceedings in Mathematics & Statistics 219, https://doi.org/10.1007/978-3-319-68103-0_16

$g_X = 1$, then the curve X admits a group structure and X acts on X by translations, hence $X \subset \mathrm{Aut}(X)$, so X is infinite as well.

When $g_X \geq 2$ the automorphism group is finite. In characteristic zero this can be proved by applying the Riemann–Roch theorem, in order to see that every automorphism fixing more than $2g_X + 2$ points is the identity, and then obtaining a faithful representation on the set of Weierstrass points. For the case of positive characteristic, the same argument does not work. In order to prove that the automorphism group is finite, we first prove that the decomposition group $G(P)$ is finite, for every $P \in X$. Then we need the existence of a finite non-empty $\mathrm{Aut}(X)$-invariant set Σ. We can use as Σ the set of Weierstrass points, since they are invariant under the action of the automorphism group. A notion of Weierstrass points in positive characteristic was given first by F.K. Schmidt [75] using the theory of Hasse derivatives. For a modern account of this topic we refer to [26, Sect. 6], [29, Chap. 11].

Moreover, it is known that for any finite group G there exists a curve X such that $\mathrm{Aut}(X) \cong G$, see [52]. Notice that most of the curves have trivial automorphism group [57, 67], since curves with non-trivial automorphisms correspond to singularities of the moduli space of curves of fixed genus. However, finding specific examples of curves without automorphisms is not easy, see [65].

Understanding the automorphism group is an interesting problem on its own and has many applications to counting points, moduli problems, etc. In Sect. 16.2 we will present results concerning the order of the group and we will give upper bounds in terms of constants of topological nature, like the genus and the p-rank of the Jacobian.

In Sect. 16.5 we will study automorphisms of relative curves $\pi : \mathscr{X} \to \mathrm{Spec} R$, where R is a discrete valued ring with algebraically closed residue field. We will restrict ourselves to maps π which have fibers X of genus $g_X \geq 2$ and which vary "nicely". Both these properties are formulated by the notion of "*stable curve*". The precise definition is given in [14, Definition I.1]. If the relative curve is stable, then the automorphism group of the special fibre contain the automorphism group of the generic fibre [14, Lemma I.12]. The study of automorphisms of relative curves is a difficult problem even at the infinitesimal level. In this section we also discuss reduction, lifting and the deformation problem. Automorphisms of relative curves are related to the representation theory of the automorphism group on several natural objects of the curve like global sections of global polydifferentials and this will be explained in Sect. 16.3. In Sect. 16.6 we study integral representations of a fibrewise action in relative holomorphic polydifferentials.

The theory of automorphisms of curves is a vast object of study and this article does not have the ambition to describe it completely. It is rather focused on subjects closer to the research interests of the authors. For more general information about automorphisms of curves in characteristic zero we refer to [15, Chap. V], while for curves defined over fields of positive characteristic we refer to [29, Chap. 11], [13, par. 14.3].

16.2 Size of the Group

The automorphism group of a curve X of genus $g_X \geq 2$ is finite. However if the genus is fixed, then the size of the groups that can appear is bounded. In characteristic zero, for $g_X \geq 2$, using the theory of Riemann–Hurwitz formula and by a case by case examination, Hurwitz [31] proved the bound:

$$|\mathrm{Aut}(X)| \leq 84(g_X - 1). \tag{16.1}$$

16.2.1 Ramification Filtration

In order to explain the situation in characteristic $p > 0$ we have to introduce the ramification filtration at a closed point $x \in X$. Let G be a subgroup of $\mathrm{Aut}(X)$ and let $m_{X,x}$ be the maximal ideal of the local ring $\mathscr{O}_{X,x}$. We will denote by $k(x)$ the residue field of x. For $i \geq 0$, the ith lower ramification subgroup $G_{x,i}$ of G at x is the subgroup of all elements $\sigma \in G$ which fix x and which act trivially on $\mathscr{O}_{X,x}/m_{X,x}^{i+1}$. These groups form a decreasing finite sequence

$$G_{x,0} \rhd G_{x,1} \rhd \cdots \rhd G_{x,n} \rhd G_{x,n+1} = \{1\}, \qquad n \in \mathbb{N}. \tag{16.2}$$

When the characteristic $p = 0$ it is known that $G_{x,1} = \{1\}$. In general $G_{x,0}/G_{x,1}$ is a cyclic group of order prime to the characteristic, while for $i \geq 1$ $G_{x,i}/G_{x,i+1}$ is an elementary abelian group, i.e. isomorphic to the direct sum of finitely many cyclic groups of order p. If $G_{x,1} = \{1\}$ for every $x \in X$, then the cover $X \to X/G$ is said to be *tame*, otherwise it is called *wild*. If $G_{x,2} = \{1\}$, then the ramification is called *weak*.

The Riemann–Hurwitz formula [82, Sect. 3.4 p. 90] relates the genera of the curves X and $X/G = Y$ as follows:

$$2(g_X - 1) = 2(g_Y - 1)|G| + \sum_{x \in X} \sum_{i=1}^{\infty} \left(|G_{x,i}| - 1\right). \tag{16.3}$$

Notice that this equation can be obtained by taking degrees on Eq. (16.11).

For tame covers the Hurwitz bound remains the same. For the general wildly ramified curve H. Stichtenoth [80, 81] proved that the following bound holds:

$$|\mathrm{Aut}(X)| \leq 16 g_X^4, \tag{16.4}$$

with the Hermitian Fermat curve as only exception, given by the equation:

$$0 = x^{p^h+1} + y^{p^h+1} + z^{p^h+1} = \underline{x}\left(\underline{x}^t\right)^{p^h}, \text{ where we have set } \underline{x} = (x, y, z). \tag{16.5}$$

Notice that for $h = 0$ the above Fermat curve is a quadratic form, while for $h > 0$ it behaves as Frobenious shifted quadratic form and has $\mathrm{PGU}(3, p^{2h})$ as automorphism group, [43, 51]. The result of H. Stichtenoth was improved by H. Henn [27] who proved that

$$|\mathrm{Aut}(X)| \leq 8g_X^3, \tag{16.6}$$

with a finite list of exceptions. The result of Henn contained a gap which was filled by M. Giulietti and Gábor Korchmáros, see [20].

All exceptions in the list of Henn, have a large p-subgroup compared to its genus. C. Lehr and M. Matignon [50] defined the notion of "big action", when the $\mathrm{Aut}(X)$ contains a p-subgroup P, such that

$$|P| > \frac{2p}{p-1} g_X. \tag{16.7}$$

M. Matignon and M. Rocher [55, 70, 71], and M. Giulietti and G. Korchmáros [19] studied and classified "big actions" defined by an equation similar to Eq. (16.7).

In characteristic $p > 0$ the p-rank of the Jacobian γ_X plays a role analogous to the rank of the homology group and as a matter of fact $0 \leq \gamma_X \leq g_X$. Curves with $g_X = \gamma_X$ are called ordinary and they form a Zariski-dense set in the moduli space of curves of fixed genus. For such curves S. Nakajima [62] proved the bound:

$$|\mathrm{Aut}(X)| \leq 84(g_X - 1)g_X. \tag{16.8}$$

He further notices that his bound could not be best possible and by studying the Artin–Schreier–Mumford curve

$$(x^{p^h} - x)(y^{p^h} - y) = c, \qquad c \in k \tag{16.9}$$

he conjectured that the best possible bound is given by a cubic polynomial in $\sqrt{g_X}$.

16.2.2 Mumford Curves

It is well known that an algebraic curve X, defined over $\mathbb{C}$ can be uniformized by a discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$, i.e. $X \cong \Gamma \backslash \mathbb{H}$, and the Hurwitz upper bound given in Eq. (16.1) is equivalent to Siegel's lower bound $\pi/21$ on the volume of the fundamental domain of a Fuchsian group ([15, Exercise 6 p. 245], [49]).

