Available online at www.sciencedirect.com

ScienceDirect

Journal of Pure and Applied Algebra 210 (2007) 551–558

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Polydifferentials and the deformation functor of curves with automorphisms

A. Kontogeorgis

Department of Mathematics, University of the Ægean, 83200 Karlovassi, Samos, Greece

Received 18 July 2006; received in revised form 6 September 2006
Available online 1 December 2006
Communicated by E.M. Friedlander

Abstract

We give a relation between the dimension of the tangent space of the deformation functor of curves with automorphisms and the Galois module structure of the space of 2-holomorphic differentials. We prove a homological version of the local–global principle similar to the one of J. Bertin and A. Mézard. Let G be a cyclic subgroup of the group of automorphisms of a curve X, so that the order of G is equal to the characteristic. By using the results of S. Nakajima on the Galois module structure of the space of 2-holomorphic differentials, we compute the dimension of the tangent space of the deformation functor.

MSC: 14H37; 14H10

1. Introduction

Let X be a non-singular complete curve of genus $g \geq 2$ defined over an algebraic closed field k of positive characteristic p, and let G be a subgroup of the automorphism group of X. In [1] Bertin and Mézard proved that the equivariant cohomology of Grothendieck $H^1(G, \mathcal{T}_X)$ is the tangent space of the global deformation functor of smooth curves with automorphisms. The dimension of the k-vector space $H^1(G, \mathcal{T}_X)$ is a measure of the directions in which a curve can be deformed together with a subgroup of the automorphism group.

Since the genus g of X is $g \geq 2$, the edge homomorphisms of the spectral sequence of Grothendieck [4, 5.2.7] give us

$$H^1(G, \mathcal{T}_X) = H^1(X, \mathcal{T}_X)^G, \tag{1}$$

where $\mathcal{T}_X$ is the tangent sheaf of the curve X and $H^1(X, \mathcal{T}_X)$ is the first Čech cohomology group. It is known that $H^1(X, \mathcal{T}_X)$ is the tangent space of the deformation functor of smooth curves (without considering any group action) [5, p. 89]. Eq. (1) tells us that the first equivariant cohomology equals the G-invariant space of the "tangent space" to the moduli space at the point-curve X. J. Bertin and A. Mézard were successful in computing $H^1(G, \mathcal{T}_X)$, using a version of equivariant Čech theory, and proved a local–global theorem.

E-mail address: kontogar@aegean.gr.
URL: http://eloris.samos.aegean.gr.

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2006.10.015

In order to compute the G-invariants of the space $H^1(G, \mathcal{T}_X)$, it is tempting to use Serre's duality to transfer the computation to the space $H^0(G, \Omega_X^{\otimes 2})$. One should be careful using this approach, because we are now considering the dual space of $H^1(X, \mathcal{T}_X)$ and it is not the functor of invariants we have to compute, but the adjoint functor, i.e., the functor of covariants.

We begin our study, by examining the relation between V^G and V_G^* when G is a p-group, V is a k-vector space acted on by G and V^* denotes the dual space of V. This point of view will allow us to prove that $\dim_k H^1(G, \mathcal{T}_X) = \dim_k H^0(G, \Omega_X^{\otimes 2})_G$. Furthermore, in order to compute the covariant elements of $H^0(X, \Omega_X^{\otimes 2})$, we use the normal basis theorem for Galois extensions and the explicit form of Serre duality in terms of repartitions [6, 7.14.2], [12, I.5]. This leads us to a homological version of the local–global theorem of J. Bertin and A. Mézard (8).

Finally, we apply known results on the Galois module structure of the space of holomorphic differentials, in order to compute covariant elements. We have to notice here, that, as far as the author knows, if the characteristic p divides $|G|$, the Galois module structure of $H^0(X, \Omega^{\otimes s})$ is far from being understood, and there are only partial results in the case of tame ramification [7,11,9], in the case of ordinary curves and $s = 1$ and when G is a cyclic group of order equal to the characteristic [10].

