ON THE GALOIS-MODULE STRUCTURE OF POLYDIFFERENTIALS OF ARTIN-SCHREIER-MUMFORD CURVES, MODULAR AND INTEGRAL REPRESENTATION THEORY

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Abstract. We study the Galois-module structure of polydifferentials for Mumford curves, defined over a field of positive charactersitic, using the theory of harmonic cocycles. For the case of Artin-Schreier-Mumford curves the structure of holomorphic polydifferentials is explicitly computed.

1. Introduction

Let X be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field K of characteristic $p > 0$, and G a group of automorphisms of X. The group G acts on X from the left, by our convention, and hence on the space of n-polydifferentials $H^0(X, \Omega_X^{\otimes n})$ from the right. The so-called *Galois-module structure problem* for X asks for the direct sum decomposition of $H^0(X, \Omega_X^{\otimes n})$ into G-indecomposable pieces. In characteristic zero, the $n = 1$ case is a classical result ([15]), which can be easily generalized for $n \geq 1$.

In positive characteristic, the Galois-module structure is unknown in general. There are only some partial results known. Let us give a brief overview. If the cover $X \to G\backslash X$ is unramified or if $(|G|, p) = 1$, Tamagawa [33] determined the G-module structure of $H^0(X, \Omega_X)$. Valentini [35] generalized this result to unramified extensions with G being a p-group. In the p-group case, moreover, Salvador and Bautista [24] determined the semi-simple part of the representation with respect to the Cartier operator. For the cyclic-group case, Valentini and Madan [36] and S. Karanikolopoulos [16] determined the structure of $H^0(X, \Omega_X)$ in terms of indecomposable modules. A similar study has been done for the elementary abelian case by Calderón, Salvador and Madan [28]. Finally, N. Borne [2] developed a theory, using advanced techniques from both modular representation theory and K-theory, for computing in some cases the $K[G]$-module structure of the space of polydifferentials $H^0(X, \Omega_X^{\otimes n})$.

Let us point out that the determination of the Galois-module structure as above has several applications. For example, in [21], [20], the second author connected the $K[G]$-module structure of $H^0(X, \Omega_X^{\otimes 2})$ to the computation of the tangent space of the global deformation functor of curves.

In this paper, we consider the Galois-module structure problem for the so-called *Artin-Schreier-Mumford curves* (see below). We give for these curves explicit bases of the space of polydifferentials, and apply the theory of B. Köck [18] to complete spaces of polydifferentials by admitting controlled poles at certain points in order to obtain projective modules. By this way, we can prove that all indecomposable

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$K[G]$-modules admit $K[G]$ itself as an injective hull, and finally arrive at our main results.

Over a complete discrete valuation field K, D. Mumford [26] has shown that a smooth projective curve with the split multiplicative reduction is isomorphic to the algebraization of a rigid analytic curve over K of the form $\Gamma\backslash(\mathbb{P}^{1,\mathrm{an}}_K - \mathcal{L}_\Gamma)$. Here, Γ is a finitely generated torsion-free discrete subgroup of $\mathrm{PGL}(2,K)$, called a *Schottky group*, and $\mathcal{L}_\Gamma$ is the set of limit points. A smooth projective curve obtained in this way, denoted by X_Γ, is called a *Mumford curve*, and the uniformization just described provides us with a set of tools similar to those coming from the uniformization theory of Riemann surfaces. It is known that the subgroup Γ is always a free group of finite rank, and the rank is equal to the genus of X_Γ. The authors together with G. Cornelissen have used this technique in order to bound the automorphism groups of Mumford curves in [7]. In fact, the automorphism group $\mathrm{Aut}(X_\Gamma)$ is isomorphic to the quotient N_Γ/Γ of the normalizer of Γ in $\mathrm{PGL}(2,K)$ by Γ; cf. [7, 1.3] and [11, VII.1)]. Also the equivariant deformation theory of such curves was studied by the first author and G. Cornelissen in [3].

One of the tools we will use is the explicit description of polydifferentials in terms of harmonic cocycles. P. Schneider and J. Teitelbaum [29][34], defined the space of modular forms (or harmonic measures as they are known in the literature) $C_{\mathrm{har}}(\Gamma,n)$ on the reduction graph, and they showed that it is naturally isomorphic to $H^0(X_\Gamma,\Omega^{\otimes n}_{X_\Gamma})$. Moreover, the space $C_{\mathrm{har}}(\Gamma,n)$ can be described by the Galois cohomology $C_{\mathrm{har}}(\Gamma,n)\cong H^1(\Gamma,P_n)$, where P_n denotes the space of polynomials of one variable of degree $\leq 2n-2$ (cf. §2 for more details).

Now let us state our main results of this paper. We first give the definition of Artin-Schreier-Mumford curves:

Definition 1. Let K be a complete non-archimedean valued field of characteristic $p>0$, and q a power of p. For $\lambda\in K$ with $0<|\lambda|<1$, the smooth projective model of the affine plane curve defined by the equation

$$(x^q-x)(y^q-y)=\lambda$$

will be called an *Artin-Schreier-Mumford curve.*

These curves are special from quite a few points of view. For example, they are the Mumford curves with maximal automorphism group (and hence their Schottky groups are the analogue of classical Hurwitz groups), cf. [7] and [4]. They were first studied by D. Subrao [32], Valentini-Madan [36], and S. Nakajima [27]. M. Matignon has studied their equivariant liftability to characteristic zero [25], and these curves play a special role when studying the 'field of definition versus field of moduli' question for cyclic covers of the projective line (cf. [22]).

In this paper, we only deal with Artin-Schreier-Mumford curves as in Definition 1 with $q=p$. For the proof of the following facts, we refer to [7, §9] and [8, p. 347]:

Proposition 2. *The Artin-Schreier-Mumford curves are Mumford curves of the form X_Γ, where the group Γ is, up to conjugacy, given by the commutator group $\Gamma=[A,B]$ of the cyclic subgroups $A,B\subset\mathrm{PGL}(2,K)$ of order p generated by*

$$\epsilon_A=\begin{pmatrix}1&1\\0&1\end{pmatrix}\quad\textit{and}\quad\epsilon_B=\begin{pmatrix}1&0\\s&1\end{pmatrix},\tag{1}$$

respectively, where $s\in K^\times$ and $|s|>1$. The groups A and B generate a discrete subgroup $N\subseteq\mathrm{PGL}(2,K)$, which is isomorphic to the free product $A*B$. Moreover:

(a) *Γ is a normal subgroup of N and $N/\Gamma\cong A\times B$;*
(b) *Γ is a free group of rank $(p-1)^2$ with the basis given by the commutators $e_{i,j}=[\epsilon_A^i,\epsilon_B^j]$ $(=\epsilon_A^i\epsilon_B^j\epsilon_A^{-i}\epsilon_B^{-j})$ for $i,j=1,\dots,p-1$.* □

Remark 3. The relation between the parameter λ in Definition 1 and the parameter s in Proposition 2 has been studied in [8].

It has been shown in [7, §9] that the automorphism group of the Artin-Schreier-Mumford curve X_Γ contains $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, generated by the images of ϵ_A and ϵ_B in $\mathrm{Aut}(X_\Gamma) \cong N_\Gamma/\Gamma$.

The first result of this paper gives the $K[G]$-module structure of the space of 1-differentials:

Theorem 4. (1) *As a $K[A]$-module, we have*

$$H^0(X_\Gamma, \Omega_X) \cong L^{p-1} \otimes_{\mathbb{Z}} K,$$

where L is the integral representation of $A \cong \mathbb{Z}/p\mathbb{Z}$ with the minimal rank $p-1$ (corresponding to the matrix M in (8) below).

(2) *As a $K[G]$-module, $H^0(X_\Gamma, \Omega_X)$ is indecomposable.*

Notice that, since the space of 1-differentials has a lot of combinatorial feature, the $K[A]$-module structure actually comes from an integral representation as in Theorem 4 (1), which is, however, not the case for higher polydifferentials.

Theorem 5. *Suppose $p \neq 2$. For $n > 1$, let r $(0 \le r < p)$ be the remainder of $2n-1$ modulo p.*

(1) *As a $K[A]$-module, the following decomposition holds:*

$$H^0(X_\Gamma, \Omega_{X_\Gamma}^{\otimes n}) \cong K[A]^{(p-1)(2n-1)-p\lceil \frac{2n-1}{p} \rceil} \oplus \left(K[A]/(\epsilon_A - 1)^{p-r}\right)^p.$$

A similar result holds for the group B.

(2) *As a $K[G]$-module, the following decomposition holds:*

$$H^0(X_\Gamma, \Omega_{X_\Gamma}^{\otimes n}) \cong K[G]^{2n-1-2\lceil \frac{2n-1}{p} \rceil} \oplus K[G]/(\epsilon_A - 1)^{p-r} \oplus K[G]/(\epsilon_B - 1)^{p-r}.$$

Let us now describe the structure of this paper. The next section (§2) recalls the description of the space of polydifferentials of Mumford curves in terms of the group cohomologies. In §3 we focus on 1-differentials. As a side result, we obtain a bound for the order of an automorphism acting on them (Corollary 12). We also give in this section a criterion for a module to be indecomposable, based on the dimension of the space of invariant elements. From §4 onward, we proceed to the study of the space of polydifferentials. In §5 we first study $K[A]$-module structure using a combinatorial approach. Then we also show how results of S. Nakajima [27] can be applied without the usage of the theory of Mumford curves. For the $K[G]$-module structure, we employ both the theory of projective covers and the theory of B. Köck on the Galois-module structure of weakly ramified covers.

Conventions. For a ring R and a group G, we denote by $R[G]$ the group ring over R. As for $R[G]$-modules, we always consider *right* $R[G]$-modules, unless otherwise clearly stated. If an $R[G]$-module V is finite free as an R-module, then, for any $\gamma \in G$, the *matrix representation* of γ by an R-basis $\{v_1, \ldots, v_r\}$ of V is the matrix $M_\gamma \in \mathrm{GL}(r, R)$ whose i-th *row* is given by $(a_1, \ldots, a_r)$, where $v_i^\gamma = \sum_{j=1}^r a_j v_j$. Notice that, by this way, the map $G \to \mathrm{GL}(r, R)$ by $\gamma \mapsto M_\gamma$ is a group homomorphism. Accordingly, Jordan matrices in our sense will be the transpose of the conventional ones, having 1's on the *lower* subdiagonal entries.

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2. Preliminaries

2.1. **Invariants and direct factors.** Let K be an algebraically closed field of characteristic $p > 0$, $G \cong \mathbb{Z}/p\mathbb{Z}$ a cyclic group of order p, and $\sigma \in G$ a generator. For each $1 \le \mu \le p$, consider the $\mu \times \mu$ Jordan matrix $J_\mu \in \mathrm{GL}(\mu, K)$ (with the 1's on the lower subdiagonal entries) of eigenvalue 1. Then the μ-dimensional K-vector space K^μ can be regarded as a right $K[G]$-module by $\sigma \mapsto J_\mu$. By a slight abuse of notation, we denote thus obtained $K[G]$-module by the same notation J_μ; notice that J_μ is an indecomposable $K[G]$-module, isomorphic to $K[G]/((\sigma-1)^\mu)$ (cf. [36, §1 p. 107]).

