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Biduality and reflexivity in positive characteristic

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ABSTRACT
The biduality and reflexivity theorems are known to hold for projective varieties defined over fields of characteristic zero, and to fail in positive characteristic. In this article we construct a notion of reflexivity and biduality in positive characteristic by generalizing the ordinary tangent space to the notion of h-tangent spaces. The ordinary reflexivity theory can be recovered as the special case $h = 0$, of our theory. Several varieties that are not ordinary reflexive or bidual become reflexive in our extended theory.

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1. Introduction

The aim of this article is to study the biduality theorem and the stronger notion of reflexivity of varieties in positive characteristic. Let k be an algebraically closed field of characteristic $p \geq 0$, it is a very old observation that points in the projective space $\mathbb{P}_k^n$ correspond to hyperplanes in the dual projective space and vice versa. This notion of duality can be generalized to closed irreducible varieties $M \subset \mathbb{P}_k^n$ and gives rise to a dual variety M^* in the dual projective space.

The biduality theorem (known to hold over fields of characteristic 0) asserts that $(M^*)^* = M$. One of the proofs of this fact, [9, p. 29] uses the notion of the conormal bundle, the natural symplectic structure on the cotangent bundle of a manifold. Wallace [29] was the first to consider the theory of projective duality over fields of positive characteristic. For a nice introduction to projective duality independent of the characteristic of the base field we refer to [22].

Let $M \subset \mathbb{P}_k^n$ be a projective variety and denote by M_{sm} the set of smooth points of M. The classical conormal variety $\mathrm{Con}(M)$ is defined by

$$\mathrm{Con}(M) := \overline{\{(P, H) \in M_{\mathrm{sm}} \times \mathbb{P}_k^{n*} : T_P M \subset H\}} \subset M \times \mathbb{P}_k^{n*} \subset \mathbb{P}_k^n \times \mathbb{P}_k^{n*},$$

i.e., the Zariski closure of the algebraic set consisted of pairs (P, H), $P \in M_{\mathrm{sm}}$, $H \in \mathbb{P}^{*n}$ such that $T_P M \subset H$.

Let π_2 be the second projection $\mathrm{Con}(M) \to \pi_2(\mathrm{Con}(M)) := M^* \subset \mathbb{P}^{*n}$, which will be called the conormal map. It is known that M^* is an algebraic variety of $\mathbb{P}^{*n}$. If $\mathrm{Con}(M) = \mathrm{Con}(M^*)$, then M is called reflexive. Equivalently, in terms of isomorphisms, M is reflexive if the natural isomorphism from $\mathbb{P}_k^n$ to $(\mathbb{P}_k^{n^*})^*$ induces the isomorphism

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$$\begin{tikzcd}
\mathbb{P}_k^{n^*} \times \left(\mathbb{P}_k^{n^*}\right)^* \arrow[r, equal] & \mathbb{P}_k^{n^*} \times \mathbb{P}_k^{n} \\
\mathrm{Con}(M^*) \arrow[u, hook] \arrow[r, leftrightarrow, "\cong"] & \mathrm{Con}(M) \arrow[u, hook]
\end{tikzcd}$$

It is known that reflexivity implies biduality, but there are examples known of bidual varieties that are not reflexive. Reflexivity also holds for all projective varieties in characteristic zero, while in characteristic $p>0$, reflexivity can fail, see the Fermat-curve example in [29]. In positive characteristic there is the following criterion for reflexivity, whose proof may be found in [14].

Theorem 1 (Monge-Segre-Wallace). *A projective variety M is reflexive if and only if the conormal map π_2 is separable.*

The problems of biduality and reflexivity of a projective variety $M \subset \mathbb{P}^n$ have been addressed by several authors via the use of the Gauss map, i.e., the rational map from M to the Grassmann variety $\mathbb{G}(n,m)$, which sends a smooth point $P \in M$ to the m-dimensional tangent space $T_PM \in \mathbb{P}^n$—in the case of a hypersurcace, the Gauss map is just a map $\gamma : M \to \mathbb{P}^{*n}$. As proved in [19], the separability of the Gauss map and the reflexivity of a variety are equivalent in the one-dimensional case, i.e. for projective curves. For higher dimensions, the authors in [20] showed that the Gauss map of a projective variety M is separable if M is reflexive. On the other hand, the converse of this result, i.e. whether the reflexivity of a projective variety implies the separability of the Gauss map, was answered recently negatively, since there are specific examples (such as the Segre varieties) for which this assumption is not true. These examples and further analysis is found in [7] and the references therein. The previous work and results are, to the best of our knowledge, the most recent with regard to the study of biduality and reflexivity and are focused on weather and when they fail or not, in positive characteristic.

The aim of this article is to extend the notions of biduality and reflexivity in the case of positive characteristic. We will make appropriate definitions which will make some important examples of varieties reflexive. We generalize the theory of Lagrange varieties presented in [9, p. 29] for projective varieties in the zero characteristic case, by introducing the respective h-cotagent bundle and h-Lagrangian subvarieties. The case of hypersurfaces is illuminating and straightforward calculations can be made in terms of the implicit-inverse function theorem approach of Wallace, see [29].

Reflexivity has many important applications to enumerative geometry, computations with discriminants and resultants, invariant theory, combinatorics, etc. We hope that our construction will find some similar applications to positive characteristic algebraic geometry.

From now on k is an algebraically closed field of positive characteristic p and $q=p^h$ is a power of p. Instead of tangent hyperplanes, we will consider generalized hyperplanes, i.e. hypersurfaces of the form $V(\sum_{i=0}^n a_i x_i^{p^h})$ and the duality will be expressed in terms of these generalized hyperplanes.

Let V be a finite dimensional vector space over k. Consider $M \subset \mathbb{P}(V)$ an irreducible projective variety and consider the cone $M' \subset V$ seen as an affine variety in V. Assume that the homogeneous ideal of M' is generated by the homogeneous polynomials $F_1, \ldots, F_r$. Fix a natural number h and consider the $n+1$-upple

$$\nabla^{(h)} F_i = \left(D_0^{(h)}\Big|_P F_i, D_1^{(h)}\Big|_P F_i, \ldots, D_n^{(h)}\Big|_P F_i \right),$$

where $D_i^{(h)}$ denotes the h-Hasse derivative which will be defined in definition 5. Each F_i defines a p^h-linear form given by

$$L_i^{(h)} := \sum_{\nu=0}^{n} \left(D_\nu^{(h)} \Big|_P F_i \right) x_\nu^{p^h}.$$

For the precise definition of p^h-linear forms and their space V^{*h} see Sec. 2.2.