More precisely, Nakajima in [10] studied the Galois module of $H^0(G, \mathcal{L}(D))$ if G is isomorphic to a cyclic group of order p, and D is a G-invariant divisor of degree $> 2g_X - 2$, where g_X denotes the genus of the curve X. We apply the results of Nakajima in order to compute $\dim_k H^0(G, \Omega_X^{\otimes 2})_G = \dim_k H^1(G, \mathcal{T}_X)$ and we are able to recover the result of J. Bertin and A. Mézard concerning the computation of the dimension of the later space.

Recently, tools from modern representation theory were used in order to extend the results of Nakajima, [2,8]. We hope that these tools can be applied to give new information on the dimension of the deformation functor of curves with automorphisms.

2. Dual actions

Let V be a k-vector space equipped with a left G-action. Denote by $V^G = \{v \in V : gv = v\}$ the space of invariant elements. Let also $V^* = \mathrm{Hom}_k(V, k)$ denote the dual space of V. The group G acts on V^* in terms of the contragradient action $gf : v \mapsto f(g^{-1}v)$. It is wrong to assume in general that $(V^*)^G = (V^G)^*$.

We consider the restriction map

$$V^* = \mathrm{Hom}_k(V, k) \to \mathrm{Hom}_k(V^G, k) \to 0,$$

sending $f : V \to k$ to the restriction $f|_{V^G}$. The kernel A of the restriction map consists of

$$V^* \ni f \text{ such that } f(v) = 0 \quad \text{for all } v \in V^G.$$

We have the following short exact sequence:

$$1 \to A \to V^* \to (V^G)^* \to 0. \tag{2}$$

For a $k[G]$-module M let us denote by $M_G = M/\langle (g-1)m \rangle = M \otimes_{k[G]} k$, i.e. the M_G is the module M if we factor out the module generated by elements of the form $gm - m$ for all $g \in G$ and for all $m \in M$. The module M_G will be called the module of covariants of the G-action. If we consider covariants in (2) we have the following long exact sequence:

$$\cdots \to H_1(G, (V^G)^*) \to A_G \to V_G^* \to ((V^G)^*)_G \to 0.$$

Observe that $(V^G)^* = \mathrm{Hom}_k(V^G, k)$ is a G-invariant space, therefore $((V^G)^*)_G = (V^G)^*$.

Lemma 2.1. *If G is a p-group then the image of A_G in V_G^* is zero.*

Proof. Let W be a k-vector space of characteristic $p > 0$, such that $V = V^G \oplus W$. Notice that this is a direct sum in the category of vector spaces. One of the difficulties that arise in modular representation theory is that W can not in general be selected so that W is a G-module, i.e., the above direct sum of vector spaces is not a direct sum of $k[G]$-modules.

Consider an element g of order p in the p-group G. Using Jordan decomposition, we have that for a suitable basis of V, g acts on V in terms of a block diagonal matrix $\mathrm{diag}(B_1, \ldots, B_t)$ where each component B_i is an $r_i \times r_i$ matrix that acts on a subspace V_i of V and B_i is of the form:

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} = I_r + N_r,$$

where I_r is the identity and N_r is a nilpotent element, such that $N_r^{\ell} = 0$ if and only if $\ell \geq r$. Since the elements B_i have order p, and by the above remark we have that $r \leq p$.

From the special form of the matrices B_i we conclude that there is an eigenvector of every V_i that is left invariant under the action of the cyclic group generated by g. Let $e_{i,1}, \ldots, e_{i,r_i}$ be a basis of the vector space V_i so that the action of g on V_i is expressed by B_i. Then $e_{i,1}$ is a generator of the g-invariant subspace, while $ge_{i,\nu} = e_{i,\nu} + e_{i,\nu-1}$ for all other elements (set $e_{i,0} = 0$). Thus, for every $\kappa \geq \nu$

$$g^{\kappa} e_{i,\nu} = \sum_{\mu=0}^{\nu} \binom{\kappa}{\mu} e_{i,\nu-\mu}.$$

Let us consider the g-invariant element $e_{i,1}$. Pick an arbitrary

$$V \ni v = \sum_{i=1}^{t} \sum_{\nu=1}^{r_i} \lambda_{i,\nu}(v) e_{i,\nu},$$

where $\lambda_{i,\nu}$ is the dual basis of V^*, i.e., the coefficients of v in the expression of v is a linear combination of the basis elements $e_{i,\nu}$. Fix $i_0 \in \{1, \ldots, t\}$. Then

$$g^{-1}\lambda_{i_0,1}(v) - \lambda_{i_0,1}(v) = \lambda_{i_0,1}(gv) - \lambda_{i_0,1}(v) = \lambda_{i_0,2}(v).$$