Let K be a non-archimedean valued field. D. Mumford [59] showed that curves defined over K, whose stable reduction is split multiplicative, i.e. a union of rational curves intersecting at $\bar{K}$-rational points, are isomorphic to an analytic space of the form $\Gamma \backslash (\mathbb{P}^1_K - \mathscr{L}_\Gamma)$, where Γ is a discontinuous group in $\mathrm{PGL}(2, K)$ and $\mathscr{L}_\Gamma$ is the set of limit points. The automorphism group of the curve X is then isomorphic to the group N/Γ, where N equals the normalizer of $\Gamma \in \mathrm{PGL}(2, K)$, [12], [18, p. 216].

We will call such curves *Mumford curves*. Notice that not all curves defined over K admit such a uniformisation. For example the Artin–Schreier Mumford curve has split multiplicative reduction and is a Mumford curve only if $|c| < 1$. The uniformization theory can give stronger results when applied to Mumford curves.

Herrlich [28] has shown that for p-adic Mumford curves of genus $g_X \geq 2$ and $p \geq 7$ the Hurwitz bound can be strengthened to $12(g_X - 1)$.

Notice that by the work of Manin-Drinfeld [53] and Gerritzen [17], Mumford curves are known to be ordinary, therefore the Nakajima bound given in Eq. (16.8) holds. For Mumford curves defined over non-archimedean fields of positive characteristic G. Cornelissen, F. Kato and the second author [12], proved that Nakajima's conjecture was correct for Mumford curves and the following bound holds:

$$|\mathrm{Aut}(X)| \leq \max\left\{12(g_X - 1), 2\sqrt{g_X}(\sqrt{g_X} + 1)^2\right\}. \tag{16.10}$$

They also classified those curves for which $|\mathrm{Aut}(X)| \geq 12(g_X - 1)$. Moreover, the above bound is best possible since it is attained for the Artin–Schreier–Mumford curves given by Eq. (16.9).

This theorem can also be reformulated in the style of Siegel lower bound as follows: the $\mu(N)$ invariant [37, Eq. 2] of its normalizer N is bounded from below by

$$\mu(N) \geq \min\left\{1/12, \frac{\sqrt{g_X} - 1}{2\sqrt{g_X}(\sqrt{g_X} + 1)}\right\}.$$

Notice that $\mu(N)$ plays the role of a Gauss–Bonnet "volume" and the index $[N : \Gamma]$ which equals the order of automorphism group can be evaluated in terms of theorems of HNN groups as in Theorem 2 in [37].

Concerning the Nakajima conjecture for ordinary curves X over a field of characteristic $p > 0$, R. Guralnik and M. Zieve in a Workshop in Leiden on Automorphisms of curves in 2004, announced that there exists a sharp bound of the order of $g_X^{8/5}$ for $|\mathrm{Aut}(X)|$.

For automorphisms groups of Mumford curves with a specific structure we can have better bounds. For example S. Nakajima in [61] used the Hasse-Arf theorem in order to prove that

$$|\mathrm{Aut}(X)| \leq 4g_X + 4,$$

and this bound has been further improved for abelian automorphisms groups of Mumford curves by V. Rotger and the second author in [47], to the bound

$$|\mathrm{Aut}(X)| \leq 4(g_X - 1).$$

16.3 Representation Theory

The next step is to understand representations of $\mathrm{Aut}(X)$ in some naturally defined vector spaces. Let Ω_X denote the sheaf of relative differentials of X over k and by $H^0(X, \Omega_X^{\otimes m})$ the space of global holomorphic polydifferentials of X. The automorphism group acts on both Ω_X and $H^0(X, \Omega_X^{\otimes m})$, therefore $H^0(X, \Omega_X^{\otimes m})$ becomes a $k[G]$-module of k-dimension equal to $(2m-1)(g-1)$ if $m \neq 1$ or g if $m = 1$. By the work of B. Köck and J. Tait [41] we know that this action is faithful, unless $\mathrm{Aut}(X)$ contains a hyperelliptic involution and either $m = 1$ and $p = 2$ or $m = 2$ and $g_X = 2$.

It is a classical problem proposed first by Hecke [25], to analyse the $k[G]$-module structure of $H^0(X, \Omega_X^{\otimes m})$, i.e. analyse the indecomposable components together with their multiplicities. If the characteristic does not divide $|G|$, this problem was solved by Chevalley and Weil [9].

If the ramification of $X \to X/G$ is tame, then Nakajima [60, Theorem 2] and, independently, Kani [32, Theorem 3] determined the $k[G]$-module structure of $H^0(X, \Omega_X)$. B. Köck in [40] studied weakly ramified covers, he generalized Kani's and Nakajima's work and corrected a criterion for the projectivity of the space of holomorphic differentials given by Kani, see remark 2.4b.

K. Ward in [85] studied the Galois module structure of holomorphic differentials for the cyclotomic function fields obtained by the torsion points of Carlitz modules C_M for a totally split polynomial $M \in \mathbb{F}_q(T)$.

The case when G is a cyclic group was first studied by Valentini and Madan [84, Theorem 1] who considered cyclic p-groups (and also revisited cyclic groups of order prime to the characteristic, [84, Theorem 2]). The case of a general cyclic G was treated by S. Karanikolopoulos and the second author [35, Theorem 7]. A different, general approach to determining the decomposition of global sections of coherent $\mathscr{O}_X - G$-modules into decomposable direct summands was developed by Borne in [7], using the notion of rings with several objects. Some formulas concerning the case of cyclic groups and curves are given in [7, Sect. 7.2].

The situation in positive characteristic is more difficult, because phenomena of modular representation theory appear; for example, the notion of irreducible representation is different than the notion of indecomposable representation. Moreover wild ramification appears: the decomposition groups are not cyclic groups and higher ramification groups appear, see Eq. (16.2). Also the classification of non-cyclic p-groups even for the simplest group $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, for a prime $p > 2$ is considered to be impossible [3, p.13 Sect. 1.2].

For each closed point $x \in X$, let $m_{X,x}$ be the maximal ideal of the local ring $\mathscr{O}_{X,x}$ and let $k(x)$ be the residue field of x. The fundamental character of the inertia group $G_{x,0}$ of x is the character $\theta_x : G_{x,0} \to k(x)^* = \mathrm{Aut}(m_{X,x}/m_{X,x}^2)$ giving the action of $G_{x,0}$ on the cotangent space of x. Here θ_x factors through the maximal p'-quotient $G_{x,0}/G_{x,1}$ of $G_{x,1}$.

In [5] F. Bleher, T. Chinburg and the second author studied the structure of $H^0(X, \Omega_X)$, when G is any group such that the p-Sylow subgroup of G is cyclic. It turns out that the $k[G]$-module structure depends only on the ramification data and the fundamental characters of closed points of X, ramified under the action of G.

16.4 Bases of Holomorphic Differentials

16.4.1 Boseck Theory

A strategy for studying the $k[G]$-module structure is to first write explicit bases of the spaces $H^0(X, \Omega_X^{\otimes m})$. Usually, a curve with a non-trivial automorphism group comes with a natural Galois cover $\pi : X \to X/H = Y$, where Y is a known curve (usually $\mathbb{P}^1$ or an elliptic curve) and H is a subgroup of the full automorphism group G. In this way the divisor of the H-invariant differential $\mathrm{div}\pi^* dx$ can be computed in terms of the pullback formula [24, prop. 2.3 p. 301]

$$\mathrm{div}\pi^* dx \cong \pi^* \mathrm{div}(dx) + R_{X/Y}, \tag{16.11}$$

where $R_{X/Y}$ denotes the ramification divisor of the cover.

