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The equivariant Hilbert series of the canonical ring of Fermat curves

Hara Charalambous[a], Kostas Karagiannis[b], Sotiris Karanikolopoulos[b], Aristides Kontogeorgis[b,*]

[a] Department of Mathematics, Aristotle University of Thessaloniki School of Sciences, 54124, Thessaloniki, Greece

[b] Department of Mathematics, National and Kapodistrian University of Athens Panepistimioupolis, 15784 Athens, Greece

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Abstract

We consider a Fermat curve $F_n : x^n + y^n + z^n = 0$ over an algebraically closed field K of characteristic $p \geq 0$ and study the action of the automorphism group $G = (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \rtimes S_3$ on the canonical ring $R = \bigoplus H^0(F_n, \Omega_{F_n}^{\otimes m})$ when $p > 3$, $p \nmid n$ and $n-1$ is not a power of p. In particular, we explicitly determine the classes $[H^0(F_n, \Omega_{F_n}^{\otimes m})]$ in the Grothendieck group $K_0(G, K)$ of finitely generated $K[G]$-modules, describe the respective equivariant Hilbert series $H_{R,G}(t)$ as a rational function, and use our results to write a program in Sage that computes $H_{R,G}(t)$ for an arbitrary Fermat curve.

Keywords: Hilbert series; Equivariant; Canonical ring; Group action; Curves; Automorphisms; Holomorphic differentials

1. Introduction

1.1. Graded representations

Let K be an algebraically closed field of characteristic $p \geq 0$, let G be a finite group and let $K_0(G, K)$ denote the Grothendieck group of the category of finitely generated

* Corresponding author.

E-mail addresses: hara@math.auth.gr (H. Charalambous), konstantinos.karagiannis@manchester.ac.uk (K. Karagiannis), sotiriskaran@gmail.com (S. Karanikolopoulos), kontogar@math.uoa.gr (A. Kontogeorgis).

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$K[G]$-modules; it is well-known, see for example [25, Part III] that $K_0(G, K)$ is generated by the irreducible representations of G over K and that it becomes a commutative ring with unit with respect to $\otimes_K$. If V is a $K[G]$-module, we denote by $[V]$ its image in $K_0(G, K)$ and recall that if $\mathrm{char}(k) \nmid |G|$, then $[V]$ determines uniquely the isomorphism class of V, whereas if $\mathrm{char}(k) \mid |G|$ this is no longer true.

Next, we consider a finitely generated, $\mathbb{N}$-graded K-algebra $R = \bigoplus_{d=0}^{\infty} R_d$ whose graded components R_d are finite dimensional K-vector spaces acted on by G. The respective representations $\rho_d : G \to \mathrm{GL}(R_d)$ give rise to a formal power series

$$H_{R,G}(t) = \sum_{d=0}^{\infty} [R_d] t^d, \text{ where } [R_d] \in K_0(G, K),$$

which is called the *equivariant Hilbert series* of the action of G on R. Note that if G is trivial, then $[R_d]$ is just the dimension of R_d as a K-vector space, and thus $H_{R,G}(t)$ generalizes the classic, non-equivariant Hilbert series of R. At the same time, it encodes all the invariants of the action of G on R, which are infinitely many, in a finite, rational expression, see [9,27]. To explicitly compute $H_{R,G}(t)$ when $\mathrm{char}(K) \nmid G$, one can follow the approach described in Stanley's exposition [23].

Each $[R_d] \in K_0(G, K)$ is uniquely determined by its decomposition $R_d = \bigoplus n_{d,V} V$ as a direct sum of irreducible $K[G]$-modules; this in turn gives rise to the following decomposition of the graded K-algebra R:

$$R = \bigoplus_{d=0}^{\infty} \bigoplus_{V \in \mathrm{Irrep(G)}} n_{d,V} V = \bigoplus_{V \in \mathrm{Irrep(G)}} \bigoplus_{d=0}^{\infty} n_{d,V} V.$$

The graded K-algebras $R_V^G := \bigoplus_{d=0}^{\infty} n_{d,V} V$ are called the *isotypical components* of the action of G on R and we obtain the identity

$$H_{R,G}(t) = \sum_{V \in \mathrm{Irrep}(G)} \sum_{d=0}^{\infty} n_{d,V} [V] t^d.$$

Thus, the computation of the equivariant Hilbert series $H_{R,G}(t)$ is reduced to determining the multiplicities $n_{d,V} \in \mathbb{N}$ and studying the convergence of the respective power series for each $V \in \mathrm{Irrep}(G)$. This is theoretically doable using character theory, at least in the case of ordinary representations. However, computations become increasingly hard and thus one needs to take into account specific properties of R and G to get concrete results.

The most well studied case is when R is the polynomial ring in n variables over some algebraically closed field K of characteristic 0. The isotypical component corresponding to the trivial representation is then by definition the ring of invariants R^G and every isotypical component R_V^G for $V \in \mathrm{Irrep}(G)$ becomes naturally an R^G-module. Thus, the study of the equivariant Hilbert series in this context falls under classic invariant theory, while a beautiful result of Molien provides an explicit formula for $H_{R,G}(t)$; for an overview of the subject and some striking applications to combinatorics the reader may refer to Stanley's exposition [23].

Himstedt and Symonds in [9] studied equivariant Hilbert series in a generalized setting by dropping the assumption on $\mathrm{char}(K)$ and considering finitely generated graded R-modules M where R in turn is a finitely generated K-algebra. This generalized setting gives a geometric flavor to graded representation theory, as the results of Himstedt and Symonds are applicable to line bundles $\mathscr{L}$ on projective curves which are equivariant under finite group actions.

From this viewpoint, the study of equivariant Hilbert series relates to that of equivariant Euler characteristics, as expected from the non-equivariant case; the latter is an active area of research and has far-reaching applications, from normal integral bases of sheaf cohomology [6], to Dedekind zeta functions [28] to ramification theory [13,20], to mention a few.

This approach naturally gives a connection between equivariant Hilbert series and the classic problem of determining the Galois module structure of polydifferentials on projective curves. The problem was originally posed by Hecke and settled by Chevalley–Weil in [5] for characteristic 0 curves; their results were generalized by Ellinsgrud and Lonsted in [7] when $\mathrm{char}(K) = p \nmid |G|$ while for modular representation theory the general case remains open and there exist only partial results. Finally, the case of integral representations, which naturally contains both ordinary and modular representation theory, has been studied only in very specific cases, see for example [12,22].

To make things more concrete, we consider a pair (X, G), where X is a smooth, projective curve of genus $g \geq 4$ over K which is not hyperelliptic, and G is a finite subgroup of its automorphism group. If $\Omega_{X/K}$ denotes the sheaf of holomorphic differentials on X, then a classical result of Max Noether ensures that the *canonical map*

$$\phi : \mathrm{Sym}\left(H^0(X, \Omega_{X/K})\right) \twoheadrightarrow \bigoplus H^0(X, \Omega_{X/K}^{\otimes d})$$

is surjective, giving rise to the *canonical embedding* $X \hookrightarrow \mathbb{P}_k^{g-1}$; for a modern treatment of the subject see [24]. The respective homogeneous coordinate ring $R = \bigoplus H^0(X, \Omega_{X/K}^{\otimes d})$, called *the canonical ring*, is a graded K-algebra acted on by G. In this setting, the equivariant Hilbert series of interest is

$$H_{R,G}(t) = \sum_{d=0}^{\infty} [H^0(X, \Omega_{X/K}^{\otimes d})] t^d$$

and its computation requires determining the $K[G]$-module structure of the K-vector spaces $H^0(X, \Omega_{X/K}^{\otimes d})$ of global holomorphic polydifferentials.

This paper settles the problem in the case when $X = F_n$ is a *Fermat curve* given by the equation $x^n + y^n + z^n = 0$ and G is its automorphism group. If $n - 1$ is not a power of the characteristic of the ground field K, then the automorphism group G is isomorphic to $(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \rtimes S_3$, see [16,30]. The case $\mathrm{char}(K) \mid n$ needs to be excluded as well, since in this case the Fermat curve is not reduced. We also exclude the characteristics 2,3 so the representation we consider is ordinary. We can‘t resist to point out that the group G appears as the analogue of $\mathrm{GL}_3(\mathbb{F}_{1^n})$, that is the general linear group with entries in the degree n extension $\mathbb{F}_{1^n}$ of the mythical "field" with one element, see [11]. These groups play a significant role in knot theory [8], and are isomorphic to the complex reflection groups $G(d, 1, n)$, see [18].

Our results are original, as there only exist explicit formulas for the $K[G]$-module structure of global holomorphic m-differentials on Fermat curves when $m = 1$, see [17, Section 6]. The author of [17] makes essential use of the results of [5,7] and the computations are based on ramification data and information on the local characters of the corresponding stabilizers. This is a well-studied technique, which has been further developed by several authors, see for example [10,13,19,21]. It would be probably possible to obtain the $K[G]$-module structure of global holomorphic m-differentials on Fermat curves using the results of Chevalley–Weil, Ellinsgrud–Lonsted, Kani, Nakajima and Köck, however our approach in this paper is different;

we explicitly compute the characters of the irreducible representations of G and the characters of the K-vector spaces $H^0(X, \Omega_{X/K}^{\otimes d})$, then take their inner product to obtain our results. In the process, we study sums of roots of unity which resemble Kloosterman sums and apply combinatorial techniques to count lattice points that satisfy modular congruences. It would definitely be interesting to inspect whether one could arrive at the same results by applying the techniques of the above mentioned authors; this is one of the directions we would like to explore in the future.

1.2. Outline

Our main result is the explicit computation of the Galois module structure of holomorphic polydifferentials on Fermat curves, the calculation of the respective equivariant Hilbert series $H_{R,G}(t)$ and a computer program in Sage [26] that computes $H_{R,G}(t)$ for an arbitrary Fermat curve.

In Section 2 we review some preliminaries on Fermat curves F_n and their automorphism groups G. Since the latter are given as semidirect products, we use a classic result of Serre in Proposition 1, to construct all the irreducible representations of G. The list and the respective character table are given in Proposition 3. We proceed in Section 3 to obtain the characters of the action of G on the global sections of holomorphic m-differentials. The standard bases which are given in the bibliography, see Proposition 6, are not suitable for computations; motivated by our work in [4] on the canonical embedding of smooth, projective curves, we rewrite m-differentials as a quotient of two K-vector spaces which are easier to manipulate and give the respective characters in Proposition 11.

The inner products necessary to obtain the decomposition of m-differentials as a direct sum of the irreducible representations of Proposition 3 reduce to computing various sums of roots of unity which are interesting in their own sake. We thus devote Section 4 to their explicit computation and in Proposition 14 we obtain an equivalent characterization of these sums as the number lattice points inside a triangle which satisfy certain modular congruences. We then proceed with counting the cardinality of these lattices in Proposition 16 and obtain the exact values necessary for the subsequent sections in Corollary 17.

Section 5 contains the explicit formulas for the Galois module structure of the global sections of holomorphic differentials. We compute the inner products of the irreducible characters of Section 2 with the characters of Section 3 using the results of Section 4. The main formulas are summarized in Theorem 25, and we verify our results in Table 2 by giving the explicit decomposition of m-differentials for $m \in \{0, \dots, 9\}$ in the case of Fermat curves F_4, F_5 and F_6. Finally, in Section 6 we obtain an explicit expression for the equivariant Hilbert series as a rational function: the main results are summarized in Theorem 26 and we once more verify the computations by applying the results to get the equivariant Hilbert function in the case of the Fermat curve F_6. We remark that even though the formulas of Theorems 25 and 26 are complicated, they are appropriate for computations as they have allowed us to write a program in Sage which takes as input the value of n that determines the Fermat curve F_n and outputs the equivariant Hilbert function of its canonical ring. The code is uploaded in one of the authors' website.

2. The irreducible representations of the automorphism group

Let K be an algebraically closed field of characteristic $p \geq 0$ and let $n \in \mathbb{N}$. In this section we will study the representation theory of the semidirect product $G = A \rtimes S_3$ over K, where

$$S_3 = \langle s, t : s^3 = t^2 = 1,\ tst = s^{-1}\rangle = \{1, s, s^2, t, st, ts\}$$
$$A = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} = \{\sigma_{\alpha,\beta} : 0 \le \alpha, \beta \le n-1\},$$

and the action of S_3 on A is given by

$$s \cdot \sigma_{\alpha,\beta} = \sigma_{\beta-\alpha,-\alpha},\ s^2 \cdot \sigma_{\alpha,\beta} = \sigma_{-\beta,\alpha-\beta},\ t \cdot \sigma_{\alpha,\beta} = \sigma_{-\alpha,\beta-\alpha},\ ts \cdot \sigma_{\alpha,\beta} = \sigma_{\beta,\alpha},$$
$$st \cdot \sigma_{\alpha,\beta} = \sigma_{\alpha-\beta,-\beta}.$$

In what follows, we assume that $n \ge 4$ and that $n \ne 1 + p^h$ for all $h \in \mathbb{N}$. Motivation for the construction of G and the assumptions on p and n comes from the fact that G can be realized as the automorphism group of a particular class of Fermat curves; more details will be given in the next section, and the reader may consult [16,30] and [2, Prop. 2.1.2] for the formation of G.

To describe the irreducible representations of G we will use the technique described by Serre in [25, section 8.2]. Let Ξ be the group of irreducible characters of A, with elements denoted by $\chi_{\kappa,\lambda},\ 0 \le \kappa, \lambda \le n-1$. Then S_3 acts on Ξ and we have

$$h\chi_{\kappa,\lambda}(\sigma_{\alpha,\beta}) = \chi_{\kappa,\lambda}(h^{-1}\sigma_{\alpha,\beta}h),$$

where $\chi_{\kappa,\lambda} \in \Xi, \sigma_{\alpha,\beta} \in A,\ h \in S_3$ and $\chi_{\kappa,\lambda}(\sigma_{\alpha,\beta}) = \zeta^{\alpha\kappa+\beta\lambda}$; we write $H_{\kappa,\lambda}$ for the stabilizer subgroup of S_3 with respect to $\chi_{\kappa,\lambda}$ and define $G_{\kappa,\lambda} = A \rtimes H_{\kappa,\lambda}$. Let ρ be an irreducible representation of $H_{\kappa,\lambda}$ and let $\widetilde{\rho}$ be the representation of $G_{\kappa,\lambda}$ obtained by composing ρ with the natural projection $G_{\kappa,\lambda} \to G_{\kappa,\lambda}/A = H_{\kappa,\lambda}$. Every character $\chi_{\kappa,\lambda} \in \Xi$ of A can be extended to a character of $G_{\kappa,\lambda}$ by defining $\chi_{\kappa,\lambda}(ah) = \chi_{\kappa,\lambda}(a)$, for $a \in A, h \in H_{\kappa,\lambda}$, and thus we may form the tensor product $\chi_{\kappa,\lambda} \otimes \widetilde{\rho}$.