If V is an arbitrary finite dimensional $K[G]$-module, then, by taking Jordan normal forms, one has the indecomposable decomposition of the form $V \cong \bigoplus_{i=1}^r J_{\mu_i}$ as $K[G]$-module, where $r \ge 0$ and $1 \le \mu_i \le p$ for each $i = 1, \ldots, r$.

Proposition 6. *Let G be a finite cyclic p-group, and V a finite dimensional $K[G]$-module. Then the number of indecomposable $K[G]$-summands of V is equal to $\dim_K V^G$.*

Proof. The indecomposable $K[G]$-summands of V are in one-to-one correspondence with the blocks of the Jordan normal form of a generator σ of G, seen as an element of $\mathrm{GL}(V)$. Since a Jordan block has the one-dimensional invariant subspace, every direct summand contributes exactly an one dimensional invariant subspace. $\square$

Remark 7. The assumption that G is cyclic is necessary. See, for example, the $K[\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}]$-module given by Heller and Reiner in [38, Example 1.4, p. 157].

Corollary 8. *Let G be an abelian p-group acting on a finite dimensional non-zero K-vector space V. Then we have $V^G \neq \{0\}$.*

Proof. The group G is isomorphic to a direct product of cyclic p-groups. The proof follows by induction with respect to the number of the direct factors, aided with the fact $M^{H_1 \times H_2} = (M^{H_1})^{H_2}$ and Proposition 6. $\square$

Corollary 9. *In the situation as in Corollary 8, if $\dim_K V^G = 1$, then V is indecomposable.* $\square$

2.2. **Derivations and the group cohomology.** Let K be a field, G a group, and P a right $K[G]$-module. A *derivation* (or 1-*cocycle*) of G to P is a map $d\colon \Gamma \to P$ satisfying

$$d(\gamma\gamma') = (d\gamma)^{\gamma'} + d\gamma' \tag{2}$$

for any $\gamma, \gamma' \in G$. In particular, for $\gamma \in G$ and a integer $k \ge 0$, we have

$$d(\gamma^k) = (d\gamma)^{1+\gamma+\cdots+\gamma^{k-1}}. \tag{3}$$

The set of all derivations $\mathrm{Der}(G, P)$ is naturally a K-linear space. A *principal derivation* (or 1-*coboundary*) is a derivation of the form

$$G \ni \gamma \mapsto F^\gamma - F,$$

by an element $F \in P$. Principal derivations form a subspace $\mathrm{PDer}(G, P)$ of $\mathrm{Der}(G, P)$. The quotient is the (1st) *group cohomology*:

$$H^1(G, P) = \mathrm{Der}(G, P)/\,\mathrm{PDer}(G, P).$$

2.3. **Polydifferentials on Mumford curves.** Now, let K be a complete non-archimedean valued field, $\Gamma \subseteq \mathrm{PGL}(2,K)$ a Schottky subgroup, and X_Γ the Mumford curve obtained from Γ. If $N = N_\Gamma \subseteq \mathrm{PGL}(2,K)$ is the normalization of Γ in $\mathrm{PGL}(2,K)$, the quotient group $G = N/\Gamma$, which acts on X_Γ from the left, is the automorphism group $\mathrm{Aut}(X_\Gamma)$ of X_Γ over K. Notice that the group Γ is a free group of finite rank, whose rank, say g, is equal to the genus of X_Γ. We fix a free generating set $\{\gamma_1, \ldots, \gamma_g\}$ of Γ.

For any right $K[\Gamma]$-module P, each derivation $d\colon \Gamma \to P$ is uniquely determined by its values $h_i = d(\gamma_i)$ for $1 \le i \le g$, and conversely, since Γ is free, such values $h_i \in P$ can be freely chosen to obtain a derivation d; indeed, once h_i's are chosen, then $d(w)$ for any $w \in \Gamma$ is uniquely determined by the recursive application of (2).

For a positive integer n, we consider the space of polynomials $P_n \subseteq K[T]$ of degree $\le 2(n-1)$, which is a K-vector space of dimension $2n-1$. The group $\mathrm{PGL}(2,K)$ acts on P_n from the right as follows: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,K)$ and $F \in P_n$, we define

$$F^\gamma(T) := \frac{(cT+d)^{2(n-1)}}{(ad-bc)^{n-1}} F\left(\frac{aT+b}{cT+d}\right) \in K[T]. \tag{4}$$

Now, consider the space $\mathrm{Der}(\Gamma, P_n)$ of derivations. By what we have seen above, this is a K-linear space of dimension $(2n-1)g$. The space $\mathrm{Der}(\Gamma, P_n)$ admits a right action of N (and hence of the group algebra $K[N]$) as follows: for $\delta \in N$ and $d \in \mathrm{Der}(\Gamma, P_n)$, define

$$(d^\delta)(\gamma) = [d(\delta\gamma\delta^{-1})]^\delta \tag{5}$$

for $\gamma \in \Gamma$. We have thus a well-defined right action of $G = N/\Gamma$ on the group cohomology $H^1(\Gamma, P_n)$, since Γ acts trivially modulo principal derivations:

$$[d(\delta\gamma\delta^{-1})]^\delta = d(\delta)^\gamma - d(\delta) + d(\gamma) \text{ for } \delta, \gamma \in \Gamma.$$

Theorem 10 ([34, Theorem 1])**.** *For any $n \ge 1$, the space $H^0(X_\Gamma, \Omega_{X_\Gamma}^{\otimes n})$ of n-differentials on the curve X_Γ is naturally isomorphic to the space group cohomology $H^1(\Gamma, P_n)$. Moreover, this identification is G-equivariant with respect to the natural right G-action on $H^0(X_\Gamma, \Omega_{X_\Gamma}^{\otimes n})$.* □

3. The space of 1-differentials

3.1. ***G*-action on 1-differentials.** We continue to work with the notation as in §2.3, and suppose K is of characteristic $p > 0$. By Theorem 10, we have

$$\begin{aligned} H^0(X_\Gamma, \Omega_{X_\Gamma}) &\cong H^1(\Gamma, P_0) = H^1(\Gamma, K) = \mathrm{Hom}(\Gamma, K) = \mathrm{Hom}(\Gamma, \mathbb{Z}) \otimes K \\ &\cong \mathrm{Hom}(\Gamma^{\mathrm{ab}}, \mathbb{Z}) \otimes K, \end{aligned} \tag{6}$$

where $\Gamma^{\mathrm{ab}} \cong \mathbb{Z}^g$ denotes the maximal abelian quotient of the free group Γ. Since $G = N/\Gamma$ acts on Γ^{ab} from the left (by conjugation), we have the right action of G on $\mathrm{Hom}(\Gamma^{\mathrm{ab}}, \mathbb{Z})$, and hence on $\mathrm{Hom}(\Gamma^{\mathrm{ab}}, \mathbb{Z}) \otimes K$. The isomorphism $H^0(X_\Gamma, \Omega_{X_\Gamma}) \cong \mathrm{Hom}(\Gamma^{\mathrm{ab}}, \mathbb{Z}) \otimes K$ by (6) is easily seen to be G-equivariant. In particular, the G-action on $H^0(X_\Gamma, \Omega_{X_\Gamma})$ comes from an integral representation $\rho\colon G \to \mathrm{GL}(g, \mathbb{Z})$. Then, by [19], we have:

Proposition 11. *The integral representation ρ is injective, i.e., G can be seen as a subgroup of $\mathrm{GL}(g, \mathbb{Z})$, unless the cover $X \to G\backslash X = Y$ is not tamely ramified, the characteristic of K is 2, and the genus of Y is 0.* □

Corollary 12. *Suppose $p \ne 2$. If the order of an element $g \in G$ is a prime number q, then $q \le g + 1$.*

Proof. This follows from Proposition 11 and a special case of [23, Theorem 2.7]. □

3.2. **Proof of Theorem 4.** Let $A = \langle \epsilon_A \rangle$, $B = \langle \epsilon_B \rangle$, Γ, N, and $G = N/\Gamma$ be as in Proposition 2. The set $\{e_{i,j} = [\epsilon_A^i, \epsilon_B^j] \mid 1 \le i, j \le p-1\}$ gives a basis of Γ (cf. [31, p. 6, Prop. 4]). By an easy calculation, we have

$$\epsilon_A e_{i,j} \epsilon_A^{-1} = [\epsilon_A^{i+1}, \epsilon_B^j][\epsilon_A, \epsilon_B^j]^{-1} \quad \text{for} \quad 1 \le i, j \le p-1,$$

which describes the left action of A on Γ.

For any $\gamma \in \Gamma$, let us denote by $\overline{\gamma}$ the image of γ in the maximal abelian quotient $\Gamma^{\mathrm{ab}} \cong \mathbb{Z}^g$. The free abelian group Γ^{ab} has the $\mathbb{Z}$-basis consisting of $\overline{e}_{i,j}$'s. Let $\{f_{i,j}\}$ be the dual basis in $V = \mathrm{Hom}(\Gamma^{\mathrm{ab}}, \mathbb{Z}) \otimes K \cong H^0(X_\Gamma, \Omega_{X_\Gamma})$ of $\{\overline{e}_{i,j}\}$. Since, written additively in $\Gamma^{\mathrm{ab}} \cong \mathbb{Z}^g$, we have

$$\overline{\epsilon_A e_{i,j} \epsilon_A^{-1}} = \begin{cases} \overline{e}_{i+1,j} - \overline{e}_{1,j} & \text{for } 1 \le i \le p-2, \\ -\overline{e}_{1,j} & \text{if } i = p-1, \end{cases}$$

for $1 \le j \le p-1$, we have

$$f_{i,j}^{\epsilon_A} = \begin{cases} -\displaystyle\sum_{k=1}^{p-1} f_{k,j} & \text{if } i = 1, \\ f_{i-1,j} & \text{for } 2 \le i \le p-1. \end{cases} \tag{7}$$

Remark 13. Usually on the dual space we act in terms of the contragredient representation in order to have a left action on dual elements as well. Here the action on $f : V \to k$ is given by $f \mapsto f^g$, where f^g is the function sending $v \mapsto f(gv)$.

Hence he matrix representation of ϵ_A with respect to the basis $\{f_{i,j}\}$ is given by the block diagonal matrix $\mathrm{diag}(M, \ldots, M)$ consisting of $p-1$ copies (indexed by $j = 1, \ldots, p-1$) of

$$M = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathrm{GL}(p-1, \mathbb{Z}) \tag{8}$$

(cf. our convention for the matrix representation in Introduction). Notice that the matrix M has characteristic polynomial $\frac{x^p-1}{x-1} = 1 + x + \cdots + x^{p-1}$, of which M is the companion matrix. The integral representation of $\mathbb{Z}/p\mathbb{Z}$ on $\mathbb{Z}^{p-1}$ given by the matrix M, denoted by L in the sequel, is the one with the minimal degree $p-1$; cf. [23]. We thus have proven the first part of Theorem 4.