Once the divisor $\mathrm{div}(dx)$ is computed, finding the space of holomorphic (poly) differentials is the same as computing the Riemann–Roch space $L(\mathrm{div} dx)$. This method was used by H. Boseck in [8], who gave precise formulas for both Kummer and Artin–Schreier extensions of the projective line. Once a basis is constructed, one has to identify the indecomposable summands. For the case of cyclic group action the last computation essentially is equivalent to the computation of the Jordan normal form. Notice that Boseck's article has an error concerning the computation of Weierstrass points, see the article of A. Garcia [16] for more details.

This method was used in [35, 84] and also by articles of Rzedowski–Calderón Villa-Salvador and Madan [73] and Marques and Ward [54] for some other groups under additional hypotheses on the cover $X \to X/G$.

16.4.2 Mumford Curves

For the case of Mumford curves there is a pure group theoretic approach to the determination of global sections of holomorphic differentials initiated by the work of V. Drinfeld and Y. Manin [53]. For holomorphic polydifferentials there is also a group theoretic approach, the theory of harmonic measures studied by J. Teitelbaum and P. Schneider see [76, 83].

Let K be a complete non-archimedean valued field, $\Gamma \subseteq \mathrm{PGL}(2, K)$ be a Schottky subgroup, and X_Γ the Mumford curve obtained from Γ. We will denote by N the normalizer of Γ in $\mathrm{PGL}(2, K)$. The quotient group $G = N/\Gamma$, which acts on X_Γ from the left, is the automorphism group $\mathrm{Aut}(X_\Gamma)$ of X_Γ over K. Recall that Γ is a free group of finite rank, whose rank, say g, is equal to the genus of X_Γ. Let us fix a free generating set $\{\gamma_1, \ldots, \gamma_g\}$ of Γ. For any right $K[\Gamma]$-module P, each derivation $d\colon \Gamma \to P$ is uniquely determined by its values $h_i = d(\gamma_i)$ for $1 \le i \le g$, and conversely, since Γ is free, such values $h_i \in P$ can be freely chosen to obtain a derivation d; indeed, once h_i's are chosen, then $d(w)$ for any $w \in \Gamma$ is uniquely determined by the derivation rules.

For a positive integer n, we consider the $2n-1$ dimensional vector space of polynomials $P_n \subseteq K[T]$ of degree $\le 2(n-1)$. The group $\mathrm{PGL}(2, K)$ acts on P_n from the right as follows: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2, K)$ and $F \in P_n$, we define

$$F^\gamma(T) := \frac{(cT+d)^{2(n-1)}}{(ad-bc)^{n-1}} F\left(\frac{aT+b}{cT+d}\right) \in K[T]. \tag{16.12}$$

Now, consider the $(2n-1)g_{X_\Gamma}$-dimensional space $\mathrm{Der}(\Gamma, P_n)$ of derivations, which can be seen as an N-module as follows: for $\delta \in N$ and $d \in \mathrm{Der}(\Gamma, P_n)$, define

$$(d^\delta)(\gamma) = [d(\delta\gamma\delta^{-1})]^\delta \tag{16.13}$$

for $\gamma \in \Gamma$. There is then a well defined $G = N/\Gamma$ action on the group cohomology $H^1(\Gamma, P_n)$, since Γ acts trivially modulo principal derivations.

Theorem 16.1 ([83, Theorem 1]) *For any $n \ge 1$, the space $H^0(X_\Gamma, \Omega_{X_\Gamma}^{\otimes n})$ of n-differentials on the curve X_Γ is naturally isomorphic to the space group cohomology $H^1(\Gamma, P_n)$. Moreover, this identification is G-equivariant with respect to the natural right G-action on $H^0(X_\Gamma, \Omega_{X_\Gamma}^{\otimes n})$.* □

F. Kato and the second author [38] used this approach to study the $K[G]$-module structure of polydifferentials for the case of Artin–Schreier–Mumford curves, where $N = A * B$, $\Gamma = [A, B]$ and $A, B \subset \mathrm{PGL}(2, K)$ are cyclic groups of order p generated by

$$\varepsilon_A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \varepsilon_B = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, \tag{16.14}$$

respectively, where $s \in K^\times$ and $|s| > 1$.

16.5 Curves in Families

16.5.1 Stable Curves

Let $\mathscr{X} \to R$ be a family of curves of genus $g \geq 2$ over a base scheme $S := \mathrm{Spec}R$, where R is a discrete valuation ring with algebraically closed residue field. For every point $P : \mathrm{Spec}k \to S$, we will consider the *absolute* automorphism group of the fibre P to be the automorphism group $\mathrm{Aut}_{\bar{k}}(\mathscr{X} \times_S \mathrm{Spec}\bar{k})$ where $\bar{k}$ is the algebraic closure of k. Any automorphism σ acts like the identity on $\bar{k}$ so in our setting there is no $\mathrm{Gal}(\bar{k}/k)$ contribution to the automorphism group of any special fibre. The following theorem due to P. Deligne and D. Mumford [14, Lemma I.12] compares the automorphism groups of the generic and special fibres:

Theorem 16.2 *Consider a stable curve $\mathscr{X} \to S$ and let $\mathscr{X}_\eta$ denote its generic fibre. Every automorphism $\phi : \mathscr{X}_\eta \to \mathscr{X}_\eta$ can be extended to an automorphism $\phi : \mathscr{X} \to \mathscr{X}$.*

The example of the Fermat curve given in Eq. (16.5), shows that the automorphism group of the special fibre can be strictly bigger. A special fibre $\mathscr{X}_p := \mathscr{X} \times_S S/p$ with $\mathrm{Aut}(\mathscr{X}_p) > \mathrm{Aut}(\mathscr{X}_\eta)$ will be called *exceptional*. In general we know that there are finite many exceptional fibres and it is an interesting problem to determine exactly all of them.

There are some results towards this problem for some curves of arithmetic interest. A. Adler [1] and C.S. Rajan [68] proved for the modular curves $X(N)$, that $X(11)_3 := X(11) \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}\mathbb{F}_3$ has the Mathieu group M_{11} as the full automorphism group. C. Ritzenthaler in [69] and P. Bending, A. Carmina, R. Guralnick in [2] studied the automorphism groups of the reductions $X(q)_p$ of modular curves $X(q)$ for various primes p. It turns out that the reduction $X(7)_3$ of $X(7)$ at the prime 3 has an automorphism group $\mathrm{PGU}(3, 3)$, and $X(7)_3$ and $X(11)_3$ are the only cases where $\mathrm{Aut}X(q)_p > \mathrm{Aut}X(q) \cong \mathrm{PSL}(2, p)$. Also Y. Yang together with the second author in [48] studied special fibers of hyperelliptic modular curves.

In this spirit, a particular interesting problem is the lifting of automorphisms in characteristic zero: Let X be a curve defined over a field of characteristic p and a group $G \subset \mathrm{Aut}(X)$. Is there a smooth family $\mathscr{X} \to \mathrm{Spec}S$, where S is a local ring with closed point k and generic point a field of characteristic zero, such that G acts fibrewise on the family and the special fibre is the initial curve X?

These types of lifting problems where initiated by J.P. Serre in [79] in his attempt (before étale cohomology was invented) to define an appropriate cohomology theory, which could solve the Weil conjectures.

The answer is no for general G. For example in zero characteristic the Hurwitz bound holds, while in positive characteristic there are known examples of automorphism groups that exceed this bound. Frans Oort in 1987 conjectured that such a lift always exists if the group G is cyclic. This was known in the literature as the Oort Conjecture until recently. Florian Pop proved in [66] that this conjecture is true in a stronger sense: in the case where G has only cyclic groups as inertia groups. We must mention that Pop's proof is based on recent results by Obus and Wewers [64]. For a survey article and for a complete list of the protagonists for this effort see [63] and the historical note in [66].

