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On the non-existence of exceptional automorphisms on Shimura curves

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Abstract

We study the group of automorphisms of Shimura curves $X_0(D,N)$ attached to an Eichler order of square-free level N in an indefinite rational quaternion algebra of discriminant $D > 1$. We prove that, when the genus g of the curve is greater than or equal to 2, $\mathrm{Aut}(X_0(D,N))$ is a 2-elementary abelian group which contains the group of Atkin–Lehner involutions $W_0(D,N)$ as a subgroup of index 1 or 2. It is conjectured that $\mathrm{Aut}(X_0(D,N)) = W_0(D,N)$ except for finitely many values of (D,N) and we provide criteria that allow us to show that this is indeed often the case. Our methods are based on the theory of complex multiplication of Shimura curves and the Čerednik–Drinfeld theory on their rigid analytic uniformization at primes $p \mid D$.

1. *The automorphism group of Shimura curves*

1.1. *Congruence subgroups of* $\mathrm{PSL}_2(\mathbb{R})$ *and automorphisms*

Let Γ be a congruence subgroup of $\mathrm{PSL}_2(\mathbb{R})$. As explained in [16, § 4], we see that Γ is a congruence subgroup of $\mathrm{PSL}_2(\mathbb{R})$ if there exist

- a quaternion algebra B/F over a totally real number field F of degree $d \geqslant 1$;
- an embedding $\varphi : B \hookrightarrow \mathrm{M}_2(\mathbb{R}) \times D \times \overset{(d-1)}{\cdots} \times D$;
- an integral two-sided ideal I of a maximal order $\mathcal{O}$ of B;

such that Γ contains $\varphi(\{\alpha \in \mathcal{O}^1 : \alpha \in 1 + I\})$.

Here, we let D denote Hamilton's skew-field over $\mathbb{R}$ and $n : B \to F$ stand for the reduced norm. We write $\mathcal{O}^1 = \{\alpha \in \mathcal{O} : n(\alpha) = 1\}$. We refer the reader to [25] for generalities on quaternion algebras. Examples of congruence subgroups of $\mathrm{PSL}_2(\mathbb{R})$ with $F = \mathbb{Q}$ will be described in detail below.

Let X_Γ denote the compactification of the Riemann surface $\Gamma \backslash \mathcal{H}$. Let $N = \mathrm{Norm}_{\mathrm{PSL}_2(\mathbb{R})}(\Gamma)$ denote the normalizer of Γ in $\mathrm{PSL}_2(\mathbb{R})$. The group $B_\Gamma = N/\Gamma$ is a finite subgroup of $\mathrm{Aut}(X_\Gamma)$.

When the genus of X_Γ is 0 or 1, $\mathrm{Aut}(X_\Gamma)$ is not a finite group and necessarily $\mathrm{Aut}(X_\Gamma) \supsetneq B_\Gamma$. However, there exist finitely many congruence groups Γ for which $g(X_\Gamma) \leqslant 1$. One actually expects much more, as we claim in the following conjecture.

Conjecture 1.1. $\mathrm{Aut}(X_\Gamma) = B_\Gamma$ for all but finitely many congruence groups $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$.

We call *exceptional* those congruence groups Γ for which the genus of X_Γ satisfies $g \geqslant 2$ and $\mathrm{Aut}(X_\Gamma) \supsetneq B_\Gamma$.

The conjecture as we have stated remains widely open, although it is based on several positive partial results in its favour. The first general statement is the following classical result of Riemann surfaces, of which we briefly recall a proof.

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Proposition 1.2. *Let Γ be a congruence subgroup of $\mathrm{PSL}_2(\mathbb{R})$ that contains no elliptic or parabolic elements. Then* $\mathrm{Aut}(X_\Gamma) = B_\Gamma$.

Proof. Since Γ contains no parabolic elements, the quotient $\Gamma\backslash\mathcal{H}$ is already compact. The absence of elliptic elements implies that the natural projection $\mathcal{H} \to X_\Gamma = \Gamma\backslash\mathcal{H}$ is the universal cover of the curve. Thus all automorphisms of X_Γ lift to a Möbius transformation of $\mathcal{H}$ which, by construction, normalizes Γ. The result follows. □

Besides this, the question has been settled for certain families of modular curves, as we now review.

Let $D \geqslant 1$ be the square-free product of an even number of prime numbers, and let $N \geqslant 1$, $(D, N) = 1$, be an integer coprime to D.

Let B be a quaternion algebra over $\mathbb{Q}$ of reduced discriminant D such that there exists a monomorphism $B \xrightarrow{\varphi} \mathrm{M}_2(\mathbb{R})$. Let $\mathcal{O}$ be a maximal order in B. Regard $\mathcal{O}^1$ as a subgroup of $\mathrm{SL}_2(\mathbb{R})$ by means of φ.

For any prime $p \mid N$ fix isomorphisms $\mathcal{O} \otimes \mathbb{Z}_p \simeq \mathrm{M}_2(\mathbb{Z}_p)$. If $p^e \,||\, N$ is the exact power of p which divides N, then let $\pi_p : \mathcal{O} \otimes \mathbb{Z}_p \simeq \mathrm{M}_2(\mathbb{Z}_p) \to \mathrm{M}_2(\mathbb{Z}/p^e\mathbb{Z})$ denote the natural reduction map.

Define the congruence groups $\Gamma(D, N) \subseteq \Gamma_1(D, N) \subseteq \Gamma_0(D, N)$ as follows:

- $\Gamma(D, N) = \{\gamma \in \mathcal{O}^1 : \pi_p(\gamma) = \mathrm{Id} \in \mathrm{GL}_2(\mathbb{Z}/p^e\mathbb{Z})$ for all $p \mid N\}$;
- $\Gamma_1(D, N) = \{\gamma \in \mathcal{O}^1 : \pi_p(\gamma) = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/p^e\mathbb{Z})$ for all $p \mid N\}$;
- $\Gamma_0(D, N) = \{\gamma \in \mathcal{O}^1 : \pi_p(\gamma) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/p^e\mathbb{Z})$ for all $p \mid N\}$.

Let $X(D, N)$, $X_1(D, N)$ and $X_0(D, N)$ denote the corresponding Shimura modular curves of discriminant D and level N (cf. [**2**]).