Proposition 1 (*[25, Prop. 25]*). *The induced representation $\theta_{\kappa,\lambda,\rho} = \mathrm{Ind}_{G_{\kappa,\lambda}}^{G}\left(\chi_{\kappa,\lambda} \otimes \widetilde{\rho}\right)$ is an irreducible representation of G and all irreducible representations of G can be obtained in this manner, up to isomorphism. Further, it suffices to consider only one representative $\chi_{\kappa,\lambda}$ for each orbit of S_3 in Ξ, and different orbits give different irreducible representations.*

We will use the following explicit description of the action of S_3 on Ξ:

$h \in S_3$	$h^{-1}\sigma_{\alpha,\beta}h$	$\chi_{\kappa,\lambda}\left(h^{-1}\sigma_{\alpha,\beta}h\right)$	$h \cdot \chi_{\kappa,\lambda}$
s	$\sigma_{\beta-\alpha,-\alpha}$	$\zeta^{\alpha(-\kappa-\lambda)+\beta\kappa}$	$\chi_{-\kappa-\lambda,\kappa}$
s^2	$\sigma_{-\beta,\alpha-\beta}$	$\zeta^{\alpha\lambda+\beta(-\kappa-\lambda)}$	$\chi_{\lambda,-\kappa-\lambda}$
t	$\sigma_{-\alpha,\beta-\alpha}$	$\zeta^{\alpha(-\kappa-\lambda)+\beta\lambda}$	$\chi_{-\kappa-\lambda,\lambda}$
ts	$\sigma_{\beta,\alpha}$	$\zeta^{\alpha\lambda+\beta\kappa}$	$\chi_{\lambda,\kappa}$
st	$\sigma_{\alpha-\beta,-\beta}$	$\zeta^{\alpha\kappa+\beta(-\kappa-\lambda)}$	$\chi_{\kappa,-\kappa-\lambda}$

(1)

Thus, a character $\chi_{\kappa,\lambda} \in \Xi$ is fixed by s or s^2 if and only if $\kappa = \lambda = 0$ or $\kappa = \lambda = \frac{n}{3}$ or $\kappa = \lambda = \frac{2n}{3}$ when $3 \mid n$. It is fixed by t if and only if $\lambda = -2\kappa$, it is fixed by ts if and only if $\lambda = \kappa$ and it is fixed by st if and only if $\kappa = -2\lambda$. The trivial character is fixed by all S_3. This is summarized in the following table.

$\chi_{\kappa,\lambda} \in \Xi$	Stabilizer $H_{\kappa,\lambda}$	Orbit	Condition
$\chi_{0,0}$	S_3	$\{\chi_{0,0}\}$	
$\chi_{\frac{n}{3},\frac{n}{3}}$	S_3	$\{\chi_{\frac{n}{3},\frac{n}{3}}\}$	$3 \mid n$
$\chi_{\frac{2n}{3},\frac{2n}{3}}$	S_3	$\{\chi_{\frac{2n}{3},\frac{2n}{3}}\}$	$3 \mid n$
$\chi_{\kappa,\kappa}$	$\langle ts \rangle$	$\{\chi_{\kappa,\kappa}, \chi_{\kappa,-2\kappa}, \chi_{-2\kappa,\kappa}\}$	$\kappa \neq 0,\ \kappa \neq \frac{n}{3},\ \kappa \neq \frac{2n}{3}$
$\chi_{\kappa,-2\kappa}$	$\langle t \rangle$	$\{\chi_{\kappa,\kappa}, \chi_{\kappa,-2\kappa}, \chi_{-2\kappa,\kappa}\}$	$\kappa \neq 0,\ \kappa \neq \frac{n}{3},\ \kappa \neq \frac{2n}{3}$
$\chi_{-2\kappa,\kappa}$	$\langle st \rangle$	$\{\chi_{\kappa,\kappa}, \chi_{\kappa,-2\kappa}, \chi_{-2\kappa,\kappa}\}$	$\kappa \neq 0,\ \kappa \neq \frac{n}{3},\ \kappa \neq \frac{2n}{3}$
$\chi_{\kappa,\lambda}$	$\langle 1 \rangle$	$\{\chi_{\kappa,\lambda}, \chi_{-\kappa-\lambda,\kappa}, \chi_{\lambda,-\kappa-\lambda}, \chi_{-\kappa-\lambda,\lambda}, \chi_{\lambda,\kappa}, \chi_{\kappa,-\lambda-\kappa}\}$	$\kappa \neq \lambda, \lambda \neq -2\kappa, \kappa \neq -2\lambda$

Thus, a system of representatives for the orbits of S_3 in Ξ is given by $\{\chi_{0,0}, \chi_{\frac{n}{3},\frac{n}{3}}, \chi_{\frac{2n}{3},\frac{2n}{3}} \chi_{\kappa,\kappa}, \chi_{\kappa,\lambda}\}$ under the restrictions given in the last column of the above table. By Proposition 1, all irreducible representations of G are obtained by tensoring the above representatives with the irreducible representations of the three stabilizers $\{\langle 1\rangle, \langle ts\rangle, S_3\}$. Recall that S_3 has three irreducible representations denoted by $\rho_{\text{triv}}, \rho_{\text{sgn}}$ and ρ_{stan}.

• For $\nu \in \{0, 1, 2\}$, the stabilizer $H_{\frac{\nu n}{3},\frac{\nu n}{3}}$ of $\chi_{\frac{\nu n}{3},\frac{\nu n}{3}}$ equals the group S_3. Thus we have the representations $\theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{triv}}}, \theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{sgn}}}, \theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{stan}}}$. We write $g = \sigma_{\alpha,\beta}h,\ \sigma_{\alpha,\beta} \in A, h \in S_3$ for the arbitrary element $g \in G$, and obtain the corresponding characters

$$\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}\left(\sigma_{\alpha,\beta}h\right) = \zeta^{\frac{\nu n}{3}(\alpha+\beta)}\chi_\rho(h), \text{ where } \nu \in \{0, 1, 2\} \text{ and } \rho \in \{\rho_{\text{triv}}, \rho_{\text{sgn}}, \rho_{\text{stan}}\}.$$

• The stabilizer $H_{\kappa,\kappa} = \langle ts\rangle$ of $\chi_{\kappa,\kappa}$ for $\kappa \neq 0, \kappa \neq \frac{n}{3}, \kappa \neq \frac{2n}{3}$ has two one dimensional representations, ρ_{triv} and ρ_{sgn}, which give rise to the representations $\theta_{\kappa,\kappa,\rho_{\text{triv}}}$ and $\theta_{\kappa,\kappa,\rho_{\text{sgn}}}$ respectively. To compute the respective characters, recall that by Proposition 1, for $\rho \in \{\rho_{\text{triv}}, \rho_{\text{sgn}}\}$, we have that $\theta_{\kappa,\kappa,\rho} = \mathrm{Ind}_{G_{\kappa,\kappa}}^G\left(\theta_{\kappa,\kappa} \otimes \rho\right)$ and thus the induced representation character formula [25, Section 7.2] gives

$$\chi_{\kappa,\kappa,\rho}(g) = \frac{1}{|G_{\kappa,\kappa}|} \sum_{\substack{r\in G \\ r^{-1}gr\in G_{\kappa,\kappa}}} \chi_{\kappa,\kappa}\left(r^{-1}gr\right)\chi_\rho\left(r^{-1}gr\right), \text{ where } \rho \in \{\rho_{\text{triv}}, \rho_{\text{sgn}}\}. \tag{2}$$

Lemma 2. *Let $g = \sigma_{\alpha,\beta}h,\ r = \sigma_{\gamma,\delta}z \in G = A \rtimes S_3$ where $\sigma_{\alpha,\beta}, \sigma_{\gamma,\delta} \in A$ and $h, z \in S_3$. Then:*

(1) $r^{-1}gr \in G_{\kappa,\kappa} = A \rtimes \langle ts\rangle \Leftrightarrow z^{-1}hz \in \langle ts\rangle$.
(2) $\chi_{\kappa,\kappa}\left(r^{-1}gr\right) = z \cdot \chi_{\kappa,\kappa}\left(\sigma_{\alpha,\beta}\right)$.

Proof. For (1) we write

$$r^{-1}gr = z^{-1}\sigma_{\gamma,\delta}^{-1}\sigma_{\alpha,\beta}h\sigma_{\gamma,\delta}z = z^{-1}\sigma_{\gamma,\delta}^{-1}\sigma_{\alpha,\beta}h\sigma_{\gamma,\delta}h^{-1}zz^{-1}hz$$

and observe that $h\sigma_{\gamma,\delta}h^{-1} \in A \Rightarrow \sigma_{\gamma,\delta}^{-1}\sigma_{\alpha,\beta}h\sigma_{\gamma,\delta}h^{-1} \in A \Rightarrow z^{-1}\sigma_{\gamma,\delta}^{-1}\sigma_{\alpha,\beta}h\sigma_{\gamma,\delta}h^{-1}z \in A$, i.e.

$$r^{-1}gr = \underbrace{z^{-1}\overbrace{\sigma_{\gamma,\delta}^{-1}\sigma_{\alpha,\beta}\underbrace{h\sigma_{\gamma,\delta}h^{-1}}_{\in A}}^{\in A}z}_{\in A}z^{-1}hz$$

and so $r^{-1}gr \in A \rtimes \langle ts\rangle \Leftrightarrow z^{-1}hz \in \langle ts\rangle$.
(2) Follows from extending the action of S_3 on the irreducible characters of A to $G_{\kappa,\kappa}$. □

Thus, we may rewrite Eq. (2) as

$$\begin{aligned}\chi_{\kappa,\kappa,\rho}\left(\sigma_{\alpha,\beta}h\right) &= \frac{1}{2n^2}\sum_{\substack{z\in S_3\\ z^{-1}hz\in\langle ts\rangle}} z\chi_{\kappa,\kappa}\left(\sigma_{\alpha,\beta}\right)\chi_\rho\left(z^{-1}hz\right)\\ &= \frac{1}{2n^2}\sum_{\substack{z\in S_3\\ z^{-1}hz\in\langle ts\rangle}} z\chi_{\kappa,\kappa}\left(\sigma_{\alpha,\beta}\right)\chi_\rho\left(ts\right),\end{aligned}$$

If $h \in \{s, s^2\}$, then $z^{-1}hz$ has order 3 and therefore $z^{-1}hz \notin \langle ts\rangle$. If $h = 1$ then trivially $z^{-1}hz = 1$ for all z. Finally, for each $h \in S_3 - \langle s\rangle = \{t, ts, st\}$ there are two elements $z \in S_3$ such that $z^{-1}hz = ts$: $z = s^2$ or st for $h = t$, $z = 1$ or ts for $h = ts$, $z = s$ or t for $h = st$. Combining with table (1):

$$\chi_{\kappa,\kappa,\rho}(\sigma_{\alpha,\beta}h) = \begin{cases}\zeta^{\kappa(\alpha+\beta)} + \zeta^{\kappa(\alpha-2\beta)} + \zeta^{\kappa(-2\alpha+\beta)} & \text{, if } h = 1\\ \zeta^{\kappa(\alpha+\beta)}\chi_\rho(ts) & \text{, if } h = ts\\ \zeta^{\kappa(\alpha-2\beta)}\chi_\rho(ts) & \text{, if } h = t\\ \zeta^{\kappa(-2\alpha+\beta)}\chi_\rho(ts) & \text{, if } h = st\\ 0 & \text{, if } h = s, s^2\end{cases}$$

• Finally the generic $\chi_{\kappa,\lambda}$ has trivial stabilizer which admits only the trivial representation, so we have a unique representation $\theta_{\kappa,\lambda,\rho_{\text{triv}}} = \text{Ind}_{G_{\kappa,\lambda}}^{G}\left(\chi_{\kappa,\lambda}\right)$, with character

$$\chi_{\kappa,\lambda,\rho_{\text{triv}}}(g) = \frac{1}{n^2}\sum_{\substack{r\in G\\ r^{-1}gr\in A}} \chi_{\kappa,\lambda}(r^{-1}gr), \text{ or equivalently}$$

$$\chi_{\kappa,\lambda,\rho_{\text{triv}}}(\sigma_{\alpha,\beta}h) = \begin{cases}\chi_{\kappa,\lambda}(\sigma_{\alpha,\beta}) + \chi_{-\kappa-\lambda,\kappa}(\sigma_{\alpha,\beta}) + \chi_{\lambda,-\kappa-\lambda}(\sigma_{\alpha,\beta}) + \\ \chi_{\lambda,\kappa}(\sigma_{\alpha,\beta}) + \chi_{-\kappa-\lambda,\lambda}(\sigma_{\alpha,\beta}) + \chi_{\kappa,-\lambda-\kappa}(\sigma_{\alpha,\beta}) & \text{, if } h = 1\\ 0 & \text{, if } h \neq 1.\end{cases}$$

To summarize the above, the irreducible representations of G are given in Table 1

Table 1
The irreducible representations of G.

Case 1: If $n \nmid 3$:	
	1.1. $\theta_{0,0,\rho}$, where $\rho \in \{\rho_{\text{triv}}, \rho_{\text{sgn}}, \rho_{\text{stan}}\}$.
	1.2. $\theta_{\kappa,\kappa,\rho}$, where $\kappa \neq 0$ and $\rho \in \{\rho_{\text{triv}}, \rho_{\text{sgn}}\}$.
	1.3. $\theta_{\kappa,\lambda,\rho_{\text{triv}}}$, where $\kappa \neq \lambda, \lambda \neq -2\kappa, \kappa \neq -2\lambda$.
Case 2: If $n \mid 3$:	
	2.1. $\theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}$, where $\nu \in \{0, 1, 2\}$ and $\rho \in \{\rho_{\text{triv}}, \rho_{\text{sgn}}, \rho_{\text{stan}}\}$.
	2.2. $\theta_{\kappa,\kappa,\rho}$, where $\kappa \neq 0$, $\kappa \neq \frac{n}{3}$, $\kappa \neq \frac{2n}{3}$ and $\rho \in \{\rho_{\text{triv}}, \rho_{\text{sgn}}\}$.
	2.3. $\theta_{\kappa,\lambda,\rho_{\text{triv}}}$, where $\kappa \neq \lambda, \lambda \neq -2\kappa, \kappa \neq -2\lambda$.

and the character table of G is given by:

Proposition 3. *The irreducible characters of the group G are given in the following table*

Representation	***Degree***	***Character value*** $\chi(\sigma_{\alpha,\beta}h)$, *where* $h \in S_3, \sigma_{\alpha,\beta} \in A$	***Cases***
$\theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}$	1	$\zeta^{\frac{\nu n}{3}(\alpha+\beta)}\chi_\rho(h)$	1.1, 2.1
$\theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{stan}}}$	2	$\zeta^{\frac{\nu n}{3}(\alpha+\beta)}\chi_{\text{stan}}(h)$	1.1, 2.1
$\theta_{\kappa,\kappa,\rho}$	3	$\begin{cases} \zeta^{\kappa(\alpha+\beta)} + \zeta^{\kappa(\alpha-2\beta)} + \zeta^{\kappa(\beta-2\alpha)} & , \text{ if } h = 1 \\ \zeta^{\kappa(\alpha+\beta)}\chi_\rho(ts) & , \text{ if } h = ts \\ \zeta^{\kappa(\alpha-2\beta)}\chi_\rho(t) & , \text{ if } h = t \\ \zeta^{\kappa(\beta-2\alpha)}\chi_\rho(st) & , \text{ if } h = st \\ 0 & , \text{ if } h = s, s^2 \end{cases}$	1.2, 2.2
$\theta_{\kappa,\lambda,\rho_{\text{triv}}}$	6	$\begin{cases} \begin{pmatrix} \zeta^{\kappa\alpha+\lambda\beta} + \zeta^{-(\kappa+\lambda)\alpha+\kappa\beta} \\ + \zeta^{\lambda\alpha-(\kappa+\lambda)\beta} + \\ \zeta^{\lambda\alpha+\kappa\beta} + \zeta^{-(\kappa+\lambda)\alpha+\lambda\beta} \\ + \zeta^{\kappa\alpha-(\kappa+\lambda)\beta} \end{pmatrix} & , \text{ if } h = 1 \\ 0 & , \text{ if } h \neq 1 \end{cases}$	1.3, 2.3

3. Character tables for m-differentials

Let $F_n : x_1^n + x_2^n + x_0^n = 0$ be a Fermat curve defined over the algebraically closed field K of characteristic $p \geq 0$ introduced in the previous section. Recall that we assume that $p \nmid n$, $n \geq 4$ and $n \neq 1 + p^h$ for all $h \in \mathbb{N}$. The former assumption is to ensure that we deal with Fermat curves of genus $g \geq 2$, since it is well known that F_n has genus $g = \frac{(n-1)(n-2)}{2}$, see for example [29, Prop. 1]; we need the latter assumption to ensure that the automorphism group of F_n is given by the group $G = \mathrm{Aut}_K(F_n) = (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \rtimes S_3$, introduced in the previous section. This result is proven in [30], extending a preprint of Leopoldt, which was eventually published in [16].