This proves that for all i the functions $\lambda_{i,2} \in A_G$ have image 0 in V_G^*. We compute in the same way that

$$g^{-\kappa}\lambda_{i_0,1}(v) - \lambda_{i_0,1}(v) = \sum_{\mu=1}^{k} \binom{k}{\mu} \lambda_{i_0,\mu+1}(v).$$

Notice that for all $k < p$ the binomial coefficient $\binom{k}{\mu} \neq 0$ so an induction argument gives that all $\lambda_{i,\nu}$ have image zero in V_G^*. □

3. (Co)Homological computations

It is proved by J. Bertin and A. Mézard that the tangent space to the global deformation functor of smooth curves with automorphisms is given in terms of Grothendieck equivariant cohomology [1,3,4], i.e., $H^1(G, \mathcal{T}_X)$, while the tangent space of the deformation functor of smooth curves is $H^1(X, \mathcal{T}_X)$.

Proposition 3.1. *There is an isomorphism* ψ

$$\psi : H^1(G, \mathcal{T}_X) \to H^1(X, \mathcal{T}_X)^G \subset H^1(X, \mathcal{T}_X).$$

Proof. We will follow the computation of Bertin and Mézard [1, 3.1] of $H^1(G, \mathcal{T}_X)$ in terms of Čech cohomology. Let $\{U_i\}$ be an open affine covering consisting of G-stable open sets U_i. Let ζ_i^{σ} be a family of G-derivations, i.e., elements in $\Gamma(U_i, \mathcal{T}_X)$, and let δ_{ij} be Čech-cocycles in $\Gamma(U_i \cap U_j, \mathcal{T}_X)$. Then the equivariant cohomology is given by

$$H^1(G, X) = \frac{\{\{\zeta_i^{\sigma}\}, \{\delta_{ij}\}\}}{\{\{\sigma\gamma_i - \gamma_i\}, \{\gamma_j - \gamma_i\}\}}, \tag{3}$$

where $\sigma\gamma_i - \gamma_i$ is a family of principal G-derivations and $\gamma_j - \gamma_i$ is a family of 1-Čech coboundaries, and moreover

$$\zeta_j^\sigma - \zeta_i^\sigma = \sigma(\delta_{ij}) - \delta_{ij}. \tag{4}$$

The desired function $H^1(G, \mathcal{T}_X) \to H^1(X, \mathcal{T}_X)$ is formed by forgetting the family of derivations $\{f_i^\sigma\}$, i.e., by sending

$$\frac{\{\{\zeta_i^\sigma\}, \{\delta_{ij}\}\}}{\{\{\sigma\gamma_i - \gamma_i\}, \{\gamma_j - \gamma_i\}\}} \mapsto \frac{\{\delta_{ij}\}}{\{\gamma_j - \gamma_i\}} \in H^1(X, \mathcal{T}_X).$$

This is a well defined morphism and moreover (4) implies that the image is G-invariant.

This map is onto $H^1(X, \mathcal{T}_X)^G$ because if $\{\delta_{ij}\}$ is a G-invariant Čech cocycle, this means that for every $\sigma \in G$, $\sigma(\delta_{ij}) - \delta_{ij}$ is a Čech coboundary, i.e., there are elements ζ_i^σ such that $\sigma(\delta_{ij}) - \delta_{ij} = \zeta_i^\sigma - \zeta_j^\sigma$, and one can check that the function $\sigma \mapsto \zeta_i^\sigma$ is a G-derivation.