16.5.2 Deformations of Curves

We will now explain infinitesimal deformation problems from the viewpoint of M. Schlessinger [74]. A deformation of the curve X is a relative curve $\mathscr{X} \to \mathrm{Spec}(R)$ (proper, smooth) over a local ring R with maximal ideal m and $R/m \cong k$, such that $X \cong \mathscr{X} \times_{\mathrm{Spec} R} \mathrm{Spec} R/m$, i.e. we have the following commutative diagram:

$$\begin{tikzcd} X \cong \mathscr{X} \times_{\mathrm{Spec} R} \mathrm{Spec} R/m \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ \mathrm{Spec}(k) \cong R/m \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

Two deformations $\mathscr{X}_1$ $\mathscr{X}_2$ are considered to be equivalent if there is an isomorphism $\psi : \mathscr{X}_1 \to \mathscr{X}_2$ making the diagram

$$\begin{tikzcd} \mathscr{X}_1 \arrow[rr, "\psi"] \arrow[dr] & & \mathscr{X}_2 \arrow[dl] \\ & \mathrm{Spec} R & \end{tikzcd}$$

commutative, such that ψ gives the identity on the special fibres.

Definition 16.1 We consider a deformation functor from the category $\mathscr{C}$ of local Artin algebras R with $R/m_R \cong k$, to the category of sets:

$$D : \mathscr{C} \to \text{Sets},$$

$$R \mapsto \left\{ \begin{array}{l} \text{Equivalence classes of} \\ \text{deformations of } X \text{ over } R \end{array} \right\}.$$

We define the tangent space to the deformation functor to be $D(k[\varepsilon]/\langle \varepsilon^2 \rangle)$.

The space $D(k[\varepsilon]/\langle \varepsilon^2 \rangle)$ is known to be a vector space [74] and by Chech theory and affine triviality we can show [23, p. 89] that:

$$D(k[\varepsilon]/\langle \varepsilon^2 \rangle) = H^1(X, T_X), \tag{16.15}$$

where $T_X \cong \Omega_X^*$ is the tangent sheaf of the curve X.

We now fix a pair (X, G) of curves together with a subgroup G of the automorphism group. A deformation of (X, G) over the local ring R is a deformation of the curve X over R together with a group isomorphism $G \to \mathrm{Aut}_R(\mathscr{X})$, such that there is a G-equivariant isomorphism ϕ from the fibre over the closed point of A to the original curve X:

$$\phi : \mathscr{X} \otimes_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \to X.$$

The notion of equivalence of (X, G) deformations is similar to the non equivariant case, but we now assume that ψ is also G-equivariant. A deformation functor is then defined:

$$D_{(X,G)} : \mathscr{C} \to \mathrm{Sets},$$

$$R \mapsto \left\{ \begin{array}{l} \text{Equivalence classes} \\ \text{of deformations of} \\ \text{couples } (X, G) \text{ over } R \end{array} \right\}$$

J. Bertin and A. Mézard [4] proved that there is an equivariant analogon of Eq. (16.15)

$$T := D_{(X,G)}(k[\varepsilon]/\langle \varepsilon^2 \rangle) \cong H^1(G, X, T_X),$$

where $H^1(G, X, T_X)$ is Grothendiecks's equivariant cohomology as defined in [21]. Cohomology theories appear as derived functors of appropriate left exact functors. For example for group cohomology we apply the functor of invariant elements of G-modules, and for Zarisky cohomology the functor of global sections. Grothendieck's equivariant cohomology [21] appears naturally when we consider the composition of two left exact functors. In this setting we consider both the functor of global sections and the functor of group invariants.

Geometrically the space $H^1(X, T_X)$ can be interpreted as the tangent space to the moduli space of curves of genus g_X, computed at the point-curve X. It consists of equivalence classes of infinitesimal deformations of the curve X. Similarly the space $H^1(G, X, T_X)$ can be interpreted as the subspace of $H^1(X, T_X)$ consisted of G-invariant elements, which give rise to infinitesimal deformations acted on by G (Fig. 16.1).

16.5.3 Dimension of the Tangent Space to the Deformation Functor

The study of the space $D_{(X,G)}(k[\varepsilon])$ can be reduced to the short exact sequence [4]:

$$0 \to H^1\left(X/G, \pi_*^G(T_X)\right) \to H^1(G, X, T_X) \to H^0\left(X/G, R^1\pi_*^G(T_X)\right) \to 0.$$

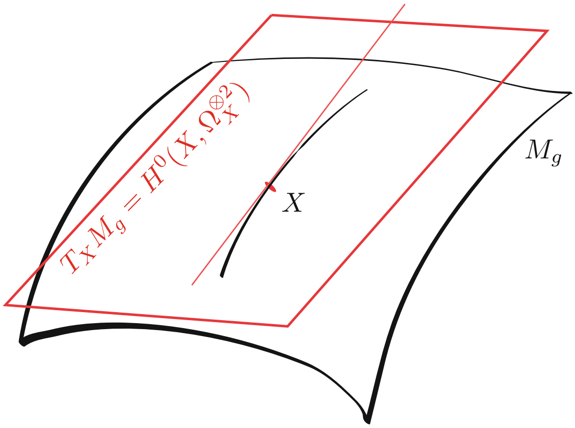

Fig. 16.1 Tangent space to the deformation functor

Suppose that in the cover $X \to X/G$ there are r ramified points $x_1, \ldots, x_r$ and set $e_i^{(\mu)} = |G_{x_i,\mu}|$, for $i = 1, \ldots, r$, $\mu \in \mathbb{N}$. The first factor can be computed using Riemann–Roch theorem [4]

$$\dim H^1(X/G, \pi_*^G(T_X)) = 3g_{X/G} - 3 + \sum_{\mu=1}^{r} \left\lceil \sum_{i=0}^{n_\mu} \frac{e_i^{(\mu)} - 1}{e_0^{(\mu)}} \right\rceil.$$

The second functor can be expressed in terms of group cohomology:

$$H^0\left(X/G, R^1\pi_*^G(T_X)\right) \cong \bigoplus_{i=1}^{r} H^1(G_{0,x_i}, \hat{T}_{X,x_i}), \tag{16.16}$$

where the later sum runs over all wildly ramified points and by $H^1(G_{0,x_i}, \hat{T}_{X,x_i})$ we mean the first cohomology groups, and $\hat{T}_{X,x_i} = k[[t]]d/dt$ is the local tangent space at x_i, while the action of G is the adjoint action:

$$\left(f(t)\frac{d}{dt}\right)^{\sigma} = f(t)^{\sigma}\sigma\frac{d}{dt}\sigma^{-1} = f(t)^{\sigma}\sigma\left(\frac{d\sigma^{-1}(t)}{dt}\right)\frac{d}{dt}.$$

The computation of group cohomology in Eq. (16.16) is manageable only for explicit covers, in particular for Artin–Schreier extensions [11]. One idea exploited by the second author in [44] is to use that the decomposition group G_{x_i} admits a ramification filtration given in Eq. (16.2), where the successive quotients are elementary abelian groups given by Artin–Schreier extensions.

Therefore one can use the Lyndon-Hochshild-Serre spectral sequence [30] which connects the cohomology of the extensions of groups

$$1 \to H \to G \to G/H \to 1,$$

giving rise to a 5-term exact sequence:

$$0 \to H^1(G/H, A^H) \xrightarrow{\text{inf}} H^1(G, A) \xrightarrow{\text{res}} H^1(H, A)^{G/H} \xrightarrow{\text{tg}} H^2(G/H, A^H) \xrightarrow{\text{inf}} H^2(G, A)$$

Unfortunately, the transgression map can be effectively computed only in special cases like the next theorem [44], which limits the usage of this method.