In what follows, we will work with the affine model $F_n : x^n + y^n + 1 = 0$, obtained by setting $x = \frac{x_1}{x_0}$ and $y = \frac{x_2}{x_0}$. The action of G on F_n is then explicitly given by $\sigma_{\alpha,\beta}(x, y) = \left(\zeta^\alpha x, \zeta^\beta y\right)$ and

$$\begin{aligned} &s(x, y) = \left(\frac{y}{x}, \frac{1}{x}\right), \ t(x, y) = \left(\frac{1}{x}, \frac{y}{x}\right), \ st(x, y) = \left(\frac{x}{y}, \frac{1}{y}\right), \ ts(x, y) = (y, x), \\ &s^2(x, y) = \left(\frac{1}{y}, \frac{x}{y}\right). \end{aligned} \tag{3}$$

Let Ω_{F_n} denote the sheaf of holomorphic differentials on the Fermat curve F_n. More generally, for $m \geq 1$, we write $\Omega_{F_n}^{\otimes m}$ for the sheaf of holomorphic m-differentials. The global sections $H^0(F_n, \Omega_{F_n}^{\otimes m})$ form a vector space over the ground field K; its dimension is given by the Riemann–Roch Theorem

$$\dim_K H^0(F_n, \Omega_{F_n}^{\otimes m}) = \begin{cases} g = \dfrac{(n-1)(n-2)}{2}, & \text{if } m = 1, \\ (2m-1)(g-1) = (2m-1)\dfrac{n(n-3)}{2}, & \text{if } m \geq 2. \end{cases}$$

We proceed with an auxiliary lemma that will allow us to obtain explicit bases for the spaces $H^0(F_n, \Omega_{F_n}^{\otimes m})$.

Lemma 4. *For each pair $(i, j) \in \mathbb{Z}^2$ satisfying $0 \leq i, j, i + j \leq m(n-3)$, the differential*

$$\frac{x^i y^j}{y^{m(n-1)}} dx^{\otimes m}$$

is holomorphic.

Proof. This requires computing the divisors of x, y and dx. Such a computation can be found in the classical articles of Boseck [3, pp. 48–50] and Towse [29, pp. 3359–3360], or in the work of the fourth author [1,14]. Indeed, from [1, p. 113], we have that

$$\mathrm{div}(x) = \sum_{\nu=1}^{n} \alpha_\nu - \sum_{\nu=1}^{n} \gamma_\nu, \quad \mathrm{div}(y) = \sum_{\nu=1}^{n} \beta_n - \sum_{\nu=1}^{n} \gamma_\nu, \quad \mathrm{div}(dx) = (n-1)\sum_{\nu=1}^{n} \beta_\nu - 2\sum_{\nu=1}^{n} \gamma_\nu$$

where for $\nu = 1, \ldots, n$, α_ν are the n points on the Fermat curve F_n with x coordinate equal to zero, β_ν are the n points of the Fermat curve with y coordinate equal to zero and γ_ν are the n points at infinity that have z coordinate equal to zero. We then compute

$$\mathrm{div}\left(\frac{x^i y^j}{y^{m(n-1)}} dx^{\otimes m}\right) = \sum_{\nu=1}^{n} i\alpha_\nu + \sum_{\nu=1}^{n} j\beta_\nu - \sum_{\nu=1}^{n} \big(i + j - m(n-3)\big)\gamma_\nu,$$

and the result follows. □

Remark 5. The differentials of Lemma 4 do not form a basis for $H^0(F_n, \Omega_{F_n}^{\otimes m})$ for $m \geq 2$, as they are not linearly independent. Indeed, if $n \leq i \leq m(n-3)$, then we can write $i = qn + \upsilon$ with $0 \leq \upsilon \leq n-1$ and

$$\begin{aligned}\frac{x^i y^j}{y^{m(n-1)}} dx^{\otimes m} &= x^{qn} \frac{x^\upsilon y^j}{y^{m(n-1)}} dx^{\otimes m} = (1-y^n)^q \frac{x^\upsilon y^j}{y^{m(n-1)}} dx^{\otimes m} \\ &= \sum_{\nu=0}^{q} (-1)^\nu \binom{q}{\nu} y^{n\nu} \frac{x^\upsilon y^j}{y^{m(n-1)}} dx^{\otimes m} \\ &= \sum_{\nu=0}^{q} (-1)^\nu \binom{q}{\nu} \frac{x^\upsilon y^{j+n\nu}}{y^{m(n-1)}} dx^{\otimes m}.\end{aligned}$$

We remark that the pairs $(i', j') = (\upsilon, j + n\nu)$ satisfy the inequalities

$$0 \leq i' \leq n-1 \text{ and } 0 \leq j', i' + j' \leq m(n-3).$$

Proposition 6. *A K-basis for $V_m = H^0(F_n, \Omega_{F_n}^{\otimes m})$ is given by*

$$\mathbf{b}_m = \begin{cases} \left\{ \frac{x^i y^j}{y^{n-1}} dx : 0 \leq i, j, i+j \leq n-3 \right\}, & \text{if } m = 1 \\ \left\{ \frac{x^i y^j}{y^{m(n-1)}} dx^{\otimes m} : 0 \leq i \leq n-1,\ 0 \leq j, i+j \leq m(n-3) \right\}, & \text{if } m \geq 2. \end{cases}$$

Proof. The result for $m = 1$ is classic and a proof can be found in [29, Proposition 2]. Lemma 4 implies that the differentials in $\mathbf{b}_m$ for $m \geq 2$ are holomorphic. The fact that they are linearly independent is trivial and a simple counting argument verifies that they have the correct cardinality. □

The above basis is not suitable for computations when $m \geq 2$, since it is not symmetric in i, j.

Definition 7. Let $E_M = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i, j, i+j \leq M\}$ be the triangle with vertices $(0, 0), (0, M), (M, 0)$, as in Lemma 4. For $m \geq 1$, we will denote by W_m the K-vector space with basis the symbols

$$\left\{ w_{i,j}^{(m)} : (i, j) \in E_{m(n-3)} \right\}.$$

and, for $m \geq 2$, we will denote by I_m its K-subspace generated by the expressions

$$\left\{ \pi_{i,j}^{(m)} := w_{i,j}^{(m)} + w_{i+n,j}^{(m)} + w_{i,j+n}^{(m)} : (i, j) \in E_{m(n-3)-n} \right\}.$$

Lemma 8. $\dim_K I_m = \# E_{m(n-3)-n}$.

It suffices to show that the elements $\pi_{i,j}^{(m)}$ are linearly independent. Let $\lambda_{i,j} \in K$ be such that

$$\begin{aligned} 0 &= \sum_{(i,j) \in E_{m(n-3)-n}} \lambda_{i,j} \left(w_{i,j}^{(m)} + w_{i+n,j}^{(m)} + w_{i,j+n}^{(m)} \right) \\ &= \sum_{(i,j) \in E_{m(n-3)-n}} \lambda_{i,j} w_{i,j}^{(m)} + \sum_{(i,j) \in E_{m(n-3)-n}} \lambda_{i,j} w_{i+n,j}^{(m)} + \sum_{(i,j) \in E_{m(n-3)-n}} \lambda_{i,j} w_{i,j+n}^{(m)} \end{aligned}$$

$$= \sum_{(i,j)\in E_{m(n-3)-n}} \lambda_{i,j} w_{i,j}^{(m)} + \sum_{\substack{(i,j)\in E_{m(n-3)-n} \\ i\geq n}} \lambda_{i-n,j} w_{i,j}^{(m)} + \sum_{\substack{(i,j)\in E_{m(n-3)-n} \\ j\geq n}} \lambda_{i,j-n} w_{i,j}^{(m)}$$

$$= \sum_{(i,j)\in E_{m(n-3)-n}} \left(\lambda_{i,j} + \lambda_{i-n,j} + \lambda_{i,j-n}\right) w_{i,j}^{(m)}, \text{ where } \lambda_{i,j} = 0 \text{ if } i < n \text{ or } j < n.$$

Since the elements $w_{i,j}^{(m)}$ are linearly independent, we obtain that

$$\lambda_{i,j} + \lambda_{i-n,j} + \lambda_{i,j-n} = 0, \text{ for all } (i,j) \in E_{m(n-3)-n}. \tag{4}$$

We will use that $\lambda_{i,j} = 0$ for $i < n$ or $j < n$ and Eq. (4) to prove by induction on $m \geq 2$ that all $\lambda_{i,j} = 0$.

- If $m = 2$, then $E_{m(n-3)-n} = E_{n-6}$. Thus $\lambda_{i-n,j} = \lambda_{i,j-n} = 0$ and Eq. (4) gives $\lambda_{i,j} = 0$.
- Assume that $\lambda_{i,j} = 0$ for all $(i,j) \in E_{(m-1)(n-3)-n}$. If $(i_0, j_0) \in E_{m(n-3)-n}$ then

  $$0 \leq i_0 + j_0 \leq m(n-3) - n \Rightarrow 0 \leq i_0 + j_0 - n \leq m(n-3) - 2n < (m-1)(n-3) - n.$$

  Thus $(i_0 - n, j_0), (i_0, j_0 - n) \in E_{(m-1)(n-3)-n}$ and so $\lambda_{i_0-n,j_0} = \lambda_{i_0,j_0-n} = 0$ by the inductive hypothesis. Eq. then (4) gives $\lambda_{i_0,j_0} = 0$.

Proposition 9. *We have that*

$$H^0(F_n, \Omega_{F_n}^{\otimes m}) \cong \begin{cases} W_1 & , \text{ if } m = 1 \\ W_m / I_m & , \text{ if } m \geq 2 \end{cases}$$

Proof. When $m = 1$, the result follows trivially since

$$\dim H^0(F_n, \Omega_{F_n}^{\otimes m}) = \#\mathbf{b}_1 = \#E_{n-3} = \dim W_1.$$

For $m \geq 2$, consider the K-linear map

$$\phi : W_m \longrightarrow H^0(F_n, \Omega_{F_n}^{\otimes m}), \; w_{i,j}^{(m)} \mapsto \frac{x^i y^j}{y^{m(n-1)}} dx^{\otimes m},$$

which is well defined by Lemma 4. It is onto, since every differential in $\mathbf{b}_m$ is the image of a symbol $w_{i,j}^{(m)} \in E_{m(n-3)}$ that satisfies $0 \leq i \leq n-1$. We thus have that

$$H^0(F_n, \Omega_{F_n}^{\otimes m}) \cong W_m / \ker\phi,$$

and it remains to show that $\ker\phi = I_m$. For an arbitrary $\pi_{i,j}^{(m)} \in I_m$, we have that

$$\phi\left(\pi_{i,j}^{(m)}\right) = \frac{x^i y^j}{y^{m(n-1)}} dx^{\otimes m} + \frac{x^{i+n} y^j}{y^{m(n-1)}} dx^{\otimes m} + \frac{x^i y^{j+n}}{y^{m(n-1)}} dx^{\otimes m}$$
$$= \frac{x^i y^j}{y^{m(n-1)}} dx^{\otimes m} \left(1 + x^n + y^n\right) = 0,$$

by the defining equation of F_n, and thus $I_m \subseteq \ker\phi$. For the inverse inclusion, we first remark that the cardinality of the lattice points in the triangle E_M is given by

$$\#E_M = \frac{(M+1)(M+2)}{2}.$$

By Lemma 8 we get

$$\begin{aligned}\dim_K (W_m/I_m) &= \#E_{m(n-3)} - \#E_{m(n-3)-n} \\ &= \frac{[m(n-3)+1][m(n-3)+2]}{2} \\ &\quad - \frac{[m(n-3)-n+1][m(n-3)-n+2]}{2} \\ &= (2m-1)\frac{n(n-3)}{2} = (2m-1)(g-1) = \dim_K H^0(F_n, \Omega_{F_n}^{\otimes m}). \quad \square\end{aligned}$$

The reader may observe that this decomposition of $H^0(F_n, \Omega_{F_n}^{\otimes m})$ is closely related to the classic result of M. Noether, F. Enriques and K. Petri on the canonical ideal of non-hyperelliptic curves: indeed the graded ring $\bigoplus W_m$ is isomorphic to the quotient of $\mathrm{Sym}\left(H^0(F_n, \Omega_{F_n})\right)$ by some binomial relations, whereas the elements of I_m are the missing generators for the kernel of the canonical map

$$\mathrm{Sym}\left(H^0(F_n, \Omega_{F_n})\right) \twoheadrightarrow \bigoplus H^0(F_n, \Omega_{F_n}^{\otimes m}).$$

For more details on the explicit construction in the case of Fermat curves see [15], an application of a more general technique given in [4].

Next, we describe the appropriate action of G on the vector spaces W_m and I_m, that makes the isomorphism of Proposition 9 G-equivariant.