The kernel of the map ψ consists of equivalence classes of elements of the form $\{\{\zeta_i^\sigma\}, \{\delta_{ij}\}\}$ so that $\delta_{ij} = a_i - a_j$, i.e., δ_{ij} is cohomologous to zero, so an element in the kernel can be represented by $\{\{\zeta_i^\sigma\}, \{0\}\}$. Then Eq. (4) gives us that $\zeta_i^\sigma = \zeta_j^\sigma$, so $\sigma : G \to \{\zeta_i^\sigma\}$ can be interpreted as a function $G \to H^0(X, \mathcal{T}_X) = 0$, since $g \geq 2$. This proves that the kernel of ψ is zero. □

Denote by K_X the function field of the curve X. Let $\mathcal{K}_X$ be the constant sheaf K_X, and consider the exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{K}_X \to \frac{\mathcal{K}_X}{\mathcal{O}_X} \to 0. \tag{5}$$

The sheaf $\frac{\mathcal{K}_X}{\mathcal{O}_X}$ can be expressed in the form

$$\frac{\mathcal{K}_X}{\mathcal{O}_X} = \bigoplus_{P \in X} i_*(K_X/\mathcal{O}_P),$$

where $i : \{P\} \to X$ is the inclusion map.

We tensor the sequence (5) with the sheaf $\Omega_X^{\otimes 2}$ over $\mathcal{O}_X$ and get the sequence:

$$0 \to \Omega_X^{\otimes 2} \to \mathcal{K}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes 2} \to \bigoplus_{P \in X} i_*(K_X/\mathcal{O}_P) \otimes \Omega_X^{\otimes 2} \to 0.$$

We will denote by $\mathcal{M}^{\otimes 2} = \mathcal{K}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes 2}$ the sheaf of meromorphic 2-differentials and by $\Omega_P^{\otimes 2} = \Omega_X^{\otimes 2} \otimes_{\mathcal{O}_X} \mathcal{O}_P$. Thus we might write

$$\bigoplus_{P \in X} i_*(K_X/\mathcal{O}_P) \otimes \Omega_X^{\otimes 2} = \bigoplus_{P \in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2}).$$

We apply the global section functor:

$$0 \to \Gamma(X, \Omega_X^{\otimes 2}) \to \Gamma(X, \mathcal{K}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes 2}) \to \bigoplus_{P \in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2}) \to H^1(X, \Omega_X^{\otimes 2}) \to \cdots.$$

Since X is a curve of genus $g \geq 2$ we have that $H^1(X, \Omega_X^{\otimes 2}) = 0$ and if we denote by $\Omega = \Gamma(X, \Omega_X^{\otimes 2})$ and $M = \Gamma(X, \mathcal{M}^{\otimes 2})$ the spaces of global sections of homomorphic and meromorphic differentials we have:

$$0 \to \Omega \to M \to \Gamma\left(X, \bigoplus_{P \in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2})\right) \to 0. \tag{6}$$

Lemma 3.2. *Let $Y = X/G$ be the quotient group of the action of G on X. The G-module M as a $K_Y[G]$ module is projective.*

Proof. Let ω be a meromorphic differential of the curve $Y = X/G$, and denote by K_Y the function field of the curve Y. The lift $\pi^*\omega$ is a G-invariant meromorphic differential on X, and M can be recovered as the set of the expressions

$$M = \{f \cdot \pi^*(\omega), f \in K_X\}.$$

We want to apply the functor of covariants, i.e., to tensor with $K_Y \otimes_{K_Y[G]}$. We notice first that by the normal basis theorem [13, 6.3.7 p. 173] for the Galois extension K_X/K_Y we obtain that $K_X \cong K_Y[G]$ as a Galois module, thus M is isomorphic to $K_Y[G]$ as a $K_Y[G]$-module and the desired result follows. $\square$

We consider the long exact homology sequence arising from (6) after taking the functor of covariants:

$$\cdots \to H_1(G, M) \to H_1\left(G, \Gamma\left(X, \bigoplus_{P\in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2})\right)\right) \to \Omega_G \xrightarrow{\alpha} M_G$$
$$\to \Gamma\left(X, \bigoplus_{P\in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2})\right)_G \to 0. \tag{7}$$

Since $M \cong K_Y[G]$ we have $H_1(G, M) = 0$ and $M_G = \{f \cdot \pi_*(\omega)\}$, with $f \in K_Y$.