For a projective variety M we will define the set M^h_{sm} of smooth h-points in definition 33, which set if non-empty is dense in M, since we have assumed that M is irreducible.

Definition 2. For a projective irreducible variety M with $M^h_{\mathrm{sm}} \neq \emptyset$ we define the h-tangent space $T_P^{(h)}M$ at P to be the variety defined by the equations $L_i^{(h)} = 0$. The h-conormal space $\mathrm{Con}^{(h)}(M)$ is defined as the subset of $\mathbb{P}(V) \times \mathbb{P}(V^{*h})$

$$\mathrm{Con}^{(h)}(M) := \overline{\left\{ (P, H) : P \in M^h_{\mathrm{sm}}, H \text{ is a } p^h - \text{linear form which vanishes on } T_P^{(h)}M \right\}}. \tag{1.1}$$

By definition of the affine cone M' of M we see that every point $P = [a_0 : \cdots : a_n]$ corresponds to a line $\{\ell \cdot (a_0, ..., a_n), \ell \in k^*\} \subset M$. Following [27] we will denote by $\mathrm{Lag}^{(\mathrm{h})}(M) \subset V \times V^{*h}$ the corresponding affine set in $V \times V^{*h}$, that is

$$\mathrm{Lag}^{(h)}(M) = \overline{\left\{ (P, H) : P \in \mathrm{Cone}(M^h_{\mathrm{sm}}), H \text{ is a } p^h - \text{linear form which vanishes on } T_P^{(h)}M \right\}}.$$

Let $\pi_1 : V \times V^{*h} \to V$ be the first projection. For every $P \in M^h_{\mathrm{sm}}$ the set $\pi_1^{-1} \cap \mathrm{Lag}^{(h)}(M)$ can be identified to the space of p^h-linear forms on the h-normal space $N_P^{(h)}(M)$ defined as

$$N_P^{(h)}(M) = V^{*h}/T_P^{(h)}(M).$$

Indeed, the set of h-linear maps $\phi : N_P^{(h)}(M) \to k$ is in bijection with the set of h-linear maps $V^{*h} \to k$ which are zero on $T_P^{(h)}(M)$ and the later set is by definition $\pi_1^{-1} \cap \mathrm{Lag}^{(h)}(M)$.

If the variety is not reflexive, we might choose an appropriate h so that we can have a form of reflexivity based on $\mathrm{Con}^{(h)}(M)$. How are we going to select h? If the characteristic of the base field k is zero or if the variety M is reflexive, then $h = 0$. If the variety M is just a hypersurface then the answer is simple: If M fails to be reflexive then the second projection $\mathrm{Con}(M) \to M^*$ is a map of inseparable degree p^h, and in this way we obtain the required h.

Even in the case of hypersurfaces one has to be careful. Projective duality depends on Euler's theorem on homogeneous polynomials, since a homogeneous polynomial can be reconstructed by the values of all first order derivatives. An appropriate generalization of Euler's theorem is known, but we have to restrict ourselves to a class of polynomials which we will call h-homogeneous. Their precise definition will be given in definition 19.

Definition 3. For an irreducible projective variety $M \in \mathbb{P}(V)$ generated by h-homogeneous elements, which also has a non-empty h-nonsingular locus, as these are defined in Definition 19 and Definition 33, respectively, we can define $\mathrm{Lag}^{(h)}(M)$. Let $Z = \pi_2(\mathrm{Lag}^{(h)}(M)) \subset V^{*h}$, where $\pi_2 : V \times V^{*h} \to V^{*h}$ is the second projection.

Consider the vector space V^{*h} of p^h-linear forms. Assume also that the Z is defined by homogeneous polynomials and that the corresponding projective variety $Z \subset \mathbb{P}(V^{*h})$ has also an non-empty h-nonsingular locus so $\mathrm{Lag}^{(h)}(Z)$ can also be defined.

In Theorem 12 we will introduce the isomorphism $F : V \to (V^{*h})^{*h}$. If the map

$$\begin{aligned} \Psi : V \times V^{*h} &\to V^{*h} \times (V^{*h})^{*h} \\ (x, y) &\mapsto (y, F(x)), \end{aligned}$$

gives rise to an isomorphism

$$\Psi(\mathrm{Lag}^{(h)}(M)) = \mathrm{Lag}^{(h)}(Z) \subset V^{*h} \times (V^{*h})^{*h} = V^{*h} \times V.$$

then M will be called *h-reflexive*.

The main result of our work is the following theorem:

Theorem 4. *Let $M \in \mathbb{P}(V)$ be a projective variety satisfying the assumptions of definition 3. Assume that we can select an h so that the map $\pi_2 : V \times V^{*h} \supset \mathrm{Lag}^{(h)}(M) \to \pi_2(M) := Z \subset V^{*h}$ is separable and generically smooth and also that Z has a non-empty set of h-nonsingular points. Then M is h-reflexive.*

Notice also that in contrast to ordinary situation where the set of nonsingular points forms a dense open subset, for $h > 0$ the set of h-nonsingular points can be empty. The existence of a non-empty set of h-nonsingular points is essential for the definition of the conormal space and has to be assumed.

The explicit construction of the dual variety involves a projection map which can be computed using elimination theory, see [6, ex. 14.8, p. 315]. The algebraic set $M \subset \mathbb{P}^n$, gives rise to the conormal scheme $\mathrm{Con^{(h)}(M)} \subset \mathbb{P}^n \times (\mathbb{P}^n)^{h*}$. If $k[\xi_0, \ldots, \xi_n]$ is the polynomial ring corresponding to the dual projective space and

$$I = \langle f_1, \ldots, f_r \rangle \lhd k[x_0, \ldots, x_n],$$

the ideal corresponding to M, then the ideal $I' \lhd k[x_0, \ldots, x_n, \xi_0, \ldots, \xi_n]$ corresponding to the conormal scheme is generated by $I \cdot k[x_0, \ldots, x_n, \xi_0, \ldots, \xi_n]$ and the equations

$$\sum_{i=0}^{r} \sum_{j=0}^{n} \lambda_i D_{x_i}^{(h)} f_i \cdot \xi_i^{p^h} = 0, \ \ \lambda_i \in k.$$

The dual variety can be computed by eliminating the variables $x_0, \ldots, x_n, \lambda_1, \ldots, \lambda_r$ and by obtaining a homogeneous ideal in $k[\xi_0, \ldots, \xi_n]$. Notice that there are powerful algorithms for performing elimination using the theory of Gröbner bases, see example 39.