Theorem 16.3 *If G is an abelian group and $G/H \cong \mathbb{Z}/p$, $G \cong G/H \times H$ then the transgression map is zero.*

16.5.4 Representation Theory and the Tangent Space

Serre duality allows us to compute

$$H^1(X, T_X) \cong H^0(X, \Omega^{\otimes 2})^*, \tag{16.17}$$

and the dimension of the later space can be effectively computed using Riemann–Roch theorem to be $3g_X - 3$. In [45] the second author proposes an equivariant form of Eq. (16.17)

$$D_{\text{gl}}(k[\varepsilon]/\langle \varepsilon^2 \rangle) = H^1(X, T_X)^G \cong H^0(X, \Omega_X^{\otimes 2})_G.$$

Notice that the space of invariants becomes the space of the co-invariants on the dual space, where for a G-module A, the spaces of invariants and coinvariants respectively, are given by

$$A^G := \{a \in A : a^g = a\} \qquad A_G := A/\langle ga - a : a \in A, g \in G \rangle.$$

For $G = \mathbb{Z}/p$ we have $A^G \cong A_G$, but if G is a more complicated group like $G = \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$ we can have $A^G \ncong A_G$.

The idea of this construction is that the knowledge of $k[G]$-module structure can lead to the computation of $\dim D_{\text{gl}}(k[\varepsilon])$.

S. Karanikolopoulos [33] pursued this idea by studying elementary abelian extensions given as Artin-Schreier extensions: $F/K(x)$ with

$$y^{p^n} - y = \frac{g(x)}{(x - a_1)^{\Phi(1)} \cdots (x - a_s)^{\Phi(s)}}$$

using a modified Boseck construction in order to compute the Galois module structure of global polydifferentials. It turns out that

$$H^0(X, \Omega_X^{\otimes m}) \cong \bigoplus_{j=1}^{p^n} W_j^{d_j},$$

where

$$\Gamma_k(m) = \sum_{i=1}^{s} \left\lfloor \frac{m(p^n - 1)(\Phi(i) + 1) - k\Phi(i)}{p^n} \right\rfloor,$$

$$d_{p^n} = \Gamma_{p^n-1}(m) - 2m + 1, \qquad d_j = \Gamma_{j-1}(m) - \Gamma_j(m), \qquad j = 1, \ldots, p^n - 1,$$

$$W_j = \langle \theta_0, \ldots, \theta_{j-1} \rangle_K, \qquad \sigma_\alpha(\theta_i) = \sum_{\ell=0}^{i} \binom{i}{\ell} \alpha^{i-\ell} \theta_\ell.$$

Moreover if j has the p-adic expansion $j = \sum_{i=1}^{n} a_i p^i$ and χ be the map

$$\chi : \{0, \ldots, p-1\} \to \{0, 1\}$$

defined by:

$$\chi(a) := \begin{cases} 1 & \text{if } a \neq 0, \\ 0 & \text{if } a = 0 \end{cases}$$

then

$$\dim((W_j)_G) = \sum_{i=1}^{n} \chi(a_i).$$

Finally

$$\dim(H^1(X, G, T_X)) = \begin{cases} s(n+2) - 3 & \text{if } p > 3 \\ s(n+1) - 3 & \text{if } p = 3 \\ sn - 3 & \text{if } p = 2 \end{cases}$$

16.5.5 Weakly Ramified Covers

For the case of weakly ramified covers B. Köck [40] proved that one can extend the global section of holomorphic polydifferentials $H^0(X, \Omega_X^{\otimes m}(D))$ by considering a suitable G-invariant divisor D such that the Euler characteristic $\chi(G, X, \Omega_X(D))$ lifts to a class in the Grothendieck group of projective $k[G]$-modules. This implies that if $H^1(X, \Omega_X(D)) = 0$ vanishes then $H^0(G, \Omega_X(D))$ is projective.

B. Köck together with the second author in [42] used this idea in order to write the short exact sequence

$$0 \to H^0(X, \Omega_X^{\otimes 2}) \to H^0(X, \Omega_X^{\otimes 2}(D)) \to H^0(X, \Sigma) \to 0,$$

where D is selected so that the G-module $H^0(X, \Omega^{\otimes 2}(D))$ is projective, and Σ is a skyscraper sheaf supported at ramified points. Then, the coinvariant functor is applied and the following long exact homology sequence is obtained:

$$0 \to H_1(G, H^0(X, \Sigma)) \to H^0(X, \Omega_X^{\otimes 2})_G \to H^0(X, \Omega_X^{\otimes 2})(D))_G \to H^0(X, \Sigma)_G \to 0.$$

Using the above exact sequence, they arrived at the dimension formula, where g_Y is the genus of the quotient curve $Y = X/G$:

$$\dim H^0(X, \Omega_X^{\otimes 2})_G = 3g_Y - 2 + \sum_{j=1}^{r} \log_p |G(x_j)| + \begin{cases} 2r & \text{if } p > 3 \\ r & \text{if } p = 2 \text{ or } 3. \end{cases}$$

16.5.6 Galois Weierstrass Points and Harbater–Katz–Gabber Covers

In 1986 Ian Morrison and Henry Pinkham [58], connected the $k[G]$-structure of the space $H^0(X, \Omega_X)$ to the theory of Weierstrass semigroups of Galois Weierstrass points for the case of Riemann surfaces. A point P on a compact Riemann surface is called *Galois Weierstrass*, if for a meromorphic function f on X such that $(f)_\infty = dP$, where d is the least pole number in the Weierstrass semigroup at P the induced cover $f : X \to \mathbb{P}^1$ is Galois. Morrison and Pinkham's study was based on the monodromy representation of the Galois group at a ramified point, on the fact that the stabilizer of a point in characteristic zero is cyclic and on character theory of cyclic groups. The character of the associated representation of the group $G := \mathrm{Gal}(f)$ is called a Hurwitz character of G, and the authors were able to classify all such characters.

As a wild replacement of the Galois Weierstrass points we can consider the Harbater–Katz–Gabbercovers. A p-order Harbater-Katz-Gabber cover, which from now on will be called HKG-cover, see [22], is a Galois cover $X \to \mathbb{P}^1$ with Galois group a p-group G which has a unique totally ramified point.

Let G act on the complete local ring $k[[t]]$. The Harbater–Katz–Gabber compactification theorem [22, 39], asserts that there is a HKG-cover $X_{\mathrm{HKG}} \to \mathbb{P}^1$ ramified only at one point P of X with Galois group $G = \mathrm{Gal}(X_{\mathrm{HKG}}/\mathbb{P}^1) = G_0$ such that $G_0(P) = G_0$ and the action of G_0 on the completed local ring $\hat{\mathscr{O}}_{X_{\mathrm{HKG}},P}$ coincides with the original action of G_0 on $\mathscr{O}$. There is a lot of recent interest on HKG-covers see [6, 10].

By considering the Harbater–Katz–Gabber compactification to an action on the local ring $k[[t]]$, we have the advantage to attach global invariants, like genus, p-rank, differentials etc., in the local case. Also finite subgroups of the automorphism group $\mathrm{Aut} k[[t]]$, which is a difficult object to understand (and is a crucial object in understanding the deformation theory of curves with automorphisms, see [4]) become subgroups of $\mathrm{GL}(V)$ for a finite dimensional vector space V.

More precisely, let P be a ramified point and let $G(P)$ be the decomposition group at P. There is a representation:

$$\rho : G(P) \to \mathrm{Aut}(k[[t]]),$$

expressing the action of the decomposition group to the completed local ring at a point. The local deformation functor is defined:

$$D_P : \mathscr{C} \to \text{Sets}, \; R \mapsto \left\{ \begin{array}{l} \text{lifts } G(P) \to \text{Aut}(R[[t]]) \; of \; \rho \text{ mod}- \\ \text{ulo conjugation with an element} \\ \text{of } \ker(\text{Aut} R[[t]] \to k[[t]]) \end{array} \right\}$$

The representation ρ maps $G(P)$ inside the group of automorphisms of formal powerseries, which is a group hard to understand. The following theorem introduced by the second author in [46] gives us a linear representation instead.