To proceed, let us compute the invariant part by the action of A. It suffices to look at each block for $1 \le j \le p-1$. The condition for an element $\sum_{i=1}^{p-1} \lambda_i f_{i,j}$ to be invariant is given by

$$\sum_{i=1}^{p-1} \lambda_i f_{i,j} = \left(\sum_{i=1}^{p-1} \lambda_i f_{i,j} \right)^{\epsilon_A} = -\lambda_1 \sum_{i=1}^{p-1} f_{i,j} + \sum_{i=1}^{p-2} \lambda_{i+1} f_{i,j},$$

which is equivalent to $\lambda_i = i \cdot \lambda_1$ for $i = 1, \ldots, p-1$. Hence, each diagonal block contributes 1 to the dimension of the space of invariants. Since there are $p-1$ of them, the space of A-invariants in $V = \mathrm{Hom}(\Gamma^{\mathrm{ab}}, \mathbb{Z}) \otimes K \cong H^0(X_\Gamma, \Omega_{X_\Gamma})$ has dimension $p-1$, generated by the elements

$$f_j = \sum_{i=1}^{p-1} i \cdot f_{i,j} \text{ for } 1 \le j \le p-1.$$

We can now compute the space of $A \times B$-invariants by using the fact $V^{A\times B} = (V^A)^B$. Similarly to (7), one computes

$$f_{i,j}^{\epsilon_B} = \begin{cases} -\sum_{k=1}^{p-1} f_{i,k} & \text{if } j = 1, \\ f_{i,j-1} & \text{for } 2 \leq j \leq p-1. \end{cases} \tag{9}$$

From this, one has

$$f_j^{\epsilon_B} = \begin{cases} -\sum_{k=1}^{p-1} f_k & \text{if } j = 1, \\ f_{j-1} & \text{for } 2 \leq j \leq p-1, \end{cases} \tag{10}$$

which means that the matrix representation of ϵ_B with respect to the basis $\{f_j\}$ of the space of A-invariants coincides with the one in (8). Hence, the space $V^{A\times B}$ is one dimensional and the representation is indecomposable by Corollary 9, which finishes the proof of the second part of Theorem 4.

4. Computations on Artin-Schreier-Mumford curves continued.

We continue with the notation of the previous subsection. In this section, as a preparation for the proof of Theorem 5, we first compute the space $H^1(\Gamma, P_n)^G$, and study the $K[A]$-module structure of $\mathrm{Der}(\Gamma, P_n)$. (See §2 for the definition of P_n.)

4.1. Computation of the space $H^1(\Gamma, P_n)^G$.

We first of all prove:

Proposition 14. *We have $P_n^\Gamma = H^0(\Gamma, P_n) = \{0\}$ for $n > 1$.*

To show the proposition, we need the following lemma:

Lemma 15. *For any discrete free subgroup $\Gamma \subseteq \mathrm{PGL}(2, K)$ of finite rank ≥ 2 and any closed point x of $\mathbb{P}^1_K$, the Γ-orbit $\Gamma \cdot x = \{\gamma x \mid \gamma \in \Gamma\}$ is an infinite set.*

Proof. Suppose $\Gamma \cdot x$ is finite. If $\{\gamma_1, \ldots, \gamma_g\}$ $(g \geq 2)$ is a free basis of Γ, then there exists an integer $N \geq 1$ such that $\gamma_i^N x = x$ for any $1 \leq i \leq g$. Since the subgroup generated by $\gamma_1^N, \ldots, \gamma_g^N$ is discrete and free of rank g, we may replace Γ by this subgroup, and thus may assume that x is fixed by every element in Γ. Since $g \geq 2$, one can find two $\gamma, \delta \in \Gamma$ that share exactly one point x as their fixed points. Then one sees easily (cf. the proof of [17, 4.2 (3)]) that $[\gamma, \delta] \in \Gamma$ is a parabolic element, and hence is of order p, which is absurd. □

Proof of Proposition 14. Let $F \in P_n^\Gamma$. Notice first that, since $n > 1$, F cannot be a non-zero constant; indeed, there exists an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c \neq 0$ (e.g., $[\epsilon_A, \epsilon_B]$). Hence, if $F \neq 0$, there exists an irreducible polynomial over K that divides F. On the other hand, it can be checked by an easy calculation that, if $\rho \in \overline{K}$ is a root of F, then for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2, K)$, $\gamma^{-1}(\rho) = \dfrac{d\rho - b}{-c\rho + a}$ is a root of F^γ. Hence, by Lemma 15, F has to be divided by infinity many irreducible polynomials, which is absurd. □

Corollary 16. *For $n > 1$, we have*

$$H^1(N, P_n) \cong H^1(\Gamma, P_n)^G.$$

Proof. Consider the 5-term restriction-inflation sequence coming from the Lyndon-Hochschild-Serre spectral sequence [37, par. 6.8.3]:

$$0 \to H^1(G, P_n^\Gamma) \xrightarrow{\text{inf}} H^1(N, P_n) \xrightarrow{\text{res}} H^1(\Gamma, P_n)^G \to H^2(G, P_n^\Gamma) \to H^2(N, P_n).$$

By Proposition 14, we have $H^1(G, P_n^\Gamma) = H^2(G, P_n^\Gamma) = \{0\}$, whence the result. □

Remark 17. For $n = 1$, we have $P_0 = K$, and we compute

$$H^1(G, P_0^\Gamma) \cong H^1(N, P_0) \cong K^2. \tag{11}$$

Indeed, the action of $N \cong A * B$ on K is trivial, and hence by [37, Ex. 6.2.5, p. 171], we have

$$H^1(N, P_0) \cong H^1(A, K) \times H^1(B, K) \cong K^2.$$

On the other hand, by [10, §3.5, p. 32], the cohomology ring $H^*(G, K)$ (recall that $G \cong A \times B$) is of the form

$$H^*(G, K) \cong \bigwedge[\eta_1, \eta_2] \otimes k[\xi_1, \xi_2],$$

where $\deg \eta_i = 1$, $\deg \xi_i = 2$, $\eta_i^2 = 0$. The degree-1 part is the two dimensional vector space spanned by η_1, η_2, hence we deduce that the inflation map $H^1(G, P_0^\Gamma) \hookrightarrow H^1(N, P_0)$ is an isomorphism, obtaining the isomorphisms as in (11). Notice that $H^2(G, K)$ is of dimension three, generated by $\eta_1 \wedge \eta_2, \xi_1, \xi_2$, while the space $H^2(N, K) \cong H^2(A, K) \times H^2(B, K)$ (by [37, Cor. 6.2.10, p. 170]) is two-dimensional, being compatible with the computation of invariants done in section 3.2, and the map $H^2(G, K) \to H^2(N, K)$ is surjective.

To proceed, we need a convenient basis for the space P_n. Consider

$$\binom{T}{k} = \frac{T(T-1)(T-2)\cdots(T-k+1)}{k!} \in K[T], \tag{12}$$

for $k \geq 0$, which is a polynomial of degree k. For $k < 0$, we set $\binom{T}{k} = 0$. For each $k = 0, \ldots, 2(n-1)$, let q and r be the integers such that $k = qp + r$ and $0 \leq r < p$, and define

$$b_k = (T^p - T)^q \binom{T}{r}. \tag{13}$$

The elements b_k for $0 \leq k \leq 2(n-1)$ form a basis of P_n, and using the binomial relation, we have

$$b_k^{\epsilon_A} = \begin{cases} b_k + b_{k-1} & \text{if } p \nmid k, \\ b_k & \text{if } p | k. \end{cases} \tag{14}$$

Hence ϵ_A in terms of $\{b_k\}$ is expressed by the block diagonal matrix

$$E_A = \operatorname{diag}(J_p, \ldots, J_p, J_r) \tag{15}$$

consisting of $\left\lfloor \frac{2n-1}{p} \right\rfloor$ copies of J_p and J_r (cf. §2.1 for the notation), where $0 \leq r < p$ is the remainder of $2n - 1$ modulo p; here, we put $J_0 = \{0\}$ for convenience.

Proposition 18. *Suppose $n > 1$.*

(1) *We have* $\dim_K \operatorname{Der}(A, P_n) = \dim_K \operatorname{Der}(B, P_n) = 2n - 1 - \left\lfloor \frac{2n-1}{p} \right\rfloor$.

(2) *We have* $\dim_K H^1(N, P_n) = 2n - 1 - 2\left\lfloor \frac{2n-1}{p} \right\rfloor$.

To show the proposition, we need:

Lemma 19. *Consider the Jordan matrix J_μ for $1 \le \mu \le p$. We have*

$$1 + J_\mu + J_\mu^2 + \cdots J_\mu^{p-1} = \begin{cases} 0 & \text{if } \mu < p, \\ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} & \text{if } \mu = p. \end{cases}$$

Proof. Set $N_\mu = J_\mu - 1$, which is the nilpotent Jordan matrix. Then the assertion follows immediately from the direct calculation

$$\begin{aligned} 1 + J_\mu + J_\mu^2 + \cdots + J_\mu^{p-1} &= \sum_{i=0}^{p-1} \sum_{j=0}^{i} \binom{i}{j} N_\mu^j = \sum_{j=0}^{p-1} \sum_{i=j}^{p-1} \binom{i}{j} N_\mu^j \\ &= \sum_{j=0}^{p-1} \left[\binom{p}{j+1} - \sum_{i=0}^{j-1} \binom{i}{j} \right] N_\mu^j = N_\mu^{p-1} - \sum_{j=0}^{p-2} \sum_{i=0}^{j-1} \binom{i}{j} N_\mu^j = N_\mu^{p-1}. \end{aligned}$$

Here, we used the identity $\sum_{i=0}^{p-1} \binom{i}{j} = \binom{p}{j+1}$, which comes from $\sum_{i=0}^{p-1} (1+T)^i = [(1+T)^p - 1]/T$. ∎

Proof of Proposition 18. (1) Each derivation $\delta \in \mathrm{Der}(A, P_n)$ is completely determined by the image $F = \delta(\epsilon_A) \in P_n$ with

$$F^{1+\epsilon_A+\epsilon_A^2+\cdots\epsilon_A^{p-1}} = 0,$$

which comes from (3); that is, $\mathrm{Der}(A, P_n)$ is isomorphic to the kernel of the matrix $1 + E_A + E_A^2 + \cdots + E_A^{p-1}$ on the K-linear space P_n. Hence the desired equality for $\dim_K \mathrm{Der}(A, P_n)$ follows immediately from Lemma 19. One can similarly argue $\dim_K \mathrm{Der}(B, P)$, since $\epsilon_B = \tau\epsilon_A\tau$ by an involution $\tau = \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix} \in \mathrm{PGL}(2, K)$.

(2) Since every derivation δ on $N \cong A * B$ can be recovered from $\delta|_A$ and $\delta|_B$, one has the K-linear isomorphism $\mathrm{Der}(N, P_n) \to \mathrm{Der}(A, P_n) \times \mathrm{Der}(B, P_n)$. Thus we have the exact sequence

$$0 \to \mathrm{PDer}(N, P_n) \to \mathrm{Der}(A, P_n) \times \mathrm{Der}(B, P_n) \to H^1(N, P_n) \to 1.$$

On the other hand, due to Proposition 14, the mapping $P_n \to \mathrm{PDer}(N, P_n)$, which maps F to the principal derivation $\delta \mapsto F^\delta - F$, is bijective. Hence we have

$$\begin{aligned} \dim_K H^1(N, P_n) &= 2(2n-1) - 2\left\lfloor \frac{2n-1}{p} \right\rfloor - (2n-1) \\ &= 2n - 1 - 2\left\lfloor \frac{2n-1}{p} \right\rfloor, \end{aligned}$$

as desired. ∎

4.2. **The $K[A]$-module structure of $\mathrm{Der}(\Gamma, P_n)$.** We want to compute the action of $G \cong A \times B$ on the coholomogy group $H^1(\Gamma, P_n)$. To this end, we first calculate d^δ for $d \in \mathrm{Der}(\Gamma, P_n)$ and $\delta \in A$ (and $\delta \in B$, as well).