Proposition 10. *Let $w_{i,j}^{(m)}$ and $\pi_{i,j}^{(m)}$ be the generators of W_m and I_m respectively, as in Definition 7. We define an action of G on W_m via the first column of the table below, which induces an action of G on I_m as in the second column. Then the isomorphism of Proposition 9 is G-equivariant, where the action on $H^0(F_n, \Omega_{F_n}^{\otimes m})$ is given by Eq. (3).*

$g \in G$	$g \cdot w_{i,j}^{(m)}$	$g \cdot \pi_{i,j}^{(m)}$
$\sigma_{\alpha,\beta}$	$\zeta^{\alpha(i+m)+\beta(j+m)} w_{i,j}^{(m)}$	$\zeta^{\alpha(i+m)+\beta(j+m)} \pi_{i,j}^{(m)}$
s	$w_{m(n-3)-(i+j),i}^{(m)}$	$\pi_{m(n-3)-(i+j)-n,i}^{(m)}$
s^2	$w_{j,m(n-3)-(i+j)}^{(m)}$	$\pi_{j,m(n-3)-(i+j)-n}^{(m)}$
t	$(-1)^m w_{m(n-3)-(i+j),j}^{(m)}$	$(-1)^m \pi_{m(n-3)-(i+j)-n,j}^{(m)}$
ts	$(-1)^m w_{j,i}^{(m)}$	$(-1)^m \pi_{j,i}^{(m)}$
st	$(-1)^m w_{i,m(n-3)-(i+j)}^{(m)}$	$(-1)^m \pi_{i,m(n-3)-(i+j)-n}^{(m)}$

Proof. For $\sigma_{\alpha,\beta} \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ we have that

$$\begin{aligned}\sigma_{\alpha,\beta}\left(\frac{x^i y^j}{y^{m(n-1)}} dx^{\otimes m}\right) &= \frac{(\zeta^\alpha x)^i \left(\zeta^\beta y\right)^j}{\left(\zeta^\beta y\right)^{m(n-1)}} d\left(\zeta^\alpha x\right)^{\otimes m} = \frac{(\zeta^\alpha x)^i \left(\zeta^\beta y\right)^j}{\left(\zeta^\beta y\right)^{-m}} \zeta^{\alpha m} dx^{\otimes m} \\ &= \zeta^{\alpha(i+m)+\beta(j+m)} \frac{x^i y^j}{y^{m(n-1)}} dx^{\otimes m}\end{aligned}$$

and thus $\phi\left(\sigma_{\alpha,\beta} \cdot w_{i,j}^{(m)}\right) = \sigma_{\alpha,\beta} \cdot \phi\left(w_{i,j}^{(m)}\right)$. For the element $s \in S_3$ of order 3 we first note that differentiating the Fermat equation $x^n + y^n + 1 = 0$ gives $dy = -\frac{x^{n-1}}{y^{n-1}} dx$ and thus

$d\left(\frac{y}{x}\right)=\frac{1}{x^2y^{n-1}}dx$. We then have

$$s\left(\frac{x^iy^j}{y^{m(n-1)}}dx^{\otimes m}\right)=\frac{y^i}{x^i}\frac{1}{x^j}x^{m(n-1)}d\left(\frac{y}{x}\right)^{\otimes m}=x^{m(n-1)-(i+j)}y^i\left(\frac{1}{x^2y^{n-1}}\right)^m dx^{\otimes m}$$
$$=\frac{x^{m(n-3)-(i+j)}y^i}{y^{m(n-1)}}dx^{\otimes m}$$

and thus $\phi\left(s\cdot w_{i,j}^{(m)}\right)=s\cdot\phi\left(w_{i,j}^{(m)}\right)$. For the element $t\in S_3$ of order 2 we have

$$t\left(\frac{x^iy^j}{y^{m(n-1)}}dx^{\otimes m}\right)=\frac{1}{x^i}\frac{y^j}{x^j}\frac{x^{m(n-1)}}{y^{m(n-1)}}d\left(\frac{1}{x}\right)^{\otimes m}=\frac{x^{m(n-1)-(i+j)}y^j}{y^{m(n-1)}}(-1)^m\frac{1}{x^{2m}}dx^{\otimes m}$$
$$=(-1)^m\frac{x^{m(n-3)-(i+j)}y^j}{y^{m(n-1)}}dx^{\otimes m}$$

and thus $\phi\left(t\cdot w_{i,j}^{(m)}\right)=t\cdot\phi\left(w_{i,j}^{(m)}\right)$. The remaining cases follow similarly. □

We proceed with an explicit description of the character tables of the representations $G\to GL\left(W_m\right)$ and $G\to GL\left(I_m\right)$:

Proposition 11. *The characters χ_{W_m} and χ_{I_m} of the representations $\rho_{W_m}:G\to GL\left(W_m\right)$ and $\rho_{I_m}:G\to GL\left(I_m\right)$ respectively are given in the following table*

$g\in G$	*Character* $\chi_{W_m}(g)$	*Character* $\chi_{I_m}(g)$ $m\geq 2$
$\sigma_{\alpha,\beta}$	$\sum_{i=0}^{m(n-3)}\sum_{j=0}^{m(n-3)-i}\zeta^{\alpha(i+m)+\beta(j+m)}$	$\sum_{i=0}^{m(n-3)-n}\sum_{j=0}^{m(n-3)-n-i}\zeta^{\alpha(i+m)+\beta(j+m)}$
$\sigma_{\alpha,\beta}s$	$\begin{cases}\zeta^{(\alpha+\beta)\frac{mn}{3}}, & \textit{if } 3\mid n \textit{ or } 3\mid m\\ 0, & \textit{if } 3\nmid n \textit{ and } 3\nmid m\end{cases}$	$\begin{cases}\zeta^{(\alpha+\beta)\frac{n(m-1)}{3}}, & \textit{if } 3\mid n(m-1)\\ 0, & \textit{otherwise}\end{cases}$
$\sigma_{\alpha,\beta}s^2$	$\begin{cases}\zeta^{(\alpha+\beta)\frac{mn}{3}}, & \textit{if } 3\mid n \textit{ or } 3\mid m\\ 0, & \textit{if } 3\nmid n \textit{ and } 3\nmid m\end{cases}$	$\begin{cases}\zeta^{(\alpha+\beta)\frac{n(m-1)}{3}}, & \textit{if } 3\mid n(m-1)\\ 0, & \textit{otherwise}\end{cases}$
$\sigma_{\alpha,\beta}t$	$(-1)^m\sum_{i=0}^{\left\lfloor\frac{m(n-3)}{2}\right\rfloor}\zeta^{(\alpha-2\beta)(i+m)}$	$(-1)^m\sum_{i=0}^{\left\lfloor\frac{m(n-3)-n}{2}\right\rfloor}\zeta^{(\alpha-2\beta)(i+m)}$
$\sigma_{\alpha,\beta}ts$	$(-1)^m\sum_{i=0}^{\left\lfloor\frac{m(n-3)}{2}\right\rfloor}\zeta^{(\alpha+\beta)(i+m)}$	$(-1)^m\sum_{i=0}^{\left\lfloor\frac{m(n-3)-n}{2}\right\rfloor}\zeta^{(\alpha+\beta)(i+m)}$
$\sigma_{\alpha,\beta}st$	$(-1)^m\sum_{i=0}^{\left\lfloor\frac{m(n-3)}{2}\right\rfloor}\zeta^{(\beta-2\alpha)(i+m)}$	$(-1)^m\sum_{i=0}^{\left\lfloor\frac{m(n-3)-n}{2}\right\rfloor}\zeta^{(\beta-2\alpha)(i+m)}$

Proof. By Proposition 10, the matrix $\rho_{W_m}(\sigma_{\alpha,\beta})$ is diagonal with trace

$$\chi_{W_m}(\sigma_{\alpha,\beta}) = \sum_{(i,j)\in E_{m(n-3)}} \zeta^{\alpha(i+m)+\beta(j+m)}.$$

Similarly, the diagonal entries of the matrix $\rho(s)$ are either 0 or 1. The number of non-zero such diagonal entries equals the number of pairs (i, j) that satisfy the relations:

$$0 \le i + j \le m(n-3),\ i = m(n-3) - (i + j),\ i = j$$

which gives $3i = m(n-3)$. Thus, we have a unique non-zero diagonal entry if only if $3 \mid m(n-3)$, i.e. if $3 \mid n$ or $3 \mid m$. The formula for $\chi_{W_m}(\sigma_{\alpha,\beta}s)$ is then obtained by substituting $i = j = \frac{m(n-3)}{3}$ in the expression for $\chi_{W_m}(\sigma_{\alpha,\beta})$, and the same holds for $\chi_{W_m}(\sigma_{\alpha,\beta}s^2)$.

Again by Proposition 10 we have that the diagonal entries of the matrix $\rho_{W_m}(t)$ are either 0 or -1. The number of non-zero such diagonal entries, is given by the number of pairs (i, j) that satisfy the relations:

$$0 \le i + j \le m(n-3),\ i = m(n-3) - (i + j).$$

Thus, for each value of i we have that $j = m(n-3) - 2i$. Since both i and j must be non-negative, the number of pairs (i, j) satisfying the above relations is equal to the number of unique i-values that satisfy

$$0 \le i \le \left\lfloor \frac{m(n-3)}{2} \right\rfloor.$$

The formula for $\chi_{W_m}(\sigma_{\alpha,\beta}t)$ follows from substituting $j = m(n-3) - 2i$ in the expression for $\chi_{W_m}(\sigma_{\alpha,\beta})$. The remaining cases for the character χ_{W_m} easily follow from the computations above.

The character χ_{I_m} is obtained by replacing throughout $E_{m(n-3)}$ by $E_{m(n-3)-n}$. □

The above result and the G-equivariant isomorphism $W_m/I_m \cong H^0(F_n, \Omega_{F_n}^{\otimes m})$ allow us to obtain the characters of the spaces $V_m = H^0(F_n, \Omega_{F_n}^{\otimes m})$:

Theorem 12. *The character for the space $H^0(F_n, \Omega_{F_n})$ is equal to the character of the space W_1, while the character for $H^0(F_n, \Omega_{F_n}^{\otimes m})$, for $m \ge 2$ equals $\chi_{W_m} - \chi_{I_m}$, where χ_{W_m}, χ_{I_m} are given in Proposition 11.*

4. Computing sums of roots of unity

To determine the $K[G]$-module structure of the spaces $H^0(F_n, \Omega_{F_n}^{\otimes m})$ we will use the standard approach of computing the inner product of the irreducible characters of G, given in Proposition 3, with the characters $\chi_{W_m} - \chi_{I_m}$ given in Proposition 11. These computations require finding closed formulas for two types of sums that involve nth roots of unity:

Definition 13. For $M \in \mathbb{Z}_{\geq 0}$, let $E_M = \{(i, j) \in \mathbb{N}^2 : 0 \leq i, j, i + j \leq M\} \subseteq \mathbb{N}^2$ be the triangle with vertices $(0, 0), (0, M), (M, 0)$. For any $X, Y \in \mathbb{Z}$, we define the quantities

$$I(M, X, Y) := \sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \sum_{(i,j)\in E_M} \zeta^{\alpha(i+X)+\beta(j+Y)}, \quad J(M, X) := \sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \sum_{i=0}^{\lfloor M/2\rfloor} \zeta^{(\alpha+\beta)(i+X)}$$

For $M \in \mathbb{Z}_{<0}$, the triangle E_M is empty and thus we set $I(M, X, Y) = J(M, X) = 0$.

Proposition 14. *For $M \in \mathbb{Z}_{\geq 0}$, we have that*

$$\begin{aligned} I(M, X, Y) &= n^2 \cdot \#\{(i, j) \in E_M : \ i \equiv -X \bmod n \text{ and } j \equiv -Y \bmod n\} \\ J(M, X) &= \sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \sum_{i=0}^{\lfloor M/2\rfloor} \zeta^{(\alpha-2\beta)(i+X)} = \sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \sum_{i=0}^{\lfloor M/2\rfloor} \zeta^{(\beta-2\alpha)(i+X)} \\ &= n^2 \cdot \# \left\{ i \in \mathbb{N} : 0 \leq i \leq \left\lfloor \frac{M}{2} \right\rfloor \text{ and } i \equiv -X \bmod n \right\}. \end{aligned}$$

Proof. The formula for $I(M, X, Y)$ follows from the fact that for fixed $(i, j) \in E_M$ we have

$$\sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \zeta^{\alpha(i+X)+\beta(j+Y)} = \begin{cases} n^2, & \text{if } i \equiv -X \bmod n \text{ and } j \equiv -Y \bmod n \\ 0, & \text{otherwise.} \end{cases}$$

The formula for $J(M, X)$ follows similarly, since for fixed $0 \leq i \leq \left\lfloor \frac{M}{2} \right\rfloor$ we have that

$$\begin{aligned} \sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \zeta^{(\alpha+\beta)(i+X)} &= \sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \zeta^{(\alpha-2\beta)(i+X)} = \sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \zeta^{(\beta-2\alpha)(i+X)} \\ &= \begin{cases} n^2, & \text{if } i \equiv -X \bmod n \\ 0, & \text{otherwise.} \end{cases} \qquad \square \end{aligned}$$

To obtain explicit formulas for the above quantities, we will use the following auxiliary result.

Lemma 15. *Let $X, L \in \mathbb{N}$. Denote by υ_L, υ_{-X}, the remainder of the division of L, $-X$, by n. Then*

$$\begin{aligned} \#\{i \in \mathbb{N} : 0 \leq i \leq L \text{ and } i \equiv -X \bmod n\} &\overset{(1)}{=} \left\{ \begin{array}{ll} \left\lfloor \frac{L}{n} \right\rfloor + 1 & \text{if } \upsilon_{-X} \leq \upsilon_L \\ \left\lfloor \frac{L}{n} \right\rfloor & \text{if } \upsilon_{-X} > \upsilon_L. \end{array} \right\} \\ &\overset{(2)}{=} \left\lfloor \frac{L - \upsilon_{-X}}{n} \right\rfloor + 1 \end{aligned}$$

Proof. To prove equality (1), note that the greatest multiple of n always less than L is $\lfloor \frac{L}{n} \rfloor n$. The interval $0 \le i \le \lfloor \frac{L}{n} \rfloor n - 1$ contains exactly $\lfloor \frac{L}{n} \rfloor$ subintervals of length n, and thus it contains exactly $\lfloor \frac{L}{n} \rfloor$ solutions to the congruence $i \equiv -X \mod n$. The congruence has one more solution in the interval $\lfloor \frac{L}{n} \rfloor n \le i \le L = \lfloor \frac{L}{n} \rfloor n + \upsilon_L$ if and only if $\upsilon_{-X} \le \upsilon_L$.

To prove equality (2) we write $L = \lfloor \frac{L}{n} \rfloor n + \upsilon_L$ as above and observe that

$$\left\lfloor \frac{L-\upsilon_{-X}}{n} \right\rfloor + 1 = \left\lfloor \frac{\lfloor \frac{L}{n} \rfloor n + \upsilon_L - \upsilon_{-X}}{n} \right\rfloor + 1 = \left\lfloor \frac{L}{n} \right\rfloor + \left\lfloor \frac{\upsilon_L - \upsilon_{-X}}{n} \right\rfloor + 1$$
$$= \begin{cases} \left\lfloor \frac{L}{n} \right\rfloor + 1 & \text{if } \upsilon_{-X} \le \upsilon_L \\ \left\lfloor \frac{L}{n} \right\rfloor & \text{if } \upsilon_{-X} > \upsilon_L. \end{cases} \qquad \square$$

Recall that our motivation for computing the quantities $I(M, X, Y)$ and $J(M, X)$ is to obtain the Galois module structure of the global sections of m-differentials for $m \ge 1$. Since the respective characters are given by the differences $\chi_{W_m} - \chi_{I_m}$ we need formulas for *differences* of these quantities for different values of M. Recall that when $M \le n - 1$, $I(M-n, X, Y) = J(M-n, X) = 0$.

Proposition 16. *For $X, Y, M \in \mathbb{N}$, let*

$$\delta(M, X, Y) = \begin{cases} -1 & \text{if } \upsilon_{-X} > \upsilon_M \text{ and } \upsilon_{-X} + \upsilon_{-Y} > \upsilon_M + n \\ 1 & \text{if } \upsilon_{-X} \le \upsilon_M \text{ and } \upsilon_{-X} + \upsilon_{-Y} \le \upsilon_M \\ 0 & \text{otherwise.} \end{cases}$$

where $\upsilon_{-X}, \upsilon_{-Y}, \upsilon_M$ denote the respective remainders when dividing by n. Then

$$I(M, X, Y) - I(M-n, X, Y) = n^2 \left(\left\lfloor \frac{M}{n} \right\rfloor + \delta(M, X, Y) \right),$$
$$J(M, X) - J(M-n, X) = n^2 \left(\left\lfloor \frac{\lfloor \frac{M}{2} \rfloor - \upsilon_{-X}}{n} \right\rfloor - \left\lfloor \frac{\lfloor \frac{M-n}{2} \rfloor - \upsilon_{-X}}{n} \right\rfloor \right) \in \{0, n^2\}.$$

Proof. The formula for $J(M, X) - J(M-n, X)$ follows directly by Proposition 14 and the second equality of Lemma 15 for $L = \lfloor \frac{M}{2} \rfloor$ and $L = \lfloor \frac{M-n}{2} \rfloor$ respectively.