Thus

$$\Omega_G = H_1\left(G, \Gamma\left(X, \bigoplus_{P\in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2})\right)\right) \oplus \mathrm{Im}\alpha.$$

Remark. If the order $|G|$ of the group G is prime to the characteristic p then the order $|G|$ is invertible in the module $\Gamma(X, \bigoplus_{P\in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2}))$ and the first homology is zero, therefore

$$\Omega_G = \mathrm{Im}\alpha.$$

Proposition 3.3. *Let $b_1, \ldots, b_r$ be the set of ramification points of the cover $X \to Y$, and let $G_i = G(b_i)$ be the corresponding decomposition groups. The following holds:*

$$H_1\left(G, \bigoplus_{P\in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2})\right) = \left(\bigoplus_{i=1}^{r} H_1(G_i, \mathcal{M}^{\otimes 2}/\Omega_{b_i}^{\otimes 2})\right).$$

Proof. Let P be a point of X, and let t_P be a local uniformizer at the point P. Consider an element $a = \sum_{P\in X} a_P P \in (\bigoplus_{P\in X} i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2}))$. Let ω be a 2-holomorphic differential. Every meromorphic 2-differential can be written as a product $f\omega$, $f \in \mathcal{K}_X$. The t_P-expansion of $f\omega$ at P is given by

$$\left(\sum_{\nu=-n}^{-1} \frac{a_\nu}{t_P^\nu} + \sum_{\nu=0}^{\infty} a_\nu t_P^\nu\right)\omega, \quad \text{for some } n \geq 0.$$

The finite sum $\sum_{\nu=-n}^{-1} \frac{a_\nu}{t_P^\nu}$ is called the polar part of $f\omega$ at P and it can be identified with the coefficient a_P of a at P.

The action of an element $g \in G$ on $f\omega$ is given by $(f\omega)^g = f^g\omega^g$. Since ω^g is still a 2-holomorphic differential we have a well defined action of G on $\mathcal{M}^{\otimes 2}/\Omega^{\otimes 2}$. If we express $f\omega$ in terms of a local uniformizer t_P at P in the following form

$$f\omega = \left(\sum_{\nu=-n}^{-1} \frac{a_\nu}{t_P^\nu} + \sum_{\nu=0}^{\infty} a_\nu t_P^\nu\right) \cdot \omega,$$

then

$$\begin{aligned}(f\omega)^g &= \left(\sum_{\nu=-n}^{-1} \frac{a_\nu}{g(t_P)^\nu} + \left(\sum_{\nu=0}^{\infty} a_\nu t_P^\nu\right)^g\right) \cdot \omega^g \\ &= \left(\sum_{\nu=-n}^{-1} \frac{a_\nu}{g(t_P)^\nu}\right)\omega^g \quad \text{modulo holomorphic 2 differentials.}\end{aligned}$$

The element $g(t)$ is a local uniformizer at the point $g(P)$. Let $G(P) = \{g \in G : g(P) = P\}$ be the decomposition group at the point P and write G as a disjoint union of cosets: $G = \cup_{\nu=1}^{[G:G(P)]} g_i G(P)$. If $g = g_i h$, $h \in G(P)$ then the action of the element $g \in G$ on a is of the form

$$g\left(\sum_{P\in X} a_P P\right) = \sum_{P\in X} h(a_P) g_i(P).$$

Let $M_P = i_*(\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2})$ be the summand corresponding to the point P. We consider the induced module, seen as a subspace of $\oplus_{P\in X}\mathcal{M}^{\otimes 2}/\Omega_P^{\otimes 2}$,

$$\mathrm{Ind}_{G(P)}^G M_P = K_Y[G]\otimes_{K_Y[G(P)]} M_P = \bigoplus_{g\in G/G(P)} M_{g(P)}.$$

For every point Q of Y choose a point $P_Q \in X$. We have

$$\bigoplus_{P\in X} M_P = \bigoplus_{Q\in Y}\bigoplus_{g\in G/G(P)} M_{g(P_Q)} = \bigoplus_{Q\in Y}\mathrm{Ind}_{G(P_Q)}^G M_{P_Q},$$

where the above two direct sums are direct sums of G-modules. Shapiro's lemma [13, 6.3.2] implies that

$$H_1(G, \mathrm{Ind}_{G(P)}^G M_P) = H_1(G(P), M_P).$$

Thus if P is not a ramification point it does not contribute to the cohomology, and the desired formula comes by the sum of the contributions of the ramification points. □

We have proved that the following sequence is exact:

$$0 \to \left(\bigoplus_{i=1}^r H_1(G(b_i), \mathcal{M}^{\otimes 2}/\Omega_{b_i}^{\otimes 2})\right) \to \Omega_G \to \mathrm{Im}\alpha \to 0 \tag{8}$$

which is exactly the dual sequence of Bertin and Mézard [1, p. 206].