Recall that $e_{i,j} = [\epsilon_A^i, \epsilon_B^j]$ for $i, j = 1, \ldots, p-1$ form a free basis of Γ. For $i, j \in (\mathbb{Z}/p\mathbb{Z})^\times$ and $k = 0, \ldots, 2(n-1)$, define $d_{i,j}^{(k)} \in \mathrm{Der}(\Gamma, P_n)$ by

$$d_{i,j}^{(k)}(e_{i',j'}) = \delta_{ii'}\delta_{jj'} b_k^{\epsilon_B^{-j}}$$

for $i', j' \in (\mathbb{Z}/p\mathbb{Z})^\times$ (recall the definition of b_k in §4.1). Then $\{d^{(k)}_{i,j} \mid i, j \in (\mathbb{Z}/p\mathbb{Z})^\times,\ 0 \leq k \leq 2(n-1)\}$ gives a K-basis of $\mathrm{Der}(\Gamma, P_n)$. We calculate

$$
\begin{aligned}
(d^{(k)}_{i,j})^{\epsilon_A}(e_{i',j'}) &= [d^{(k)}_{i,j}(e_{i'+1,j'})]^{\epsilon_B^{j'}\epsilon_A\epsilon_B^{-j'}} - [d^{(k)}_{i,j}(e_{1,j'})]^{\epsilon_B^{j'}\epsilon_A\epsilon_B^{-j'}} \\
&= (\delta_{i,i'+1} - \delta_{i,1})\delta_{jj'} b_k^{\epsilon_A\epsilon_B^{-j}}.
\end{aligned} \tag{16}
$$

Here we have set $e_{0,j} = 1$ for convenience. By computation

$$
\left(d^{(k)}_{i,j}\right)^{\epsilon_A} = \begin{cases} d^{(k)}_{i-1,j} + d^{(k)}_{i-1,j} & \text{if } p \nmid k \text{ and } i \neq 1, \\ d^{(k)}_{i-1,j} & \text{if } p \mid k \text{ and } i \neq 1, \\ -\sum_{i' \neq 0} d^{(k)}_{i',j} + d^{(k-1)}_{i',j} & \text{if } p \nmid k \text{ and } i = 1, \\ -\sum_{i' \neq 0} d^{(k)}_{i',j} & \text{if } p \mid k \text{ and } i = 1. \end{cases} \tag{17}
$$

From this one can compute the matrix for ϵ_A by means of the basis $\{d^{(k)}_{ij}\}$, ordered lexicographically

$$
d^{(0)}_{1,1}, d^{(1)}_{1,1}, \ldots, d^{(2(n-1))}_{1,1}, d^{(0)}_{2,1}, d^{(1)}_{2,1}, \ldots, d^{(2(n-1))}_{2,1}, \ldots.
$$

The matrix Q_A in question is the square matrix of degree $(2n-1)(p-1)^2$, which is first of all a block diagonal

$$
Q_A = \mathrm{diag}(N, \ldots, N), \tag{18}
$$

consisting of $p-1$ copies (indexed by $j = 1, \ldots, p-1$) of a square matrix N of degree $(2n-1)(p-1)$, and the matrix N is the 'tensor product' of M in (8) and E_A in (15), i.e., the matrix obtained by replacing ± 1 in M with $\pm E_A$.

In more algebraic terms, the $K[A]$-module structure of $\mathrm{Der}(\Gamma, P_n)$ is described as follows. As in Theorem 4 (1), let L be the free $\mathbb{Z}$-module of rank $p-1$ with the $\mathbb{Z}[A]$-module structure given by $\epsilon_A \mapsto M$ (with respect to some basis $\{v_1, \ldots, v_{p-1}\}$), and W be the K-vector space of dimension $2n-1$ with the $K[A]$-module structure by $\epsilon_A \mapsto E_A$ (with respect to some basis $\{w_1, \ldots, w_{2n-1}\}$). As in §2.1, we simply denote by J_μ the K-vector space K^μ with the $K[A]$-module structure by $\epsilon_A \mapsto J_\mu$, we have

$$
W \cong J_p^{\lfloor \frac{2n-1}{p} \rfloor} \oplus J_r \tag{19}
$$

as $K[A]$-module, where r is the remainder of $2n-1$ modulo p; here, as before, we put $J_0 = \{0\}$ for convenience. Notice that, as we have seen in §4.1, we have $P_n \cong W$ as $K[A]$-module. Then, what we have shown above amounts to the $K[A]$-module isomorphism

$$
\mathrm{Der}(\Gamma, P_n) \cong (W \otimes_{\mathbb{Z}} L)^{p-1}. \tag{20}
$$

To count the number of indecomposable summands, let us consider a slightly general situation as follows: Let U be a right $K[A]$-module, and consider the $K[A]$-module $U \otimes_{\mathbb{Z}} L$. Any element $x \in U \otimes_{\mathbb{Z}} L$ can be written as $x = \sum_{i=1}^{p-1} a_i \otimes v_i$, where $a_i \in U$ $(1 \leq i \leq p-1)$. We have

changed $a_1^{\epsilon_A} \otimes v_i$ to $a_1^{\epsilon_A} \otimes v_{p-1}$

$$
x^{\epsilon_A} = \sum_{i=1}^{p-1} a_i^{\epsilon_A} \otimes v_i^{\epsilon_A} = \sum_{i=1}^{p-2} (a_{i+1}^{\epsilon_A} - a_1^{\epsilon_A}) \otimes v_i - a_1^{\epsilon_A} \otimes v_{p-1}.
$$

By a straightforward calculation, we see that $x = x^{\epsilon_A}$ holds if and only if

$$
a_i = a_1^{1+\epsilon_A^{-1}+\cdots+\epsilon_A^{1-i}} \quad (i = 1, \ldots, p-1) \quad \text{and} \quad a_1^{1+\epsilon_A^{-1}+\cdots+\epsilon_A^{1-p}} = 0. \tag{21}
$$

Hence each $x \in (U \otimes_{\mathbb{Z}} L)^A$ is determined by its first coefficient a_1, which is further subject to the second condition in (21).

If $U = J_\mu$ $(1 \le \mu \le p)$, then by a similar calculation to that in Lemma 19, we deduce that the second equality of (21) gives a non-trivial condition only when $\mu = p$, and that

$$\dim_K (J_\mu \otimes_{\mathbb{Z}} L)^A = \begin{cases} \mu & \text{if } \mu < p, \\ p-1 & \text{if } \mu = p. \end{cases} \tag{22}$$

Proposition 20. (1) *As a $K[A]$-module, we have*

$$J_p \otimes_{\mathbb{Z}} L \cong J_p^{p-1}.$$

(2) *For $1 \le r \le p-1$, we have*

$$J_r \otimes_{\mathbb{Z}} L \cong J_p^{r-1} \oplus J_{p-r}$$

as a $K[A]$-module.

Proof. (1) First notice that any direct summand has to be of the form J_μ with $1 \le \mu \le p$ (cf. §2.1); in particular, it is of dimension at most p. Since the number of indecomposable summands of $J_p \otimes_{\mathbb{Z}} L$ is $p-1$, and since $\dim_K J_p \otimes_{\mathbb{Z}} L = p(p-1)$, it has only J_p as its direct summands.

(2) Consider the obvious exact sequence

$$0 \to J_r \to J_p \to J_{p-r} \to 0.$$

Tensoring the free $\mathbb{Z}$-module L yields the exact sequence

$$0 \to J_r \otimes L \to J_p \otimes L \to J_{p-r} \otimes L \to 0,$$

from which we obtain

$$0 \to (J_r \otimes L)^A \to (J_p \otimes L)^A \to (J_{p-r} \otimes L)^A \to H^1(A, J_r \otimes L) \to H^1(A, J_p \otimes L).$$

Since $J_p \otimes L \cong J_p^{p-1}$ is a free $K[A]$-module, we have $H^1(A, J_p \otimes L) = \{0\}$. By (22), we know that $\dim_K H^1(A, J_r \otimes L) = 1$. Now if $J_r \otimes L \cong \bigoplus_{i=1}^m J_{\mu_i}$, we have

$$H^1(A, J_r \otimes L) \cong \bigoplus_{i=1}^m H^1(A, J_{\mu_i}).$$

Since $H^1(A, J_\mu)$ is zero for $\mu = p$, and is 1-dimensional for $1 \le \mu < p$, $J_r \otimes_{\mathbb{Z}} L \cong J_p^{r-1} \oplus J_{p-r}$ is the only possibility, for $\dim_K J_r \otimes L = r(p-1)$. □

Remark 21. The problem determining the indecomposable summands of tensor products in representation theory is called the Clebsch-Gordan problem. The case of Jordan normal form in modular representation theory was studied by many authors see [12] and references within.

Corollary 22. *The $K[A]$-module structure of* $\mathrm{Der}(\Gamma, P_n)$ *is given by*

$$\mathrm{Der}(\Gamma, P_n) \cong \begin{cases} \left(J_p^{(p-1)\lfloor \frac{2n-1}{p} \rfloor} \oplus J_p^{r-1} \oplus J_{p-r} \right)^{p-1} & \text{if } p \nmid 2n-1, \\ \left(J_p^{(p-1)\frac{2n-1}{p}} \right)^{p-1} & \text{if } p \mid 2n-1. \end{cases}$$

Here, in the first case, r denotes the remainder of $2n-1$ modulo p.

Proof. This follows from (19), (20), and Proposition 20. □

5. Computations on cohomology

In this section we focus on the computation of both the $K[A]$ and $K[A \times B] = K[N/\Gamma] = K\big[(A * B)/[A, B]\big]$-module structure of $H^1(\Gamma, P_n)$. We will give two different proofs of Theorem 5 (1).

5.1. **First Proof of Theorem 5(1).** In order to form the quotient we need to know exactly how the module of principal derivations sits inside the module of derivations.

Definition 23. From now on we will denote by h the endomorphism $h = (\epsilon_A - 1)$. Let V be a $K[A]$-module. We will say that an element $w \in V$ has order $\mathrm{ord}(w) = a$ if $w \in \ker h^a - \ker h^{a-1}$. We will say that u generates a module M isomorphic to J_a as a $K[A]$-module, if and only if the set $\{u, hu, \ldots, h^{a-1}u\}$ forms a K-vector space basis of M. Notice that the generator u of a module isomorphic to J_a has order a.

Generators of direct summands J_p of the space of principal derivations have order p and therefore go to generators of J_p summands of the space $\mathrm{Der}(\Gamma, P_n)$. Let $b_r = (T^p - T)^{\left\lfloor \frac{2n-1}{p} \right\rfloor}\binom{T}{r}$ be a generator of the J_r component of P_n. Let ψ be the map sending $P_n \ni b$ to the principal derivation $\gamma \mapsto b^\gamma - b$. For N_1 and N_2 given by corolllary 22, the principal derivation $\psi(b_r)$ is decomposed as a sum

$$\psi(b_r) = \sum_{i=1}^{N_1} a_i + \sum_{j=1}^{N_2} \beta_j \in J_p^{N_1} \oplus J_{p-r}^{N_2}, \tag{23}$$

where $a_i \in J_p$ and $b_j \in J_{p-r}$. Since the order of b_r is r, there is at least one summand of order r and all other summand have order $\geq r$. It is clear that if a_i in J_p has order $t \geq r$, then there is an element $a_i' \in J_p$ such that $h^{p-t}(a_i') = a_i$. We will prove that the elements β_j in eq. (23) can also be written as $h^{p-r}(\beta_j')$.