Theorem 16.4 *Let P be a fully ramified point of $X \to X/G_1(P)$. Assume that $g_X \geq 2$, $p \geq 2, 3$. Consider the Weierstrass semigroup at P up to the first pole number m_r not divisible by p:*

$$0 = m_0 < \cdots < m_{r-1} < m_r,$$

and select functions in $k(X)$ $f_0, \ldots, f_r$ with $(f_i)_\infty = m_i P$. Then the natural representation

$$\rho : G_1(P) \to \text{GL}(L(m_r P))$$

is faithful.

This theorem allows us to write in explicit form the action on the formal powerseries ring. Indeed, by Hensel's lemma we can select the uniformizer t such that $f_r = t^{-m}$, $m = m_r$. Then the action is given in closed form:

$$\sigma(t) = t \left(1 + t^m \sum_{\nu=1}^{r} a_{\nu,r} f_\nu \right)^{-1/m}.$$

This allows us to work with a general linear group instead of $\text{Aut}(k[[t]])$ and define a representation functor of linear Galois representations as used in the proof of Fermat's last theorem [56].

16.5.7 Representation Filtration

S. Karanikolopoulos and the second author in [34] defined a filtration similar to the ramification filtration, the representation filtration. More precisely for each $0 \leq i \leq r$, consider the representations:

$$\rho_i : G_1(P) \to \text{GL}(L(m_i P)),$$

which give rise to the decreasing sequence of groups:

$$G_1(P) = \ker \rho_0 \supseteq \ker \rho_1 \supseteq \ker \rho_2 \supseteq \cdots \supseteq \ker \rho_r = \{1\},$$

corresponding to the tower of function fields:

$$F^{G_1(P)} = F^{\ker \rho_0} \subseteq F^{\ker \rho_1} \subseteq \cdots \subseteq F^{\ker \rho_r} = F.$$

Theorem 16.5 *If $X \to X/G$ is a HKG-cover, then the representation and the ramification filtrations coincide.*

Select a function $f_{i_0} \in k(X)$ such that $k(X)^G = k(f_{i_0})$.

$$\operatorname{div}(df_{i_0}^{\otimes m}) = \left(-2mp^{h_0} + m\sum_{i=1}^{n}(b_i - b_{i-1})(p^{h-1} - 1)\right)P,$$

where

$$b_0 = -1,\ p^{h_0} = |G_1(P)|,\ p^{h_i} = |\ker \rho_{c_{i+1}}| = |G_{b_{i+1}}|,\ \text{ for } i \geq 1.$$

The following theorems give some information for $k[G]$-module structure of holomorphic polydifferentials for the case of HKG-covers.

Theorem 16.6 *For every pole number μ select a function f_μ such that $(f_\mu)_\infty = \mu P$. The set*

$$\{f_\mu df_{i_0}^{\otimes m} : \deg(f_i) \leq m(2g_X - 2)\}$$

forms a basis for the space of m-holomorphic (poly)differentials of X.

Theorem 16.7 *The module $H^0(X, \Omega_X^{\otimes m})$ is a direct sum of $N = \left\lfloor \frac{m(2g-2)}{p^{h_0}} \right\rfloor$ direct indecomposable summands.*

Corollary: If $|G_1(P)| \geq m(2g-2)$, then $N = 1$. In particular curves with *big-action* (in the sense of M. Matignon-M. Rocher) have one indecomposable summand.

16.6 Integral Representation Theory

Suppose that a relative curve $\mathscr{X} \to \operatorname{Spec} R$ with a fibrewise action of G is given. When R is a principal ideal domain then one can show that the spaces $M_n = H^0(\mathscr{X}, \Omega_{\mathscr{X}}^{\otimes n})$ are free R-modules.

Problem: Describe the module structure of M_n within the theory of *integral representations*. Notice that usually the term *integral representation* is reserved for $\mathbb{Z}[G]$-modules. Our situation is a little bit easier since we work over complete local rings, and we also add the eigenvalues $R = W(k)(\zeta_n)$.

S. Karanikolopoulos and the second author in [36] used the model of Bertin-Mézard [4] based on the work of Sekiguchi, Oort and Suwa theory [77, 78] in order to study this problem for cyclic groups. More precisely, the generic fibre for the Bertin-Mézard model is a Kummer extenrion defined over the Witt ring $S := W(k)(\zeta_p)$ of k with a p-th root of unity ζ_p adjoined, given by:

$$(X + \lambda^{-1})^p = x^{-m} + \lambda^{-p},$$

where $\lambda = \zeta_p - 1$ such that $\lambda \equiv 0 \mod m_S$. We set $m = pq - l$, $0 < l \leq p - 1$ and $\lambda X + 1 = y/x^q$. The model then becomes

$$y^p = (\lambda^p + x^m)x^l = \lambda^p x^l + x^{qp}.$$

More generally x^q can be replaced by $a(x) = x^q + x_1 x^{q-1} + \cdots + x_q$, where $x_q = 0$ if $l \neq 1$, and consider the Kummer extension

$$(\lambda\xi + a(x))^p = \lambda^p x^l + a(x)^p,$$

where $\xi = Xa(x)$, $y = \lambda\xi + a(x) = a(x)(\lambda X + 1)$. This more general model is given by

$$y^p = \lambda^p x^l + a(x)^p = x^l(\lambda^p + a(x)^p x^{-l}).$$

Let R denote the Oort-Sekiguchi-Suwa factor of the versal deformation ring [4]:

$$R = \begin{cases} W(k)[\zeta_p][x_1, \ldots, x_q] & \text{if } l = 1 \\ W(k)[\zeta_p][x_1, \ldots, x_{q-1}] & \text{if } l > 1 \end{cases}$$

The Bertin-Mézard model is a relative curve $\mathscr{X} \to \mathrm{Spec} R$, where the horizontal branch locus is given in Fig. 16.2.

Using the theory of Boseck for the generic fibre of R we see that the set of differentials of the form

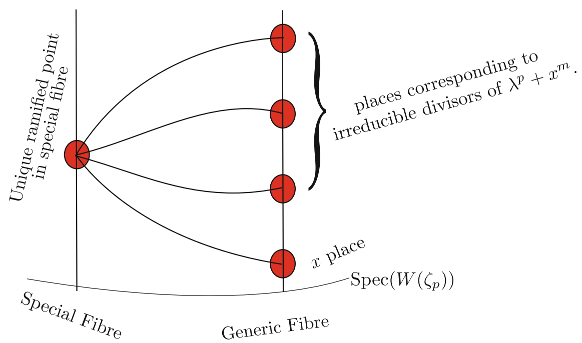

Fig. 16.2 Splitting the branch locus

$$x^N a(x)^a \frac{(\lambda X+1)^a}{a(x)^{p-1}(\lambda X+1)^{p-1}} dx, \tag{16.18}$$

where

$$0 \le a < p-1 \text{ and } l - \left\lceil \frac{(1+a)l}{p} \right\rceil \le N \le (p-1-a)q - 2, \tag{16.19}$$

forms a basis of holomorphic differentials. This base is not suitable for taking the reduction modulo the maximal ideal of the ring $S = W(k)[\zeta]$ since in the reduction $\lambda = 0$. The idea is to change the basis of the generic fibre so that no λ appears in the numerator of the differentials. Then we use the special fibre Boseck basis to show that the reductions are holomorphic, therefore the relative differentials are indeed holomorphic over Spec$R$.