The structure of the article is as follows: In Sec. 2.1 we define and describe a number of important tools, notions and results, we are going to use throughout the paper. First we start with the family of Hasse derivatives, which will be seen as derivatives with respect to some new ghost variables $x_i^{(q^h)}$. These derivatives were first introduced by Hasse and Schmidt [12, 24] in order to study Weierstrass points in positive characteristic. Afterwards, we define the so-called p^h-linear forms and their respective space. In the same section we define the q-symplectic form we are going to use in the last section, in order to create a suitable Lagrangian variety for our work. In the same section we generalize the Euler identity for homogeneous polynomials and obtain the h-homogeneous polynomial definition. In Sec. 3 we present the implicit-inverse function theorem approach of our theory, we make connections with elimination theory, and treat the hypersurface case. In the last section, we generalize all the respective notions met in Lagrangian manifold theory for biduality in characteristic zero, [9, p. 29] and using them we prove Theorem 4.

2. Tools and basic constructions

The main idea behind our approach, assuming that k has characteristic $p > 0$, is to set the quantity $x_i^{p^h}$ as a new variable $x_i^{(h)}$, for $h = 0, 1, 2, \ldots$. As it is well known, the classical partial derivatives D_{x_i} on the polynomial ring $k[x_0, \ldots, x_r]$ are zero on the polynomials of the form $f(x_0^p, \ldots, x_r^p)$, and this is the reason biduality and reflexivity fail in positive characteristic. The theory of Hasse derivatives will help us deal with this.

2.1. Hasse derivatives

Definition 5. A Hasse family of differential operators on a commutative unital k-algebra A, is a family $D_{\underline{\ell}}, \underline{\ell} \in \mathbb{N}^{r+1}$, of k-vector space endomorphisms of A satisfying the conditions:

(1) $D_{\underline{0}} = \mathrm{Id}$
(2) $D_{\underline{\ell}}(c) = 0$, for all $c \in k$ and $\underline{\ell} \neq \underline{0}$.
(3) $D_{\underline{\ell}} \circ D_{\underline{m}} = \binom{\underline{\ell}+\underline{m}}{\underline{\ell}} D_{\underline{\ell}} + \underline{m}$
(4) $D_{\underline{\ell}}(a \cdot b) = \sum_{\underline{i}+\underline{j}=\underline{\ell}} D_{\underline{i}}a \cdot D_{\underline{j}}b$,

where for $\underline{\ell} = (\ell_0, ..., \ell_n), \underline{m} = (m_0, ..., m_n) \in \mathbb{N}^{n+1}$

$$\binom{\underline{\ell}}{\underline{m}} = \binom{n_0}{m_0} \cdots \binom{n_n}{m_n}.$$

An example of a Hasse family is given as follows: For $A = k[\underline{x}] = k[x_0, ..., x_n]$, and $\underline{x}^{\underline{m}} = x_0^{m_0} \cdots x_n^{m_n}$ we define

$$D_{\underline{\ell}}\, \underline{x}^{\underline{m}} = \binom{\underline{m}}{\underline{\ell}} \underline{x}^{\underline{m}} - \underline{\ell}.$$

Let us denote by $D_i = D_{\underline{\ell}_i}$ for $\underline{\ell}_i = (0, ..., 0, 1, 0..., 0)$, i.e. there is an 1 in the ith position. For general $\underline{\ell}$ we can recover $D_{\underline{\ell}}$ by $D_{\underline{\ell}} = D_0^{\ell_0} \circ \cdots \circ D_{\ell}^{\ell_n}$, where $D_i^{\ell_i}$ denotes the composition of D_i ℓ_i times. One can prove (see [13]) that for $\ell = \sum_{j=0}^{s} \ell_j p^j$ with $0 \le \ell_j < p$ for all $j = 0, ..., s$ we have

$$D_i^{\ell} = \frac{1}{\ell_0! \cdots \ell_s!} \left(D_i^{p^s}\right)^{\ell_s} \cdots (D_i^p)^{\ell_1} (D_i^1)^{\ell_0}, \tag{2.1}$$

therefore for each i, the family (D_i^{ℓ}), $\ell \in \mathbb{N}$ is determined by the operators $D_i^1, D_i^p, D_i^{p^2}, ...$

Definition 6. We will denote by $D_{x_i}^{(h)}$ the operator $D_i^{p^h}$.

Definition 7. For two integers m, j we consider their p-adic expansions:

$$m = \sum_{\nu=0}^{\infty} \alpha_\nu p^\nu, \text{ where } 0 \le \alpha_\nu < p \text{ for all } \nu \in \mathbb{N}$$
$$j = \sum_{\nu=0}^{\infty} \beta_\nu p^\nu, \text{ where } 0 \le \beta_\nu < p \text{ for all } \nu \in \mathbb{N}$$

We will write $m \ge_p j$ if and only if

$$\alpha_\nu \ge \beta_\nu \text{ for all } \nu \in \mathbb{N}.$$

If $D_i^j a = 0$ for some a and $j \in \mathbb{N}$, then $D_i^m = 0$ for all $m \ge_p j$. In particular if $D_i^{p^\mu} = 0$, then $D_i^{p^\mu+1}(a) = \cdots = D_i^{p^{\mu+1}-1}(a) = 0$.

The following result, [13], will be used several times during derivation processes in the next sections.

Lemma 8. *Let x, t we indeterminate and $q = p^h$. If $f(t) \in k[t]$, then*

$$D_x^{\ell} f(x^q) = \begin{cases} D_t^{\ell/q}(f)(x^q) & \text{if } q \mid \ell \\ 0 & \text{if } q \nmid \ell \end{cases},$$

where D_x^n (resp. D_t^{ℓ}) are the Hasse derivatives defined on $k[x]$ (resp. $k[t]$).

Remark 9. Note that in multilinear algebra, a system of divided powers on a k-algebra A, is a collection of functions $x \mapsto x^{(d)}$ satisfying a set of axioms given in [6, p. 579]. We observe that the Hasse derivatives D_i^n form a system of divided powers on the commutative ring of differential operators $k[\partial/\partial x_i]$.

2.2. Semilinear algebra

Since first order Hasse derivatives cannot grasp the structure of p-powers, we have to generalize the notion of tangent space.