Notice that if $r = 1$ this means that $\beta_j = 0$, since $J_r \otimes J_{p-1} \cong J_p^{r-1} \oplus J_{p-r}$, and if $r = 1$, then there is no J_p direct summand in $J_1 \otimes J_{p-1} \cong J_{p-1} = J_{p-r}$.

Proposition 24. *The direct summand J_r of P_n given in eq. (15) is mapped inside a direct summand of* $\mathrm{Der}(\Gamma, P_n)$*, isomorphic to J_p^N for some $N \in \mathbb{N}$.*

In order to prove of proposition 24 we have to introduce a combinatorial point of view of the basis of $J_r \otimes J_{p-1}$. Consider a basis $b_1, \ldots, b_r$ of the module J_r such that $\epsilon_A(b_\nu) = b_\nu + b_{\nu-1}$ and a basis $\epsilon_1, \ldots, \epsilon_{p-1}$ of J_{p-1}, such that $\epsilon_A(\epsilon_i) = \epsilon_i + \epsilon_{i-1}$. Also b_i and ϵ_i are considered to be zero for $i \leq 0$. The elements $\epsilon_{i,j} = b_i \otimes \epsilon_j$ form a basis for the space $J_r \otimes J_{p-1}$. Geometrically we consider the elements $\epsilon_{i,j}$ to form an $r \times (p-1)$ grid arranged as follows:

$$\begin{array}{ccccc} e_{1,p-1} & e_{2,p-1} & \cdots & e_{r-1,p-1} & e_{r,p-1} \\ e_{1,p-2} & e_{2,p-2} & \cdots & e_{r-1,p-2} & e_{r,p-2} \\ \vdots & \vdots & & \vdots & \vdots \\ e_{1,1} & e_{2,1} & \cdots & e_{r-1,1} & e_{r,1} \end{array} \tag{24}$$

One can approach the action of ϵ_A on basis elements of $J_r \otimes J_{p-1}$ in the following way: First we compute the effect of ϵ_A:

$$\epsilon_A(e_{i,j}) = (b_i + b_{i-1}) \otimes (\epsilon_j + \epsilon_{j-1}).$$

Next, we notice that the action of a λ-power of ϵ_A is given by:

$$\epsilon_A^\lambda(e_{i,j}) = \left(\sum_{\nu=0}^{\lambda} \binom{\lambda}{\nu} b_{i-\nu} \right) \otimes \left(\sum_{\mu=0}^{\lambda} \binom{\lambda}{\mu} \epsilon_{j-\mu} \right)$$

Finally we compute:

$$h^k = (\epsilon_A - 1)^k = \sum_{\lambda=0}^{k} \binom{k}{\lambda} (-1)^{k-\lambda} \epsilon_A^\lambda.$$

Thus the action of h^k on e_{ij} is given by:

$$
\begin{aligned}
h^k(e_{i,j}) &= (\epsilon_A - 1)^k e_{i,j} \\
&= \sum_{\lambda=0}^{k} \binom{k}{\lambda} (-1)^{k-\lambda} \epsilon_A^{\lambda} e_{i,j} \\
&= \sum_{\lambda=0}^{k} \binom{k}{\lambda} (-1)^{k-\lambda} \sum_{\nu=0}^{\lambda} \binom{\lambda}{\nu} \sum_{\mu=0}^{\lambda} \binom{\lambda}{\mu} e_{i-\nu,j-\mu} \\
&= \sum_{\lambda=0}^{k} \binom{k}{\lambda} (-1)^{k-\lambda} \sum_{\nu=0}^{k} \binom{\lambda}{\nu} \sum_{\mu=0}^{k} \binom{\lambda}{\mu} e_{i-\nu,j-\mu} \\
&= \sum_{\nu=0}^{i} \sum_{\mu=0}^{j} \left(\sum_{\lambda=0}^{k} \binom{\lambda}{\mu} \binom{\lambda}{\nu} \binom{k}{\lambda} (-1)^{k-\lambda} \right) e_{i-\nu,j-\mu}. \qquad (25)
\end{aligned}
$$

In eq. (25) above, we have extended the summation up to k since $\binom{I}{J} = 0$ for $J > I$. Define

$$
\delta_{\nu,\mu}^{k} := \left(\sum_{\lambda=0}^{k} \binom{\lambda}{\mu} \binom{\lambda}{\nu} \binom{k}{\lambda} (-1)^{k-\lambda} \right)
$$

Lemma 25.

$$
\begin{aligned}
\delta_{\nu,\mu}^{k} &= \binom{k}{\nu} \binom{\nu}{\mu - k + \nu} \\
&= \binom{k}{\nu} \binom{\nu}{k - \mu}
\end{aligned}
$$

Proof. We start from formula 118 p. 36 on [13]:

$$
\sum_{\lambda=0}^{n} \binom{n}{\lambda} \binom{\lambda}{j} x^{\lambda} = \binom{n}{j} x^j (1+x)^{n-j}. \qquad (26)
$$

Let $F_j(x) = x^j(1+x)^{n-j}$. We compute the Taylor expansion at $x = -1$ using the binomial expansion of $x^j = \big((1+x) - 1\big)^j$:

$$
F_j(x) = \sum_{\phi=0}^{j} \binom{j}{\phi} (-1)^{j-\phi} (1+x)^{\phi+n-j}.
$$

This allows us to compute the derivative:

$$
F_j^{(t)}(-1) = \begin{cases} 0 & \text{if } t < n-j \\ t!\binom{j}{\phi}(-1)^{j-\phi} & \text{if } t \geq n-1, t = n-j+\phi \end{cases}.
$$

By differentiation of eq. (26) t-times we obtain:

$$
\sum_{\lambda=0}^{\infty} \binom{n}{\lambda} \binom{\lambda}{j} \lambda(\lambda-1)\cdots(\lambda-t+1) x^{\lambda-t} = \binom{n}{j} F_j^{(t)}(x)
$$

and now by multiplying with $1/t!$ and evaluating at $x = -1$ we obtain:

$$
\sum_{\lambda=0}^{\infty} \binom{n}{\lambda} \binom{\lambda}{j} \frac{\lambda!}{(\lambda-t)!t!} (-1)^{\lambda} = \binom{n}{j} \binom{j}{t-n+j} (-1)^n.
$$

∎

Lemma 25 combined with eq. (25) gives us

$$\boxed{h^k(e_{i,j}) = \sum_{\nu=0}^{i}\sum_{\mu=0}^{j} \delta^k_{\nu,\mu} e_{i-\nu,j-\mu} = \sum_{\nu=0}^{i}\sum_{\mu=0}^{j} \binom{k}{\nu}\binom{\nu}{k-\mu} e_{i-\nu,j-\mu}.} \tag{27}$$

We will follow a different approach for computing the coefficients of the basis elements which appear in the expansion of $h^k e_{ij}$. Notice first that

$$h(e_{i,j}) = e_{i-1,j} + e_{i,j-1} + e_{i-1,j-1}.$$

This means that an element $e_{i,j}$ is moved by h to the sum of three elements in the grid of equation (24) which lie just below to the right and right and below.

$$\begin{tikzcd}
e_{i-1,j} & e_{i,j} \arrow[l] \arrow[dl] \arrow[d] \\
e_{i-1,j-1} & e_{i,j-1}
\end{tikzcd} \tag{28}$$

Assume that all the above arrows have length 1. For the $h^2(e_{i,j})$ we have

$$\begin{tikzcd}
e_{i-2,j} & e_{i-1,j} \arrow[l] \arrow[dl] \arrow[d] & e_{i,j} \arrow[l] \arrow[dl] \arrow[d] \\
e_{i-2,j-1} & e_{i-1,j-1} \arrow[l] \arrow[dl] \arrow[d] & e_{i,j-1} \arrow[l] \arrow[dl] \arrow[d] \\
e_{i-2,j-2} & e_{i-2,j-2} & e_{i,j-2}
\end{tikzcd}$$

On the other hand side we compute

$$\begin{aligned}
h^2 e_{i,j} &= he_{i-1,j} + he_{i,j-1} + he_{i-1,j-1} \\
&= e_{i-2,j} + e_{i-1,j-1} + e_{i-2,j-1} + \\
&\quad + e_{i-1,j-1} + e_{i,j-2} + e_{i-1,j-2} \\
&\quad + e_{i-2,j-1} + e_{i-1,j-2} + e_{i-2,j-2} \\
&= e_{i-2,j} + 2e_{i-1,j-1} + 2e_{i-2,j-1} + e_{i,j-2} + 2e_{i-1,j-2}.
\end{aligned}$$

By induction we can prove the following

Lemma 26. *The coefficient of $e_{\nu,\mu}$ in $h^k(e_{i,j})$ is the number of paths of length k we can form from $e_{i,j}$ to $e_{\nu,\mu}$ if from each intermediate node we can go in three directions of eq. (28).*

Lemma 27. *Fix an i_0, j_0 The image of $h^s(e_{i_0,j_0})$ is contained in the space generated by vertices in the grid of eq. (24), which lie left and down of the line $i+j = i_0+j_0-s$, i.e. it is contained in the vector space generated by the elements $e_{i,j}$ where $i+j \leq i_0+j_0-s$.*

Proof. This is immediate by either the geometric viewpoint explained in lemma 26 or by the computation given in eq. (27). □

We will extend our point of view by seeing $J_r \otimes J_{p-1}$ inside $J_p \otimes J_{p-1}$. In order to do so we extend the elements $b_1, \ldots, b_r$ to elements $b_1, \ldots, b_p$, such that $\epsilon_A b_i = b_i + b_{i-1}$ for all $1 \leq i \leq p$. For elements in P_n we can do this simply by adding $p-r$ extra basis elements so that we arrive at a space of dimension $p\left\lceil \frac{2n-1}{p} \right\rceil = 2n-1+p-r$ generated by $b_i = (T^p - T)^{\lfloor \frac{i}{p} \rfloor}\binom{T}{i-1}$ for $i = 1, \ldots, 2n-1+p-r$. Notice that elements of degree $\geq 2n-2$ are not mapped to polynomials using the action given in eq. (4) but to rational functions. We therefore have to extend the module of coefficients.

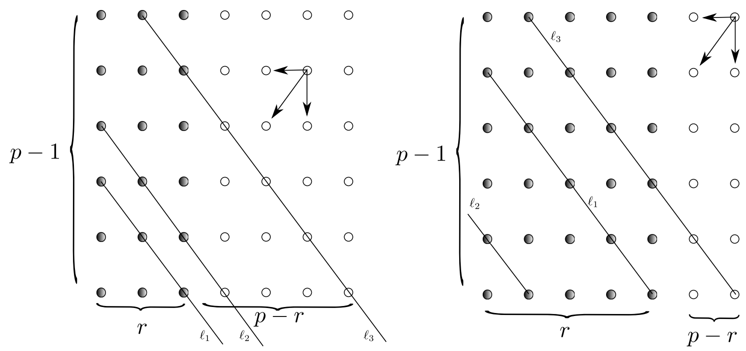

FIGURE 1. Completed $J_r \otimes J_{p-1}$ for $r < p-r$ (left) and $r > p-r$ (right).

