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## Arithmetic equivalence for function fields, the Goss zeta function and a generalisation <sup>☆</sup>

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### ABSTRACT

A theorem of Tate and Turner says that global function fields have the same zeta function if and only if the Jacobians of the corresponding curves are isogenous. In this note, we investigate what happens if we replace the usual (characteristic zero) zeta function by the positive characteristic zeta function introduced by Goss. We prove that for function fields whose characteristic exceeds their degree, equality of the Goss zeta function is the same as Gaßmann equivalence (a purely group theoretical property), but this statement can fail if the degree exceeds the characteristic. We introduce a ‘Teichmüller lift’ of the Goss zeta function and show that equality of such is *always* the same as Gaßmann equivalence.

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## 1. Introduction

The following classical theorem about number fields shows a surprising equivalence between a number theoretical, a group theoretical, and an analytical statements:

**Theorem 1.1.** *Let  $K$  and  $L$  denote two number fields. The following are equivalent:*

- (i) [Arithmetic equivalence] *For every prime number  $p$  outside a set of Dirichlet density zero, the splitting type of  $p$  (i.e., the ordered sequence of inertia degrees with multiplicities) in  $K$  and  $L$  is the same;*

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- (i') [Split equivalence] For every prime number  $p$  outside a set of Dirichlet density zero, the number of prime ideals in  $K$  and  $L$  above  $p$  is the same;
- (ii) [Gaßmann equivalence] Let  $N$  denote a Galois extension of  $\mathbf{Q}$  containing  $K$  and  $L$ ; then every conjugacy class in  $\text{Gal}(N/\mathbf{Q})$  meets  $\text{Gal}(N/K)$  and  $\text{Gal}(N/L)$  in the same number of elements;
- (iii)  $K$  and  $L$  have the same Dedekind zeta function:  $\zeta_K(s) = \zeta_L(s)$ .

For proofs, see [23,27] for the equivalence of (i) and (i') [21] or the historical source [10]. The theorem can be used to show that the zeta function of a number field does not necessarily determine the number field up to field isomorphism. We also remark that the statement in (iii) is sometimes called arithmetic equivalence (e.g., [7]), but since here, we will study various zeta functions and how the equivalences in the above theorem depend on what zeta function is used, we do not use that convention, but rather the one from [21]. Of course, the theorem shows that for number fields, the choice of terminology does not matter.

In this paper, we consider what happens to this theorem if number fields are replaced by function fields. The notational set-up is as follows. Let  $F$  denote a purely transcendental field  $\mathbf{F}_q(T)$  over a finite field  $\mathbf{F}_q$  with  $q$  elements, and  $A = \mathbf{F}_q[T]$  the corresponding polynomial ring. Assume that  $K$  and  $L$  are two subfields of a fixed algebraic closure of  $F$ , and assume that  $K$  and  $L$  are two geometric separable extensions of  $F$ . Let  $N$  denote a finite Galois extension of  $F$  that contains both  $K$  and  $L$ .

By a slight, but continuous, abuse of notation, we will not indicate the dependence of zeta functions on  $F$  and  $A$ .

There is the following classical theorem of Tate [28] and Turner [30]:

**Theorem 1.2.** *The fields  $K$  and  $L$  have the same Weil zeta function precisely if the Jacobians of the corresponding curves are  $F$ -isogenous.*

Here, by the Weil zeta function of  $K$  we mean  $\zeta_K(s) = \sum_I \mathbf{N}(I)^{-s}$ , where we sum over divisors  $I$  of  $K$ , and  $\mathbf{N}(I) = q^{\text{deg}(I)}$  is the absolute norm of the divisor  $I$ . Note that this function is absolute, i.e., only depends on  $K$ , not on the choice of a rational subfield  $F$  of  $K$ .

Mimicking the terminology above, we call  $K$  and  $L$  arithmetically, linearly or Gaßmann equivalent if the obvious function fields version of the corresponding statements in Theorem 1.1 holds. We will use a general representation theoretical argument to show that Gaßmann equivalence implies equality of Weil zeta functions (Proposition 3.2). We will show an example of function fields with the same Weil zeta function, but that are not Gaßmann equivalent over some common rational subfield (Example 3.1).

Now David Goss [13] (following preliminary work of Carlitz) has introduced a new characteristic  $p$  valued zeta function for function fields, essentially interpolating the function

$$\zeta_K^{[p]}(s) = \sum_I \frac{1}{\mathbf{N}(I)^s}$$

from positive integers  $s$  to a function field analogue  $S_\infty$  of the complex plane (cf. Section 4), which anyhow (only this is important for us) includes all integer values of  $s$ . Here, the sum ranges over nonzero ideals of the ring of integers  $\mathcal{O}_K$  (integral closure of  $A$ ) of  $K$ , and the norm is the relative norm from  $\mathcal{O}_K$  to  $A$ , so characteristic  $p$  valued, in contrast to the Weil zeta function. One may incorporate finitely many Euler factors in this definition, arising from primes above the chosen place  $\infty$ .

The aim of this note is to investigate whether equality of Goss zeta functions (for a given relative situation of  $K$  and  $L$  over  $F$ , and a choice of  $\infty$  in  $F$ ) is the same as Gaßmann equivalence of  $K$  and  $L$  over  $F$ . We will show that Gaßmann, split, and arithmetical equivalence is the same for function fields (Proposition 2.1). Concerning the relation to zeta functions, we prove

**Theorem 1.3.** *Let  $K$  and  $L$  denote two finite geometric separable field extensions of  $F = \mathbf{F}_q(T)$ .*

- (i) If  $p > [K : F]$  and  $p > [L : F]$ , then  $\zeta_K^{[p]}(s) = \zeta_L^{[p]}(s)$  means the same as arithmetical, split and Gaßmann equivalence of  $K$  and  $L$  (cf. Theorem 4.4).
- (ii) There exist infinitely many field extensions  $K/F$  that have the same Goss zeta function, but are not arithmetically equivalent. Actually, as soon as  $K$  is a Galois  $p$ -extension of  $F$  only ramified above  $\infty$ , its Goss zeta function is  $\zeta_K^{[p]}(s) = \zeta_F^{[p]}([K : F]s)$ . In particular, there is a choice of  $K$ ,  $F$  and  $\infty$  such that the corresponding Goss zeta function does not determine the Weil zeta function of  $K$  (cf. Proposition 4.6 and Corollary 4.9).