When $D = 1$, these are just other names for the elliptic modular curves $X(N)$, $X_1(N)$ and $X_0(N)$ which classify elliptic curves with various level structures. When $D > 1$, these curves still admit a moduli interpretation in terms of abelian surfaces with quaternionic multiplication (cf. [**2**]).

Finally, note that when $N = 1$ we have $X(D, 1) = X_1(D, 1) = X_0(D, 1)$, and we shall simply denote this curve by X_D.

As further examples of congruence subgroups of $\mathrm{PSL}_2(\mathbb{R})$ are the normalizers B_Γ for the groups Γ above.

The group $B_{\Gamma_0(D,N)}$ contains the subgroup $W_0(D, N)$ of Atkin–Lehner modular involutions (cf. [**18**, **19**]). It can be described as

$$W_0(D, N) = \{\omega_m : m \geqslant 1,\ m \mid D \cdot N,\ (m, DN/m) = 1\},$$

where $\omega_m^2 = w_1 = \mathrm{Id}$ and $\omega_m \cdot \omega_{m'} = \omega_{mm'/(m,m')^2}$. Hence

$$W_0(D, N) \simeq (\mathbb{Z}/2\mathbb{Z})^{|\{p \,|\, D \cdot N\}|}.$$

If N is square-free we actually have $B_{\Gamma_0(D,N)} = W_0(D, N)$.

Remark 1.3 [**16**, p. 1]. The above conjecture cannot be true for the wider family of arithmetic subgroups of $\mathrm{PSL}_2(\mathbb{R})$. As an example for which the conjecture fails, let $X(2)$ be the classical modular curve of full-level structure $\Gamma(2)$. This is a rational curve with three cusps and no elliptic points. If we choose a rational coordinate z for which the cusps are at 0, 1 and ∞, then $z^{1/n}$ is a rational coordinate for a subgroup of index n in $X(2)$. This subgroup is not congruent for n large enough, but it is arithmetic. The corresponding curve has genus 0 and the automorphism group is not finite.

Several results close to Conjecture 1.1 with regard to the above families of curves have been obtained by different authors.

THEOREM 1.4. *For the curves below, if their genus is greater than or equal to* 2, *one has:*

(i) $\mathrm{Aut}(X(N)) = B_{\Gamma(1,N)}$ *for all* N (Serre [24], Kontogeorgis [12]).

(ii) $\mathrm{Aut}(X_1(D,N)) = B_{\Gamma_1(D,N)}$ *for all* $N \geqslant 4$ (Momose (see remarks below), Buzzard [3]).

(iii) $\mathrm{Aut}(X_0(N)) = B_{\Gamma_0(1,N)}$ *for all* $N \neq 37, 63$ (Ogg [18], Kenku–Momose [11], Elkies [7]).

(iv) $\mathrm{Aut}(X_0(p)/\langle\omega_p\rangle)$ *is trivial for all primes* $p \neq 67, 73, 103, 107, 167, 191$ (Baker–Hasegawa [1]).

(v) $\mathrm{Aut}(X_D) = W_0(D,1)$, *where* $D = 2p$ *or* $3p$ *for some prime* p (Rotger [22]).

Part (i) was shown by Serre [24] for prime N and extended to composite N by Kontogeorgis in [12].

Part (ii) above was proved by Momose unpublished work, communicated to the authors in December 2005, for $D = 1$. For $D > 1$, $\Gamma_1(D,N)$ contains no parabolic (because B is division) or elliptic elements by [3, Lemma 2.2]. Thus Proposition 1.2 applies.

The result analogous to (i) in positive characteristic for the Drinfeld modular curves $X(\mathfrak{N})$ has been obtained in [4].

In (iv), for $p = 67, 73, 103, 107, 167, 191$ the genus of $X_0(p)/\langle\omega_p\rangle$ is 2 and the hyperelliptic involution is the only (exceptional) automorphism of the curve.

1.2. *Main results*

Let $D > 1$ be the square-free product of an even number of primes, and let $N \geqslant 1$ be a square-free integer coprime to D. Let $r = |\{p \mid DN\}|$.

The aim of this section is to make progress towards Conjecture 1.1 for the family of Shimura curves $X_0(D,N)$. Due to their moduli interpretation, these curves admit a canonical model over $\mathbb{Q}$, which we shall still denote by $X_0(D,N)/\mathbb{Q}$. There exists a flat proper model $M_0(D,N)/\mathbb{Z}$ of $X_0(D,N)$ which extends the moduli interpretation to arbitrary schemes over $\mathbb{Z}$ and is smooth over $\mathbb{Z}[1/DN]$.

For primes $p \nmid D$, the construction of this model over $\mathbb{Z}_p$ is very similar to that of the elliptic modular curve $X_0(N)$ as in [6]; see [3] for more details. For primes $p \mid D$ the description of $M_0(D,N) \otimes \mathbb{Z}_p$ is due to Cerednik and Drinfeld. For any prime p we let $M_0(D,N)_p$ denote the closed fibre of $M_0(D,N)$ at p.

PROPOSITION 1.5. *Let* $U \subseteq W_0(D,N)$ *be a subgroup, and let* $X_0(D,N)/U$ *denote the quotient curve. If the genus of* $X_0(D,N)/U$ *is at least* 2, *then all automorphisms of* $X_0(D,N)/U$ *are defined over* $\mathbb{Q}$ *and*

$$\mathrm{Aut}(X_0(D,N)/U) \simeq (\mathbb{Z}/2\mathbb{Z})^s$$

for some $s \geqslant r - \mathrm{rank}_{\mathbb{F}_2}(U)$.

Proof. Write $X = X_0(D,N)$. Let J_U and J denote the Jacobian varieties of X/U and X, respectively. A result essentially due to Ribet claims that J is isogenous over $\mathbb{Q}$ to $J_0(D \cdot N)^{D\text{-new}}$, the *D-new part* of the Jacobian $J_0(D,N)$ of $X_0(DN)$; see, for example, [9, Theorem 5.4] or [21] for more details. Since DN is square-free, it is well known that $J_0(DN)$ has semi-stable reduction over $\mathbb{Q}$ (cf. [6]) and $\mathrm{End}_{\mathbb{Q}}(J_0(DN)) \otimes \mathbb{Q}$ is a product of totally real fields (cf. [20]). The same thus holds for J and J_U. The proposition now follows as a direct application of [1, Proposition 2.4]. □

Let us fix some notation. Throughout, let $s \geqslant r$ be the integer such that $|\mathrm{Aut}(X_0(D,N))| = 2^s$. We have $s \geqslant r$, and Conjecture 1.1 in this setting predicts that $s = r$.