Definition 28. Consider the K-vector space Π_n generated as vector space by all elements $f^{\mathfrak{n}}$ where $\mathfrak{n} \in N$ and $f \in K[x]$, with $\deg(f) \leq p\left\lceil \frac{2n-1}{p} \right\rceil = 2n-1+(p-r)$. An arbitrary element π in Π_n is of the form $\pi = \sum_{i\in I} \lambda_i f_i^{\mathfrak{n}_i}$, where $\lambda_i \in K$, $f_i \in K[x]$ and $\mathfrak{n}_i \in N$. Observe that $P_n \subset \Pi_n$, since we can take as $\mathfrak{n}$ the identity element of N. The space Π_n becomes an N-module by defining the action of $\mathfrak{n} \in N$ on $\pi = \sum_{i\in I} \lambda_i f_i^{\mathfrak{n}_i} \in \Pi_n$ as follows:

$$\pi^{\mathfrak{n}} = \sum_{i\in I} \lambda_i f_i^{\mathfrak{n}_i \mathfrak{n}}.$$

Let Δ be the subspace of $\mathrm{Der}(\Gamma, \Pi_n)$ generated by the following derivations: For $i, j \in (\mathbb{Z}/p\mathbb{Z})^\times$ and $k = 0, \ldots, p\left\lceil \frac{2n-2}{p} \right\rceil$, define $d_{i,j}^{(k)} \in \mathrm{Der}(\Gamma, \Pi_n)$ by

$$d_{i,j}^{(k)}(e_{i',j'}) = \delta_{ii'}\delta_{jj'} b_k^{\epsilon_B^{-j}} \text{ for } i', j' \in (\mathbb{Z}/p\mathbb{Z})^\times.$$

Claim: The space Δ is an A-submodule of $\mathrm{Der}(\Gamma, \Pi_n)$ which is isomorphic as a $K[A]$-module to $J_p^{(p-1)^2\left\lceil \frac{2n-1}{p} \right\rceil}$.

Keep in mind that for $2n-2 < k \leq p\left\lceil \frac{2n-2}{p} \right\rceil$ the elements $b_k^{e_B^{-j}}$ are not polynomials but elements in Π_n. For proving the claim observe that derivations $d_{i,j}^{(k)}$ defined above are acted on by $A = \langle \epsilon_A \rangle$ in exactly the same way as given by eq. (16). So following the argument and the notation of §4.2 we see that in the $K[A]$-module structure of Δ only $J_p \otimes L$ factors appear. Using proposition 20 we see that Δ is a free $k[A]$-module of the desired rank. This finishes the proof of the claim.

Therefore the $r \times (p-1)$ grid is naturally contained inside a $p \times (p-1)$ grid corresponding to $J_p \otimes J_{p-1}$. In figure 1 the grid of $J_r \otimes J_{p-1}$ is pictorially represented by grey dots inside $J_p \otimes J_{p-1}$. Consider again the map ψ sending an element $x \in P_n$ to the principal derivation $\gamma \mapsto x^\gamma - x$.

We would like to prove that the element $b_{2n-2} \in P_n$ of order r generating the direct summand J_r of P_n is mapped to $\psi(b_{2n-2}) = h^{p-r}(y)$ for some element $y \in J_r \otimes J_{p-1}$. Notice that since the order of b_{2n-2} is r this implies that y has order p. We know that $h^{p-r}(b_{2n-2+p-r}) = b_{2n-2}$. Also the map ψ is compatible with the ϵ_A, i.e. $\psi(x^{\epsilon_A}) = \psi(x)^{\epsilon_A}$.

Therefore, the principal derivation $\psi(b_{2n-2+p-r})$ is an element of $J_p \otimes J_{p-1}$which is mapped into a linear combination of elements of the $p \times (p-1)$ grid of figure 1.

Consider the line l_3 given by vertices $e_{i,j}$ with $i + j = p + 1$, shown in figure 1. Since $h^{p-r}(\psi(b_{2n-2+p-r})) \in J_r \otimes J_{p-1}$, lemma 27 proves that basis elements $e_{i,j}$

left and below line ℓ_3, i.e. elements $e_{i,j}$ with $i+j \leq p+1$, have an image inside $J_r \otimes J_{p-1}$.

We claim that there is no element Indeed we see that an element element e_{i_0,j_0} with $i_0+j_0 > p+1$ is mapped to $h^{p-r}(e_{i_0,j_0})$. Using eq. (27) we can see that $h^{p-r}(e_{i_0,j_0})$ involves also linear combinations of basis elements, outside the $J_r \otimes J_{p-1}$ part. Keep in mind that all binomial coefficients involve integers smaller that p, hence are not zero mod p.

Therefore $\psi(b_{2n-2}) = h^{p-r}\psi(b_{2n-n+p-r})$, is in the space generated by elements left and below of line ℓ_1, given by vertices $e_{i,j}$ such that $i+j = r+1$. The element $\psi(b_{2n-2})$ is in the vector space V generated by vertices in the $r \times r$ triangle in the lower left corner of the $r \times (p-1)$ grid. Denote by $\bar{h}$ the restriction of h to $J_r \otimes J_{p-1}$. We will now prove that the image of the map $\bar{h}^{p-r}$, contains the space V. Observe that the kernel of the map $\bar{h}^{p-r}$ consists of elements which lie left and down of line ℓ_2 and are inside the $r \times (p-1)$ grid, where ℓ_2 is the line consisted by elements $e_{i,j}$ such that $i+j = p-r+1$.

Case 1 Suppose that $1 < r < p-r$. In this case the kernel of h^{p-r} has dimension:

$$(p-r) + (p-r+1) + \cdots + (p-2r+1) = (p-2r) \cdot r + \frac{r(r+1)}{2}.$$

On the other hand the image of $\bar{h}^{p-r}$ is contained in the vector space V' generated by the elements in the complement of a the upper right $(p-r) \times (p-r)$ triangle, i.e. the space with basis elements $e_{i,j}$ such that $i+j \leq 2r-1$. The dimension of the space V' equals

$$r-1+r+\cdots+2r-2 = r(r-2)+1+\cdots+r = r(r-2) + \frac{r(r+1)}{2}.$$

Since $\dim \ker \bar{h}^{p-r} + \dim \mathrm{Im} \bar{h}^{p-r} = r(p-1)$, we have that $\mathrm{Im}\bar{h}^{p-r} = V'$.

The elements of V' are on the line L_2 in figure 2 and below and to the left generate the space $\mathrm{Im}(\bar{h}^{p-r})$. These points form a trapezium where the bigest base has size $p-1-(p-2r+1) = 2r-2$. Since $r > 1$ we have $2r-2 \geq r$ therefore the lowest left $r \times r$ triangle (of points below and to the left of line L_2 in figure 2) are contained in $\mathrm{Im}\bar{h}^{p-r}$ and the result follows in this case.

Case 2 $p-r < r$. This case differs from the previous one, since here the line ℓ_2 is to the left of the line ℓ_1. The elements of the extended grid $J_p \otimes J_{p-1}$ which can be mapped under h^{p-r} inside $J_r \otimes J_{p-1}$ are the ones which are below and to the left of the line ℓ_3 in figure 1. Their images are below and to the left of the line ℓ_1. As in the previous case we will denote by $\hat{h}$ the restriction of h to the space $J_r \otimes J_{p-1}$. We will show that $\mathrm{Im}(\bar{h}^{p-r})$ contains the space generated by the vertices on the $r \times r$ triangle bellow and to the left of line ℓ_1. The kernel of the map $\bar{h}^{p-r}$ is generated by the vertices of the $(p-r) \times (p-r)$ triangle of points which are below and to the left of line ℓ_2 given by equation $i+j = p-r+1$, and has dimension $1+2+\cdots+p-r = (p-r)(p-r+1)/2$.

Let L be the line given by $i+j = 2r-2$, see figure 2. The image of $\hat{h}^{p-r}$ is contained in the vector space generated by vertices on the line L and below and to the left of line L, i.e. by vertices $i+j \leq 2r-2$. These elements form a trapezium with smaller base of size $p-1-(p-r+1) = r-2$ and since this size is positive for $r \geq 2$ we have that the vector space obtained by the above mentioned trapezium contains the space generated by the desired lower left $r \times r$ triangle.

Case 3 $r = 1$. We will now treat the $r = 1$ case. Consider the principal derivation $\psi(b_k)$ sending $\gamma_{i,j} = [e_A^i, e_B^j] \mapsto b_k^{\gamma_{i,j}} - b_k$. Fix $k = 2n-2$ and i, j and

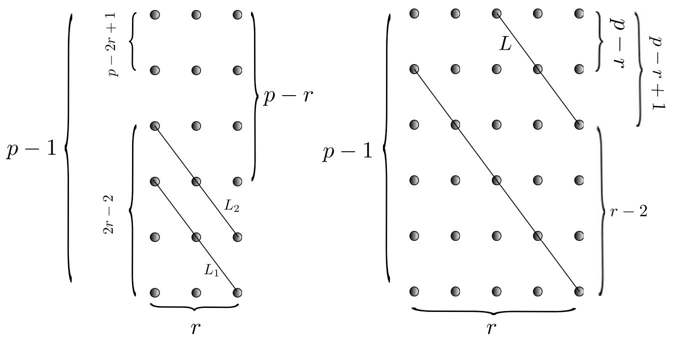

FIGURE 2. The image of $\bar{h}^{p-r}$ contains the lower left $r \times r$ triangle. The cases $r < p - r$ (left) and $r > p - r$ (right).

write:

$$b_{2n-2}^{\epsilon_A^i \epsilon_B^j \epsilon_A^{-i} \epsilon_B^{-j}} - b_{2n-2} = \sum_{\lambda=0}^{2n-2} a(i,j)^{(2n-2)} b_\lambda^{e_B^{-j}}. \tag{29}$$

We would like to find the coefficients $a(i,j)^{(\lambda)}$ By applying e_B^j in eq. (29), using that b_{2n-2} is ϵ_A-invariant we obtain

$$b_{2n-2}^{\epsilon_B^j \epsilon_A^{-i}} - b_{2n-2}^{\epsilon_B^{-j}} = \sum_{\lambda=0}^{2n-2} a(i,j)^{(2n-2)} b_\lambda. \tag{30}$$

We set $B = b_{2n-2}^{\epsilon_B^j}$ and we observe that for $i \in \mathbb{N}$, $i \geq 1$

$$B^{e_A^{i+1}} - B = \left(B^{\epsilon_A^i} - B\right)^{\epsilon_A} - \left(B^{\epsilon_A} - B\right). \tag{31}$$

Equations (30) and (31) show that for $\lambda = 2n - 2$, which in our case is divisible by p, and using induction we have

$$a_1 := a(1,j)^{(2n-2)}, a(2,j)^{(2n-2)} = 0, a(3,j)^{(2n-2)} = -a_1, a(3,j)^{(2n-2)} = -2a_1, \ldots,$$

and finally we have $a(p,j)^{(2n-2)} = -(p-2)a_1$ but this should be zero (since $e_A^p = 1$), therefore the coefficients $a_{(i,j)}^{(2n-2)}$ in eq. (29) are zero (we have assumed $p > 2$). This proves that the image of principal derivations are inside the projective part of the derivations since there is no contribution from the J_{p-r} part.