2.2.1. Frobenius actions

We consider the action of the Frobenius map F_p by acting on the coordinates of elements of V that is

$$\begin{tikzcd} V \arrow[r, "F_p"] & V \\ v = \sum_{v=0}^n \lambda_j e_i \arrow[r] \arrow[d] & F_p(v) = \sum_{v=0}^n \lambda_i^p e_i \arrow[d] \\ (\lambda_1, \ldots, \lambda_n) \arrow[r] & (\lambda_1^p, \ldots, \lambda_n^p) \end{tikzcd} \tag{2.2}$$

The polynomial ring $k[x_0, \ldots, x_n]$ is naturally attached to the vector space V since $\mathrm{Sym}(V^*) = k[x_0, \ldots, x_n]$.

Remark 10. For an element $v \in V$ we will denote by v^{p^i} the element $F_p^i(v)$ for $i \in \mathbb{Z}$. Since k is assumed to be perfect we can also define $v \mapsto v^{1/p^h}$ similarly by tanking the p^h roots of the coordinates of v.

Definition 11. An h-hyperplane H is the algebraic set given by an equation of the form:

$$\sum_{i=0}^n a_i x_i^{p^h} = 0, a_i \in k.$$

Such a hyperplane defines a p^h-linear map:

$$\phi : V \to k$$

$$\sum_{i=0}^n x_i(v) e_i = v \mapsto \phi(v) = \sum_{i=0}^n a_i x_i(v)^{p^h}.$$

The set of p^h-linear maps denoted by V^{*h} consists of functions $\phi : V \to k,$ such that

(1) $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$ for all $v_1, v_2 \in V$
(2) $\phi(\lambda v) = \lambda^{p^h} \phi(v)$ for all $\lambda \in k$ and $v \in V$.

The space V^{*h} becomes naturally a k-vector space, with basis the set $\{x_i^{p^h} : 0 \leq i \leq n\}$.

Theorem 12. *The space $(V^{*h})^{*h}$ is canonically isomorphic to the initial space V.*

Proof. The element $v \in V$ is sent by the isomorphism F to the space $(V^{*h})^{*h}$ defined by:

$$\begin{aligned} F : V &\to (V^{*h})^{*h} \\ v &\mapsto F(v), \end{aligned}$$

where $F(v)$ is the map defined by:

$$F(v) : V^{*h} \to k$$
$$\phi \;\to\; F(v)(\phi) = \phi\left(v^{1/p^{2h}}\right)^{p^h}.$$

Observe first that $F(v)$ is indeed a p^h-linear map. Indeed,

$$\begin{aligned} F(v)(\lambda_1\phi_1 + \lambda_2\phi_2) &= \left(\lambda_1\phi_1\left(v^{1/p^{2h}}\right) + \lambda_2\phi_2\left(v^{1/p^{2h}}\right)\right)^{p^h} \\ &= \lambda_1^{p^h}\phi_1\left(v^{1/p^{2h}}\right)^{p^h} + \lambda_2^{p^h}\phi_2\left(v^{1/p^{2h}}\right)^{p^h} \\ &= \lambda_1^{p^h}F(v)(\phi_1) + \lambda_2^{p^h}F(v)(\phi_2). \end{aligned}$$

Now we prove that F is linear:

$$\begin{aligned} F(\lambda_1 v_1 + \lambda_2 v_2)(\phi) &= \phi\left(\lambda_1^{1/p^{2h}} v_1^{1/p^{2h}} + \lambda_2^{1/p^{2h}} v_2^{1/p^{2h}}\right)^{p^h} \\ &= \lambda_1\phi\left(v_1^{1/p^{2h}}\right)^{p^h} + \lambda_2\phi\left(v_2^{1/p^{2h}}\right)^{p^h} \\ &= \left(\lambda_1 F(v_1) + \lambda_2 F(v_2)\right)\phi, \end{aligned}$$

i.e., $F(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 F(v_1) + \lambda_2 F(v_2),$ for all $\lambda_1, \lambda_2 \in k$ and $v_1, v_2 \in V$.

We will now prove that F is an isomorphism. Since $\dim V = \dim(V^{*h}) = \dim(V^{*h})^{*h}$ it is enough to prove that $\ker F = \{0\}$. If for a $v \in V$ we have $F(v) = 0$, then for every $\phi \in V^{*h}$ we have $F(v)(\phi) = \phi(v^{1/p^{2h}})^{p^h} = 0$. By taking as ϕ the elements e_i^{*h} of the dual basis of V^{*h} we see that $v = 0$. □

Let us work with coordinates now. Express an element $v \in V$ as $v = \sum_{i=0}^n x_i(v)e_i,$ where $\{e_i\}_{i=0,\dots,n}$ is a Frobenius invariant basis as expressed in Sec. 2.2.1, and let $\phi \in V^{*h}$ written in terms of a Frobenius dual basis as $\phi = \sum_{i=0}^n y_i(\phi)e_i^{*h}$. Set $q = p^h,$ we have

$$\phi(v) = \sum_{i=0}^n x_i(v)^q y_i(\phi), \tag{2.3}$$

while we have (recall that $F(v) \in (V^{*h})^{*h}$)

$$F(v)(\phi) = \phi\left(\sum_{i=0}^n x_i(v)^{1/2q} e_i\right)^q = \sum_{i=0}^n x_i(v) y_i(\phi)^q. \tag{2.4}$$

This means that

$$\begin{array}{ccc} V \times V^{*h} \ni (v, \phi) & \longmapsto & \phi(v) = \sum_{i=0}^n x_i(v)^q y_i(\phi) \in k \\ \downarrow \Psi & & \\ V^{*h} \times (V^{*h})^{*h} \ni (\phi, F(v)) & \longmapsto & F(v)(\phi) = \sum_{i=0}^n x_i(v) y_i(\phi)^q. \end{array} \tag{2.5}$$

In our generalized point of view duality means that a point $[v] \in \mathbb{P}(V),$ represented by the vector $v \in V,$ can be also seen as a q-hyperplane $[F(v)]$ on $\mathbb{P}((V^{*h})^{*h})$.

2.3. q-Symplectic forms

Let $\mathbb{F}$ be a field of positive characteristic p and let $q = p^h$ be a certain power of p. In order to define a suitable Lagrangian variety in the positive characteristic case, we need its respective symplectic form.