The letters p, q will stand for non-necessarily different prime numbers and $(\frac{\cdot}{p})$ will denote the Kronecker quadratic character *mod* p. For an imaginary quadratic field K, let δ_K denote its discriminant and $h(K)$ its class number. If $\delta_R = \mathrm{disc}(R)$ is the discriminant of some order R in K, then let $h(\delta_R)$ denote its class number.

If $m > 1$ is a square-free integer, then let $\delta_m = -4$ if $m = 2$, and $\delta_m = \delta_{\mathbb{Q}(\sqrt{-m})}$ otherwise.

For any prime $p \mid DN$, set

$$\varepsilon_p = \begin{cases} 1 & \text{if } p \mid D, \\ -1 & \text{if } p \mid N, \end{cases}$$

and for any $m \mid DN$,

$$h_{D,N}(m) = \begin{cases} 1 & \text{if } m = 2, \\ h(-4m) & \text{if } m \neq 2, m \not\equiv 3 \bmod 4, \\ h(-m) & \text{if } m \equiv 3 \bmod 4 \text{ and } h(-4m) > h(-m) \text{ or } 2 \mid DN, \\ 2h(-m) & \text{if } m \equiv 3 \bmod 4,\ h(-4m) = h(-m), 2 \nmid DN. \end{cases}$$

If in addition $(\delta_m/p) \neq \varepsilon_p$ for all $p \mid DN$ if $m \neq 2$, and $(-4/p) \neq \varepsilon_p$ for all $p \mid DN$ or $(-2/p) \neq \varepsilon_p$ for all $p \mid DN$ if $m = 2$, then set

$$\sigma_{D,N}(m) = \begin{cases} |\{p \mid DN, (\delta_m/p) = -\varepsilon_p\}| & \text{if } m \neq 2, \\ \min(|\{p \mid DN, (-4/p) = -\varepsilon_p\}|, |\{p \mid DN, (-2/p) = -\varepsilon_p\}|) & \text{if } m = 2. \end{cases}$$

Otherwise, set $\sigma_{D,N}(m) = \infty$.

THEOREM 1.6. *Assume that $g(X_0(D,N)) \geqslant 2$.*

(i) *If there exist $p, q \mid DN$ such that $(-4/p) = \varepsilon_p$ and $(-3/q) = \varepsilon_q$, then $s = r$.*

(ii) *Let $m \mid DN$ such that $\sigma_{D,N}(m) < \infty$. Then*

$$s \leqslant \mathrm{ord}_2\left(h_{D,N}(m)\right) + \sigma_{D,N}(m) + 1.$$

(iii) $s \leqslant \mathrm{ord}_2(g-1) + 2$.

For any two coprime square-free integers $\delta, \nu \geqslant 1$, let $h(\delta, \nu)$ denote the class number of any Eichler order of level ν in a quaternion algebra of reduced discriminant δ over $\mathbb{Q}$, which counts the number of one-sided ideals of the order up to principal ideals. An explicit formula for $h(\delta, \nu)$ is given in [**25**, p. 152].

THEOREM 1.7. *Assume that $g(X_0(D,N)) \geqslant 2$.*

(i) *Let $m = 2$ or $m = 3$. Assume that $(\delta_m/p) \neq \varepsilon_p$ for all $p \mid DN$ except for at most one prime divisor of D. If $m \mid DN$ then $s = r$; otherwise $s \leqslant r + 1$.*

(ii) *For any odd $p \mid D$, $s \leqslant \mathrm{ord}_2\, h(D/p, N) + 3$.*

(iii) *For any $\ell \nmid 2DN$, $s \leqslant \mathrm{ord}_2 \mid M_0(D,N)_\ell(\mathbb{F}_\ell) \mid +1$.*

An explicit formula for the number of rational points of $M_0(D,N)_\ell$ over $\mathbb{F}_{\ell^n}$, $n \geqslant 1$, may be found in [**23**, § 2]. As a direct consequence of Theorem 1.6(i) and Theorem 1.7(i) we derive the following result.

COROLLARY 1.8. *If $g(X_0(D,N)) \geqslant 2$ then $s = r$ or $s = r + 1$.*

We prove Theorem 1.6 in Section 2, while the proof of Theorem 1.7 is offered in Section 3.

Theorems 1.6 and 1.7 may be applied to show that there exist no automorphisms on many Shimura curves beyond the Atkin–Lehner involutions. Indeed, Theorem 1.6(ii) covers the case $e_2 = e_3 = 0$ while Theorem 1.7(ii) solves the cases $e_2 \neq 0$, $2 \mid DN$ and $e_3 \neq 0$, $3 \mid DN$. As a consequence, we have the following corollary.

Corollary 1.9. *If* $6 \mid DN$ *then* $s = r$*; that is,* $\mathrm{Aut}(X_0(D,N)) = W_0(D,N)$.

All together, Theorems 1.6 and 1.7 can be applied to many other Shimura curves. It follows, for instance, from Theorem 1.6(iii) that $\mathrm{Aut}(X_0(2p,N)) = W_0(2p,N)$ for all primes p and N are primes, $N \equiv 3 \bmod 8$. This follows because $h(2,N)$ is always odd (cf. [**25**, p. 152]). We refer the reader to Proposition 3.5 for more numerical computations.

1.3. *Overview of the article*

We devote the remainder of this note to introducing the necessary tools that we shall need eventually to prove Theorems 1.6 and 1.7.

The next section reviews the theory of complex multiplication on Shimura curves and its behaviour with respect to the Atkin–Lehner group. Following ideas borrowed from [**22**], we use this to prove Theorem 1.6.