The ‘problem’ with the Goss zeta function is that it takes values in positive characteristic, so it only determines the relative splitting types (in the extension  $K/F$ ) ‘mod  $p$ ’. We can fix this, however, by introducing a ‘Teichmüller lift’ of the Goss zeta function, which we call the *lifted Goss zeta function*.

**Definition 1.4.** The *lifted Goss zeta function* of  $K$  (w.r.t.  $F$  and  $A$ ) is

$$\zeta_K^{[0]}(s) = \sum_I \frac{1}{\chi(N(I)^s)},$$

where  $s \in S_\infty$ , and  $\chi : \mathbf{C}_F \rightarrow W$  is the Teichmüller lift from  $\mathbf{C}_F$  to its ring of Witt vectors, where  $\mathbf{C}_F$  is the completion of an algebraic closure of the completion  $F_\infty$  of  $F$  w.r.t. the place  $\infty$ . Here, the sum is over all ideals of  $\mathcal{O}_K$  and the norm  $N$  is again the relative characteristic  $p$  valued norm from  $\mathcal{O}_K$  to  $A$ .

This new zeta function has some obvious properties that it shares with the Goss zeta function, such as permitting an Euler product, but also some nice characteristic zero properties, such as subsiding to the Artin formalism of factorisation according to characters (cf. Section 5).

We prove:

**Theorem 1.5.** Let  $K$  and  $L$  denote two finite geometric field extensions of  $F = \mathbf{F}_q(T)$ .

- (i) The equality  $\zeta_K^{[0]}(s) = \zeta_L^{[0]}(s)$  means the same as arithmetical, split and Gaßmann equivalence, irrespective of any bounds on the degrees of  $K$  and  $L$  over  $F$  (cf. Theorem 5.3).
- (ii) The lifted Goss zeta function of a function field  $K$  determines its Weil zeta function and its Goss zeta function (cf. Corollary 5.4).
- (iii) For every  $q$ , there exist two function field extensions of  $F = \mathbf{F}_q(T)$  such that  $K$  and  $L$  have the same lifted Goss zeta function, but are not isomorphic; the degree of  $K/F$  and  $L/F$  has to exceed 6 for this to be possible (cf. Proposition 5.8).

**Remark 1.6.** A novel point of view on zeta functions is to view it as the partition function of a quantum statistical mechanical (QSM) system in the style of Bost and Connes [4]. For the Weil zeta function, such a system was constructed by Jacob [19], and for the Goss zeta function, such a ‘characteristic  $p$ -valued QSM-system’ was constructed by Consani and Marcolli [6]. It seems an interesting question to ask in how far these function field QSM-systems capture the isomorphism type of the underlying global field and/or what is their relation to arithmetic equivalence. One may also wonder whether there is such a  $W$ -valued QSM-system whose partition function is the lifted Goss zeta function.

**Notation 1.7.** For the convenience of the reader, we gather our notational convention for the three different zeta functions of a function field  $K$ :

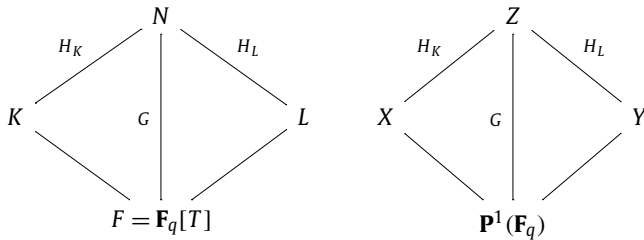
- (1)  $\zeta_K(s)$  is the Weil zeta function of  $K$  for  $s \in \mathbf{C}$  – which takes values in the complex numbers  $\mathbf{C}$ ;
- (2)  $\zeta_K^{[p]}$  is the Goss zeta function of  $K$  for  $s \in S_\infty$  – the superscript indicates that this function takes values in the characteristic  $p$  field  $\mathbf{C}_F$ ;

- (3)  $\zeta_K^{[0]}$  is the lifted Goss zeta function of  $K$  for  $s \in S_\infty$  – the superscript indicates that this function takes values in characteristic 0; actually, in the ring of Witt vectors of  $\mathbf{C}_F$ .

One should also keep in mind the following notational fact: The Weil zeta function (with infinite factors) only depends on the function field  $K$ , since it depends only on the absolute norm on  $K$  (cardinality of the residue field of the places). By contrast, the Goss zeta function and its lift depend on a choice of a rational subfield  $F$  of  $K$  and a place  $\infty$  of  $F$ ; it depends on the relative norm from  $\mathcal{O}_K$  to  $A$ . Even the domain  $S_\infty$  of these functions depends on the choice of  $\infty$ . To keep notation manageable, we will omit  $A$  and  $\infty$  from the notation.

### 2. Gaßmann, linear and arithmetical equivalence for function fields

Throughout, let  $F$  denote a purely transcendental field  $\mathbf{F}_q(T)$  over a finite field  $\mathbf{F}_q$  with  $q$  elements. Assume  $K$  and  $L$  are two subfields of a fixed algebraic closure of  $F$ , and assume that  $K$  and  $L$  are two geometric separable extensions of  $F$ . Let  $N$  denote a finite Galois extension of  $F$  that contains both  $K$  and  $L$ . Let  $G$  denote the Galois group of  $N$  over  $F$ , and for any subfield  $M$  of  $N$ , let  $H_M$  denote the Galois group of  $N$  over  $M$ . We have a diagram of function fields, and a corresponding diagram of curves:



We start with proving that the equivalence of the first three conditions in Theorem 1.1 is preserved for global function fields. This is essentially because the proofs only depend on group theory (in particular, rational representation theory of a permutation representation), and general facts from algebraic number theory that persists in global fields. We nevertheless outline the proofs with references:

**Proposition 2.1.** *The following are equivalent:*

- (i) [Arithmetic equivalence] *For every irreducible polynomial  $p$  in  $F$  outside a set of Dirichlet density zero, the splitting type of  $p$  (i.e., the ordered sequence of inertia degrees with multiplicities) in  $K$  and  $L$  is the same;*
- (i') [Split equivalence] *For every irreducible polynomial  $p$  in  $F$  outside a set of Dirichlet density zero, the number of prime ideals in  $K$  and  $L$  above  $p$  is the same;*
- (ii) [Gaßmann equivalence] *Every conjugacy class in  $G$  meets  $H_K$  and  $H_L$  in the same number of elements.*

**Proof.** We note that (i) implies (i'), since the latter is weaker (only talking about the length of the splitting type).