To obtain the formula for $I(M, X, Y) - I(M-n, X, Y)$, we first assume that $M > n - 1$. By Proposition 14, $I(M, X, Y)$ is equal to n^2 times the number of solutions to the system

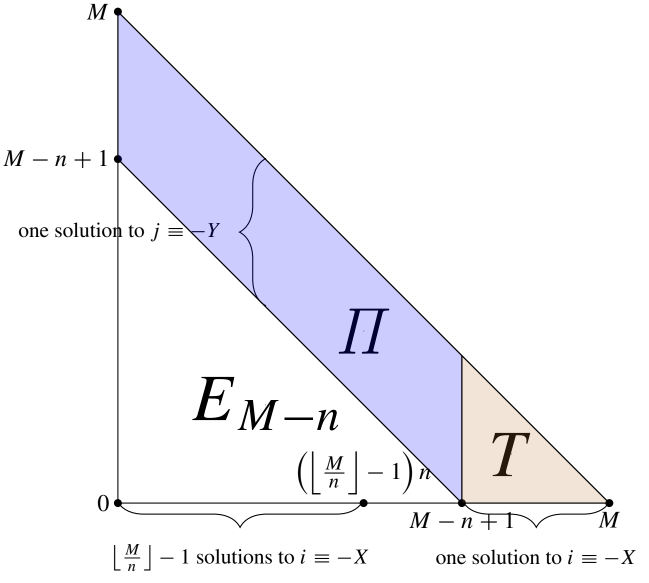
$$\begin{cases} i \equiv -X \mod n \\ j \equiv -Y \mod n \end{cases} \tag{5}$$

that lie inside the trapezoid $E_M \setminus E_{M-n}$ with vertices $(0, M-n+1), (0, M), (M-n+1, 0), (M, 0)$. We remark that $E_M \setminus E_{M-n}$ can be written as the disjoint union of the sets Π and T where Π is the parallelogram with vertices $(0, M-n+1), (0, M), (M-n, 1), (M$

$-\, n, n)$ and T is the triangle with vertices $(M-n+1, 0), (M-n+1, n-1), (M, 0)$. That is

$$\Pi = \{(i, j) : 0 \le i \le M-n, M-n-i+1 \le j \le M-i.\}$$
$$T = \{(i, j) : M-n+1 \le i \le M, 0 \le j \le M-i\}.$$

Thus, we compute the number of solutions to the system (5) inside $E_M \setminus E_{M-n}$ by computing the number of solutions inside Π and the number of solutions inside T separately:



- If $(i, j) \in \Pi$, then $0 \le i \le M-n$ and $M-n-i+1 \le j \le M-i$. Thus, for a fixed value of i satisfying $0 \le i \le M-n$ there exist exactly n values of j such that $(i, j) \in \Pi$ and exactly 1 of them satisfies the congruence $j \equiv -Y \bmod n$. Hence, the number ν_Π of solutions to the system (5) inside Π is equal to the number of solutions to the congruence $i \equiv -X \bmod n$ that satisfy $0 \le i \le M-n$. To count the cardinality of the set $\{i \in \mathbb{N} : 0 \le i \le M-n \text{ and } i \equiv -X \bmod n\}$, we apply the first equality of Lemma 15 for $L = M-n$ and observe that $\left\lfloor \frac{M-n}{n} \right\rfloor = \left\lfloor \frac{M}{n} \right\rfloor - 1$ to obtain

$$\nu_\Pi = \begin{cases} \left\lfloor \dfrac{M}{n} \right\rfloor & \text{if } \upsilon_{-X} \le \upsilon_M \\ \left\lfloor \dfrac{M}{n} \right\rfloor - 1 & \text{if } \upsilon_{-X} > \upsilon_M. \end{cases}$$

- To count the number of solutions to the system (5) inside the triangle T we observe that its base consists of n consecutive integers and so the congruence $i \equiv -X \bmod n$ has exactly one solution i_0. For this solution i_0, there exists a point (i_0, j_0) in T satisfying

$j_0 \equiv -Y \bmod n$ if and only if $i_0 + \upsilon_{-Y} \leq M$, since $0 \leq j_0 \leq n-1$. Division of i_0 by n gives

$$i_0 = \begin{cases} \left\lfloor \frac{M}{n} \right\rfloor n + \upsilon_{-X} & \text{, if } \upsilon_{-X} \leq \upsilon_M \\ \left(\left\lfloor \frac{M}{n} \right\rfloor - 1 \right) n + \upsilon_{-X} & \text{, if } \upsilon_{-X} > \upsilon_M \end{cases}$$

and thus

$$i_0 + \upsilon_{-Y} \leq M \Leftrightarrow \begin{cases} \upsilon_{-X} + \upsilon_{-Y} \leq M - \left\lfloor \frac{M}{n} \right\rfloor n = \upsilon_M & \text{, if } \upsilon_{-X} \leq \upsilon_M \\ \upsilon_{-X} + \upsilon_{-Y} \leq M - \left(\left\lfloor \frac{M}{n} \right\rfloor - 1 \right) n = \upsilon_M + n & \text{, if } \upsilon_{-X} > \upsilon_M. \end{cases}$$

We conclude that the number of solutions ν_T to the system (5) inside T is given by

$$\nu_T = \begin{cases} 1 & \text{if } i_0 + \upsilon_{-Y} \leq M \\ 0 & \text{if } i_0 + \upsilon_{-Y} < M \end{cases} = \begin{cases} 1 & \text{if } \upsilon_{-X} \leq \upsilon_M \text{ and } \upsilon_{-X} + \upsilon_{-Y} \leq \upsilon_M \\ 0 & \text{if } \upsilon_{-X} \leq \upsilon_M \text{ and } \upsilon_{-X} + \upsilon_{-Y} > \upsilon_M \\ 1 & \text{if } \upsilon_{-X} > \upsilon_M \text{ and } \upsilon_{-X} + \upsilon_{-Y} \leq \upsilon_M + n \\ 0 & \text{if } \upsilon_{-X} > \upsilon_M \text{ and } \upsilon_{-X} + \upsilon_{-Y} > \upsilon_M + n. \end{cases}$$

The formula for $I(M, X, Y) - I(M-n, X, Y)$ follows by combining the cases for ν_Π and ν_T.

For $M \leq n-1$, we have $I(M-n, X, Y) = 0$ and the trapezoid considered in the previous case degenerates to the triangle $T = E_M$ with vertices $(0,0), (0,M), (M,0)$. The system (5) has a unique solution inside E_M if and only if $(\upsilon_{-X}, \upsilon_{-Y}) \in E_M$ which is equivalent to the condition $\upsilon_{-X} + \upsilon_{-Y} \leq M$. □

We conclude this section with substituting the value of M that we will need for the computations on m-polydifferentials:

Corollary 17. *If $n \geq 4$ and $M = m(n-3)$ for $m \in \mathbb{N}$, we define $\delta^{(m)}_{X,Y} := \delta(m(n-3), X, Y)$ and*

$$I^{(m)}_{X,Y} := I(m(n-3), X, Y) - I(m(n-3)-n, X, Y),\ J^{(m)}_X := J(m(n-3), X) - J(m(n-3)-n, X).$$

Then

$$I^{(m)}_{X,Y} = n^2 \left(m - \left\lceil \frac{3m}{n} \right\rceil + \delta^{(m)}_{X,Y} \right),$$

$$\text{where } \delta^{(m)}_{X,Y} = \begin{cases} -1 & \text{if } \upsilon_{-X} > \upsilon_{-3m} \text{ and } \upsilon_{-X} + \upsilon_{-Y} > \upsilon_{-3m} + n \\ 1 & \text{if } \upsilon_{-X} \leq \upsilon_{-3m} \text{ and } \upsilon_{-X} + \upsilon_{-Y} \leq \upsilon_{-3m} \\ 0 & \text{otherwise}. \end{cases}$$

$$J^{(m)}_X = n^2 \left(\left\lfloor \frac{\left\lfloor \frac{m(n-3)}{2} \right\rfloor - \upsilon_{-X}}{n} \right\rfloor - \left\lfloor \frac{\left\lfloor \frac{m(n-3)-n}{2} \right\rfloor - \upsilon_{-X}}{n} \right\rfloor \right) \in \{0, n^2\}.$$

Proof. The formula for $J_X^{(m)}$ follows directly from the substitution $M = m(n-3)$. The formula for $I_{X,Y}^{(m)}$ follows from the observation that $\left\lfloor \frac{m(n-3)}{n} \right\rfloor = m + \left\lfloor \frac{-3m}{n} \right\rfloor = m - \left\lceil \frac{3m}{n} \right\rceil$. For the conditions defining $I_{X,Y}^{(m)}$, we remark that $\upsilon_{m(n-3)} = \upsilon_{-3m}$. □

5. The Galois module structure of holomorphic poly-differentials

For $m \geq 1$, let $V_m = H^0(F_n, \Omega_{F_n}^{\otimes m})$ denote the K-vector space of global sections of holomorphic m-differentials on the Fermat curve F_n. By Theorem 12, we have that the character of the representation $\rho_{V_m} : G \to \mathrm{GL}(V_m)$ is given by $\chi_{V_m} = \chi_{W_m} - \chi_{I_m}$, where χ_{W_m} and χ_{I_m} are given in Proposition 11. Notice that for $m = 1$ the character $\chi_{I_1} = 0$. Thus, the Galois module structure of V_m can be computed as follows:

Corollary 18. *Let $\chi_{\kappa,\lambda,\rho} \in \mathrm{Irrep}(G)$ denote any of the irreducible representations of G given in Proposition 3. Then*

$$\langle \chi_{V_m}, \chi_{\kappa,\lambda,\rho} \rangle = \frac{1}{6n^2} \sum_{\substack{\alpha,\beta \in \mathbb{Z}/n\mathbb{Z} \\ g \in S_3}} \left(\chi_{W_m}(\sigma_{\alpha,\beta} g) - \chi_{I_m}(\sigma_{\alpha,\beta} g) \right) \overline{\chi_{\kappa,\lambda,\rho}(\sigma_{\alpha,\beta} g)}$$

Hence, the computation of each $\langle \chi_{V_m}, \chi_{\kappa,\lambda,\rho} \rangle$ breaks down in computing (at most) six sums over $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, one for each element $g \in S_3$. We remark that by the results of the previous section, the sums corresponding to $\sigma_{\alpha,\beta}$ will be computed using the quantities $I_{X,Y}^{(m)}$ and the sums corresponding to $\sigma_{\alpha,\beta} t$, $\sigma_{\alpha,\beta} ts$, $\sigma_{\alpha,\beta} st$ will be computed using $J_X^{(m)}$. For the sums corresponding to $\sigma_{\alpha,\beta} s$, $\sigma_{\alpha,\beta} s^2$, which appear only in the multiplicities of the irreducible representations of degree 1 and 2, we have the following:

Lemma 19. *For $\nu \in \{0, 1, 2\}$ let*

$$\Gamma_{\frac{\nu n}{3}}^{(m)} := \begin{cases} 1 & , \text{ if } 3 \mid m - \nu \\ -1 & , \text{ if } 3 \mid m - \nu + 2 \\ 0 & , \text{ otherwise} \end{cases}$$

Then for , $i \in \{1, 2\}$ and $\rho \in \{\rho_{\mathrm{triv}}, \rho_{\mathrm{sgn}}\}$ we have that

$$\sum_{\alpha,\beta \in \mathbb{Z}/n\mathbb{Z}} \chi_{V_m}(\sigma_{\alpha,\beta} s^i) \overline{\chi_{\frac{\nu n}{3}, \frac{\nu n}{3}, \rho}(\sigma_{\alpha,\beta} s^i)} = n^2 \Gamma_{\frac{\nu n}{3}}^{(m)} \in \{-n^2, 0, n^2\}$$

and for $\rho = \rho_{\mathrm{stan}}$ we have that

$$\sum_{\alpha,\beta \in \mathbb{Z}/n\mathbb{Z}} \chi_{V_m}(\sigma_{\alpha,\beta} s^i) \overline{\chi_{\frac{\nu n}{3}, \frac{\nu n}{3}, \rho_{\mathrm{stan}}}(\sigma_{\alpha,\beta} s^i)} = -n^2 \Gamma_{\frac{\nu n}{3}}^{(m)} \in \{-n^2, 0, n^2\}$$

Proof. By Propositions 3 and 11, for $\nu \in \{0,1,2\}$, $i \in \{1,2\}$ and $\rho \in \{\rho_{\rm triv}, \rho_{\rm sgn}\}$, we have that if $3 \mid mn$ then

$$\sum_{\alpha,\beta \in \mathbb{Z}/n\mathbb{Z}} \chi_{W_m}(\sigma_{\alpha,\beta}s^i)\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}(\sigma_{\alpha,\beta}s^i)} = \sum_{\alpha,\beta \in \mathbb{Z}/n\mathbb{Z}} \zeta^{(\alpha+\beta)\frac{mn}{3}}\overline{\zeta^{(\alpha+\beta)\frac{\nu n}{3}}} = \sum_{\alpha,\beta \in \mathbb{Z}/n\mathbb{Z}} \zeta^{(\alpha+\beta)\frac{(m-\nu)n}{3}}$$
$$= \begin{cases} n^2 & , \text{ if } n \mid \frac{(m-\nu)n}{3} \\ 0 & , \text{ otherwise.} \end{cases} = \begin{cases} n^2 & , \text{ if } 3 \mid m-\nu \\ 0 & , \text{ otherwise,} \end{cases}$$

since $n \mid \frac{(m-\nu)n}{3} \Leftrightarrow \frac{(m-\nu)n}{3} = kn$ for some $k \in \mathbb{Z} \Leftrightarrow \frac{(m-\nu)}{3} = k \in \mathbb{Z} \Leftrightarrow 3 \mid m-\nu$.