Proposition 3.4. *Let $b_1, \dots, b_r$ be the ramification points of the cover $\pi : X \to Y$, and assume that the groups in the ramification filtration at each ramification point b_k have orders*

$$e_0^{(k)} \geq e_1^{(k)} \geq \cdots \geq e_{n_k}^{(k)} > 1.$$

The dimension of the space $\mathrm{Im}\alpha$ *is given by:*

$$\dim_k \mathrm{Im}\alpha = 3g_Y - 3 + \sum_{k=1}^r \left\lceil 2\sum_{i=1}^{n_k}\frac{e_i^{(k)}-1}{e_0^{(k)}}\right\rceil.$$

Proof. We are looking for elements of the form $f\pi^*(\omega)$, $f \in K_Y$, such that $\mathrm{div}_X f\pi^*(w) \geq 0$. We know that if ω is a 2-differential on Y then:

$$\mathrm{div}(\pi^*(\omega)) = \pi^*(\mathrm{div}\omega) + 2R, \tag{9}$$

where R is the ramification divisor of $\pi : X \to Y$. Therefore $\pi^*(w)$ is holomorphic if and only if $\pi^*(\mathrm{div}\omega) + 2R \geq 0$.

We will push forward again and we will use Riemann–Roch on Y. We want to compute the dimension of the space

$$L(2K_Y + \pi_*(2R)/|G|),$$

where K_Y is the canonical divisor on Y.

The ramification divisor is

$$R = \sum_{k=1}^r \sum_{i=1}^{n_k}(e_i^{(k)} - 1)b_k,$$

and the Riemann–Roch theorem implies that

$$\dim_k \mathrm{Im}\alpha = 3g_Y - 3 + \sum_{k=1}^r \left\lceil 2\sum_{i=1}^{n_k}\frac{e_i^{(k)}-1}{e_0^{(k)}}\right\rceil. \quad \square$$

4. The cyclic group case

In this section suppose that $G = \langle\sigma\rangle$, $\sigma^p = 1$. The $k[G]$-module structure of the module of 2-differentials is described by Nakajima in [10]. Following the article of Nakajima, we introduce the following notation: Assume that $X \to X^G = Y$ is a branched Galois cover with Galois group the cyclic group G of order p.

Let t_P be a normaliser of the local ring at $P \in X$ and let v_P denote the corresponding valuation. Set $N_i + 1 := v_P(\sigma(t_P) - t_P)$. Denote by V the $k[G]$-module with k-basis $\{e_1, \dots, e_p\}$ and action given by $\sigma e_\ell = e_\ell + e_{\ell-1}$, $e_0 = 0$. Let V_j be the subspace of V generated by $\{e_1, \dots, e_j\}$. The vector spaces V_j are $k[G]$-modules. Using the theory of Jordan normal forms of matrices, we can show that every $k[G]$-module is isomorphic to a direct sum of V_j. Therefore,

$$H^0(X, \Omega_X^{\otimes 2}) = \sum_{j=1}^{p} m_j V_j \tag{10}$$

as a direct sum of $k[G]$-modules. We observe that $\dim_k V_G = 1$, therefore

$$\dim_k H^0(X, \Omega_X^{\otimes 2})_G = \sum_{j=1}^{p} m_j. \tag{11}$$

There is a 2-differential on Y so that the support of $\operatorname{div}(\omega)$ has empty intersection with the branch locus. Using (9) we obtain that

$$n_i := v_{P_i}(\operatorname{div}(f^*\omega)) = v_{P_i}(2R) = 2(N_i + 1)(p - 1).$$