The implication (i')  $\Rightarrow$  (ii) is proven by Perlis and Stuart in [27] for number fields, and we outline the (similar) argument for function fields. The point is that the proof uses only general features of algebraic number theory that hold over global fields, and the theory of rational representations. Let  $S$  denote the set of Dirichlet density zero from the statement, enlarged by the finitely many ramified primes in  $N/F$ . For any  $g \in G$ , let  $C$  denote the cyclic subgroup generated by  $g$ . By function field Chebotarev (e.g., [9, Theorem 6.3.1]), the irreducible polynomials  $p$  that have a prime factor in  $N$  with decomposition group  $C$  (i.e., whose Frobenius automorphism belongs to  $C$ ), form a set  $A$  of (positive)

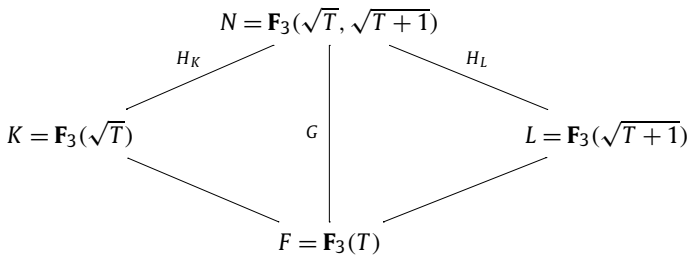
Dirichlet density  $|C|/|G|$ . Since  $S$  had Dirichlet density zero, so does  $S \cap A$ , and hence  $A - (S \cap A)$  have positive density. This means we can find an irreducible polynomial  $p$  outside  $S$  having decomposition group  $C$  in  $N$ . The splitting number of  $p$  in  $K$  is the number of double cosets  $|C \backslash G/H_K|$  and similarly for  $L$  (the number field statement is due to Hasse, it is reproven as Lemma 1 in [27], and this proof works *mutatis mutandis* for function fields). Let  $\rho_K = \text{Ind}_{H_K}^N 1$  and  $\rho_L = \text{Ind}_{H_L}^N 1$ . Now  $|C \backslash G/H_K|$  is the dimension of the  $\rho_K(C)$ -invariants of the  $\mathbf{Q}[G]$ -module  $\mathbf{Q}[G/H_K]$ , and similarly for  $L$  (cf. Lemma 2 in [27]). The hypothesis hence implies that  $\rho(C)$ -invariant subspaces of  $\mathbf{Q}[G/H_K]$  and  $\mathbf{Q}[G/H_L]$  have the same dimensions for all cyclic subgroups  $C \subseteq N$ . By [25, corollary to Theorem 30], this implies that the representations  $\rho_K$  and  $\rho_L$  are isomorphic, and this is the same as Gaßmann equivalence.

For (ii)  $\Rightarrow$  (i), Gaßmann has proved the purely group theoretical fact that if  $H_K$  and  $H_L$  are Gaßmann equivalent, then for every cyclic subgroup  $C$  of  $G$ , the coset types of  $(G, H_K, C)$  and  $(G, H_L, C)$  coincide. Here, the coset type of  $(G, H, C)$  is the list of integers  $(f_1, \dots, f_r)$  (in increasing order) defined by  $f_i := |H\tau_i C|/|H|$  if  $G = \bigcup H\tau_i C$ . It remains to show the number theoretical fact that the coset type of  $(G, H_K, C)$  is the splitting type of  $p$ , if  $C$  is a (cyclic) decomposition group of (an unramified)  $p$ . But this follows from the cited theorem of Hasse (see [27, Lemma 1, part (b)] for the case of number fields, but the proof is the same for function fields). Now one can leave out a set of Dirichlet density zero, as this only weakens the statement.  $\square$

### 3. Equivalences for the Weil zeta function

Let us first present an example that explicitly shows the failure of the analogue of the equivalence of (iii) to the other parts of Theorem 1.1 for function fields and the Weil zeta function:

**Example 3.1.** We show an explicit example where two function fields are not Gaßmann equivalent, but do have the same Weil zeta functions (see diagram below): Let  $F = \mathbf{F}_3(T)$ ,  $K = \mathbf{F}_3(\sqrt{T})$  and  $L = \mathbf{F}_3(\sqrt{T+1})$ . Then  $H_K = H_L = \mathbf{Z}/2$  and  $\text{Gal}(N/F) = H_K \times H_L$ . If we pick  $c \neq 1$  in  $H_K$ , then its conjugacy class intersects  $H_K$  in  $\{c\}$  and doesn't intersect  $H_L$ , so  $K$  and  $L$  are not Gaßmann equivalent, but since  $K$  and  $L$  are isomorphic by the map sending the element  $T$  to  $T + 1$ , their zeta functions are equal.



We now present an easy unified proof of the following statement for global fields (cf. also [21, VI.3.3]):

**Proposition 3.2.** *If  $K$  and  $L$  are Gaßmann equivalent global fields over some subfield  $F$ , then they have the same Weil zeta function.*

**Proof.**  $K$  and  $L$  are Gaßmann equivalent if and only if the representations  $\text{Ind}_{H_K}^G 1$  and  $\text{Ind}_{H_L}^G 1$  are isomorphic (e.g., [27, pp. 301–302]). By Kani and Rosen [20, p. 232, property (d)], we have an expression for the Weil zeta function (in the notation of Serre [24] for general zeta functions of schemes over  $\mathbf{Z}$ ):

$$\zeta_K(s) = \zeta(N^{H_K}, s) = L(Z, \text{Ind}_{H_K}^G 1; s) = L(Z, \text{Ind}_{H_L}^G 1; s) = \zeta(N^{H_L}, s) = \zeta_L(s). \quad \square$$

#### 4. Equivalences for the Goss zeta function

An elementary explanation of the failure of the Weil zeta function to detect arithmetic equivalence is that it provides a counting function only for the number of ideals of a given *degree* in a function field. Now David Goss in [13] (cf. [14]) has introduced a completely different zeta function for function fields, which is based on a norm that does not take values in  $\mathbf{Z}$  (or rather  $q^{\mathbf{Z}}$ ), but in the ring of integers of the ground function field  $F$  itself. So this has some more hope of capturing the splitting type of primes, and investigating this situation is the aim of this section.