We have seen that $\psi(b_{2n-2}) = \bar{h}^{p-r}(y)$ for some y of order p, therefore $\psi(b_{2n-2})$ is inside a J_p direct summand of $\mathrm{Der}(\Gamma, P_n)$. Indeed, the element y can be expressed as a linear combination

$$y = \sum_{i=1}^{N_1} \lambda_i a_i + \sum_{j_1}^{N_2} \mu b_j,$$

where a_i are generators of J_p summands and b_j are generators of J_r summands. Since y has order p at least one of the coefficients λ_{i_0} is not zero, and by basis exchange lemma of linar algebra we see that $\{a_1, \ldots, a_{i_0-1}, y, a_{i_0+1}, \ldots, \}$ are also generators of the J_p summands.

We can now proceed to the computation of the quotient $\mathrm{Der}(\Gamma, P_n)/\mathrm{PDer}(\Gamma, P_n)$ is isomorphic to

$$\begin{aligned} H^1(\Gamma, P_n) &\cong \mathrm{Der}(\Gamma, P_n)/\mathrm{PDer}(\Gamma, P_n) \\ &\cong J_p^{(p-1)\left\lfloor \frac{(2n-1)(p-1)}{p} \right\rfloor - 1 - \left\lfloor \frac{2n-1}{p} \right\rfloor} \oplus J_p/J_r \oplus J_{p-r}^{p-1} \\ &\cong J_p^{(p-1)(2n-1) - p\left\lceil \frac{2n-1}{p} \right\rceil} \oplus J_{p-r}^{p}. \end{aligned} \tag{32}$$

5.2. **Second Proof of Theorem 5 (1).** Our second proof of Theorem 5 (1) uses the algebraic theory of curves $(x^p - x)(y^p - y) = c$.

Recall that the Artin-Schreier-Mumford curves we are studying are uniformized by $\Gamma = [A, B] \cong [\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}]$, and are given by the following algebraic model

$$X_c : (x^p - x)(y^p - y) = c,$$

for some $c \in K$, $|c| < 1$. The group $\mathbb{Z}/p\mathbb{Z}$ is a subgroup of the automorphism group and acts for instance on the curve X_c by letting the generator τ of $\mathbb{Z}/p\mathbb{Z}$ to act on the curve in terms of the map $(x, y) \mapsto (x, y + 1)$. We call Y the quotient curve $X_c/\langle\tau\rangle$. Note that Y is isomorphic to $\mathbb{P}^1$, and hence the genus g_Y is zero.

Denote the function field of X_c, for a fixed value of $|c| < 1$, by F. The extension $F/K(x)$ is a cyclic extension of the rational function field $K(x)$. In this extension p-places $P_i = (x - i)$, $1 \le i \le p - 1$ of $K(x)$ are weakly ramified. The different is:

$$\mathrm{Diff}_{F/K(x)} = \sum_{i=0}^{p-1} 2(p-1)P_i.$$

We will employ the results of S. Nakajima [27]. We have the following decomposition in terms of indecomposable modules

$$H^0(X, \Omega_X^{\otimes n}) = \bigoplus_{i=1}^{p} m_i J_i,$$

and the coefficients are given by [27, th.1]:

$$m_p = (2n-1)(g_Y - 1) + \sum_{i=1}^{p} \left\lfloor \frac{n_i - (p-1)N_i}{p} \right\rfloor,$$

where $N_i = 1$ (ordinary curves) and $n_i = 2n(p-1)$, see [21, sec. 4].

Since $g_Y = 0$ we compute:

$$\begin{aligned} m_p &= (2n-1)(g_Y - 1) + p\left\lfloor \frac{2n(p-1) - (p-1)}{p} \right\rfloor \\ &= (2n-1)(g_Y - 1) + p(2n-1) - p\left\lceil \frac{2n-1}{p} \right\rceil \\ &\overset{g_Y=0}{=} (p-1)(2n-1) - p\left\lceil \frac{2n-1}{p} \right\rceil. \end{aligned}$$

For $1 \le j \le p - 1$ the coefficients m_j are given by the following formulas [27, th.1]:

$$\begin{aligned} \frac{m_j}{p} &= -\left\lfloor \frac{n_i - jN_i}{p} \right\rfloor + \left\lfloor \frac{n_i - (j-1)N_i}{p} \right\rfloor \\ &= -\left\lfloor \frac{2n(p-1) - j}{p} \right\rfloor + \left\lfloor \frac{2n(p-1) - (j-1)}{p} \right\rfloor \\ &= -\left\lfloor \frac{-2n - j}{p} \right\rfloor + \left\lfloor \frac{-2n - (j-1)}{p} \right\rfloor \\ &= \left\lceil \frac{2n + j}{p} \right\rceil - \left\lceil \frac{2n + j - 1}{p} \right\rceil. \end{aligned}$$

We now notice that for $0 \leq j \leq p-1$ the above expression is zero unless $p \mid 2n+j-1$.

We write $2n-1 = \left\lfloor \frac{2n-1}{p} \right\rfloor p + r$, and we see that $m_j = 0$ unless

$$j = p - r = p - (2n-1) + \left\lfloor \frac{2n-1}{p} \right\rfloor p.$$

Notice that if $p > 2n-1$ then $j = p-(2n-1)$. So we have that

$$H^1(\Gamma, P_n) = H^0(X, \Omega_X^{\otimes n}) = K[A]^{(p-1)(2n-1)-p\left\lceil \frac{2n-1}{p} \right\rceil} \bigoplus J_{p-r}^p. \tag{33}$$

5.3. **Using the theory of B. Köck. Study of the $K[A\times B]$-module structure.** In this section we will employ the results of B. Köck on the projectivity of the cohomology groups of certain sheaves in the weakly ramified case. Consider a p-group G and the cover $\pi : X \to X/G =: Y$. For every point P of X we consider the local uniformizer t at P, the stabilizer $G(P)$ of P and assign a sequence of ramification groups

$$G_i(P) = \{\sigma \in G(P) : v_P(\sigma(t) - t) \geq i + 1\}.$$

Notice that $G_0(P) = G(P)$ for p-groups, see [30, chap. IV]. Let $e_i(P)$ denote the order of $G_i(P)$. We use the notation X_{ram} for the set of ramification points. We will say that the cover $X \to X/G$ is weakly ramified if all $e_i(P)$ vanish for $i \geq 2$. All Mumford curves X are ordinary and in $X \to X/\mathrm{Aut}(X)$ only weak ramification is allowed [7]. We denote by Ω_X the sheaf of differentials on X and by $\Omega_X(D)$ the sheaf $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$. For a divisor $D = \sum_{P\in X} n_P P$ we denote by $D_{\mathrm{red}} = \sum_{P\in X: n_P \neq 0} P$ the associated reduced divisor. We will also denote by

$$L(D) = H^0(X, \mathcal{O}_X(D)) = \{D + (f) > 0 : f \in F_X\} \cup \{0\},$$

where F_X is the function field of the curve X. The ramification divisor equals $R = \sum_{P\in X} \sum_{i=0}^{\infty} \big(e_i(P) - 1\big)$. Finally, Σ denotes the skyscraper sheaf defined by the short exact sequence:

$$0 \to \Omega_X^{\otimes n} \to \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big) \to \Sigma \to 0. \tag{34}$$

Lemma 29. *For $n > 1$ the cohomology group $H^1(X, \Omega_X^{\otimes n}) = 0$.*

Proof. There is a correspondence of sheaves between divisors and 1-dimensional $\mathcal{O}_X$-modules, $D \mapsto \mathcal{O}_X(D)$. The divisor of any differential is a canonical divisor K and Ω_X can be identified with $\mathcal{O}(K)$.

Recall that Serre duality asserts:

$$\dim H^1(X, \mathcal{O}_X(D)) = \dim H^0(X, \Omega_X \otimes \mathcal{O}_X(D)^{-1}).$$

Hence we find that

$$\dim H^1(X, \Omega_X^{\otimes n}) = \dim H^0(X, \Omega_X \otimes \Omega_X^{-\otimes n}).$$

The sheaf $\Omega_X \otimes \Omega_X^{-n}$ corresponds to the $\mathcal{O}_X$-module $\mathcal{O}_X(K - nK)$ and since

$$H^0(X, \mathcal{O}_X(K - nK)) = L(K - nK),$$

it holds that

$$\dim H^1(X, \Omega_X^{\otimes n}) = \dim L(K - nK) = 0.$$

□

Now we apply the functor of global sections to the short exact sequence in (34) and obtain the following short exact sequence:

(35)

$$0 \to H^0(X, \Omega_X^{\otimes n}) \to H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big) \to H^0(X, \Sigma) \to H^1(X, \Omega_X^{\otimes n}) = 0.$$

Theorem 30. *The $K[G]$-module $H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big)$ is a free $K[G]$-module of rank $(2n-1)(g_Y-1+r_0)$, where r_0 denotes the cardinality of $X_{\mathrm{ram}}^G = \{P \in X/G : e(P) > 1\}$, and g_Y denotes the genus of the quotient curve $Y = X/G$.*

Proof. Since G is a p-group a module is free if and only if it is projective. So we have to show that $H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big)$ is projective. B. Köck proved [18, Th. 2.1] that if $D = \sum_{P\in X} n_p P$ is a G-equivariant divisor, the map $\pi : X \to Y := X/G$ is weakly ramified, $n_P \equiv -1 \bmod e_P$ for all $P \in X$ and $\deg(D) \geq 2g_X - 2$, then the module $H^0(X, \mathcal{O}_X(D))$ is projective.

We have to check the conditions for the divisor $D = nK_X + (2n-1)R_{\mathrm{red}}$, where K_X is a canonical divisor on X. Notice that $K_X = \pi^* K_Y + R$ and $R = \sum_{P\in X} 2\big(e_0(P)-1\big)$, therefore

$$D = n\pi^* K_Y + \sum_{P\in X : e_0(P)>1} \left(2ne_0(P) - 2n + 2n - 1\right) P.$$

Therefore, the condition $n_P \equiv -1 \bmod e_0(P)$ is satisfied.

We will now compute the dimension of $H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big)$ using Riemann–Roch theorem (keep in mind that $H^1\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big) = 0$)

$$\begin{aligned}\dim_K H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big) &= n(2g_X-2) + (2n-1)|X_{\mathrm{ram}}| + 1 - g_X \\ &= (2n-1)(g_X - 1 + |X_{\mathrm{ram}}|) \\ &= |G|(2n-1)(g_Y - 1 + r_0),\end{aligned}$$

where in the last equality we have used the Riemann–Hurwitz formula [14, 7, Cor. IV 2.4]

$$g_X - 1 = |G|(g_Y-1) + \sum_{P\in X_{\mathrm{ram}}} (e_0(P)-1).$$

□

Remark 31. This method was applied by the second author and B. Köck in [20] for the $n = 2$ case in order to compute the dimension of the tangent space to the deformation functor of curves with automorphisms. Deformations of curves with automorphisms for Mumford curves were also studied by the first author and G. Cornelissen in [3].