Definition 13. A q-symplectic form Ω on V is a function:

$$\Omega : V \times V^{*h} \to k$$

which is additive, i.e. for all $v_1, v_2 \in V, w_1, w_2 \in V^{*h}$ we have

$$\Omega(v_1 + v_2, w_1) = \Omega(v_1, w_1) + \Omega(v_2, w_1), \qquad \Omega(v_1, w_1 + w_2) = \Omega(v_1, w_1) + \Omega(v_1, w_2),$$

such that there is a symplectic basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ so that

$$\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j), \Omega(e_i, f_j) = \delta_{ij}, \Omega(f_i, e_j) = -\delta_{ij}.$$

Moreover for arbitrary elements

$$v = \sum_{i=0}^{n} \lambda_i e_i + \sum_{j=0}^{n} \mu_j f_j$$

and

$$w = \sum_{i=0}^{n} \lambda'_i e_i + \sum_{j=0}^{n} \mu'_j f_j$$

the symplectic form is computed:

$$\Omega(v, w) = \sum_{i=0}^{n} \left(\lambda_i^{p^h} \mu'_i - \mu_i \lambda_i'^{p^h} \right).$$

Remark 14. As in [4, p. 8], the notions of p^h-orthogonality, p^h-symplectic, p^h-isotropic and p^h-Langrangian subvector spaces can be defined. Since these notions are not needed in this note, we will not develop their theory here.

2.4. Powers of Frobenius as ghost variables

In this section we will add extra ghost variables $x_i^{(h)}$ for $0 \leq i \leq n$ and for $h = 1, ..., \infty$. This is an idea coming from the similarities of the p-power Frobenius map and differential equations [10, sec. I.1.9] and the ring of differential polynomials see [3, exam. 5.2.5].

Lemma 15. *Consider a term* $\underline{x}^{\underline{i}}$, *where* $\underline{i} = (i_0, ...i_n) \in \mathbb{N}^{n+1}$, *and the p-adic expansions of each index:*

$$i_\nu = \sum_{\mu=0}^{\infty} i_\nu(\mu) p^\mu, \qquad 0 \leq i_\nu(\mu) < p.$$

Therefore, a term $\underline{x}^{\underline{i}}$ *can be written as*

$$\underline{x}^{\underline{i}} = \prod_{\mu_0=0}^{\infty} \cdots \prod_{\mu_n=0}^{\infty} x_0^{i_0(\mu_0) p^{\mu_0}} \cdots x_n^{i_n(\mu_n) p^{\mu_n}}. \tag{2.6}$$

Consider the ring

$$R := k\left[x_0, ..., x_n, x_0^{(1)}, ..., x_n^{(1)}, ..., x_0^{(h)}, ..., x_n^{(h)}, ...\right] \tag{2.7}$$

and define the degree $\deg x_\nu^{(i)} = p^i$. *We also define the homomorphism*

$$\begin{aligned} \phi : R &\to k[x_0, ..., x_n] \\ x_i^{(j)} &\mapsto x_i^{p^j} \; for \; all \; 0 \le i \le n, 0 \le j \le h. \end{aligned} \tag{2.8}$$

The map ϕ *is onto, and moreover*

$$\phi\left(D_{x_i^{(j)}} f\right) = D_{x_i}^{(h)} \phi(f). \tag{2.9}$$

Proof. Let $f \in k[x_0, ..., x_n]$. If we write every term of f as in Eq. (2.6) and replace $x_j^{i_j(\mu)p^\mu}$ by $(x_j^{(\mu)})^{i_j(\mu)}$, we get a polynomial $\tilde{f} \in R$ such that $\phi(\tilde{f}) = f$. The relation given in Eq. (2.9) follows by the property of the Hasse derivative

$$D_{x_i}^{(h)}(x_j^{p^\ell}) = \delta_{ij}\delta_{h,\ell},$$

and the differentiation rules. □

In other words, this lemma shows that if we set the quantity $x_i^{p^h}$ which appears in the related varieties, as a new variable $x_i^{(h)}$, with the use of suitable expansions, the partial derivation $D_{x_i^{(h)}}$ with respect to the variables $x_i^{(h)}$ will coincide with the Hasse derivatives $D_{x_i}^{(h)}$.

Remark 16. The kernel of the map ϕ of Eq. (2.8) is the ideal generated by $x_i^{p^h} - x_i^{(h)}$, which is a homogeneous ideal by the definition of the degrees $\deg_\nu x^{(i)}$. Therefore, we have the following compatible diagram of vector spaces, rings and derivations:

$$\begin{array}{ccc} \tilde{V} & R = \mathrm{Sym}(\tilde{V}^*) & \{D_{x_i^{(h)}} \quad 0 \le i \le n, 0 \le h \le N\} \\ \downarrow & \downarrow \phi & \Vert \\ V & k[x_0, \ldots, x_n] = \mathrm{Sym}(V^*) & \{D_{x_i}^{(h)} \quad 0 \le i \le n, 0 \le h \le N\} \end{array}$$

In the above diagram we have a vector space, the natural ring of polynomial functions on it and the natural set of derivations. When taking the quotient by the ideal $\ker\phi$, the set of derivations is not altered and the derivations corresponding to the dual basis of $\tilde{V}$ survive, giving rise to Hasse derivations on the quotient.

Remark 17. The definition of the ring R in this subsection, could provide an alternative way to force separability and therefore reflexivity to hold, for a class of weighted projective varieties, which we may call bihomogeneous.

Consider an ideal I of $k[x_0, ..., x_n]$ generated by elements $F_1, ..., F_t$. Instead of working with the polynomial ring R, of infinite Krull dimension we restrict ourselves to the ring

$$R_N := k\left[x_0, ..., x_n, x_0^{(1)}, ..., x_n^{(1)}, ..., x_0^{(N)}, ..., x_n^{(N)}\right],$$

where N is big enough so that the map $\phi_h : R_N \to k[x_0, ..., x_n]$ is onto I. Essentially this means that every term of all polynomials F_i is of the form $x_0^{i_0} \cdots x_n^{i_n}$ and the p-adic expansions of i_j, $0 \le j \le n$ do not involve p-powers p^h with $N < h$. For example for $p = 3$ and the polynomial $x^{10} + x^{21}$ we have to take $N = 2$ since

$$x^{10} + x^{21} = x^{1+3^2} + x^{3\cdot 7} = xx^{3^2} + (x^3)^7 = x\phi(x^{(2)}) + \phi(x^{(1)})^7.$$

Let now $\tilde{I}$ be the ideal of R_N defined by $\phi^{-1}(I)$, then $\tilde{I}$ is generated by the polynomials $\tilde{F}_i \in R_N$ defined in the proof of Lemma 15. Since, the procedure of Lemma 15 replaces all powers of the form $x_i^{p^h}$ by the new coordinates $x_i^{(h)}$, which still have degree p^h, if I is a homogeneous ideal of $k[x_0, ..., x_n]$, then it is generated by homogeneous elements $F_1, ..., F_t$ and the corresponding polynomials in new variables are still homogeneous. In other words, if I is a homogeneous ideal of $k[x_0, ..., x_n]$, then $\tilde{I}$ is a homogeneous ideal of R.