Fix the place  $\infty = T^{-1}$  of  $F$  of degree one, let  $A = \mathbf{F}_q[T]$  denote the corresponding ring of integers in  $F$ , and let  $\mathcal{O} := \mathcal{O}_K$  denote the integral closure of  $A$  in  $K$  (and similarly for  $L$ ). Let  $N$  denote the “relative norm map” from ideals of  $\mathcal{O}$  to ideals of  $A$  defined as follows: If  $\mathfrak{P}$  is a prime ideal of  $\mathcal{O}$  lying above the monic irreducible polynomial  $p \in A$  given by  $(p) = \mathfrak{P} \cap A$ , then we set  $N(\mathfrak{P}) := p^{\deg(\mathfrak{P})/\deg(p)}$ , and extend this multiplicatively to all ideals of  $\mathcal{O}$ . Now Goss’ zeta function is

$$\zeta_K^{[p]}(s) := \sum_I \frac{1}{N(I)^s}.$$

This is defined for  $s$  in a certain “plane”  $S_\infty = \mathbf{C}_F^* \times \mathbf{Z}_p$ , where  $\mathbf{C}_F$  is the completion of an algebraic closure of the completion  $F_\infty$  of  $F$  w.r.t.  $T^{-1}$ , and  $\mathbf{Z}_p$  is the ring of  $p$ -adic integers. The Goss zeta function takes values in  $\mathbf{C}_F$ .

**Remark 4.1.** Note that strictly speaking, the Goss zeta function, and even its domain  $S_\infty$ , depend on the choice of the place  $\infty$  and the subfield  $F \subseteq K$ , hence on the ring of integers  $\mathcal{O}$ , but we will not indicate this in the notation.

For now, it is not important to know how to define the power  $N(I)^s$  for such general  $s$ , it suffices for what follows to define it for  $s \in \mathbf{Z}_{>0}$ . Such functions were considered earlier by Carlitz (e.g., [5]) and Goss [12]. One can actually show the following:

**Proposition 4.2.** (See Goss [14, p. 260].) *A general function field Dirichlet series  $D(s) := \sum_I c(I)N(I)^{-s}$  for some  $c(I) \in F_\infty$ , with nontrivial half-plane of convergence in  $s$ , is uniquely determined by its values  $D(j)$  at sufficiently large integers  $j$ .*

This proposition, of which the analogue for general complex valued Dirichlet series is well known [18, Theorem 6], implies that in our situation, we have:

**Corollary 4.3.** *For both number fields and function fields, and for Dirichlet, respectively for both the Weil and Goss zeta functions, we have that the condition*

(iii)  $\zeta_K(s) = \zeta_L(s)$  for all  $s$

is equivalent to the condition

(iii')  $\zeta_K(j) = \zeta_L(j)$  for all sufficiently large integers  $j$ .

**Theorem 4.4.** *If  $[L : F] < p$  and  $[K : F] < p$ , then equality of the Goss zeta functions of  $K$  and  $L$  is the same as arithmetic equivalence of  $K$  and  $L$ .*

**Proof.** We observe that the Goss zeta function can be written as

$$\zeta_K^{[p]}(s) = \sum_{n \in A} \frac{B(n)}{n^s}, \tag{1}$$

where we sum over monic  $n \in A$ , and  $B(n)$  is the number of ideals of  $\mathcal{O}_K$  of norm  $n \in A$ , but viewed modulo  $p$ , since the value is in  $F_\infty$ . For an integer  $f$  and an irreducible element  $\mathfrak{p}$  of  $A$ , we then know the integer  $C(\mathfrak{p}^f)$  modulo  $p$ , where  $C(\mathfrak{p}^f)$  is the number of prime ideals of  $\mathcal{O}_K$  with norm  $\mathfrak{p}^f$ . Indeed, doing the combinatorics (e.g., Klingen [21, proof of Theorem 2.1 on p. 16]), we find:

$$B(\mathfrak{p}^f) = \sum_{\substack{r_1 f_1 + \dots + r_s f_s = f \\ 1 \leq f_1 < \dots < f_s \\ 1 \leq r_1, \dots, r_s}} \prod_{i=1}^s \binom{C(\mathfrak{p}^{f_i}) + r_i - 1}{r_i} \tag{2}$$

$$= C(\mathfrak{p}^f) + [\text{function of } C(\mathfrak{p}^{f'}) \text{ for } f' < f]. \tag{3}$$

For  $f = 1$ , we find from the first line of this formula that  $C(\mathfrak{p}) = B(\mathfrak{p})$ , and then we can use the second line and complete induction w.r.t.  $f$  to compute all  $C(\mathfrak{p}^f)$  as a function of  $B(n)$  for various  $n$ .

Now since  $C(\mathfrak{p}^f) < [K : F] < p$ , we know the integer  $C(\mathfrak{p}^f)$  on the nose. Now knowing all integers  $C(\mathfrak{p}^f)$  implies knowing the splitting types of all  $\mathfrak{p} \in A$  (namely, for  $1 \leq f \leq [K : F]$ , put  $f$  in the splitting type of  $\mathfrak{p}$  exactly  $C(\mathfrak{p}^f)$  times). Hence equality of Goss zeta functions implies equality of splitting types.  $\square$

**Remark 4.5.** The formula in [23, p. 346], expressing the corresponding  $C$ 's in terms of the  $B$ 's should be replaced by (2), but this does not affect the argument in [23], since it only uses that the  $C$ 's are polynomially expressible in terms of the  $B$ 's.