Section 3 recalls the theory of Cerednik–Drinfeld, which provides an explicit description of the reduction mod p of Shimura curves $X_0(D,N)$ at primes $p \mid D$. The main result of Cerednik and Drinfeld describes $X_0(D,N) \times \mathbb{Q}_p$ as a quadratic twist of a Mumford curve over $\mathbb{Q}_p$ which is rigid and analytically uniformized by a certain discrete finitely generated subgroup Γ_+ of $\mathrm{PGL}_2(\mathbb{Q}_p)$. The group Γ_+ is constructed by means of an *interchange of invariants* of the quaternion algebra of reduced discriminant D over $\mathbb{Q}$.

In turn, this allows us to interpret the dual graph of the special fibre of a suitable model of $X_0(D,N)$ over $\mathbb{Z}_p$ as the quotient of the Bruhat–Tits tree at p by Γ_+. We use this to give a combinatorial description of the stable model and minimal regular model of $X_0(D,N)$ at primes $p \mid D$; see Proposition 3.2.

In Subsection 3.1 we make use of this material to complete the proof of Theorem 1.7. Finally, in Subsection 3.2 we offer a numerical result that shows how the methods of this note apply to prove the non-existence of exceptional automorphisms on most Shimura curves $X_0(D,N)$ for $D \leqslant 1500$ and $N = 1$.

2. *Automorphisms and points of complex multiplication*

Let (D,N) be a pair as in the previous section. For an order R in an imaginary quadratic field K, let c_R be its conductor in K, and let $\mathrm{CM}(\delta_R)$ denote the set of complex multiplication (CM) points on $X_0(D,N)$ by the order R. A fundamental result of Shimura states that the coordinates of a CM-point $P \in \mathrm{CM}(\delta_R)$ on $X_0(D,N)$ generate the ring class field H_R over K (cf., for example, [**9**, § 5]). That is,

$$K \cdot \mathbb{Q}(P) = H_R. \tag{2.1}$$

The cardinality of $\mathrm{CM}(\delta_R)$ is given in [**19**, § 1]:

$$|\,\mathrm{CM}(\delta_R)\,| = \begin{cases} 0 & \text{if } (\delta_K/p) = \varepsilon_p \text{ for some } p \mid DN \text{ or } (c_R, DN) \neq 1, \\ h(R) \cdot 2^{|\{p \,\mid\, DN, (\delta_K/p) = -\varepsilon_p\}|} & \text{otherwise.} \end{cases} \tag{2.2}$$

CM-points arise in a natural way as fixed points of Atkin–Lehner involutions on $X_0(D,N)$. Indeed, for any $m \mid DN$, the set $\mathcal{F}_m$ of fixed points of ω_m on $X_0(D,N)$ is

$$\mathcal{F}_m = \begin{cases} \mathrm{CM}(-4) \cup \mathrm{CM}(-8) & \text{if } m = 2, \\ \mathrm{CM}(-m) \cup \mathrm{CM}(-4m) & \text{if } m \equiv 3 \mod 4, \\ \mathrm{CM}(-4m) & \text{otherwise.} \end{cases} \tag{2.3}$$

Under our assumptions on D and N, the groups $\Gamma_0(D,N)$ contain no parabolic elements and the only elliptic points on these curves are of order 2 or 3. In fact, the set of elliptic points of order $i = 2,3$ is $\mathrm{CM}(\delta_i)$. Thus, its cardinality e_i is

$$e_i = \begin{cases} 0 & \text{if there exists a } p \mid DN, (\delta_i/p) = \varepsilon_p, \\ 2^{r-1} & \text{if } i \mid DN \text{ and } (\delta_i/p) \neq \varepsilon_p \text{ for any } p \mid DN, \\ 2^r & \text{otherwise.} \end{cases} \tag{2.4}$$

By [19, pp. 280, 301] the genus of $X_0(D,N)$ is

$$g = g(X_0(D,N)) = 1 + \frac{DN}{12} \cdot \prod_{p \mid D} \left(1 - \frac{1}{p}\right) \cdot \prod_{p \mid N} \left(1 + \frac{1}{p}\right) - \frac{e_3}{3} - \frac{e_2}{4}.$$

The next lemma is a particular case of [18, § 1, Hilfsatz 1].

Lemma 2.1. *Let X be an irreducible curve over a field k with $\mathrm{char}(k) \neq 2$. If $\mathrm{Aut}(X) \simeq (\mathbb{Z}/2\mathbb{Z})^s$ for some $s \geqslant 1$ and $P \in C(k)$ is a regular point, then*

$$\mathrm{Stab}(P) = \{\omega \in \mathrm{Aut}(X) : \omega(P) = P\}$$

has order 1 or 2.

2.1. *Proof of Theorem* 1.6

Let $X = X_0(D,N)$ and $A = \mathrm{Aut}(X_0(D,N))$.

(i) By (2.4) the group $\Gamma_0(D,N) \subset \mathrm{PSL}_2(\mathbb{R})$ has neither elliptic nor parabolic elements. According to Proposition 1.2, $A = B_{\Gamma_0(D,N)} = W_0(D,N)$.

(ii) As is clear from (2.2) and (2.3), the assumptions of part (ii) imply that the set $\mathcal{F}_m$ of fixed points of ω_m is non-empty. Since all automorphisms of X commute with ω_m by Proposition 1.5, we deduce that A acts on $\mathcal{F}_m$.
Assume that either $m = 2$ or $m \equiv 3 \bmod 4$ and $h(-4m) > h(-m)$ or $2 \mid DN$. Set $S_1 = \mathrm{CM}(-4)$, $S_2 = \mathrm{CM}(-8)$ if $m = 2$; $S_1 = \mathrm{CM}(-m)$, $S_2 = \mathrm{CM}(-4m)$ otherwise. By (2.3), $\mathcal{F}_m = S_1 \cup S_2$. Moreover, (2.2) guarantees that at least one of S_1 and S_2 is non-empty. In fact, when $m \equiv 3 \bmod 4$, we have $S_1 \neq \emptyset$. When $m = 2$, $S_1 \subseteq X(\mathbb{Q}(\sqrt{-1}))$ and $S_2 \subseteq X(\mathbb{Q}(\sqrt{-2}))$ by (2.1). As all automorphisms of X are defined over $\mathbb{Q}$ by Proposition 1.5, A leaves both S_1 and S_2 invariant. When $m \equiv 3 \bmod 4$, by (2.1) any point in S_1 generates the Hilbert class field of $K = \mathbb{Q}(\sqrt{-m})$, which is an abelian extension of K of degree $h(-m)$. Similarly, $S_2 = \emptyset$ if $2 \mid DN$ by (2.2); otherwise, any point in S_2 generates an extension of K of degree $h(-4m)$. Since $h(-4m) > h(-m)$, we conclude as above that A fixes the sets S_1 and S_2.
Hence, in any case, A acts on a non-empty set $S\,(= S_1,\ S_2$ or $\mathcal{F}_m)$ with $|S| = h_{D,N}(m) \cdot 2^{\sigma_{D,N}(m)}$. By Lemma 2.1 the stabilizer of any of the elements of S in A is exactly $\langle \omega_m \rangle$. Thus $A/\langle \omega_m \rangle$ acts freely on S, and (ii) follows.