In this way, we arrive at:

Theorem 16.8 *Let σ be an automorphism of $\mathscr{X}$ of order $p \neq 2$ and conductor m with $m = pq - l$, $1 \le q$, $1 \le l \le p-1$. Consider the modules*

$$V_{a_0,a_1} :=_S \langle (\lambda X+1)^{a_0} X^i : 0 \le i < a_1 \rangle.$$

which are indecomposable $S[G]$-modules and define $V_a := V_{1-p,a}$.

The free R-module $H^0(\mathscr{X}, \Omega_{\mathscr{X}})$ has the following $R[G]$ structure:

$$H^0(\mathscr{X}, \Omega_{\mathscr{X}}) = \bigoplus_{\nu=0}^{p-2} V_\nu^{\delta_\nu},$$

where

$$\delta_\nu = \begin{cases} q + \left\lceil \frac{(\nu+1)l}{p} \right\rceil - \left\lceil \frac{(2+\nu)l}{p} \right\rceil & \text{if } \nu \le p-3, \\ q-1 & \text{if } \nu = p-2. \end{cases}$$

Acknowledgements The authors were supported by the European Union (European Social Fund ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) Research Funding Program: THALES: Reinforcement of the interdisciplinary and/or interinstitutional research and innovation.
The authors would like to thank the referees for their suggestions on the improvement of the text. We would like to also thank Professor P. Roquette for introducing us to the fascinating subject of automorphisms of curves 25 years ago.

References

1. Adler, A.: The Mathieu group M_{11} and the modular curve $X(11)$. Proc. Lond. Math. Soc. (3), **74**(1), 1–28 (1997)
2. Bending, P., Camina, A., Guralnick, R.: Automorphisms of the modular curve. In: Progress in Galois Theory, Dev. Math. vol. 12, pp. 25–37. Springer, New York (2005)

3. Benson, D.: Representations of Elementary Abelian p-Groups and Vector Bundles (2015)
4. Bertin, J., Mézard, A.: Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques. Invent. Math. **141**(1), 195–238 (2000)
5. Bleher, F., Chinburg, T., Kontogeorgis, A.: On holomorphic differentials of curves (In preparation)
6. Bleher, F.M., Chinburg, T., Poonen, B., Symonds, P.: Automorphisms of Harbater-Katz-Gabber curves. Mathematische Annalen (to appear)
7. Borne, N.: Cohomology of G-sheaves in positive characteristic. Adv. Math. **201**(2), 454–515 (2006)
8. Boseck, H.: Zur Theorie der Weierstrasspunkte. Math. Nachr. **19**, 29–63 (1958)
9. Chevalley, C., Weil, A., Hecke, E.: Über das verhalten der integrale 1. gattung bei automorphismen des funktionenkörpers. Abh. Math. Semin. Univ. Hambg. (1934)
10. Chinburg, T., Guralnick, R., Harbater, D.: Oort groups and lifting problems. Compos. Math. **144**(04), 849–866 (2008)
11. Cornelissen, G., Kato, F.: Equivariant deformation of Mumford curves and of ordinary curves in positive characteristic. Duke Math. J. **116**(3), 431–470 (2003)
12. Cornelissen, G., Kato, F., Kontogeorgis, A.: Discontinuous groups in positive characteristic and automorphisms of Mumford curves. Math. Ann. **320**(1), 55–85 (2001)
13. Daniel, G., Salvador, V.: Mathematics: Theory & Applications. Topics in the Theory of Algebraic Function Fields. Birkhäuser, Boston (2006)
14. Deligne, P., Mumford, D.: The irreducibility of the space of curves of given genus. Inst. Hautes Études Sci. Publ. Math. **36**, 75–109 (1969)
15. Farkas, H.M., Kra, I.: Graduate Texts in Mathematics. Riemann surfaces, vol. 71. Springer, New York (1980)
16. García, A.: On Weierstrass points on Artin-Schreier extensions of $k(x)$. Math. Nachr. **144**, 233–239 (1989)
17. Gerritzen, L.: On non-archimedean representations of abelian varieties. Math. Ann. **196**(4), 323–346 (1972)
18. Gerritzen, L., van der Put, M.: Lecture Notes in Mathematics. Schottky groups and Mumford curves, vol. 817. Springer, Berlin (1980)
19. Giulietti, M., Korchmáros, G.: Algebraic curves with a large non-tame automorphism group fixing no point. Trans. Amer. Math. Soc. **362**(11), 5983–5983 (2010)
20. Giulietti, M., Korchmáros, G.: Nakajima's remark on Henn's proof. Electronic Notes Discret. Math. **40**, 135–138 (2013)
21. Grothendieck, A.: Sur quelques points d'algèbre homologique. Tôhoku Math. J. **2**(9), 119–221 (1957)
22. Harbater, D.: Moduli of p-covers of curves. Comm. Algebra **8**(12), 1095–1122 (1980)
23. Harris, J., Morrison, I.: Graduate Texts in Mathematics. Moduli of curves, vol. 187. Springer, New York (1998)
24. Hartshorne, R.: Graduate Texts in Mathematics. Algebraic geometry, vol. 52. Springer, New York (1977)
25. Hecke, E.: Über ein Fundamentalproblem aus der Theorie der elliptischen Modulfunktionen. Abh. Math. Sem. Univ. Hamburg **6**(1), 235–257 (1928)
26. Heinrich Matzat, B.: Ein Vortrag Ueber Weierstasspunkte (1975)
27. Henn, H.-W.: Funktionenkörper mit grosser Automorphismengruppe. J. Reine Angew. Math. **302**, 96–115 (1978)
28. Herrlich, F.: Die Ordnung der Automorphismengruppe einer p-adischen Schottkykurve. Math. Ann. **246**(2), 125–130 (1979/1980)
29. Hirschfeld, J.W.P., Korchmáros, G., Torres, F.: Algebraic Curves over a Finite Field. Princeton Series in Applied Mathematics. Princeton University Press, Princeton (2008)
30. Hochschild, G., Serre, J.-P.: Cohomology of group extensions. Trans. Amer. Math. Soc. **74**, 110–134 (1953)
31. Hurwitz, A.: Ueber algebraische gebilde mit eindeutigen transformationen in sich. Mathematische Annalen **41**(3), 403–442 (1892)