In general, if the degree condition in the above theorem is not met, then the conclusion is false. For example, we have:

**Proposition 4.6.** *Suppose that  $K$  is a Galois extension of  $F$  with Galois group  $G$  a  $p$ -group, only ramified above  $\infty$ . Then the Goss zeta function of  $K$  satisfies*

$$\zeta_K^{[p]}(s) = \zeta_F^{[p]}(|G|s).$$

*In particular, equality of Goss zeta functions does not imply arithmetic equivalence in general, not even in the case of Galois extensions.*

**Proof.** This is proven in [14, Remark 8.18.7.2] using Artin factorisation, but we present an elementary computation. The set of ideals of  $\mathcal{O}_K$  of norm  $\mathfrak{p}^f$  admits an action of  $G$ . The number of elements in the orbit of a given ideal  $I$  is congruent to 0 or 1 mod  $p$ , depending on whether the action of  $G$  on  $I$  is nontrivial or trivial, respectively. Since the action of  $G$  permutes the prime ideals above  $\mathfrak{p}$  transitively, if  $I = \mathfrak{P}_1^{a_1} \dots \mathfrak{P}_s^{a_s}$  for some prime ideals  $\mathfrak{P}_i$  and some integers  $a_i$ , the action of  $G$  on  $I$  is only trivial if  $I = (\mathfrak{P}_1 \dots \mathfrak{P}_s)^r = (\mathfrak{p} \mathcal{O}_K)^r$  for some integer  $r$ . But then  $N(I) = \mathfrak{p}^{s|G|}$  so we conclude that

$$B(\mathfrak{p}^f) = \begin{cases} 0 \pmod p & \text{if } |G| \nmid f, \\ 1 \pmod p & \text{if } |G| \mid f. \end{cases}$$

Now  $B$  is multiplicative on coprime ideals, so we can compute

$$\begin{aligned} \zeta_L^{[p]}(s) &= \sum_{n \in A - \{0\}} \frac{B(n)}{n^s} = \prod_{\mathfrak{p}} \sum_{k=0}^{\infty} \frac{B(\mathfrak{p}^k)}{\mathfrak{p}^{ks}} = \prod_{\mathfrak{p}} \sum_{k=0}^{\infty} \frac{1}{\mathfrak{p}^{|G|ks}} \\ &= \prod_{\mathfrak{p}} \frac{1}{1 - \mathfrak{p}^{-|G|s}} = \sum_n \frac{1}{n^{|G|s}} = \zeta_F^{[p]}(|G|s). \end{aligned}$$

For the last claim, just choose two  $p$ -extensions ramified above  $\infty$  with different Galois groups of the same order, e.g.  $\mathbf{Z}/p \times \mathbf{Z}/p$  versus  $\mathbf{Z}/p^2$  given as  $X^{p^2} - X = T$  and the corresponding equation in truncated Witt vectors, respectively.  $\square$

**Remark 4.7.** The above proposition is really about the zeta functions *without* the infinite (here, ramified) factors, as should also become clear from the proof. It is easy to include the correct factors at  $\infty$  depending on the ramification, cf. also Section 8.10 in [14].

**Remark 4.8.** In the above example, the degree of  $K/F$  is divisible by  $p$ . It would be interesting to investigate whether the conclusion of Theorem 4.4 still holds if, for example, the degrees of  $K/F$  and  $L/F$  are merely coprime to  $p$ .

**Corollary 4.9.** *There exist choices of  $K, F$  and  $\infty$  whose Goss zeta function does not determine the Weil zeta function of  $K$ .*

**Proof.** If we construct two  $p$ -group extensions of  $F$  only ramified above  $\infty$ , with different genera but the same degree, we are done, since the Weil zeta function as rational function in  $q^{-s}$  has a numerator of degree twice the genus of the extension. This is clearly possible, for example by varying the Hasse conductor (e.g., the cyclic  $p$ -extension  $X^p - X = T^m$  has genus  $\frac{1}{2}((mp - p - m + 1)$ , varying with  $m$ ).  $\square$

**Remark 4.10.** Recall that the Weil zeta function is absolute, i.e., only depends on  $K$ , whereas our Goss zeta function depends on a rational function field  $F$  inside  $K$  and a place  $\infty$  of  $F$ . It would be interesting to know whether the Weil zeta function of  $K$  is determined by knowing *all* Goss zeta functions for different choices of  $F$  and  $\infty$ .

**Remark 4.11.** If we denote an exponent  $s$  in the full Goss plane  $S_\infty = \mathbf{C}_F^* \times \mathbf{Z}_p$  as  $s = (x, y)$  with  $x \in \mathbf{C}_F^*$  and  $y \in \mathbf{Z}_p$ , then the definition of  $s$ -exponentiation implies that

$$N(I)^{(x,0)} = x^{\deg(N(I))} = x^{\deg((I \cap A)^{\deg(I)/\deg(I \cap A)})} = x^{\deg(I)}.$$

Note that although  $N(I)$  is a *relative* norm, this expression depends only on the *absolute* degree of  $I$ . Using this expression, we can write

$$\zeta_K^{[p]}((x, 0)) = \sum_I x^{-\deg(I)}$$

as a function of  $x \in \mathbf{C}_F^*$ , which is now an absolute function (not depending on the choice of  $F$ ), and we can find a link with the (absolute) Weil zeta function. Namely, the Weil zeta function is a function of  $u = q^{-s}$ :

$$\zeta_K(s) = Z_K(u) = \sum_I u^{\deg(I)}.$$

So if we let the formal variable  $u$  take values in  $\mathbf{C}_F^*$ , too, we find an identity

$$\zeta_K^{[p]}((x, 0)) = Z_K(x^{-1}) \pmod{p},$$

so the Goss zeta function encodes the Weil zeta function “modulo  $p$ ” (we thank David Goss for this observation).

In the above reasoning, one may or may not include infinite Euler factors.



Since the numerator of  $Z_K(u)$  (with infinite factors) is the characteristic polynomial of Frobenius, this implies that the Goss zeta function encodes this polynomial mod  $p$ . In particular, for an elliptic curve, the Goss zeta function does determine the Weil zeta function, since both are determined by the trace of Frobenius, which is bounded by  $\sqrt{p}$  in absolute value. However, in higher genus, there are curves for which not even the trace of Frobenius is determined by its value mod  $p$ . Also observe in connection with Corollary 4.9 that the Weil zeta function modulo  $p$  doesn't even determine the genus of the function field; an elliptic curve for which all traces of Frobenius are zero has the same Weil zeta function modulo  $p$  as the rational function field, for example,  $X^3 - X = T^2$  and  $X^3 - X = T$  for  $p = 3$  have the numerator of their Weil zeta functions equal to, respectively,  $3u^2 + 1$  and  $1$ , which are congruent modulo 3.