Recall that a weighted projective space is the quotient $\mathbb{P}(a_0, ..., a_n) = (\mathbb{A}^{n+1} - \{0\})/k^*$ under the equivalence relation $(x_0, ..., x_n) \sim (\lambda^{a_0} x_0, ..., \lambda^{a_n} x_n)$, for $\lambda \in k^*$.

In our case, in order to form algebraic sets corresponding to ideals $\phi^{-1}(I)$, we have to consider the weighted projective spaces, $\mathbb{P}(1, ..., 1, p, ..., p, p^2, ..., p^2, ..., p^N, ..., p^N)$. In a weighted projective space linear equations of the form

$$\sum_{h=0}^{N} \sum_{i=0}^{n} a_{h,i} x_i^{(h)} = 0,$$

do not give rise to homogeneous ideals unless they are of the form

$$\sum_{i=0}^{n} a_{h_0,i} x_i^{(h)} = 0,$$

and it is not entirely clear what projective duality will mean for weighted projective varieties. Of course, it is known that every weighted projective variety M is isomorphic to an ordinary projective variety $\tilde{M} \in \mathbb{P}^\ell$ for some big enough element ℓ, [18, th. 4.3.9]. The homogeneous ideal $\tilde{I}$ corresponding to M is generated by polynomials of degree smaller than p, therefore it is reflexive.

We will not pursue here the theory of duality of weighted projective varieties, but we can see something interesting for some of them; if we consider the polynomial ring

$$R_{0,N} = k\left[x_0, ..., x_n, x_0^{(1)}, ..., x_n^{(1)}, ..., x_0^{(N)}, ..., x_n^{(N)}\right],$$

but now $\deg(x_0)^{(i)} = 1$ for all $1 \leq i \leq h$, the ideal $\phi^{-1}(I) \in R_{0,N}$, of a homogeneous ideal I of $k[x_0, ..., x_n]$ does not need to be homogeneous in $R_{0,N}$ with this grading. If it is homogeneous, then we can define it as bihomogeneous. For example, the hypersurface defined by the polynomial $\sum_{i=0}^n x_i^{p^h+1}$ gives rise to the ideal generated by the polynomial $x_0^{(h)} x_0 + x_1^{(h)} x_1 + \cdots + x_n^{(h)} x_n$, which is bihomogeneous. On the other hand, the hypersurface defined by the homogeneous polynomial $x_0^{p+1} - x_1 x_2 \cdots x_{p+1}$, is not bihomogeneous, i.e., the polynomial $x_0^{(1)} x_0 - x_1 x_2 \cdots x_{p+1}$ is homogeneous in the graded ring R_N but not in the graded ring $R_{0,N}$. Observe now that the projective algebraic set $V(\phi^{-1}(I)) \subset \mathbb{P}^{h(n+1)}$ defined by the bihomogeneous ideal $\phi^{-1}(I) \subset R_{0,N}$ does not have a variable raised to a power of p, therefore it is reflexive.

2.4.1. Example: Generalized quadratic forms

Let $\underline{x} = (x_0, ..., x_n)^t$ and consider the homogeneous polynomial

$$f_A := \underline{x}^t A \underline{x}^q = \sum_{i,j=0}^{n} x_i a_{ij} x_j^q,$$

where $A = (a_{i,j})$ is an $(n+1) \times (n+1)$ matrix, and $q = p^h$. If $A = \mathbb{I}_{n+1}$, then F is the diagonal Fermat hypersurface also called Hermitian hypersurface. For $q = 1$ the polynomial f_A is just a quadratic form.

We compute that for a point $P = [a_0 : \dots : a_n] \in V(f_A)$

$$D^{(0)}_{x_\ell} f_A(P) = \sum_{j=0}^{n} a_{\ell,j} a_j^{p^h}$$

and

$$D^{(h)}_{x_\ell} f_A(P) = \sum_{i=0}^{n} a_i a_{i,\ell}.$$

We write the coordinates of P as a column vector $a = (a_0, \dots, a_n)^t$ and we compute both ∇f_A, $\nabla^q f_A$,

$$\nabla f_A = (D_{x_0} f_A, \dots, D_{x_n} f_A) = A a^q = (A^{1/q} a)^q$$

and

$$\nabla^q f_A = (D^{(q)}_{x_0} f_A, \dots, D^{(q)}_{x_n} f_A) = a^t \cdot A.$$

The Gauss map $a \mapsto (A^{1/q} a)^q$ is inseparable.

Define $\xi = (\xi_0, \dots, \xi_n)^t$ and $\xi^{(q)} = (\xi^{(q)}_0, \dots, \xi^{(q)}_n)^t$, given by

$$\xi = \nabla f_A = A a^q \text{ and } \xi^{(q)} = (\nabla^q f_A)^t = (a^t \cdot A)^t = A^t a.$$

We will now introduce ghost variables in order to force reflexivity. Here we consider the variables $x^q = (x_0^q, \dots, x_n^q)^t = y = (y_0, \dots, y_n)^t$ as a set of new variables y and we write the homogeneous polynomial defining the variety as

$$F_A = x^t A y = \sum_{i,j=0}^{n} x_i a_{ij} y_j.$$

The Gauss map in this case is given by:

$$(a, b) \mapsto \nabla F_A = (A \cdot b, A^t a).$$

If for a point $(a, b)^t \in V(F_A)$ satisfying $a^t F_A b = 0$ we introduce the variables $\xi = A \cdot b, \xi_1 = A^t \cdot a$, then the point (ξ, ξ_1) satisfies the equation:

$$\xi_1^t A^{-1} \xi = 0$$

since

$$\xi_1^t A^{-1} \xi = a^t A A^{-1} A b = a^t A b = 0.$$

Observe that the value $\xi_1^q = A^{qt} \cdot a^q$ can be explicitly expressed in terms of the variables ξ by the equation:

$$A A^{-tq} \xi_1^q = A A^{-tq} \cdot (A^{tq}) a^q = A a^q = \xi.$$

Notice also that the map $\phi : (X, Y) \mapsto (A^t Y, AX) = (\xi_1, \xi)$ and similarly the map $\psi : (\xi_1, \xi) \mapsto (A^{-1}\xi, A^{-t}\xi_1)$ and $\psi \circ \phi = \phi \circ \psi = \mathrm{Id}$.