(iii) Let $Y = X/A$ and let $\pi : X \to Y$ be the natural projection map. By Proposition 1.5, π is a finite morphism of degree 2^s which ramifies precisely at the set $\mathcal{F}$ of all fixed points of automorphisms of X. Riemann–Hurwitz's formula applied to π says that

$$g(X) - 1 = 2^s(g(Y) - 1) + \tfrac{1}{2} \cdot |\mathcal{F}|.$$

Hence, it suffices to show that $\mathrm{ord}_2(|\mathcal{F}|) \geqslant s-1$. However, this fact readily follows from an induction argument, since for any $u \in A$ one has

$$|\mathcal{F}| = 2 \cdot |\mathcal{F}'|, \quad \text{where } \mathcal{F}' = \{[x] \in X/\langle u\rangle : \omega([x]) = [x] \text{ for some } \omega \in A/\langle u\rangle\}.$$

3. *Cerednik–Drinfeld theory*

Fix a prime $p \mid D$. Let k_p denote the quadratic unramified extension of $\mathbb{Q}_p$, $\mathbb{Q}_p^{\mathrm{unr}}$ the maximal unramified extension of $\mathbb{Q}_p$ and $\mathbb{Z}_p^{\mathrm{unr}}$ its ring of integers.

Let $\mathcal{O}^{(p)}$ be an Eichler order of level N in a definite quaternion algebra of discriminant D/p and fix an immersion $\mathcal{O}^{(p)} \hookrightarrow \mathrm{M}_2(\mathbb{Q}_p)$. Let

$$\Gamma_+ = \{\gamma \in (\mathcal{O}^{(p)} \otimes \mathbb{Z}[1/p])^* : \mathrm{ord}_p(\det(\gamma)) \in 2\mathbb{Z}\}/\mathbb{Z}[1/p]^* \hookrightarrow \mathrm{PGL}_2(\mathbb{Q}_p),$$

which is a finitely generated discontinuous subgroup of $\mathrm{PGL}_2(\mathbb{Q}_p)$.

We warn the reader that Γ_+ may not be torsion-free: for $m = 2, 3$, any embedding

$$\mathbb{Z}[\sqrt{\delta_m}] \overset{\varphi}{\hookrightarrow} \mathcal{O}^{(p)}$$

produces an element $\gamma = \varphi(\sqrt{\delta_m})$ in Γ_+ of order m. However, by [8, p. 19] there exists a torsion-free normal subgroup Γ_+^0 of finite index in Γ_+. The group Γ_+^0 is thus a Schottky group; let $X_{\Gamma_+^0} = \Gamma_+^0 \backslash (\mathbb{P}^1_{\mathbb{Q}_p} - \mathcal{L}_{\Gamma_+^0})$ denote the Mumford curve over $\mathbb{Q}_p$ attached to Γ_+^0 as in [8]. If its genus g is at least 2, $X_{\Gamma_+^0}$ has stable totally split reduction over $\mathbb{Q}_p$.

Since $A = \Gamma_+^0 \backslash \Gamma_+$ is a finite group that lies naturally in $\mathrm{Aut}(X_{\Gamma_+^0})$, there exists an algebraic curve X_{Γ_+} over $\mathbb{Q}_p$ which is the quotient of $X_{\Gamma_+^0}$ by A.

Theorem 3.1 (Cerednik–Drinfeld). *Let $\chi : \mathrm{Gal}(k_p/\mathbb{Q}_p) \to \mathrm{Aut}(X_{\Gamma_+} \otimes k_p)$, let $\mathrm{Fr} \mapsto \omega_p$ and let $X_{\Gamma_+}^{\chi}$ be the quadratic twist of X_{Γ_+} by χ. Then*

$$X_0(D,N) \times \mathbb{Q}_p \simeq X_{\Gamma_+}^{\chi}.$$

We refer the reader to [2] for a proof. See also [10].

Let $\mathcal{T}$ denote the Bruhat–Tits tree attached to $\mathbb{Q}_p$. Following [14, § 4, 5] and [10, § 3], the special fibre of $M_0(D,N) \otimes \mathbb{Z}_p$ is described up to a quadratic twist by the finite graph $\mathcal{G} = \Gamma_+ \backslash \mathcal{T}$, regarded as a *graph with lengths*.

Each vertex v and edge e of $\mathcal{G}$ is decorated with the order $\ell(v)$ or $\ell(e)$ of the stabilizer of v or e, respectively, in Γ_+, which we call its length. Geometrically, a vertex v corresponds to an irreducible rational component C_v of $M_0(D,N)_p$. An edge e of length ℓ joining v and v' corresponds to an intersection point P_e of $C_v \cap C_{v'}$ locally at which the scheme $M_0(D,N) \times \mathbb{Z}_p^{\mathrm{unr}}$ is isomorphic to $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{unr}}[X,Y]/(XY - p^{\ell}))$. In particular, any automorphism of $X_0(D,N)$ induces an automorphism of $\mathcal{G}$ that leaves the set of edges of given length invariant.