32. Kani, E.: The Galois-module structure of the space of holomorphic differentials of a curve. J. Reine Angew. Math. **367**, 187–206 (1986)
33. Karanikolopoulos, S.: On holomorphic polydifferentials in positive characteristic. Math. Nachr. **285**(7), 852–877 (2012)
34. Karanikolopoulos, S., Kontogeorgis, A.: Automorphisms of Curves and Weierstrass semigroups for Harbater-Katz-Gabber covers. Submitted
35. Karanikolopoulos, S., Kontogeorgis, A.: Representation of cyclic groups in positive characteristic and Weierstrass semigroups. J. Number Theory **133**(1), 158–175 (2013)
36. Karanikolopoulos, S., Kontogeorgis, A.: Integral representations of cyclic groups acting on relative holomorphic differentials of deformations of curves with automorphisms. Proc. Amer. Math. Soc. **142**(7), 2369–2383 (2014)
37. Karrass, A., Pietrowski, A., Solitar, D.: Finite and infinite cyclic extensions of free groups. J. Austral. Math. Soc. **16**, 458–466 (1973). Collection of articles dedicated to the memory of Hanna Neumann, IV
38. Kato, F., Kontogeorgis, A.: On the Galois-module structure of polydifferentials of Artin-Schreier-Mumford curves, modular and integral representation theory (In preparation)
39. Katz, N.M.: Local-to-global extensions of representations of fundamental groups. Ann. Inst. Fourier (Grenoble) **36**(4), 69–106 (1986)
40. Köck, B.: Galois structure of Zariski cohomology for weakly ramified covers of curves. Amer. J. Math. **126**(5), 1085–1107 (2004)
41. Köck, B., Tait, J.: Faithfulness of actions on Riemann-Roch spaces. Canad. J. Math. **67**(4), 848–869 (2015)
42. Koeck, B., Kontogeorgis, A.: Quadratic differentials and equivariant deformation theory of curves. Annales de L'institut Fourier **62**(3), 1015–1043 (2012)
43. Kontogeorgis, A.: Automorphisms of Fermat-like varieties. Manuscripta Math. **107**(2), 187–205 (2002)
44. Kontogeorgis, A.: On the tangent space of the deformation functor of curves with automorphisms. Algebra Number Theory **1**(2), 119–161 (2007)
45. Kontogeorgis, A.: Polydifferentials and the deformation functor of curves with automorphisms. J. Pure Appl. Algebra **210**(2), 551–558 (2007)
46. Kontogeorgis, A.: The ramification sequence for a fixed point of an automorphism of a curve and the Weierstrass gap sequence. Math. Z. **259**(3), 471–479 (2008)
47. Kontogeorgis, A., Rotger, V.: On abelian automorphism groups of Mumford curves. Bull. Lond. Math. Soc. **40**(3), 353–362 (2008)
48. Kontogeorgis, A., Yang, Y.: Automorphisms of hyperelliptic modular curves $X_0(N)$ in positive characteristic. LMS J. Comput. Math. **13**, 144–163 (2010)
49. Lehner, J.: Discontinuous Groups and Automorphic Functions. Mathematical Surveys and Monographs. American Mathematical Society, Providence (1964)
50. Lehr, C., Matignon, M.: Automorphism groups for p-cyclic covers of the affine line. Compos. Math. **141**(5), 1213–1237 (2005)
51. Leopoldt, H.-W.: Über die Automorphismengruppe des Fermatkörpers. J. Number Theory **56**(2), 256–282 (1996)
52. Madden, D.J., Valentini, R.C.: The group of automorphisms of algebraic function fields. J. Reine Angew. Math. **343**, 162–168 (1983)
53. Manin, Yu., Drinfeld, V.: Periods of p-adic Schottky groups. J. Reine Angew. Math. **262/263**, 239–247 (1973). Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday
54. Marques, S., Ward, K.: Holomorphic differentials of solvable galois towers of curves over a perfect field. Preprint (2015). arXiv:1507.07023v2
55. Matignon, M., Rocher, M.: On smooth curves endowed with a large automorphism p-group in characteristic $p > 0$. Algebra Number Theory **2**, 887–926 (2008)
56. Mazur, B.: Deformation theory of Galois representations (in Modular forms and Fermat's last theorem), pp. xx+582 (1997). Papers from the Instructional Conference on Number Theory and Arithmetic Geometry held at Boston University, Boston, MA, 9–18 Aug 1995

57. Monsky, P.: The automorphism groups of algebraic curves. Ph.D. Thesis University of Chicago (1962)
58. Morrison, I., Pinkham, H.: Galois Weierstrass points and Hurwitz characters. Ann. Math. (2). **124**(3), 591–625 (1986)
59. Mumford, D.: An analytic construction of degenerating curves over complete local rings. Compositio Math. **24**, 129–174 (1972)
60. Nakajima, S.: Galois module structure of cohomology groups for tamely ramified coverings of algebraic varieties. J. Number Theory **22**(1), 115–123 (1986)
61. Nakajima, S.: On abelian automorphism groups of algebraic curves. J. Lond. Math. Soc. (2). **36**(1), 23–32 (1987)
62. Nakajima, S.: p-ranks and automorphism groups of algebraic curves. Trans. Amer. Math. Soc. **303**(2), 595–607 (1987)
63. Obus, A.: The (local) lifting problem for curves. 8 May 2011. arXiv:1105.1530
64. Obus, A., Wewers, S.: Cyclic extensions and the local lifting problem. Ann. Math. (2). **180**(1), 233–284 (2014)
65. Poonen, B.: Varieties without extra automorphisms I: Curves. Math. Res. Lett. **7**, 67–76 (2000)
66. Pop, F.: The Oort conjecture on lifting covers of curves. Ann. Math. (2). **180**(1), 285–322 (2014)
67. Popp, H.: The singularities of the moduli schemes of curves. J. Number Theory. **1**(1), 90–107 (1969)
68. Rajan, C.S.: Automorphisms of $X(11)$ over characteristic 3 and the Mathieu group M_{11}. J. Ramanujan Math. Soc. **13**(1), 63–72 (1998)
69. Ritzenthaler, C.: Automorphismes des courbes modulaires $X(n)$ en caractéristique p. Manuscripta Math. **109**(1), 49–62 (2002)
70. Rocher, M.: Large p-groups actions with $|g|/g^2>4/(p^2-1)^2$. 22 Apr 2008. arXiv:0804.3494
71. Rocher, M.: Large p-group actions with a p-elementary abelian derived group. J. Algebra **321**(2), 704–740 (2009)
72. Roquette, P.: The Riemann hypothesis in characteristic p, its origin and development I. The formation of the zeta-functions of Artin and of F.K. Schmidt. Hamburger Beitraege zur Gesichte der Mathematik, Mitt. Math. Ges. Hamburg. **21**, 79–157 (2002)
73. Rzedowski-Calderón, M., Villa-Salvador, G., Madan, M.L.: Galois module structure of holomorphic differentials in characteristic p. Arch. Math. (Basel) **66**(2), 150–156 (1996)
74. Schlessinger, M.: Functors of Artin rings. Trans. Amer. Math. Soc. **130**, 208–222 (1968)
75. Schmid, H.L.: Über die automorphismen eines algebraischen funktionenkörpers von primzahlcharakteristik. J. Reine Angew. Math. **179**, 5–15 (1938)
76. Schneider, P.: Rigid-analytic L-transforms. Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983). Lecture Notes in Mathematics, vol. 1068, pp. 216–230. Springer, Berlin (1984)
77. Sekiguchi, T., Suwa, N.: Théories de Kummer-Artin-Schreier-Witt. C. R. Acad. Sci. Paris Sér. I Math. **319**(2), 105–110 (1994)
78. Sekiguchi, T., Suwa, N.: Théorie de Kummer-Artin-Schreier et applications. J. Théor. Nombres Bordeaux. **7**(1), 177–189 (1995). Les Dix-huitièmes Journées Arithmétiques (Bordeaux, 1993)
79. Serre, J.-P.: Sur la topologie des variétés algébriques en caractéristique p. In: Symposium Internacional de Topología algebraica International Symposium on Algebraic Topology, pp. 24–53. Universidad Nacional Autónoma de México and UNESCO, Mexico City (1958)
80. Stichtenoth, H.: Über die Automorphismengruppe eines algebraischen Funktionenkörpers von Primzahlcharakteristik. I. Eine Abschätzung der Ordnung der Automorphismengruppe. Arch. Math. (Basel) **24**, 527–544 (1973)
81. Stichtenoth, H.: Über die Automorphismengruppe eines algebraischen Funktionenkörpers von Primzahlcharakteristik. II. Ein spezieller Typ von Funktionenkörpern. Arch. Math. (Basel) **24**, 615–631 (1973)
82. Stichtenoth, H.: Graduate Texts in Mathematics. Algebraic Function Fields and Codes, vol. 254, 2nd edn. Springer, Berlin (2009)

83. Teitelbaum, J.T.: Values of p-adic L-functions and a p-adic Poisson kernel. Invent. Math. **101**(2), 395–410 (1990)
84. Valentini, R.C., Madan, M.L.: Automorphisms and holomorphic differentials in characteristic p. J. Number Theory **13**(1), 106–115 (1981)
85. Ward, K.: Explicit galois representations of automorphisms on holomorphic differentials in characteristic p. Finite Fields Appl. **44**, 34–55 (2017)