**5. The lifted Goss zeta function, and equivalences**

The problem with the Goss zeta function is that the coefficients are in a field of characteristic  $p$ . Recall that  $F_\infty = \mathbf{F}_q((T^{-1}))$  is the  $T^{-1}$ -adic completion of  $F$ , and  $\mathbf{C}_F$  is the completion of an algebraic closure of  $F_\infty$ . Let us denote by  $W = W(\mathbf{C}_F)$  the ring of Witt vectors of  $\mathbf{C}_F$ . Let  $\mathfrak{m}$  denote its maximal ideal, and  $\pi$  a fixed uniformizer. Now  $W$  is a ring of characteristic zero and there is a natural injection  $\mathbf{Z} \rightarrow W$ . Let  $\chi : \mathbf{C}_F \rightarrow W$  denote the Teichmüller character, sending  $\mathbf{C}_F \ni x \mapsto \chi(x) = (x, 0, \dots)$ . Recall that  $\chi$  induces a group homomorphism on multiplicative groups. Since  $\mathbf{C}_F$  is perfect, all these constructions make sense and are canonical.

For any  $s \in S_\infty = \mathbf{C}_F^* \times \mathbf{Z}_p$  (the Goss complex plane), we define the *lifted Goss zeta function* as

$$\zeta_K^{[0]}(s) = \sum_I \frac{1}{\chi(N(I)^s)}. \tag{4}$$

Here, the sum is again over ideals of  $\mathcal{O}_K$ , and the norm  $N$  is the relative norm from  $\mathcal{O}_K$  to  $A$ . Note that the multiplicativity of  $\chi$  implies that

$$\zeta_K^{[0]}(s) = \sum_I \frac{1}{\chi(N(I)^s)} = \sum_I \chi\left(\frac{1}{N(I)^s}\right).$$

We now discuss a few properties of this function. If  $s$  is an integer, we also have

$$\zeta_K^{[0]}(s) = \sum_I \frac{1}{(\chi(N(I)))^s}. \tag{5}$$

Next, we note that

$$\zeta_K^{[p]}(s) \equiv \zeta_K^{[0]}(s) \pmod{\mathfrak{m}}, \tag{6}$$

so that the lifted Goss zeta function contains all information that the Goss zeta function contains. Also, since  $\chi \circ N$  from ideals of  $\mathcal{O}_K$  to  $A$  is multiplicative, the lifted Goss zeta function has an Euler product

$$\zeta_K^{[0]}(s) = \prod_{\mathfrak{P}} \frac{1}{1 - \chi(N(\mathfrak{P}))^{-s}}.$$

Let  $\mathbf{Z} \ni B(n) = \#\{I : N(I) = n\}$  denote the number of ideals of  $\mathcal{O}_K$  of norm  $n$ , and let

$$B(n) = \sum_{\nu=0}^{\infty} B_\nu(n) \pi^\nu$$

be the  $\pi$ -adic expansion of  $B(n)$ . Then we can write a ‘ $\pi$ -adic expansion’ of the lifted Goss zeta function as follows

$$\zeta_K^{[0]}(s) = \sum_{n \in A} \frac{B(n)}{\chi(n)^s} = \sum_{\nu=0}^{\infty} \pi^\nu \zeta_K^{[p^\nu]}(s), \quad \text{where } \zeta_K^{[p^\nu]}(s) := \sum_{n \in A} \frac{B_\nu(n)}{n^s} \in K_\infty, \quad (7)$$

again summing over monic  $n \in A$ .

**Remark 5.1.** There is no problem with the convergence of this lifted Goss zeta function since it is a Witt vector of convergent sequences.

**Remark 5.2.** At the time of writing, it is not so clear what the correct formulation of a “Riemann hypothesis” for the Goss zeta function  $\zeta_K^{[p]}(s)$  is. A discussion of this question can be followed through [15–17,29,11]; it involves the hugely nonclassical theory of nontrivial zeros and functional equation(s) (note: for rational function fields, there is a clear statement, proven in [31,26]). Whatever the contents of this mysterious “Riemann hypothesis” is, it is clear from (6) that  $\zeta_K^{[0]}(s) = 0$  implies  $\zeta_K^{[p]}(s) = 0$ . Hence if the Riemann hypothesis for  $\zeta_K^{[p]}$  states that its zeros  $s \in S_\infty$  belong to some “special subset”  $Z \subseteq S_\infty$ , then the *same* statement immediately holds for the lifted Goss zeta function  $\zeta_K^{[0]}(s)$ . But zeros of  $\zeta_K^{[p]}$  need not extend to zeros of  $\zeta_K^{[0]}$ , i.e.,  $\zeta_K^{[0]}$  can have less zeros than  $\zeta_K^{[p]}$ . Actually, a zero of  $\zeta_K^{[0]}$  is a *simultaneous* zero of all  $\zeta_K^{[p^\nu]}$  for all  $\nu$  (cf. Eq. (7)). Maybe this possibly smaller set of zeros of the lifted Goss zeta function is easier to understand and less susceptible to influences of the trivial zeros than its characteristic  $p$  counterpart? We believe this point deserves more attention, especially in connection with simultaneously ‘lifting’ the theory of Drinfeld modular forms and associated Galois representations to characteristic zero.

**Theorem 5.3.** Let  $K$  and  $L$  be arbitrary geometrical extensions of  $F = \mathbf{F}_q(T)$ . The condition  $\zeta_K^{[0]}(s) = \zeta_L^{[0]}(s)$  is the same as arithmetical equivalence of  $K$  and  $L$ .

**Proof.** Since

$$\zeta_K^{[0]}(s) = \sum_{n \in A} \frac{B(n)}{\chi(n)^s},$$

knowledge of the lifted Goss zeta function implies knowledge of the *integers* (not just mod  $p$ )  $B(n)$ . By formula (2), this implies knowledge of the splitting type of all (unramified) prime ideals of  $K$ .  $\square$

**Corollary 5.4.** Knowledge of the lifted Goss zeta function implies knowledge of the Weil zeta function.

**Proof.** The previous theorem says that if two function fields have the same lifted Goss zeta function (for some choice of  $F$  and  $\infty$ ), then they are arithmetically equivalent over that field  $F$ . The result then follows from Proposition 3.2.  $\square$

**Remark 5.5.** Along the lines of Remark 4.11, there is the following direct relation between the lifted Goss zeta function and the Weil zeta function:

$$Z_K(u) = \zeta_K^{[0]}((\chi^{-1}(u), 0)),$$

where  $u \in W$ , and this formula gives an alternative proof of the previous corollary, and of Proposition 3.2.

**Remark 5.6.** The Weil zeta function is absolute, but the lifted Goss zeta function depends on the choice of  $F$  and  $\infty$ . Nevertheless, the above corollary shows that knowing the Goss zeta function for one of these relative choices suffices to determine the Weil zeta function.