Let $h = h(D/p, N)$. The number of vertices of $\mathcal{G}$ is $2h$ and that of the edges is $h(D/p, Np)$. (According to [14, formula (4.1)] and its notation, the number of vertices of $\Gamma_0 \backslash \Delta$ is $h(D/p, 1)$. Since the index of Γ_+ in Γ_0 is 2, the number of vertices of $\Gamma_+ \backslash \Delta$ is $2h(D/p, 1)$. These formulas extend to the case of non-trivial level N without difficulty.) The set $\mathrm{Ver}(\mathcal{G})$ of vertices of $\mathcal{G}$ may be written as

$$\mathrm{Ver}(\mathcal{G}) = V \cup V', \quad V = \{v_1, \dots, v_h\}, \quad V' = \{v'_1, \dots, v'_h\}, \tag{3.1}$$

in such a way that the Atkin–Lehner involution $\omega_p \in \mathrm{Aut}(X_0(D,N))$ acts on $\mathcal{G}$ as $\omega_p(v_i) = v'_i$. There are no edges in $\mathcal{G}$ joining two vertices from the same set V or V', and hence there are no loops in $\mathcal{G}$. We have $\ell(v_i) = \ell(v'_i)$ and the number of edges of given length joining a vertex v_i with v'_j coincides with that of v'_i with v_j.

For a vertex v, it holds that $\ell(e) \mid \ell(v)$ for all edges e in its star and

$$\sum_{e\in\mathrm{Star}(v)} \frac{\ell(v)}{\ell(e)} = p+1. \tag{3.2}$$

When $(D/p, N)$ equal to $(2,1)$ or $(3,1)$, $\mathcal{G}$ consists of two vertices v, v' of length 12 and 6, respectively, joined by $g+1$ edges. Assume otherwise that $(D/p, N)$ equal to $(2,1), (3,1)$. Then all lengths of vertices and edges are 1, 2 or 3. By [14, (4.2)], for $\ell = 2, 3$, the cardinality h_ℓ of vertices in V of length ℓ is

$$h_\ell = \frac{1}{2}\prod_{q \mid \frac{DN}{p}} \left(1 - \varepsilon_q \cdot \left(\frac{\delta_\ell}{q}\right)\right). \tag{3.3}$$

If v is a vertex of length $\ell = 2, 3$, by [14, Proposition 4.2] the number of edges of length ℓ in its star is as follows.

$$|\mathrm{Star}_\ell(v)| = 1 + \left(\frac{\delta_\ell}{p}\right) \in \{0, 1, 2\}. \tag{3.4}$$

The scheme $M_0(D, N)$ is regular if and only if $\ell(e) = 1$ for all edges of $\mathcal{G}$. In general, a desingularization $\tilde{M}_0(D, N)$ of $M_0(D, N)$ is obtained by blowing -up, $\ell(e) - 1$ times, each singular point P_e. The resulting dual graph $\tilde{\mathcal{G}}$ is constructed from $\mathcal{G}$ by replacing each edge e of length $\ell(e) \geqslant 2$, by a chain of $\ell(e)$ edges of length 1 each (cf. [10, Proposition 3.6]).

In general, $M_0(D, N)$ is neither a minimal regular model nor a stable model of $X_0(D, N)$. The next proposition, which may be of independent interest, shows how to construct these two models.

Proposition 3.2. *Assume that $g = g(X_0(D, N)) \geqslant 1$.*

(i) *Let $M_0(D, N)_{\min}$ denote the minimal regular model of $X_0(D, N)$.*

- *If $p > 2$ or $h_3 = 0$, then $M_0(D, N)_{\min} = \tilde{M}_0(D, N)$.*
- *If $p = 2$ and $h_3 \geqslant 1$, then $M_0(D, N)_{\min}$ is the blow-down of all components C_v of $\tilde{M}_0(D, N)$ with $\ell(v) = 3$.*

Its dual graph $\mathcal{G}_{\min}$ is obtained from $\tilde{\mathcal{G}}$ by erasing v and $\mathrm{Star}(v)$ for all vertices of length 3.

(ii) *Assume that $g \geqslant 2$, and let $M_0(D, N)_{\mathrm{st}}$ denote the stable model of $X_0(D, N)$.*

- *If $p \neq 2, 3$ or $h_2 = h_3 = 0$, then $M_0(D, N)_{\mathrm{st}} = M_0(D, N)$.*
- *If $p = 2$, then $M_0(D, N)_{\mathrm{st}}$ is the blow-down of all components C_v of $M_0(D, N)$ with $\ell(v) = 2, 3$.*
- *If $p = 3$, then $M_0(D, N)_{\mathrm{st}}$ is the blow-down of all components C_v of $M_0(D, N)$ with $\ell(v) = 2, 3$.*

Its dual graph $\mathcal{G}_{\mathrm{st}}$ is obtained from $\mathcal{G}$ by

- *if $p = 2$, erasing v and $\mathrm{Star}(v)$ for all v with $\ell(v) = 3$;*
- *if $p = 2$ or 3, replacing each chain*

$$v \overset{e}{-} v' \overset{e'}{-} v''$$

such that $\ell(v') = 2$ or 3, by $v \overset{(e'')}{-} v''$ with $\ell(e'') = \ell(e) + \ell(e')$.

Proof. (i) By construction, $\tilde{M}_0(D, N)$ is regular. By Castelnuovo's criterion (cf. [15, pp. 416–417]), $\tilde{M}_0(D, N)$ is minimal over $\mathbb{Z}_p$ if and only if there exist no irreducible rational components E in $\tilde{M}_0(D, N)_p$ which intersect the remaining components at a *single* point. This is equivalent to saying that there exists no vertex v in $\tilde{\mathcal{G}}$ with $|\mathrm{Star}(v)| = 1$. Since $|\mathrm{Star}(v)| = 2$ for those vertices that were newly created when constructing $\tilde{\mathcal{G}}$ from $\mathcal{G}$, we can work directly with $\mathcal{G}$.

If $(D/p,N)=(2,1)$ or $(D/p,N)=(3,1)$, then the vertices v and v' are joined by $g+1\geqslant 2$ edges. Thus $\widetilde{M}_0(D,N)$ is minimal.

Assume that $(D/p,N)\neq(2,1),(3,1)$. By (3.2) and (3.4), $|\mathrm{Star}(v)|=1$ for a vertex in $\mathcal{G}$ exactly when $p=2$ and $\ell(v)=3$. The minimal regular model is then obtained by blowing down the corresponding components C_v.