The sequence in eq. (35) leads to the following short exact sequence of modules:

$$0 \to H^0(X, \Omega_X^{\otimes n}) \to K[G]^{(2n-1)(g_Y-1+r_0)} \to H^0(X, \Sigma) \to 0. \tag{36}$$

Since Σ is a skyscraper sheaf the space $H^0(X,\Sigma)$ is the direct sum of the stalks of Σ

$$H^0(X,\Sigma) = \bigoplus_{P\in X_{\mathrm{ram}}} \Sigma_P \cong \bigoplus_{j=1}^{r_0} \mathrm{Ind}_{G(P_j)}^G(\Sigma_{P_j}),$$

where, for a subgroup H of G, $\mathrm{Ind}_H^G V$ denotes the induced representation of an $K[H]$-module V to a $K[G]$-module, i.e., $\mathrm{Ind}_H^G V = V \otimes_{K[H]} K[G]$.

5.4. **Return to Artin-Schreier-Mumford curves: proof of Theorem 5 (2).** Recall that we are in the case $N = A * B$ and $\Gamma = [A, B]$, where $A \cong B \cong \mathbb{Z}/p\mathbb{Z}$. Set $G = N/\Gamma = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Lemma 32. *The indecomposable summands of the module $\mathrm{Ind}_{G(P_j)}^G(\Sigma_{P_j})$ are either $K[G]$ or $K[G]/\langle(\sigma-1)^\lambda\rangle$, where $\sigma = \epsilon_A$ or ϵ_B and $1 \leq \lambda \leq p-1$.*

Proof. It follows from the ramification of the function field of Artin-Schreier-Mumford curves, seen as a double Artin-Schreier extension of the rational function field, where $r_0 = 2$, i.e., only two points p_1, p_2 of $X/(A \times B)$ are ramified in the cover $X \to X/(A \times B)$. Another way of obtaining this result is by using the theory of graphs of Mumford curves developed by the first author, and by noticing that the subgroup of the normalizer of the Artin-Schreier-Mumford curve giving rise to the $A \times B$ cover is just $A * B$ corresponding to a graph with two ends, see [5], [17, prop. 5.6.2]. Select a point P_1 of X which lies above p_1 and a point P_2 of X which lies above p_2. Let $G(P_1) = A$ and $G(P_2) = B$.

We will use an approach similar to [20] in order to study the Galois module structure of the stalk Σ_{P_j} as a $K[G_{P_j}]$-module. Let $P = P_j$ for $j = 1$ or $j = 2$. Notice first that $nK_X = n\pi^* K_Y + nR$, so if the multiplicity of the divisor K_Y at $\pi(P)$ is m, then the multiplicity of nK_X at P is $mnp+2n(p-1)$ and the multiplicity of $nK_X+(2n-1)R_{\rm red}$ at P is $mnp+2n(p-1)+(2n-1)$. So if t is a local uniformizer at P and s is a local uniformizer at $\pi(P)$ we have that:

$$\Sigma_P = \left\langle \frac{t}{s^{mn+2n}}, \frac{t^2}{s^{mn+2n}}, \ldots, \frac{t^{2n-1}}{s^{mn+2n}}_K, \right\rangle$$

which is $G(P)$-equivariant isomorphic to the K-vector space generated by:

$$\Sigma_P = \left\langle t, t^2, \ldots, t^{2n-1} \right\rangle_K.$$

The action of $G(P)$ on Σ_p is given by the transformation $\sigma(1/t) = 1/t + 1$ for a generator σ of the cyclic group $G(P)$, or equivalently $\sigma(t) = \frac{t}{1+t}$, see [6]. Notice, that the element $t^{-p} - t^{-1} = \frac{1-t^{p-1}}{t^p}$ is invariant and so is its inverse $t^p(1-t^{p-1})^{-1}$. Here the unit $(1-t^{p-1})^{-1}$, can be seen as a polynomial modulo t^{2n}, if we expand it in terms of a geometric series and truncate the terms of degree $\geq 2n$. Now we analyse the $G(P)$-module structure of Σ_P using Jordan blocks. Observe that for $0 \leq k \leq p-1$:

$$\sigma\binom{1/t}{k} = \binom{1/t}{k} + \binom{1/t}{k-1},$$

where

$$\binom{1/t}{k} = \frac{1}{k!}\prod_{\nu=0}^{k-1}\left(\frac{1}{t} - \nu\right) = \frac{1}{k!t^k}\prod_{\nu=0}^{k-1}(1-t\nu).$$

Note that $\binom{1/t}{k}$ is a rational function, where the denominator is $k!t^k$. So if we multiply it by the invariant element $t^p(1-t^{p-1})^{-1}$ we obtain a polynomial of degree $p-k$. Another K-vector space basis of Σ_P is given by:

$$\left(\frac{t^p}{(1-t^{p-1})}\right)^i \binom{1/t}{k}, \quad \begin{array}{l} \text{where } 1 \leq i \leq \left\lfloor \frac{2n-1}{p} \right\rfloor \text{ and } 0 \leq k \leq p-1 \\ \text{or } i = \left\lfloor \frac{2n-1}{p} \right\rfloor + 1 \text{ and } p-r \leq k \leq p-1 \end{array}.$$

The above defined elements are seen as polynomials by expanding them as powerseries and truncate the powers of t greater than $2n$. These polynomials, depending on i and k, have degree $pi-k$. Their degrees start from degree one ($i = 1, k = p-1$) to $2n-1$ ($i = \left\lfloor \frac{2n-1}{p} \right\rfloor + 1, k = p-r$).

For fixed i, $i = 1, .., \left\lfloor \frac{2n-1}{p} \right\rfloor$, and by allowing k to vary from $0 \leq k \leq p-1$, we obtain a Jordan block J_p. The remaining block $i = \left\lfloor \frac{2n-1}{p} \right\rfloor + 1$, $p-r \leq k \leq p-1$ is J_r.

So the structure of Σ_P is given by

$$\Sigma_P = J_p^{\left\lfloor \frac{2n-1}{p} \right\rfloor} \bigoplus J_r. \tag{37}$$

Recall [9, 12.16 p.74] that if H is a subgroup of G and $g_1, \ldots, g_\ell$ is a set of coset representatives of G in H, then for an $K[H]$-module M the induced module can be written as

$$\mathrm{Ind}_H^G M = \bigoplus_{\nu=1}^{\ell} g_\nu \otimes M.$$

Using the above equation for $G = A \times B$ and $H = G(P_1) = A$ (resp. $G(P_2) = B$) we have

$$\mathrm{Ind}_{G(P_j)}^G (J_p) = K[G] \text{ and } \mathrm{Ind}_{G(P_1)}^G (J_r) = \frac{K[G]}{(\epsilon_A - 1)^r}.$$

Similarly

$$\mathrm{Ind}_{G(P_2)}^G (J_r) = \frac{K[G]}{(\epsilon_B - 1)^r}$$

and both of the above $K[G]$-modules are indecomposable. □

Proposition 33. *The indecomposable summands V_i of $H^0(X, \Omega_X^{\otimes n})$ are either $K[G]$ or $K[G]/\langle(\sigma - 1)^{p-r}\rangle$, for $\sigma = \epsilon_A$ or $\sigma = \epsilon_B$ and r is the remainder of the division $2n - 1$ by p.*

Proof. Let V_i be a indecomposable summand of $H^0(X, \Omega_X^{\otimes n})$. Consider the injective hull of V_i. This is the smallest injective module containing V_i, and it is of the form $K[G]^a$. Keep in mind that for group algebras of finite groups the notions of injective and projective modules coincide [9, th. 62.3].

Therefore we have to consider the smallest a such that $V_i \subset K[G]^a$. We have the short exact sequence:

$$0 \to V_i \to K[G]^a \to \Omega^{-1}(V_i) \to 0, \tag{38}$$

where $\Omega^{-1}(M)$ for a a $K[G]$-module denotes the cokernel of the embedding of M inside its injective hull. Since the algebra $K[G]$ is self injective (i.e., $K[G]$ is injective) we have (for some appropriate natural number t)

$$V_i \cong \Omega(\Omega^{-1})(V_i) \bigoplus K[G]^t, \tag{39}$$

where $\Omega(\Omega^{-1}(V_i))$ denotes the loop space of $\Omega^{-1}(V_i)$, see [1, exer. 1 p.12]. Since V_i is indecomposable, one of the two direct summands of eq. (39) is zero, so either $V_i \cong K[G]$ or $V_i = \Omega(\Omega^{-1})(V_i)$.

In the second case, we can consider the following diagram, where the first row comes from eq. (36) and the second by eq. (38):

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \Omega_X^{\otimes n}) \arrow[r] & K[G]^{(2n-1)(g_Y - 1 + r_0)} \arrow[r] & H^0(X, \Sigma) \arrow[r] & 0 . \\
0 \arrow[r] & V_i \arrow[r] \arrow[u] & K[G]^a \arrow[r] \arrow[u] & \Omega^{-1}(V_i) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Notice that since V_i is a direct summand of $H^0(X, \Omega_X^{\otimes n})$ which is contained in $K[G]^{(2n-1)(g_Y-1+r_0)}$ we can assume that the injective hull $K[G]^a$ of V_i is a submodule of $K[G]^{(2n-1)(g_Y-1+r_0)}$. The module $\Omega^{-1}(V_i)$ is a non-zero indecomposable non-projective factor of $H^0(X, \Sigma)$ and is isomorphic to $\mathrm{Ind}_{G(P_i)}^G (J_r) = K[G]/\langle(\sigma - 1)^r\rangle$. It can not be $K[G]$ since $K[G]$ is projective. We compute

$$V_i = \Omega(\mathrm{Ind}_{G(P_i)}^G (J_r)) = \Omega(K[G]/\langle(\sigma - 1)^r\rangle)) \cong K[G]/\langle(\sigma - 1)^{p-r}\rangle.$$

□

Corollary 34. *The space $H^0(X, \Omega_X^{\otimes n})^G$ has dimension equal to the number of indecomposable summands.*

Proof. Notice that each indecomposable summand V_i is contained in a $K[G]$. □

Corollary 35. *If* $2n-1 \equiv 0 \bmod p$ *then* $H^0(X, \Omega^{\otimes n})$ *is projective.*

Now we finish the proof of 5 (2). Using the sequence given in eq. (36) and the fact that only two points of Y are ramified in $X \to Y$, i.e., $g_Y = 0, r_0 = 2$, together with eq. (37) we obtain that the number of summands which are isomorphic to $K[G]$ in $H^0(X, \Omega^{\otimes n})$ is $2n-1-2\left\lceil \frac{2n-1}{p} \right\rceil$. There are two indecomposible summands in $H^0(X, \Omega_X^{\otimes n})$, V_1, V_2 such that

$$K[G]/V_1 = K[G]/h^r \text{ and } K[G]/V_2 = K[G]/(\epsilon_B - 1)^r.$$

We see that

$$V_1 = K[G]/h^{p-r} \text{ and } V_2 = K[G]/(\epsilon_B - 1)^{p-r}.$$

Adding all these together we obtain:

$$H^0(X, \Omega_X^{\otimes n}) = K[G]^{2n-1-2\left\lceil \frac{2n-1}{p} \right\rceil} \bigoplus K[G]/h^{p-r} \bigoplus K[G]/(\epsilon_B - 1)^{p-r}.$$

The Proof of Theorem 5 (2) is now complete.

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