Similarly, if $3 \mid (m-1)n$ then Proposition 11 gives

$$\sum_{\alpha,\beta \in \mathbb{Z}/n} \chi_{I_m}(\sigma_{\alpha,\beta}s^i)\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}(\sigma_{\alpha,\beta}s^i)} = \sum_{\alpha,\beta \in \mathbb{Z}/n} \zeta^{(\alpha+\beta)\frac{(m-1)n}{3}}\overline{\zeta^{(\alpha+\beta)\frac{\nu n}{3}}} = \sum_{\alpha,\beta \in \mathbb{Z}/n} \zeta^{(\alpha+\beta)\frac{(m-\nu-1)n}{3}}$$
$$= \begin{cases} n^2 & , \text{ if } n \mid \frac{(m-\nu-1)n}{3} \\ 0 & , \text{ otherwise.} \end{cases} = \begin{cases} n^2 & , \text{ if } 3 \mid m-\nu+2 \\ 0 & , \text{ otherwise.} \end{cases}$$

The result follows by Theorem 12: for $\rho \in \{\rho_{\rm triv}, \rho_{\rm sgn}\}$ we subtract the two sums while for $\rho = \rho_{\rm stan}$ everything needs to be multiplied by $\chi_{\rm stan}(s) = \chi_{\rm stan}(s^2) = -1$. $\square$

Next, we apply the results of the previous section for computing the part of $\langle \chi_{V_m}, \chi_{\kappa,\lambda,\rho}\rangle$ that corresponds to $\sigma_{\alpha,\beta}$:

Lemma 20. *For $\nu \in \{0,1,2\}$ and $\rho \in \{\rho_{\rm triv}, \rho_{\rm sgn}, \rho_{\rm stan}\}$ we have that*

$$\sum_{\alpha,\beta \in \mathbb{Z}/n\mathbb{Z}} \chi_{V_m}(\sigma_{\alpha,\beta})\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}(\sigma_{\alpha,\beta})} = \dim(\rho)\cdot n^2\left(m - \left\lceil \frac{3m}{n} \right\rceil + A^{(m)}_{\frac{\nu n}{3}}\right)$$

where $A^{(m)}_{\frac{\nu n}{3}} := \delta^{(m)}_{m-\frac{\nu n}{3}, m-\frac{\nu n}{3}} \in \{-1,0,1\}$ is as defined in Corollary 17.

Proof. By Propositions 3 and 11, for $\nu \in \{0,1,2\}$, $i \in \{1,2\}$ and $\rho \in \{\rho_{\rm triv}, \rho_{\rm sgn}\}$, we have that

$$\sum_{\alpha,\beta \in \mathbb{Z}/n\mathbb{Z}} \chi_{W_m}(\sigma_{\alpha,\beta})\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}(\sigma_{\alpha,\beta})} = \sum_{\alpha,\beta \in \mathbb{Z}/n\mathbb{Z}}\sum_{(i,j)\in E_{m(n-3)}} \zeta^{\alpha(i+m)+\beta(j+m)}\overline{\zeta^{(\alpha+\beta)\frac{\nu n}{3}}}$$
$$= \sum_{\alpha,\beta \in \mathbb{Z}/n\mathbb{Z}}\sum_{(i,j)\in E_{m(n-3)}} \zeta^{\alpha(i+m-\frac{\nu n}{3})+\beta(j+m-\frac{\nu n}{3})}$$
$$= I\left(m(n-3), m-\frac{\nu n}{3}, m-\frac{\nu n}{3}\right)$$

and similarly

$$\sum_{\alpha,\beta \in \mathbb{Z}/n} \chi_{I_m}(\sigma_{\alpha,\beta})\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}(\sigma_{\alpha,\beta})} = I\left(m(n-3)-n, m-\frac{\nu n}{3}, m-\frac{\nu n}{3}\right).$$

The result follows by Theorem 12 and Corollary 17. For $\rho = \rho_{\text{stan}}$ we multiply the sum by $\dim \rho_{\text{stan}} = 2$. □

Finally, we apply the results of the previous section for computing the parts of $\langle \chi_{V_m}, \chi_{\kappa,\lambda,\rho} \rangle$ that correspond to $\sigma_{\alpha,\beta}t, \sigma_{\alpha,\beta}ts, \sigma_{\alpha,\beta}st$:

Lemma 21. *For $\nu \in \{0, 1, 2\}$ let*

$$B^{(m)}_{\frac{\nu n}{3}} := (-1)^m \frac{1}{n^2} J^{(m)}_{m-\frac{\nu n}{3}} \in \{-1, 0, 1\}$$

Then for $g \in \{t, ts, st\}$ we have that

$$\sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \chi_{V_m}(\sigma_{\alpha,\beta}g)\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{triv}}}(\sigma_{\alpha,\beta}g)} = n^2 B^{(m)}_{\frac{\nu n}{3}} \in \{-n^2, 0, n^2\}$$
$$\sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \chi_{V_m}(\sigma_{\alpha,\beta}g)\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{sgn}}}(\sigma_{\alpha,\beta}g)} = -n^2 B^{(m)}_{\frac{\nu n}{3}} \in \{-n^2, 0, n^2\}$$

Proof. Recall from Proposition 14, that when computing the above sums, we may replace the term $\zeta^{\alpha+\beta}$ that appears in $\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}\left(\sigma_{\alpha,\beta}g\right)$ in Proposition 3 by any of $\zeta^{\alpha-2\beta}$ or $\zeta^{\beta-2\alpha}$. Hence, by Propositions 3 and 11, for $\nu \in \{0, 1, 2\}$, we have that

$$\begin{aligned}
\sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \chi_{W_m}(\sigma_{\alpha,\beta}t)\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{triv}}}(\sigma_{\alpha,\beta}t)} &= (-1)^m \sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \sum_{i=0}^{\left\lfloor\frac{m(n-3)}{2}\right\rfloor} \zeta^{(\alpha-2\beta)(i+m)}\overline{\zeta^{(\alpha-2\beta)\frac{\nu n}{3}}} \\
&= (-1)^m \sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \sum_{i=0}^{\left\lfloor\frac{m(n-3)}{2}\right\rfloor} \zeta^{(\alpha-2\beta)(i+m-\frac{\nu n}{3})} \\
&= (-1)^m J\left(m(n-3), m - \frac{\nu n}{3}\right)
\end{aligned}$$

and similarly

$$\sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \chi_{I_m}(\sigma_{\alpha,\beta}t)\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{triv}}}(\sigma_{\alpha,\beta}t)} = (-1)^m J\left(m(n-3) - n, m - \frac{\nu n}{3}\right).$$

Theorem 12 combined with the definition of $J^{(m)}_{m-\frac{\nu n}{3}}$ given in Corollary 17 give

$$\sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \chi_{V_m}(\sigma_{\alpha,\beta}t)\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{triv}}}(\sigma_{\alpha,\beta}t)} = (-1)^m J^{(m)}_{m-\frac{\nu n}{3}}.$$

The same arguments give that

$$\begin{aligned}
\sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \chi_{V_m}(\sigma_{\alpha,\beta}ts)\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{triv}}}(\sigma_{\alpha,\beta}ts)} &= \sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \chi_{V_m}(\sigma_{\alpha,\beta}st)\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{triv}}}(\sigma_{\alpha,\beta}st)} \\
&= (-1)^m J^{(m)}_{m-\frac{\nu n}{3}}.
\end{aligned}$$

For $\rho = \rho_{\text{sgn}}$ we multiply everything by $\chi_{\text{sgn}}(t) = \chi_{\text{sgn}}(ts) = \chi_{\text{sgn}}(st) = -1$. □

We collect the above results in the following:

Proposition 22. *For $\nu \in \{0,1,2\}$, we have that*

$$\begin{aligned}
\langle \chi_{V_m}, \chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\mathrm{triv}}}\rangle &= \frac{1}{6}\left(m - \left\lceil \frac{3m}{n}\right\rceil + A^{(m)}_{\frac{\nu n}{3}} + 3B^{(m)}_{\frac{\nu n}{3}} + 2\Gamma^{(m)}_{\frac{\nu n}{3}}\right)\\
\langle \chi_{V_m}, \chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\mathrm{sgn}}}\rangle &= \frac{1}{6}\left(m - \left\lceil \frac{3m}{n}\right\rceil + A^{(m)}_{\frac{\nu n}{3}} - 3B^{(m)}_{\frac{\nu n}{3}} + 2\Gamma^{(m)}_{\frac{\nu n}{3}}\right)\\
\langle \chi_{V_m}, \chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\mathrm{stan}}}\rangle &= \frac{1}{3}\left(m - \left\lceil \frac{3m}{n}\right\rceil + A^{(m)}_{\frac{\nu n}{3}} - \Gamma^{(m)}_{\frac{\nu n}{3}}\right)
\end{aligned}$$

where each of $A^{(m)}_{\frac{\nu n}{3}}$, $B^{(m)}_{\frac{\nu n}{3}}$, $\Gamma^{(m)}_{\frac{\nu n}{3}}$ take values in $\{-1,0,1\}$ as in Lemmata 19, 20, 21.

Proof. For $\rho \in \{\rho_{\mathrm{triv}}, \rho_{\mathrm{sgn}}, \rho_{\mathrm{stan}}\}$ and $\nu \in \{0,1,2\}$, Corollary 18 gives

$$\langle \chi_{V_m}, \chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}\rangle = \frac{1}{6n^2}\sum_{\substack{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}\\ g\in S_3}} \left(\chi_{V_m}(\sigma_{\alpha,\beta}g)\right)\overline{\chi_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}(\sigma_{\alpha,\beta}g)}.$$

The summand corresponding to $g = 1$ was computed in Lemma 20, the summand corresponding to $g \in \{s, s^2\}$ in Lemma 19, and the summand corresponding to $g \in \{t, ts, st\}$ in Lemma 21. Note that the absence of a $B^{(m)}_{\frac{\nu n}{3}}$ term in the formula for ρ_{stan} is due to the fact that $\chi_{\rho_{\mathrm{stan}}}(g) = 0$ for $g \in \{t, ts, st\}$. $\square$

We proceed with the multiplicities of the 3-dimensional irreducible representations:

Proposition 23. *For $\kappa \notin \{0, \frac{n}{3}, \frac{2n}{3}\}$, we have that*

$$\begin{aligned}
\langle \chi_{V_m}, \chi_{\kappa,\kappa,\rho_{\mathrm{triv}}}\rangle &= \frac{1}{2}\left(m - \left\lceil \frac{3m}{n}\right\rceil + A^{(m)}_{\kappa} + B^{(m)}_{\kappa}\right),\\
\langle \chi_{V_m}, \chi_{\kappa,\kappa,\rho_{\mathrm{sgn}}}\rangle &= \frac{1}{2}\left(m - \left\lceil \frac{3m}{n}\right\rceil + A^{(m)}_{\kappa} - B^{(m)}_{\kappa}\right)
\end{aligned}$$

where

$$A^{(m)}_{\kappa} = \frac{1}{3}\left(\delta^{(m)}_{m-\kappa,m-\kappa} + \delta^{(m)}_{m-\kappa,m+2\kappa} + \delta^{(m)}_{m+2\kappa,m-\kappa}\right),\quad B^{(m)}_{\kappa} = (-1)^m \frac{1}{n^2} J^{(m)}_{m-\kappa}$$

and $\delta^{(m)}_{X,Y}$, $J^{(m)}_{m-\kappa}$ are as defined in Corollary 17.

Proof. As in the proofs of Lemmata 20, 21, Corollary 17 gives that

$$\begin{aligned}
\sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \chi_{V_m}(\sigma_{\alpha,\beta})\overline{\chi_{\kappa,\kappa,\rho_{\mathrm{triv}}}(\sigma_{\alpha,\beta})} &= I^{(m)}_{m-\kappa,m-\kappa} + I^{(m)}_{m-\kappa,m+2\kappa} + I^{(m)}_{m+2\kappa,m-\kappa}\\
&= n^2\left[3\left(m - \left\lceil \frac{3m}{n}\right\rceil\right) + \delta^{(m)}_{m-\kappa,m-\kappa}\right.\\
&\qquad \left. + \delta^{(m)}_{m-\kappa,m+2\kappa} + \delta^{(m)}_{m+2\kappa,m-\kappa}\right]
\end{aligned}$$

and that for $g \in \{t, ts, st\}$

$$\sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \chi_{V_m}(\sigma_{\alpha,\beta} g)\overline{\chi_{\kappa,\kappa,\rho_{\mathrm{triv}}}(\sigma_{\alpha,\beta} g)} = (-1)^m J^{(m)}_{m-\kappa}.$$

Thus, we obtain that

$$\begin{aligned}\langle \chi_{V_m}, \chi_{\kappa,\kappa,\rho_{\mathrm{triv}}}\rangle &= \frac{1}{6n^2} \sum_{\substack{\alpha,\beta\in\mathbb{Z}/n\\ g\in S_3}} \chi_{V_m}(\sigma_{\alpha,\beta} g)\overline{\chi_{\kappa,\kappa,\rho}(\sigma_{\alpha,\beta} g)} \\ &= \frac{1}{6}\left(3\left(m - \left\lceil \frac{3m}{n}\right\rceil\right) + \delta^{(m)}_{m-\kappa,m-\kappa} + \delta^{(m)}_{m-\kappa,m+2\kappa}\right. \\ &\qquad \left. + \delta^{(m)}_{m+2\kappa,m-\kappa} + 3(-1)^m \frac{1}{n^2} J^{(m)}_{m-\kappa}\right) \\ &= \frac{1}{2}\left(m - \left\lceil \frac{3m}{n}\right\rceil\right) + \frac{1}{3}\left(\delta^{(m)}_{m-\kappa,m-\kappa} + \delta^{(m)}_{m-\kappa,m+2\kappa} + \delta^{(m)}_{m+2\kappa,m-\kappa}\right. \\ &\qquad \left. + (-1)^m \frac{1}{n^2} J^{(m)}_{m-\kappa}\right)\end{aligned}$$

Substituting

$$A^{(m)}_{\kappa} = \frac{1}{3}\left(\delta^{(m)}_{m-\kappa,m-\kappa} + \delta^{(m)}_{m-\kappa,m+2\kappa} + \delta^{(m)}_{m+2\kappa,m-\kappa}\right), \quad B^{(m)}_{\kappa} = (-1)^m \frac{1}{n^2} J^{(m)}_{m-\kappa}$$

gives the desired formula for ρ_{triv}. The result for ρ_{sgn} follows in the same manner. □

Finally, we obtain the multiplicities of the 6-dimensional representations:

Proposition 24. *For $\kappa \notin \{0, \frac{n}{3}, \frac{2n}{3}\}$, we have that*

$$\langle \chi_{V_m}, \chi_{\kappa,\lambda,\rho_{\mathrm{triv}}}\rangle = m - \left\lceil \frac{3m}{n}\right\rceil + A^{(m)}_{\kappa,\lambda}$$

where

$$\begin{aligned}A^{(m)}_{\kappa,\lambda} = \frac{1}{6}&\left(\delta^{(m)}_{m-\kappa,m-\lambda} + \delta^{(m)}_{m-\lambda,m-\kappa} + \delta^{(m)}_{m-\kappa,m+\kappa+\lambda} + \delta^{(m)}_{m+\kappa+\lambda,m-\kappa}\right. \\ &\left. + \delta^{(m)}_{m-\lambda,m+\kappa+\lambda} + \delta^{(m)}_{m+\kappa+\lambda,m-\lambda}\right)\end{aligned}$$

and $\delta^{(m)}_{X,Y} \in \{0, 1\}$ is as defined in Corollary 17.