(ii) By definition, $M_0(D,N)$ is stable if $|\mathrm{Star}(v)|\geqslant 3$ for all vertices v of $\mathcal{G}$. By (3.2) and (3.4), $|\mathrm{Star}(v)|<3$ exactly when $p=2,3$ and $\ell(v)=2,3$. When this holds, the stable model is achieved by blowing down all these irreducible components. □

The incidence matrix of $\mathcal{G}$ can be recovered (and explicitly computed) from the theory of Brandt modules and matrices. That is, let $M:=M_{\mathcal{O}^{(p)};p}\in\mathrm{M}_h(\mathbb{Z})$ denote the Brandt matrix attached to $\mathcal{O}^{(p)}$ and the prime number p. Let I_i, $i=1,\dots,h$, be a set of representatives of left ideals of $\mathcal{O}^{(p)}$ up to principal ideals. By definition, $M(i,j)$ is the number of integral ideals of reduced norm p, which are equivalent on the right to $I_i^{-1}\cdot I_j$.

Recall that there exist no edges joining vertices v_i, v_j (and the same holds for v_i', v_j'). The number of edges e joining two given vertices v_i and v_j' (which equals that of edges joining v_j and v_i') can be computed by means of (3.4) and the formula

$$M(i,j)=\sum_{v_i\overset{e}{\to}v_j'}\frac{\ell(v_i)}{\ell(e)}. \tag{3.5}$$

In particular it always holds that $M(i,j)/\ell(v_i)=M(j,i)/\ell(v_j)$. This completely determines $\mathcal{G}$. Note that (3.2) implies that $\sum_{j=1}^{h}M(i,j)=p+1$ for each row $i=1,\dots,h$.

3.1. *Proof of Theorem* 1.7

Definition 3.3. An automorphism $\omega\in\mathrm{Aut}(\mathcal{G})$ is *admissible* if there exists no vertex v in $\mathcal{G}$ fixed by ω such that $\mathrm{Star}(v)$ contains three different edges e_1,e_2,e_3 also fixed by ω. A subgroup $A\subseteq\mathrm{Aut}(\mathcal{G})$ is *admissible* if any $\omega\in A$, $\omega\neq\mathrm{Id}$, is admissible.

Proposition 3.4. *Assume that* $g(\,X_0(D,N)\,)\geqslant 2$. *Then there exists a monomorphism* $\varrho:\mathrm{Aut}(X_0(D,N))\hookrightarrow\mathrm{Aut}(\mathcal{G})$ *that embeds* $\mathrm{Aut}(X_0(D,N))$ *into an admissible subgroup of* $\mathrm{Aut}(\mathcal{G})$.

Proof. By [**5**, I.12; **15** chapter IX], there exists a natural monomorphism

$$\mathrm{Aut}(X_0(D,N))\hookrightarrow\mathrm{Aut}(M_0(D,N)_{\mathrm{st}}\times\mathbb{F}_p).$$

Since $M_0(D,N)$ is the blow-up of $M_0(D,N)_{\mathrm{st}}$ over $\mathbb{Z}_p$ at finitely many points, there is a birational morphism $M_0(D,N)\to M_0(D,N)_{\mathrm{st}}$ that induces an embedding

$$\mathrm{Aut}(M_0(D,N)_{\mathrm{st}}\times\mathbb{F}_p)\hookrightarrow\mathrm{Aut}(M_0(D,N)\times\mathbb{F}_p).$$

By considering the action on the irreducible components and singular points of the special fibre at p, any $\omega\in\mathrm{Aut}(X_0(D,N))$ induces through the above embeddings an automorphism $\varrho(\omega)$ of $\mathcal{G}$ as a graph with lengths. Let $\varrho:\mathrm{Aut}(X_0(D,N))\hookrightarrow\mathrm{Aut}(\mathcal{G})$ be the resulting map. By construction, it is clearly a group homomorphism.

Assume that $\varrho(\omega)=\mathrm{Id}$. Then, the action of ω on $M_0(D,N)_{\mathrm{st}}\times\mathbb{F}_p$ would fix all its irreducible components and intersection points. Since a non-trivial automorphism of the projective line has at most two fixed points, we conclude that $\omega=\mathrm{Id}$. Hence ϱ is a monomorphism and $\varrho(\omega)$ is admissible for any $\omega\neq\mathrm{Id}$. □

As we mentioned above, the Atkin–Lehner involution ω_p acts on $\mathcal{G}$, as $\omega_p(v_i)=v_i'$.

Proposition 3.4 provides an explicit method for proving in many instances that $s = r$. Indeed, the graph $\mathcal{G}$ can be computed by means of David Kohel's *Brandt modules* package implemented in MAGMA. Namely, for a given $p \mid D$, Brandt's matrix $M := M_{\mathcal{O}^{(p)};p}$ can be computed by MAGMA V:=BrandtModule($D/p, N$); M:=HeckeOperator(V,p);. For a prime $q \mid D/p$, the action of the Atkin-Lehner involution ω_q on V is obtained by MAGMA W_q:=HeckeOperator(V,q);. The action of ω_q on the set E of edges of $\mathcal{G}$ is obtained by MAGMA E:=BrandtModule($D/p, Mp$); W'_q:=HeckeOperator(E,q);.

Proof of Theorem 1.7. Throughout we may assume that $(D, N) \neq (2p, 1), (3p, 1)$, where p denotes a prime number, as these cases are covered by Theorem 1.4(v).

(i) Let $m = 2$. Fix a prime $q \mid D$. Choose q to be the (single) prime such that $(-4/q) = 1$ in case it exists. Let $\mathcal{G}$ be the dual graph of $M_0(D, N)_q$. In Ver($\mathcal{G}$) there exist $2 \cdot h_2 = 2^{r-1}$ vertices of length 2 if $2 \nmid DN$ (and 2^{r-2} if $2 \mid DN$). Since $q \neq 2$ and the maximal 2-elementary subgroup of

$$\mathrm{Aut}(\mathbb{P}^1_{\mathbb{F}_{q^2}}) = \mathrm{PGL}_2(\mathbb{F}_{q^2})$$

is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ (cf., for example, [**13**, Lemma 1.3]), it follows that the stabilizer of any vertex has order at most 4. Hence $s \leqslant r + 1$ if $2 \nmid DN$; $s = r$ if $2 \mid DN$.
The proof for $m = 3$ follows along the same lines and one again obtains that if $(-3/p) \neq \varepsilon_p$ for all $p \mid DN$ except at most for one prime divisor of D then $s \leqslant r + 1$ if $3 \nmid DN$; $s = r$ if $3 \mid DN$.