Let g_Y denote the genus of the curve Y and let $[\cdot]$ denote the integer part function, i.e., for every $x \in \mathbb{R}$, $[x] \in \mathbb{Z}$, $[x] \le x < [x] + 1$. The integers m_j that appear in (10) are computed by Nakajima in [10, p. 90]

$$m_p := 3g_Y - 3 + \sum_{i=1}^{p} \left[\frac{n_i - (p-1)N_i}{p}\right],$$

and for $j = 1, \dots, p - 1$,

$$m_j = \sum_{i=1}^{r} \left\{-\left[\frac{n_i - jN_i}{p}\right] + \left[\frac{n_i - (j-1)N_i}{p}\right]\right\}.$$

Using the above values of m_j and (11) we obtain that

$$\begin{aligned}
\dim_k H^0(X, \Omega_X^{\otimes 2})_G &= \sum_{j=1}^{p} m_j \\
&= 3g_Y - 3 - \sum_{i=1}^{p} \left[\frac{n_i - (p-1)N_i}{p}\right] + \sum_{j=1}^{p-1}\sum_{i=1}^{r} \left\{-\left[\frac{n_i - jN_i}{p}\right] + \left[\frac{n_i - (j-1)N_i}{p}\right]\right\} \\
&= 3g_Y - 3 - \sum_{i=1}^{r} \left[\frac{n_i - (p-1)N_i}{p}\right] + \sum_{i=1}^{r} \left[\frac{n_i}{p}\right] - \sum_{i=1}^{r} \left[\frac{n_i - (p-1)N_i}{p}\right] \\
&= 3g_Y - 3 + \sum_{i=1}^{r} \left[\frac{n_i}{p}\right] = 3g_Y - 3 + \sum_{i=1}^{r} \left[\frac{2(N_i + 1)(p-1)}{p}\right].
\end{aligned}$$

The later result coincides with the result of Bertin and Mézard in [1]. (Notice that on pages 235–236 of [1] $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ have to be interchanged everywhere.)

Acknowledgement

The author would like to thank the anonymous referee for his remarks and corrections.

References

[1] José Bertin, Ariane Mézard, Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques, Invent. Math. 141 (1) (2000) 195–238. MR 2001f:14023.

[2] Niels Borne, Cohomology of G-sheaves in positive characteristic, Adv. Math. 201 (2) (2006) 454–515. MR MR2211535.

[3] Gunther Cornelissen, Fumiharu Kato, Equivariant deformation of Mumford curves and of ordinary curves in positive characteristic, Duke Math. J. 116 (3) (2003) 431–470. MR 1 958 094.

[4] Alexander Grothendieck, Sur quelques points d'algèbre homologique, Tôhoku Math. J. (2) 9 (1957) 119–221. MR 21 #1328.

[5] Joe Harris, Ian Morrison, Moduli of Curves, in: Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998. MR MR1631825 (99g:14031).

[6] Robin Hartshorne, Algebraic Geometry, in: Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York, 1977.

[7] Ernst Kani, The Galois-module structure of the space of holomorphic differentials of a curve, J. Reine Angew. Math. 367 (1986) 187–206. MR 88f:14024.

[8] Bernhard Köck, Galois structure of Zariski cohomology for weakly ramified covers of curves, Amer. J. Math. 126 (5) (2004) 1085–1107. MR MR2089083 (2005i:11163).

[9] Shōichi Nakajima, On Galois module structure of the cohomology groups of an algebraic variety, Invent. Math. 75 (1) (1984) 1–8. MR 85j:14021.

[10] Shōichi Nakajima, Action of an automorphism of order p on cohomology groups of an algebraic curve, J. Pure Appl. Algebra 42 (1) (1986) 85–94. MR MR852320 (88d:14018).

[11] Shōichi Nakajima, Galois module structure of cohomology groups for tamely ramified coverings of algebraic varieties, J. Number Theory 22 (1) (1986) 115–123. MR 87i:14010.

[12] Henning Stichtenoth, Algebraic Function Fields and Codes, Springer-Verlag, Berlin, 1993. MR 94k:14016.

[13] Charles A. Weibel, An Introduction to Homological Algebra, Cambridge University Press, Cambridge, 1994. MR 95f:18001.