**Remark 5.7.** The formalism of Artin of factorisation of zeta functions of a Galois extension according to characters of the Galois group applies to the lifted Goss zeta function (the situation is unclear for the (unlifted) Goss zeta function). Since we have chosen  $\chi$  to be the multiplicative Teichmüller lift,  $\chi(N(I)^{-s})$  is multiplicative in  $I$ , hence  $\zeta_K^{[0]}$  admits an Euler product. For  $K/F$  Galois with Galois group  $G$ , let  $(V, \rho_G)$  denote the regular representation of  $G$  over an algebraic closure of the fraction field of  $W$ , then we have

$$\zeta_K^{[0]}(s) = \prod_{\mathfrak{P}} \det(1 - \rho(\text{Frob}_{\mathfrak{P}})\chi(N(\mathfrak{P})^{-s}) \mid V^{I_{\mathfrak{P}}}),$$

product over prime ideals  $\mathfrak{P}$  of  $\mathcal{O}_K$ . Let  $L/K$  denote a Galois extension with Galois group  $G_{L/K}$ , then

$$\zeta_L^{[0]}(s) = \prod_{\rho} L_K^{[0]}(\rho, s)^{\chi_{\rho}(1)},$$

for  $\rho$  running through the irreducible representations of  $G_{L/K}$ , and where the lifted Goss  $L$ -series  $L_K^{[0]}(\rho, s)$  is defined as

$$L_K^{[0]}(\rho, s) = \prod_{\mathfrak{P}} \det(1 - \rho(\text{Frob}_{\mathfrak{P}})\chi(N(\mathfrak{P})^{-s}) \mid V^{I_{\mathfrak{P}}}).$$

An application of this formalism is a conceptual proof of Proposition 4.6, as sketched in [14, p. 296].

Finally, we prove

**Proposition 5.8.**

- (i) Let  $F = \mathbf{F}_q(T)$  for any prime power  $q$ . There exist two geometric extensions  $K$  and  $L$  over  $F$  such that  $\zeta_K^{[0]}(s) = \zeta_L^{[0]}(s)$  (so  $K$  and  $L$  are arithmetically equivalent), but  $K$  and  $L$  are not isomorphic as field extensions of  $F$ .
- (ii) Let  $F = \mathbf{F}_q(T)$  with  $q$  a square if  $p \equiv 2 \pmod{3}$ . There exist two geometric extensions  $K$  and  $L$  of degree 7 over  $F$  such that  $\zeta_K^{[0]}(s) = \zeta_L^{[0]}(s)$  (so  $K$  and  $L$  are arithmetically equivalent), but  $K$  and  $L$  are not isomorphic as field extensions of  $F$ . A counterexample in smaller degree does not exist.

**Proof.** For (i), we pick up the construction of Komatsu [22]: Let  $\ell > 2$  denote a prime number, let  $H_1 = (\mathbf{Z}/\ell)^3$  and

$$H_2 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}(3, \mathbf{F}_{\ell}) \right\}.$$

Then  $H_1$  and  $H_2$  are Gaßmann equivalent as subgroups (via the Cayley map) of the symmetric group  $S_{\ell^3}$ . Note that  $H_2$  is not commutative for  $\ell > 2$ , so in particular, not isomorphic to  $H_1$ .

Now  $S_{\ell^3}$  is realized over the field  $\mathbf{F}_q(x_1, \dots, x_{\ell^3})$  as Galois group of the general polynomial, and since  $\mathbf{F}_q(x_1)$  is Hilbertian, Hilbert’s irreducibility (cf. [9, Proposition 16.1.5]) implies that there exists a specialisation of  $x_2, \dots, x_{\ell^3}$  to  $\mathbf{F}_q(x_1)$  that gives a (geometrical)  $S_{\ell^3}$  Galois extension of  $\mathbf{F}_q(T)$  (setting  $T = x_1$ ).

The extensions  $K$  and  $L$  corresponding to this construction are huge, of degree  $(\ell^3 - 1)!$  over  $F$  (a 27-digit number for  $\ell = 3$ ). To construct an example of much smaller degree, we rely on modular curves, but at the cost of assuming  $q$  to be a square:

We prove (ii) as follows. The argument in [23, Theorem 3] is purely group theoretical, and shows that there are no Gaßmann equivalent subgroups in  $S_n$  for  $n \leq 6$ , hence also no nontrivially Gaßmann equivalent function fields over  $F$  of degree  $\leq 6$ .

We know from [23, p. 358] that  $G = \mathrm{PSL}(2, 7)$  has two Gaßmann equivalent subgroups of index 7, so it remains to realize  $G$  as Galois group over  $F$ .

For this, first assume  $p \neq 7$ . Then let  $N$  denote the function field of the Klein quartic over  $\mathbf{F}_p$ , a.k.a. the modular curve  $X(7)/\mathbb{Z}$  reduced modulo  $p$ . We know that  $\mathrm{Gal}(N\overline{\mathbf{F}}_q/F\overline{\mathbf{F}}_q)$  contains  $\mathrm{PSL}(2, 7)$  (actually, is equal to it unless  $p = 3$ , cf. [2,8], when it is a group of order 6048 instead of 168). However, the action of  $\mathrm{PSL}(2, 7)$  is only defined over  $\mathbf{F}_q(\zeta_3)$  with  $\zeta_3$  a third root of unity. Now if  $p \equiv 1 \pmod{3}$ , then  $-3$  is a square mod  $p$ , so  $\zeta_3 \in \mathbf{F}_q$ , and in the other case, we assume  $F \supseteq \mathbf{F}_{p^2}$ , so we are fine, too. We don't know whether  $K/F$  and  $L/F$  can be descended further to  $\mathbf{F}_p$  in general.