Proof. The result follows as in the proofs of Lemma 20 and Proposition 23:

$$\begin{aligned}\sum_{\alpha,\beta\in\mathbb{Z}/n\mathbb{Z}} \chi_{V_m}(\sigma_{\alpha,\beta})\overline{\chi_{\kappa,\lambda,\rho_{\mathrm{triv}}}(\sigma_{\alpha,\beta})} &= I^{(m)}_{m-\kappa,m-\lambda} + I^{(m)}_{m-\lambda,m-\kappa} + I^{(m)}_{m-\kappa,m+\kappa+\lambda} \\ &\quad + I^{(m)}_{m+\kappa+\lambda,m-\kappa} + I^{(m)}_{m-\lambda,m+\kappa+\lambda} + I^{(m)}_{m+\kappa+\lambda,m-\lambda} \\ &= n^2\left[6\left(m - \left\lceil \frac{3m}{n}\right\rceil\right) + \delta^{(m)}_{m-\kappa,m-\lambda}\right.\end{aligned}$$

$$+ \delta^{(m)}_{m-\lambda,m-\kappa} + \delta^{(m)}_{m-\kappa,m+\kappa+\lambda}$$

$$+ \delta^{(m)}_{m+\kappa+\lambda,m-\kappa} + \delta^{(m)}_{m-\lambda,m+\kappa+\lambda} + \delta^{(m)}_{m+\kappa+\lambda,m-\lambda}\Big] \quad \square$$

Theorem 25.

$$\langle \chi_{V_m}, \chi_{\kappa,\lambda,\rho}\rangle = \frac{\dim \theta_{\kappa,\lambda,\rho}}{6}\left(m - \left\lceil \frac{3m}{n}\right\rceil + A^{(m)}_{\kappa,\lambda,\rho}\right) + \frac{1}{2}B^{(m)}_{\kappa,\lambda,\rho} + \frac{1}{3}\Gamma^{(m)}_{\kappa,\lambda,\rho}$$

where $A^{(m)}_{\kappa,\lambda,\rho}$, $B^{(m)}_{\kappa,\lambda,\rho}$ *are defined using the quantities* $\delta^{(m)}_{X,Y}$ *and* $J^{(m)}_X$ *of* Corollary 17 *as follows:*

κ, λ, ρ	$A^{(m)}_{\kappa,\lambda,\rho}$	$B^{(m)}_{\kappa,\lambda,\rho}$
$\frac{\nu n}{3}, \frac{\nu n}{3}, \rho_{\text{triv}}$	$\delta^{(m)}_{m-\frac{\nu n}{3},m-\frac{\nu n}{3}}$	$\frac{(-1)^m}{n^2} J^{(m)}_{m-\frac{\nu n}{3}}$
$\frac{\nu n}{3}, \frac{\nu n}{3}, \rho_{\text{sgn}}$	$\delta^{(m)}_{m-\frac{\nu n}{3},m-\frac{\nu n}{3}}$	$\frac{(-1)^{m+1}}{n^2} J^{(m)}_{m-\frac{\nu n}{3}}$
$\frac{\nu n}{3}, \frac{\nu n}{3}, \rho_{\text{stan}}$	$\delta^{(m)}_{m-\frac{\nu n}{3},m-\frac{\nu n}{3}}$	0
$\kappa, \kappa, \rho_{\text{triv}}$	$\frac{1}{3}\left(\delta^{(m)}_{m-\kappa,m-\kappa} + \delta^{(m)}_{m-\kappa,m+2\kappa} + \delta^{(m)}_{m+2\kappa,m-\kappa}\right)$	$\frac{(-1)^m}{n^2} J^{(m)}_{m-\kappa}$
$\kappa, \kappa, \rho_{\text{sgn}}$	$\frac{1}{3}\left(\delta^{(m)}_{m-\kappa,m-\kappa} + \delta^{(m)}_{m-\kappa,m+2\kappa} + \delta^{(m)}_{m+2\kappa,m-\kappa}\right)$	$\frac{(-1)^{m+1}}{n^2} J^{(m)}_{m-\kappa}$
$\kappa, \lambda, \rho_{\text{triv}}$	$\frac{1}{6}\Big(\delta^{(m)}_{m-\kappa,m-\lambda} + \delta^{(m)}_{m-\lambda,m+\kappa+\lambda} + \delta^{(m)}_{m-\kappa,m+\kappa+\lambda}$ $+\delta^{(m)}_{m+\kappa+\lambda,m-\kappa} + \delta^{(m)}_{m-\lambda,m-\kappa} + \delta^{(m)}_{m+\kappa+\lambda,m-\lambda}\Big)$	0

and

$$\Gamma^{(m)}_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{triv}}} = \Gamma^{(m)}_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{sgn}}} = -\Gamma^{(m)}_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{stan}}} = \begin{cases} 1 & , \text{ if } 3 \mid m-\nu \\ -1 & , \text{ if } 3 \mid m-\nu+2 \\ 0 & , \text{ otherwise} \end{cases}$$

$$\Gamma^{(m)}_{\kappa,\kappa,\rho_{\text{triv}}} = \Gamma^{(m)}_{\kappa,\kappa,\rho_{\text{sgn}}} = \Gamma^{(m)}_{\kappa,\lambda,\rho_{\text{triv}}} = 0$$

Note that in the above Theorem, we have extended the definition of $\Gamma^{(m)}_{\frac{\nu n}{3}}$ given in Lemma 19 by setting

$$\Gamma^{(m)}_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{triv}}} = \Gamma^{(m)}_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{sgn}}} = -\Gamma^{(m)}_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{stan}}} = \Gamma^{(m)}_{\frac{\nu n}{3}},$$

the definition of $B_\kappa^{(m)}$ given in Lemma 21 and Proposition 23 by setting

$$B^{(m)}_{\kappa,\kappa,\rho_{\text{triv}}} = -B^{(m)}_{\kappa,\kappa,\rho_{\text{sgn}}} = B^{(m)}_{\kappa}, \ \left(\text{including the case } \kappa = \frac{\nu n}{3}\right)$$

and that of $A_\kappa^{(m)}$ and $A_{\kappa,\lambda}^{(m)}$ given in Lemma 21, Propositions 23 and 24 by setting

$$A^{(m)}_{\kappa,\kappa,\rho_{\text{triv}}} = A^{(m)}_{\kappa,\kappa,\rho_{\text{sgn}}} = A^{(m)}_{\kappa} \ \left(\text{including the case } \kappa = \frac{\nu n}{3}\right),$$
$$A^{(m)}_{\kappa,\lambda,\rho_{\text{triv}}} = A^{(m)}_{\kappa,\lambda,\rho_{\text{sgn}}} = A^{(m)}_{\kappa,\lambda}.$$

To convince the reader that the above information can give explicit results, we have included Table 2, which treats Fermat curves corresponding to $n = 4, 5, 6$. The first 9 columns contain the multiplicities of one-dimensional and two-dimensional representations. For the 3 dimensional representations $\theta_{\kappa,\kappa,\rho}$, $\rho \in \{\rho_{\text{triv}}, \rho_{\text{sgn}}\}$ if the symbol $[\kappa, t]$ appears then this means that the representation $\theta_{\kappa,\kappa,\rho}$ appears with multiplicity t. Similarly if $[(\kappa, \lambda), t]$ appears then $\theta_{\kappa,\lambda,\rho_{\text{triv}}}$ appears with multiplicity t. For example the 6th row indicates that for the curve $F_4 : x^4 + y^4 + z^4 = 0$ we have the decomposition

$$H^0(F_4, \Omega_{F_4}^{\otimes 6}) = \theta_{0,0,\text{triv}} \oplus \theta_{1,1,\rho_{\text{triv}}} \oplus \theta_{2,2,\rho_{\text{triv}}} \oplus \theta_{3,3,\rho_{\text{triv}}} \oplus \theta_{2,2,\rho_{\text{sgn}}} \oplus \theta_{3,3,\rho_{\text{sgn}}} \oplus \theta_{0,1,\rho_{\text{triv}}}.$$

We remark that the last column of Table 2 serves as an extra verification that the multiplicities add to the expected K-dimension of $H^0(F_n, \Omega_{F_n}^{\otimes m})$.

6. Generating functions

Let $R = \bigoplus_{m=0}^{\infty} H^0(F_n, \Omega_{F_n}^{\otimes m})$ denote the canonical ring of the Fermat curve $F_n : x^n + y^n + z^n = 0$ and let $G = (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \rtimes S_3$ be the automorphism group of F_n. The equivariant Hilbert function of the action of G on R is defined as

$$H_{R,G}(t) = \sum_{m=0}^{\infty} [H^0(F_n, \Omega_{F_n}^{\otimes m})] t^m$$

where $[H^0(F_n, \Omega_{F_n}^{\otimes m})]$ denotes the class in the Grothendieck group $K_0(G, K)$. For each irreducible representation $\theta_{\kappa,\lambda,\rho}$ of G we denote by $H_{\kappa,\lambda,\rho}(t)$ the equivariant Hilbert series of the respective isotypical component of the action of G on R, so that

$$H_{R,G}(t) = \sum_{\kappa,\lambda,\rho} H_{\kappa,\lambda,\rho}(t).$$

We then have the following:

Theorem 26. *Let $\theta_{\kappa,\lambda,\rho} \in \mathrm{Irrep}(G)$ be as in Proposition 3, let $A^{(m)}_{\kappa,\lambda,\rho}$ and $B^{(m)}_{\kappa,\lambda,\rho}$ be as in Theorem 25 and let*

$$n_{0,\kappa,\lambda,\rho} := \begin{cases} 1 & , \text{ if } \theta_{\kappa,\lambda,\rho} = \theta_{0,0,\rho_{\mathrm{triv}}} \\ 0 & , \text{ otherwise.} \end{cases}$$

The equivariant Hilbert function $H_{\kappa,\lambda,\rho}(t)$ is given by the following rational function

$$\begin{aligned} H_{\kappa,\lambda,\rho}(t) = &\left(n_{0,\kappa,\lambda,\rho} - \frac{\dim \theta_{\kappa,\lambda,\rho}}{6} A^{(n)}_{\kappa,\lambda,\rho} - \frac{1}{2} B^{(2n)}_{\kappa,\lambda,\rho} \right) \\ &+ \frac{\dim \theta_{\kappa,\lambda,\rho}}{6} \left[\frac{t}{(1-t)^2} - \frac{1}{1-t^n} \left(\frac{3t^n}{1-t} + F(t) - G_{A_{\kappa,\lambda,\rho}}(t) \right) \right] \\ &+ \frac{1}{1-t^{2n}} G_{B_{\kappa,\lambda,\rho}}(t) + \frac{1}{3} G_{\Gamma_{\kappa,\lambda,\rho}}(t) \end{aligned}$$

where

$$F(t) = \sum_{m=0}^{n-1} \left\lceil \frac{3m}{n} \right\rceil t^m$$

$$G_{A_{\kappa,\lambda,\rho}}(t) = \sum_{m=0}^{n-1} A^{(m+n)}_{\kappa,\lambda,\rho} t^m$$

$$G_{B_{\kappa,\lambda,\rho}}(t) = \sum_{m=0}^{2n-1} B^{(m+2n)}_{\kappa,\lambda,\rho} t^m$$

$$\begin{aligned} G_{\Gamma_{\kappa,\lambda,\rho_{\mathrm{triv}}}}(t) = G_{\Gamma_{\kappa,\lambda,\rho_{\mathrm{sgn}}}}(t) &= -G_{\Gamma_{\kappa,\lambda,\rho_{\mathrm{stan}}}}(t) \\ &= \begin{cases} \dfrac{t^\nu - t^{\nu+1}}{1-t^3} & , \text{ if } \kappa = \lambda = \frac{\nu n}{3} \text{ and } \nu \in \{0,1,2\} \\ 0 & , \text{ otherwise.} \end{cases} \end{aligned}$$

Proof. For $m \geq 0$, let $V_m = H^0(F_n, \Omega_{F_n}^{\otimes m})$. Observe that $V_0 = K$ and so $\langle \chi_{V_0}, \chi_{\kappa,\lambda,\rho} \rangle = n_{0,\kappa,\lambda,\rho}$, whereas for $m \geq 1$, $\langle \chi_{V_m}, \chi_{\kappa,\lambda,\rho} \rangle$ is given by Theorem 25. Thus

$$\begin{aligned} H_{\kappa,\lambda,\rho}(t) &= \sum_{m=0}^{\infty} \langle \chi_{V_m}, \chi_{\kappa,\lambda,\rho} \rangle t^m \\ &= n_{0,\kappa,\lambda,\rho} + \sum_{m=1}^{\infty} \left[\frac{\dim \theta_{\kappa,\lambda,\rho}}{6} \left(m - \left\lceil \frac{3m}{n} \right\rceil + A^{(m)}_{\kappa,\lambda,\rho} \right) + \frac{1}{2} B^{(m)}_{\kappa,\lambda,\rho} + \frac{1}{3} \Gamma^{(m)}_{\kappa,\lambda,\rho} \right] t^m. \end{aligned}$$

Note that the quantities $A^{(m)}_{\kappa,\lambda,\rho}$ and $B^{(m)}_{\kappa,\lambda,\rho}$ are defined only for $m \geq 1$ and have no meaning for $m = 0$. However, we observe that by Theorem 25 and Corollary 17, the quantity $A^{(m)}_{\kappa,\lambda,\rho}$ depends only on the value of m modulo n and thus we extend the definition by setting $A^{(0)}_{\kappa,\lambda,\rho} := A^{(n)}_{\kappa,\lambda,\rho}$. Further, recall that for n large enough and for any $\kappa \in \mathbb{Z}/n\mathbb{Z}$, by Corollary 17

$$J^{(m)}_{m-\kappa} = n^2 \left(\left\lfloor \frac{\left\lfloor \frac{m(n-3)}{2} \right\rfloor - \upsilon_{-m+\kappa}}{n} \right\rfloor - \left\lfloor \frac{\left\lfloor \frac{m(n-3)-n}{2} \right\rfloor - \upsilon_{-m+\kappa}}{n} \right\rfloor \right)$$

Thus, if $m \equiv m' \bmod 2n$ then $J^{(m)}_{m-\kappa} = J^{(m')}_{m'-\kappa}$ and, by Theorem 25, $B^{(m)}_{\kappa,\lambda,\rho} = B^{(m')}_{\kappa,\lambda,\rho}$. This allows us to extend the definition by setting $B^{(0)}_{\kappa,\lambda,\rho} := B^{(2n)}_{\kappa,\lambda,\rho}$; finally, we observe that by Theorem 25 $\Gamma^{(0)}_{\kappa,\lambda,\rho} = 0$ and thus we set

$$H'_{\kappa,\lambda,\rho}(t) = \sum_{m=0}^{\infty}\left[\frac{\dim\theta_{\kappa,\lambda,\rho}}{6}\left(m - \left\lceil\frac{3m}{n}\right\rceil + A^{(m)}_{\kappa,\lambda,\rho}\right) + \frac{1}{2}B^{(m)}_{\kappa,\lambda,\rho} + \frac{1}{3}\Gamma^{(m)}_{\kappa,\lambda,\rho}\right]t^m$$
$$= \left(\frac{\dim\theta_{\kappa,\lambda,\rho}}{6}A^{(n)}_{\kappa,\lambda,\rho} + \frac{1}{2}B^{(2n)}_{\kappa,\lambda,\rho}\right) + \sum_{m=1}^{\infty}\left[\frac{\dim\theta_{\kappa,\lambda,\rho}}{6}\left(m - \left\lceil\frac{3m}{n}\right\rceil + A^{(m)}_{\kappa,\lambda,\rho}\right) + \frac{1}{2}B^{(m)}_{\kappa,\lambda,\rho} + \frac{1}{3}\Gamma^{(m)}_{\kappa,\lambda,\rho}\right]t^m.$$