Let $M = V(f_A) \subset \mathbb{P}(V)$ and $\tilde{M} = V(F_A) \subset \mathbb{P}(\tilde{V})$. The conormal variety $\mathrm{Con}(\tilde{M}) \subset \mathbb{P}(\tilde{V}) \times \mathbb{P}(\tilde{V}^*)$ is given by the pairs $(a, b; \xi, \xi_1) = (a, b; A \cdot b, A^t \cdot a)$. In order to compute the conormal variety $\mathrm{Con}(M) \subset \mathbb{P}(V) \times \mathbb{P}(V^*)$ we pass from $\tilde{M}$ to M by imposing the relation $b = a^q$ and we obtain $(a, a^q; A \cdot a^q, A^t \cdot a)$. Observe that ξ, ξ_1 satisfy the equation of the dual

$$\xi^{qt} A^{-1} \xi = 0.$$

2.5. Variants of Euler theorem

The Euler identity for homogeneous polynomials implies that for a homogeneous polynomial $F(x_0,\ldots,x_n) \in k[x_0,\ldots,x_n]$, of degree $\deg F$ we have

$$\sum_{i=0}^{n} x_i D_{x_i}^{p^0} F(x_0,\ldots,x_n) = \deg F \cdot F(x_0,\ldots,x_n).$$

If $p \mid \deg F$, a lot of information is lost. In particular the first order partial derivations $D_{x_i}^{p^0} F$ can be zero. Next proposition allows us to get some information, from the higher derivatives $D_{x_i}^{p^i}$. We need the following

Proposition 18. *Let $q = p^h$ be a power of the characteristic. Let $P_j(x_0,\ldots,x_n)$, $Q_j(x_0,\ldots,x_n)$ be polynomials in $k[x_0,\ldots,x_n]$, $j = 1,\ldots,s$, where P_j are homogeneous of degree $\deg_h(f)$, and Q_j have no indeterminate raised to a power bigger than or equal to a power of q. If*

$$f(x_0,\ldots,x_n) = \sum_{j=1}^{s} P_j(x_0^q,\ldots,x_n^q) Q_j(x_0,\ldots,x_n), \tag{2.10}$$

then

$$\sum_{i=0}^{n} x_i^q D_{x_i}^q f(x_0,\ldots,x_n) = \deg_h(f) \cdot f(x_0,\ldots,x_n).$$

Proof. [13, prop. 3.10] □

Definition 19. We will call a polynomial h-homogeneous of degree $\deg_h(f)$ if it is a linear combination of polynomials given in Eq. (2.10) of the same degree.

Remark 20. A polynomial which is homogeneous and h-homogeneous is bihomogeneous according to Remark 17.

2.6. h-Tangent and h-cotangent spaces and bundles

In order to compare our definition of h-tangent space we recall here the classical definition.

Let M be a projective variety defined by h-homogeneous polynomials $F_1,\ldots,F_t$, as these were defined in Definition 19, generating the homogeneous ideal I. Let S be the algebra $k[x_0,\ldots,x_n]/I$.

Definition 21. Let $P = [a_0 : \ldots : a_n]$ be a point on M. The tangent space $T_P M$ of M at P, is defined as the zero space of the differentials (we will denote by $D_{x_\nu}^{(0)}$ the classical derivative according to definition 6). In other words,

$$\begin{aligned} dF_i &= \sum_{\nu=0}^{n} D_{x_\nu}^{(0)} F_i(P) x_\nu \text{ for all } 1 \le i \le t, \\ T_P M &= V(\langle dF_1,\ldots,dF_t\rangle). \end{aligned} \tag{2.11}$$

Definition 22. For every $f \in R$, define the differential form on the tangent space $T_P M$:

$$df := \sum_{\nu=1}^{n} D_{x_\nu}^{(0)} f(P) x_\nu, \tag{2.12}$$

which gives rise to elements in the dual space T_P^*M, by sending a solution $(x_0 : \cdots : x_r) \in T_P M$ of system (2.11) to the value df given in Eq. (2.12).

The element df is well defined, since if $f_1 - f_2 \in \langle F_1, ..., F_t \rangle$, then the differentials df_1, df_2 introduce the same linear form on $T_P M$, see [25, chap II. sec. 1]. Let $\mathcal{O}_M(P)$ be the ring of functions defined at P. The map

$$d : k[x_0, ..., x_n] \to (T_P M)^*$$

defines an isomorphism of $\mathfrak{m}_P/\mathfrak{m}_P^2$ to $(T_P M)^*$, [25, chap II. th. 2.1]. This fact implies that the dimension of the tangent space is invariant under isomorphism, see [25, chap II. Cor. 2.1].

2.6.1. h-Tangent bundles

For every $F \in R$ we define the h-linear form:

$$L_F^{(h)} : V \to k$$
$$\sum_{\nu=0}^{n} x_i(v) e_i = v \mapsto \sum_{\nu=0}^{n} D_{x_\nu}^{(h)} F(P) (x_\nu(v))^{p^h}.$$

Definition 23. Let M be defined in terms if the homogeneous ideal $\langle F_1, ..., F_t \rangle$. For $h \geq 0$, the h-tangent space $T_P^{(h)} M$ at $P \in M$ is defined by

$$T_P^{(h)} M = \bigcap_{i=1}^{t} \ker L_{F_i}^{(h)} \subseteq V.$$

It is clear from the definition that $T_P^{(h)} M$ is a k-vector space.

Remark 24. The notion of classical tangent space is independent of the isomorphism class of a variety. If $\Phi : M \to Y$ is a local isomorphism from a Zariski neighborhood U of P to a Zariski neighborhood V of $\Phi(P)$, then $\dim_k T_P^{(0)} M = \dim_k T_{\Phi(P)}^{(0)} Y$.

This does not hold for the case of the h-tangent spaces, the space $T_P^{(h)} M$ depends on the embedding of M in an ambient space. For example the affine space $\mathbb{A}^1 = \mathrm{Spec}(k[x])$ has one dimensional tangent space $T_P^{(h)} \mathbb{A}^1$ for all $h > 1$, while its isomorphic image $\mathrm{Spec}(k[x, y]/\langle x \rangle) \subset \mathbb{A}^2$ has 2-dimensional h-tangent space for all $h > 1$.