Now if  $p = 7$ , we use instead the result of Abhyankar [1] that the polynomial  $X^8 - TX + 1$  has Galois group over  $\overline{\mathbf{F}}_7 F$  equal to  $\mathrm{PSL}(2, 7)$ . One may compute directly, by hand or by Magma [3] that in fact the automorphism group is defined already over  $\mathbf{F}_7$ .  $\square$

## References

- [1] Shreeram S. Abhyankar, Galois theory on the line in nonzero characteristic, *Bull. Amer. Math. Soc. (N.S.)* 27 (1) (1992) 68–133, MR MR1118002 (94a:12004).
- [2] Peter Bending, Alan Camina, Robert Guralnick, Automorphisms of the modular curve, in: *Progress in Galois Theory*, in: *Dev. Math.*, vol. 12, Springer, New York, 2005, pp. 25–37, MR MR2148458 (2006c:14050).
- [3] Wieb Bosma, John Cannon, Catherine Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* 24 (3–4) (1997) 235–265, Computational algebra and number theory, London, 1993, MR MR1484478.
- [4] Jean-Benoît Bost, Alain Connes, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory, *Selecta Math. (N.S.)* 1 (3) (1995) 411–457, MR MR1366621 (96m:46112).
- [5] Leonard Carlitz, An analogue of the Staudt–Clausen theorem, *Duke Math. J.* 7 (1940) 62–67, MR MR0002995 (2,146e).
- [6] Caterina Consani, Matilde Marcolli, Quantum statistical mechanics over function fields, *J. Number Theory* 123 (2) (2007) 487–528, MR MR2301227 (2008d:11096).
- [7] Bart de Smit, On arithmetically equivalent fields with distinct  $p$ -class numbers, *J. Algebra* 272 (2) (2004) 417–424, MR MR2028064 (2005f:11252).
- [8] Noam D. Elkies, The Klein quartic in number theory, in: *The Eightfold Way*, in: *Math. Sci. Res. Inst. Publ.*, vol. 35, Cambridge Univ. Press, Cambridge, 1999, pp. 51–101, MR MR1722413 (2001a:11103).
- [9] Michael D. Fried, Moshe Jarden, *Field Arithmetic*, third ed., *Ergeb. Math. Grenzgeb. (3)*, vol. 11, Springer-Verlag, Berlin, 2008, Revised by Jarden, MR MR2445111.
- [10] Fritz Gaßmann, Bemerkungen zur Vorstehenden Arbeit von Hurwitz: Über Beziehungen zwischen den Primidealen eines algebraischen Körpers und den Substitutionen seiner Gruppen, *Math. Z.* 25 (1926) 661–675 (in German).
- [11] David Goss,  $\zeta$ -phenomenology, preprint, 2009.
- [12] David Goss, von Staudt for  $\mathbf{F}_q[T]$ , *Duke Math. J.* 45 (4) (1978) 885–910, MR MR518110 (80a:12019).
- [13] David Goss,  $v$ -adic zeta functions,  $L$ -series and measures for function fields, *Invent. Math.* 55 (2) (1979) 107–119. With an addendum, MR MR553704 (81c:12024).
- [14] David Goss, *Basic Structures of Function Field Arithmetic*, *Ergeb. Math. Grenzgeb. (3)*, vol. 35, Springer-Verlag, Berlin, 1996, MR MR1423131 (97i:11062).
- [15] David Goss, A Riemann hypothesis for characteristic  $p$   $L$ -functions, *J. Number Theory* 82 (2) (2000) 299–322, MR MR1761624 (2001k:11171).
- [16] David Goss, The impact of the infinite primes on the Riemann hypothesis for characteristic  $p$  valued  $L$ -series, in: *Algebra, Arithmetic and Geometry with Applications*, West Lafayette, IN, 2000, Springer, Berlin, 2004, pp. 357–380, MR MR2037101 (2005g:11165).
- [17] David Goss, Zeros of  $L$ -series in characteristic  $p$ , *Int. J. Appl. Math. Stat.* 11 (N07) (2007) 69–80, MR MR2374918 (2008j:11115).
- [18] Godfrey Harold Hardy, Marcel Riesz, *The General Theory of Dirichlet's Series*, *Cambridge Tracts in Math. Math. Phys.*, vol. 18, Stechert-Hafner, Inc., New York, 1964, MR MR0185094 (32 #2564).
- [19] Benoît Jacob, Bost–Connes type systems for function fields, *J. Noncommut. Geom.* 1 (2) (2007) 141–211, MR MR2308304 (2009a:11185).
- [20] Ernst Kani, Michael Rosen, Idempotent relations among arithmetic invariants attached to number fields and algebraic varieties, *J. Number Theory* 46 (2) (1994) 230–254, MR MR1269254 (95c:11080).
- [21] Norbert Klingens, *Arithmetical Similarities*, *Oxford Math. Monogr.*, Clarendon Press Oxford University Press, New York, 1998. Prime decomposition and finite group theory, *Oxford Science Publications*, MR MR1638821 (99k:11163).
- [22] Keiichi Komatsu, On the adèle rings of algebraic number fields, *Kōdai Math. Sem. Rep.* 28 (1) (1976) 78–84, MR MR0424760 (54 #12718).

- [23] Robert Perlis, On the equation  $\zeta_K(s) = \zeta_{K'}(s)$ , *J. Number Theory* 9 (3) (1977) 342–360, MR MR0447188 (56 #5503).
- [24] Jean-Pierre Serre, Zeta and  $L$  functions, in: *Arithmetical Algebraic Geometry, Proc. Conf. Purdue Univ.*, 1963, Harper & Row, New York, 1965, pp. 82–92, MR MR0194396 (33 #2606).
- [25] Jean-Pierre Serre, *Représentations linéaires des groupes finis*, revised ed., Hermann, Paris, 1978, MR MR543841 (80f:20001).
- [26] Jeffrey T. Sheats, The Riemann hypothesis for the Goss zeta function for  $\mathbf{F}_q[T]$ , *J. Number Theory* 71 (1) (1998) 121–157, MR MR1630979 (99b:11127).
- [27] Donna Stuart, Robert Perlis, A new characterization of arithmetic equivalence, *J. Number Theory* 53 (2) (1995) 300–308, MR MR1348765 (96e:11151).
- [28] John Tate, Endomorphisms of abelian varieties over finite fields, *Invent. Math.* 2 (1966) 134–144, MR MR0206004 (34 #5829).
- [29] Dinesh S. Thakur, On characteristic  $p$  zeta functions, *Compos. Math.* 99 (3) (1995) 231–247, MR MR1361740 (96m:11045).
- [30] Stuart Turner, Adele rings of global field of positive characteristic, *Bol. Soc. Brasil. Mat.* 9 (1) (1978) 89–95, MR MR516311 (80c:12017).
- [31] Daqing Wan, On the Riemann hypothesis for the characteristic  $p$  zeta function, *J. Number Theory* 58 (1) (1996) 196–212, MR MR1387735 (97c:11064).