To compute $H'_{\kappa,\lambda,\rho}(t)$, we rewrite

$$H'_{\kappa,\lambda,\rho}(t) = \frac{\dim\theta_{\kappa,\lambda,\rho}}{6}\sum_{m=0}^{\infty}\left(m - \left\lceil\frac{3m}{n}\right\rceil + A^{(m)}_{\kappa,\lambda,\rho}\right)t^m + \frac{1}{2}\sum_{m=0}^{\infty}B^{(m)}_{\kappa,\lambda,\rho}t^m + \frac{1}{3}\sum_{m=0}^{\infty}\Gamma^{(m)}_{\kappa,\lambda,\rho}t^m,$$

then observe that

$$\sum_{m=0}^{\infty} m t^m = \frac{t}{(1-t)^2},$$

and similarly

$$\sum_{m=0}^{\infty}\left\lceil\frac{3m}{n}\right\rceil t^m = \sum_{\upsilon=0}^{n-1}\sum_{\mu=0}^{\infty}\left\lceil\frac{3\mu n + 3\upsilon}{n}\right\rceil t^{\mu n+\upsilon} = \sum_{\upsilon=0}^{n-1}t^{\upsilon}\sum_{\mu=0}^{\infty}3\mu t^{\mu n} + \sum_{\upsilon=0}^{n-1}\left\lceil\frac{3\upsilon}{n}\right\rceil t^{\upsilon}\sum_{\mu=0}^{\infty}t^{\mu n}$$
$$= \frac{3t^n}{(1-t^n)^2}\sum_{\upsilon=0}^{n-1}t^{\upsilon} + \frac{1}{1-t^n}\sum_{\upsilon=0}^{n-1}\left\lceil\frac{3\upsilon}{n}\right\rceil t^{\upsilon}$$
$$= \frac{1}{1-t^n}\left(\frac{3t^n}{1-t} + \sum_{\upsilon=0}^{n-1}\left\lceil\frac{3\upsilon}{n}\right\rceil t^{\upsilon}\right).$$

Next, we recall that by Theorem 25, $\Gamma_{\kappa,\lambda,\rho}$ is non-zero only if $\kappa = \lambda = \frac{\upsilon n}{3}$, in which case

$$\sum_{m=0}^{\infty}\Gamma^{(m)}_{\frac{\upsilon n}{3},\frac{\upsilon n}{3},\rho}t^m = \sum_{\mu=0}^{\infty}t^{3\mu+\nu} - \sum_{\mu=0}^{\infty}t^{3\mu+\nu+1} = \frac{t^{\nu} - t^{\nu+1}}{1-t^3}$$

The arguments used above that $A^{(m)}_{\kappa,\lambda,\rho}$ depends only on the value of m modulo n and $B^{(m)}_{\kappa,\lambda,\rho}$ depends only on the value of m modulo $2n$ give that:

$$\sum_{m=0}^{\infty}A^{(m)}_{\kappa,\lambda,\rho}t^m = \sum_{\mu=0}^{\infty}\sum_{m=0}^{n-1}A^{(m)}_{\kappa,\lambda,\rho}t^{\mu n+m} = \frac{1}{1-t^n}\sum_{m=0}^{n-1}A^{(m)}_{\kappa,\lambda,\rho}t^m = \frac{1}{1-t^n}\sum_{m=0}^{n-1}A^{(m+n)}_{\kappa,\lambda,\rho}t^m$$
$$\sum_{m=0}^{\infty}B^{(m)}_{\kappa,\lambda,\rho}t^m = \sum_{\mu=0}^{\infty}\sum_{m=0}^{2n-1}B^{(m)}_{\kappa,\lambda,\rho}t^{\mu\cdot 2n+m} = \frac{1}{1-t^{2n}}\sum_{m=0}^{2n-1}B^{(m)}_{\kappa,\lambda,\rho}t^m$$
$$= \frac{1}{1-t^{2n}}\sum_{m=0}^{2n-1}B^{(m+2n)}_{\kappa,\lambda,\rho}t^m. \quad \square$$

Although the expression for $H_{\kappa,\lambda,\rho}(t)$ given in Theorem 26 is complicated, it serves well for computations as it has allowed us to write a program that computes the equivariant Hilbert function of any Fermat curve; see the example below:

Example 27. We consider the Fermat curve $F_6 : x^6 + y^6 + z^6 = 0$, which has genus $g = 10$ and automorphism group $G = (\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}) \rtimes S_3$. By Proposition 3, G has the following 19 irreducible representations:

Dimension 1	$\theta_{0,0,\rho_{\text{triv}}},\ \theta_{2,2,\rho_{\text{triv}}},\ \theta_{4,4,\rho_{\text{triv}}}$ $\theta_{0,0,\rho_{\text{sgn}}},\ \theta_{2,2,\rho_{\text{sgn}}},\ \theta_{4,4,\rho_{\text{sgn}}}$
Dimension 2	$\theta_{0,0,\rho_{\text{stan}}},\ \theta_{2,2,\rho_{\text{stan}}},\ \theta_{4,4,\rho_{\text{stan}}}$
Dimension 3	$\theta_{1,1,\rho_{\text{triv}}},\ \theta_{3,3,\rho_{\text{triv}}},\ \theta_{5,5,\rho_{\text{triv}}}$ $\theta_{1,1,\rho_{\text{sgn}}},\ \theta_{3,3,\rho_{\text{sgn}}},\ \theta_{5,5,\rho_{\text{sgn}}}$
Dimension 6	$\theta_{0,1,\rho_{\text{triv}}},\ \theta_{0,2,\rho_{\text{triv}}},\ \theta_{1,2,\rho_{\text{triv}}},\ \theta_{3,4,\rho_{\text{triv}}}$

We use our computer code[1] to explictly compute the equivariant Hilbert functions $H_{\kappa,\lambda,\rho}(t)$ of the respective isotypical components, which are given in the following table:

(κ, λ)	ρ_{triv}	ρ_{sgn}	ρ_{stan}
(0, 0)	$\frac{t^{14}-t^{13}+t^8-t^7+t^6-t+1}{t^{13}-t^{12}-t+1}$	$\frac{t^{11}-t^{10}+t^9-t^7+t^5-t^4+t^3}{t^{13}-t^{12}-t+1}$	$\frac{t^4}{t^7-t^6-t+1}$
(2, 2)	$\frac{t^{12}-t^{11}+t^{10}-t^9+t^6-t^5+t^4}{t^{13}-t^{12}-t+1}$	$\frac{t^9-t^8+t^7-t^2+t}{t^{13}-t^{12}-t+1}$	$\frac{t^5-t^3+t^2}{t^7-t^6-t+1}$
(4, 4)	$\frac{t^{10}-t^9+t^8-t^5+t^4-t^3+t^2}{t^{13}-t^{12}-t+1}$	$\frac{t^{13}-t^{12}+t^7-t^6+t^5}{t^{13}-t^{12}-t+1}$	$\frac{t^6-t^5+t^3}{t^7-t^6-t+1}$
(1, 1)	$\frac{t^{12}-t^{11}+t^{10}+t^6+t^4}{t^{13}-t^{12}-t+1}$	$\frac{t^{11}+t^9-t^8+t^7+t^3-t^2+t}{t^{13}-t^{12}-t+1}$	–
(3, 3)	$\frac{t^{12}+t^8-t^7+t^6+t^2}{t^{13}-t^{12}-t+1}$	$\frac{t^{11}-t^{10}+t^9+t^7+t^5-t^4+t^3}{t^{13}-t^{12}-t+1}$	–
(5, 5)	$\frac{t^{10}+t^8+t^4-t^3+t^2}{t^{13}-t^{12}-t+1}$	$\frac{t^{13}-t^{12}+t^{11}+t^7-t^6+t^5+t^3}{t^{13}-t^{12}-t+1}$	–
(0, 1)	$\frac{t^3}{t^5-2t^4+t^3+t^2-2t+1}$	–	–
(0, 2)	$\frac{t^2}{t^3-t^2-t+1}$	–	–
(1, 2)	$\frac{t^4-t^2+t}{t^5-2t^4+t^3+t^2-2t+1}$	–	–
(3, 4)	$\frac{t^5-t^4+t^2}{t^5-2t^4+t^3+t^2-2t+1}$	–	–

[1] File: FinalCodeFermatReps.ipynb, url: https://tinyurl.com/3hvart2d.

Table 2

The $K[G]$-module structure of $H^0(X, \Omega_X^{\otimes m})$ for $n \in \{4, 5, 6\}$ and $m \in \{1, 2 \ldots, 9\}$.

n	m	ρ_{triv}			ρ_{sgn}			ρ_{stan}			$\kappa, \kappa, \rho_{\text{triv}}$	$\kappa, \kappa, \rho_{\text{sgn}}$	κ, λ	$\dim H^0(X, \Omega_X^{\otimes m})$
4	1	0	–	–	0	–	–	0	–	–		[1, 1]		3
4	2	0	–	–	0	–	–	0	–	–	[2, 1], [3, 1]			6
4	3	0	–	–	1	–	–	0	–	–		[3, 1]	[(0, 1), 1]	10
4	4	0	–	–	0	–	–	1	–	–	[1, 1], [2, 1]		[(0, 1), 1]	14
4	5	0	–	–	0	–	–	0	–	–	[1, 1]	[1, 1], [2, 1], [3, 1]	[(0, 1), 1]	18
4	6	1	–	–	0	–	–	0	–	–	[1, 1], [2, 1], [3, 1]	[2, 1], [3, 1]	[(0, 1), 1]	22
4	7	0	–	–	0	–	–	1	–	–	[3, 1]	[1, 1], [2, 1], [3, 1]	[(0, 1), 2]	26
4	8	1	–	–	0	–	–	1	–	–	[1, 1], [2, 1], [3, 1]	[1, 1], [2, 1]	[(0, 1), 2]	30
4	9	0	–	–	1	–	–	0	–	–	[1, 1], [2, 1], [3, 1]	[1, 2], [2, 1], [3, 1]	[(0, 1), 2]	34
4	10	0	–	–	0	–	–	1	–	–	[1, 1], [2, 2], [3, 2]	[1, 1], [2, 1], [3, 1]	[(0, 1), 2]	38
4	11	0	–	–	1	–	–	1	–	–	[1, 1], [2, 1], [3, 1]	[1, 1], [2, 1], [3, 2]	[(0, 1), 3]	42
4	12	1	–	–	1	–	–	1	–	–	[1, 2], [2, 2], [3, 1]	[1, 1], [2, 1], [3, 1]	[(0, 1), 3]	46
4	13	0	–	–	0	–	–	1	–	–	[1, 2], [2, 1], [3, 1]	[1, 2], [2, 2], [3, 2]	[(0, 1), 3]	50
5	1	0	–	–	0	–	–	0	–	–		[1, 1], [2, 1]		6
5	2	0	–	–	0	–	–	0	–	–	[2, 1], [3, 1], [4, 1]		[(0, 2), 1]	15
5	3	0	–	–	1	–	–	0	–	–	[3, 1]	[1, 1], [3, 1], [4, 1]	[(0, 1), 1], [(0, 2), 1]	25
5	4	0	–	–	0	–	–	1	–	–	[1, 1], [2, 1], [3, 1], [4, 1]	[4, 1]	[(0, 1), 2], [(0, 2), 1]	35
5	5	0	–	–	1	–	–	1	–	–	[1, 1], [2, 1]	[1, 1], [2, 1], [3, 1], [4, 1]	[(0, 1), 2], [(0, 2), 2]	45
5	6	1	–	–	0	–	–	0	–	–	[1, 2], [2, 2], [3, 1], [4, 1]	[1, 1], [2, 1], [3, 1], [4, 1]	[(0, 1), 2], [(0, 2), 2]	55
5	7	0	–	–	0	–	–	1	–	–	[1, 1], [2, 1], [3, 1], [4, 1]	[1, 1], [2, 2], [3, 2], [4, 2]	[(0, 1), 2], [(0, 2), 3]	65
5	8	1	–	–	0	–	–	1	–	–	[1, 2], [2, 1], [3, 2], [4, 2]	[1, 1], [2, 1], [3, 2], [4, 1]	[(0, 1), 3], [(0, 2), 3]	75
5	9	1	–	–	1	–	–	1	–	–	[1, 1], [2, 1], [3, 1], [4, 2]	[1, 2], [2, 2], [3, 2], [4, 2]	[(0, 1), 4], [(0, 2), 3]	85
5	10	1	–	–	0	–	–	2	–	–	[1, 2], [2, 2], [3, 2], [4, 2]	[1, 2], [2, 2], [3, 1], [4, 1]	[(0, 1), 4], [(0, 2), 4]	95
5	11	0	–	–	1	–	–	1	–	–	[1, 2], [2, 2], [3, 2], [4, 2]	[1, 3], [2, 3], [3, 2], [4, 2]	[(0, 1), 4], [(0, 2), 4]	105
5	12	1	–	–	1	–	–	1	–	–	[1, 2], [2, 3], [3, 3], [4, 3]	[1, 2], [2, 2], [3, 2], [4, 2]	[(0, 1), 4], [(0, 2), 5]	115
5	13	0	–	–	1	–	–	2	–	–	[1, 2], [2, 2], [3, 3], [4, 2]	[1, 3], [2, 2], [3, 3], [4, 3]	[(0, 1), 5], [(0, 2), 5]	125
6	1	0	0	0	0	1	0	0	0	0		[1, 1]	[(1, 2), 1]	10
6	2	0	0	0	0	0	1	1	0	0	[3, 1], [5, 1]		[(0, 2), 1], [(1, 2), 1], [(3, 4), 1]	27
6	3	0	1	0	0	0	0	0	0	1	[3, 1]	[1, 1], [3, 1], [5, 1]	[(0, 1), 1], [(0, 2), 1], [(1, 2), 1], [(3, 4), 2]	45
6	4	0	0	1	1	0	0	1	0	1	[1, 1], [3, 1], [5, 1]	[1, 1], [5, 1]	[(0, 1), 2], [(0, 2), 2], [(1, 2), 1], [(3, 4), 2]	63
6	5	0	1	1	0	0	1	0	1	0	[1, 1], [3, 1], [5, 1]	[1, 1], [3, 1], [5, 2]	[(0, 1), 3], [(0, 2), 2], [(1, 2), 2], [(3, 4), 2]	81
6	6	1	1	1	1	0	1	0	0	1	[1, 2], [3, 2], [5, 1]	[1, 1], [3, 1], [5, 1]	[(0, 1), 3], [(0, 2), 3], [(1, 2), 3], [(3, 4), 2]	99
6	7	0	0	1	1	1	1	0	1	1	[1, 2], [3, 1], [5, 1]	[1, 2], [3, 2], [5, 2]	[(0, 1), 3], [(0, 2), 3], [(1, 2), 4], [(3, 4), 3]	117
6	8	1	0	1	1	0	2	1	1	1	[1, 2], [3, 2], [5, 2]	[1, 1], [3, 2], [5, 2]	[(0, 1), 3], [(0, 2), 4], [(1, 2), 4], [(3, 4), 4]	135
6	9	1	1	1	0	1	1	0	1	2	[1, 2], [3, 2], [5, 2]	[1, 2], [3, 3], [5, 2]	[(0, 1), 4], [(0, 2), 4], [(1, 2), 4], [(3, 4), 5]	153

As further evidence that our computations are correct, the sum of the above functions weighted with the dimensions of the corresponding representations is computed in Sage and yields the rational function

$$\frac{t^3+8t^2+8t+1}{t^2-2t+1}$$

whose Taylor expansion is

$$1+10t+\sum_{m=2}^{\infty} 9(2m-1)t^m.$$

We note that the coefficient of t is $g = 10$, whereas for $m \geq 2$ the coefficient of t^m is $(2m-1)(g-1)$ and thus we retrieve the classic, non-equivariant Hilbert series $H_R(t)$, as expected.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.indag.2022.06.001.

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