(ii) Let $h = h(D/p, N)$. There is a well-defined action of $\mathrm{Aut}(X_0(D, N))/\langle\omega_p\rangle$ on the subset $V = \{v_1, \dots, v_h\}$ of vertices of the dual graph $\mathcal{G}$ of $M_0(D, N)_p$. As above, the stabilizer of each of these vertices has order at most 4 and the statement follows.

(iii) As we mentioned, there is a canonical embedding of $\mathrm{Aut}(X_0(D, N))$ into $\mathrm{Aut}(M_0(D, N)_\ell)$. Since $\mathrm{ord}_2(0) = \infty$, we can assume that $M_0(D, N)_\ell(\mathbb{F}_\ell) \neq \emptyset$. Our claim now follows immediately from Lemma 2.1 applied to any point $P \in M_0(D, N)_p(\mathbb{F}_\ell)$. □

3.2. *A numerical result*

Let us illustrate how our results above serve to prove that there exist no exceptional automorphisms in many Shimura curves.

Example. Let $D = 5 \cdot 41$. We have $g(X_D) = 13$. None of the items of Theorems 1.6 or 1.7 applies to show that all automorphisms of X_D are modular. Since $h(41, 1) = 4$, the dual graph $\mathcal{G}$ of the special fibre of X_D at $p = 5$ consists of eight vertices $\{v_1, \dots, v_4, v'_1, \dots, v'_4\}$, of which v_1 and v'_1 have length 3 while the remainder have length 1. Moreover, all edges have length 1. One computes that

$$M = \begin{pmatrix} 0 & 3 & 0 & 3 \\ 1 & 0 & 3 & 2 \\ 0 & 3 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad W_{41} = \mathrm{Id}_V.$$

In fact, the action of any involution $\omega \in \mathrm{Aut}(\mathcal{G})/\langle\omega_5\rangle$ fixes each vertex $v \in V$, because $\mathrm{Star}\,(v_i)$ are pairwise different for $i = 1, \dots, 4$. Looking at the action of ω on the rational component C_{v_1}, the two points P, P' corresponding to the edges joining v_1 with v'_2 and v'_4 are necessarily fixed points of $\tilde{w}$. Since $\mathrm{Aut}(\tilde{C}_{v_1}, P, P') \simeq \mathbb{Z}/2\mathbb{Z}$, it follows that $\mathrm{Aut}(X_D) = \mathrm{Aut}(\mathcal{G}) = \langle\omega_5, \omega_{41}\rangle$.

Proposition 3.5. *For $D \leqslant 1500$, the only automorphisms of X_D are the Atkin–Lehner involutions, provided that $g(X_D) \geqslant 2$ and $D \neq 493$, 583, 667, 697, 943.*

Proof. If the number of prime factors of D is $r \geqslant 4$, then Theorem 1.6(i) and Corollary 1.9 show that $s = r$.

Assume now that $D = p \cdot q$ is the product of two primes. By applying Theorems 1.4(v), 1.6(i) and 1.7(iii) for the 300 first primes $\ell \neq p, q$, we prove the statement for all such D except for a list L of 44 values of D; we do not reproduce the list here for the sake of brevity.

For the values of D in L, we apply Proposition 3.4 as in the above example. This way we are able to prove that $s = 2$ for all D except for $D = 85$, 145, 493, 583, 667, 697, 943. Let us illustrate what is going on with some examples.

When $D = 697$, let $\mathcal{G}$ denote the dual graph of the reduction mod $p = 17$. With the notation above, it turns out that $V = \{v_1, v_2, v_3, v_4\}$, with $\ell(v_1) = 3$, $\ell(v_i) = 1$ for $i = 2, 3, 4$ and one computes that the permutation ω of the vertices v_2 and v_3 is an admissible involution which commutes with the Atkin–Lehner involutions. Hence there exists an admissible subgroup of $\mathrm{Aut}(\mathcal{G})$ which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ and we cannot prove that $s = 2$.

When $D = 1057$, let $\mathcal{G}$ denote the dual graph of the reduction mod $p = 7$. One computes that $|V| = 13$. Note that any automorphism of X_{1057} must induce a permutation $\omega \in S_{13}$ of the vertices in V which commutes with the matrices M and W_{151}. When computing M and W_{151}, one shows that there exist exactly sixteen such permutations ω_k, $k = 1, \dots, 16$. However, for each of them, it turns out that either ω_k or $\omega_k \cdot W_{151}$ is *not* admissible. Thus $s = 2$.

It remains to prove our proposition for $D = 85$, 145. For $D = 145$, it has already been shown in the proof of [**22**, Theorem 7] that $\mathrm{Aut}(X_{145}) \simeq \langle \omega_5, \omega_{29} \rangle$. One shows the same similarly for $D = 85$: as obtained from William Stein's data basis, there exist isogenies

$$\mathrm{Jac}(X_{85}) \sim \mathrm{Jac}(X_0(85))^{\mathrm{new}} \sim E \times S_1 \times S_2,$$

defined over $\mathbb{Q}$, where E is a (modular) elliptic curve and S_1, S_2 are modular abelian surfaces over $\mathbb{Q}$.

The modularity implies that $\mathrm{End}(S_i)$ are orders in a real quadratic field. Hence, the only automorphisms of finite order of E, S_1 or S_2 are $\pm\mathrm{Id}$. Composing with this isogeny, we obtain a monomorphism

$$\mathrm{Aut}(X_{85}) \hookrightarrow A := \{\pm\mathrm{Id}_E\} \times \{\pm\mathrm{Id}_{S_1}\} \times \{\pm\mathrm{Id}_{S_2}\}.$$

By [**19**], X_{85} is not hyperelliptic, and this is saying that $(-\mathrm{Id}_E, -\mathrm{Id}_{S_1}, -\mathrm{Id}_{S_2})$ does not lie in $\mathrm{Aut}(X_{85})$. Thus its index in A is at least 2 and we conclude that $s = 2$.

These ideas appear to be insufficient to prove the same result for $D = 493$, 583, 667, 697 or 943. For these D, we can just claim that $s \leqslant 3$, by Corollary 1.8. □

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