Of course, in order to correct this, one can strengthen the notion of isomorphism $\Phi : X \to Y$, by requiring that Φ induces an isomorphism to h-tangent spaces as well.

Remark 25. As R. Vakil observes [28, chap. 12], the quantity $\sum_{i=0}^{n} D_{x_i}^{(0)} F \cdot x_i$ is the linear part of a given polynomial $F \in k[x_0, ..., x_n]$. In a similar fashion $\sum_{i=0}^{n} D_{x_i}^{(h)} F \cdot x_i^{p^h}$ is the p^h-linear part of the polynomial F, that is all terms that can be written as $(\sum_{i=0}^{n} a_i x_i)^{p^h}$, $a_i \in k$.

Definition 26. The h-cotangent space $T_P^{(*h)} M$ for $h \geq 0$ at P is defined as the vector space generated by the elements (set $q = p^h$)

$$d^{(h)} f = \sum_{\nu=0}^{n} D_{x_\nu}^{(h)} f(P) x_\nu^q \tag{2.13}$$

for elements $f \in k[x_0, ..., x_n]/I(M)$. Notice that the expression $d^{(h)} f$ defined for f as above gives rise to a well defined form on the tangent space. Moreover $d^{(h)} x_i^q$ is an element in $T_P^{(*h)} M$.

Remark 27. Similar to ordinary differentials, the map given in Eq. (2.13) is well defined, i.e., if $f_1 - f_2 \in I(M)$, then $d^{(h)}f_1 - d^{(h)}f_2$ is the zero map on the tangent space $T^{(h)}M$. In this way the differentials in Eq. (2.13) define functions

$$\phi : T_P^{(h)}M \to k.$$

Let us consider the example $M = \mathrm{Spec}k[x,y]/\langle x\rangle$. In order to compute the h-cotangent space $T_0^{(*h)}M$ let us compute $d^{(h)}f(x,y)$ for $f(x,y) \in k[x,y]$. The possible outcomes are all expressions of the form $ax^q + by^q$, $a, b \in k$. Notice that for $f \in \langle x\rangle$ we have $d^{(h)}x = 0$, so the h-cotangent space is two dimensional.

Let R be a finite presented k-algebra. A Frobenius map on R is a ring homomorphism $\Phi : R \to \Phi(R) \subset R$, such that $\Phi(\lambda x) = \lambda^p \Phi(x)$, for all $\lambda \in k$ and $x \in R$. The image $\Phi(R)$ is a subring of R. In this way we form a sequence of nested subrings of R,

$$R \supset \Phi(R) \supset \Phi^2(R) \supset \Phi^3(R) \supset \cdots$$

If R is a local ring with maximal ideal $\mathfrak{m}$ then all rings $\Phi^h R$ are local rings as well, with maximal ideals $\mathfrak{m}^{(h)} = \Phi^h(\mathfrak{m})$. If $f : R_1 \to R_2$ is a ring homomorphism of two rings equipped with Frobenius maps Φ_1, Φ_2 respectively then we require

$$f(\Phi_1^h R_1) \subset \Phi_2^h(R_2).$$

If moreover f is a local homomorphism of local rings R_i with corresponding maximal ideals $\mathfrak{m}_i$, for $i = 1$, 2 then $f(\mathfrak{m}_1^{(h)}) \subset \mathfrak{m}_2^{(h)}$. In what follows we will consider the polynomial ring $k[x_0, \ldots, x_r]/I$, and the localizations at certain maximal ideals of the ring $k[x_0, \ldots, x_n]/I$.

Definition 28. The intrinsic h-cotangent space $\Theta_P^{(*h)}M$ is defined to be the space $\mathfrak{m}_P^{(h)}/\mathfrak{m}_P^{(h)^2}$ and equals to the cotangent space of the local ring $\Phi^h(\mathcal{O}_P)$.

Lemma 29. *The space $\Theta_P^{(*h)}M \cong \mathfrak{m}_P^{(h)}/(\mathfrak{m}_P^{(h)})^2$ is a subspace of $T_P^{(*h)}M$.*

Proof. Consider the map

$$d^{(h)} : \frac{\mathfrak{m}_P^{(h)}}{\left(\mathfrak{m}_P^{(h)}\right)^2} \to T_P^{(*h)}(M),$$

we will prove it is injective.

Let $G \in k[x_0^{p^h}, \ldots, x_n^{p^h}]$ be a polynomial representative of an element in $\mathfrak{m}_P^{(h)}$ so that $d^{(h)}G$ is the zero form on $T_P^{(h)}M$. Assume that the homogeneous ideal of M is generated by the polynomials $F_1, \ldots, F_r$. Then $d^{(h)}G$ is a linear combination $\sum_{\nu=0}^r \lambda_\nu d^{(h)}F_\nu$ of the forms $d^{(h)}F_\nu$ for F_ν, $1 \le \nu \le r$, generating the homogeneous ideal of M. This means that

$$G(x_0^{p^h}, \ldots, x_n^{p^h}) = a_0 x_0^{p^h} + \cdots + a_n x_n^{p^h} + \sum_{0 \le i, j \le n} a_{ij} x_i^{p^h} x_j^{p^h} + \text{ higher order terms}$$

for certain elements $a_i, a_{ij} \in k$. The linear combination $\sum_{\nu=0}^r \lambda_\nu d^{(h)}F_\nu$ cancels out by assumption the $\sum_{i=0}^n a_i x_i^{p^h}$ part, so the difference $G - \sum_{\nu=0}^t \lambda_\nu d^{(h)}F_\nu \in (\mathfrak{m}_P^{(h)})^2$, and the result follows. □

Definition 30. Define $\Theta_P^{(h)}(M)$ to be the dual space of $\Theta_P^{(*h)}M$, that is

$$\Theta_P^{(h)}(M) = \mathrm{Hom}_k(\Theta_P^{(*h)}(M)), k).$$

Remark 31. By Eq. (2.13) we have that $d^{(h)}(x_i^{p^h})$ on the tangent space acts like the q-form $x_i \mapsto x_i^q$.

Corollary 32. *The dimension of $\Theta_P^{(*h)}M$ is an invariant of the isomorphism class of a variety, i.e. if $\Phi : M \to Y$ is a local isomorphism from a Zariski neighborhood U of P to a Zariski neighborhood V of $\Phi(P)$, then* $\dim_k \Theta_P^{(h)}M = \dim_k \Theta_{\Phi(P)}^{(h)} Y$.

Let $M \subset V$ be an irreducible variety. Consider the algebraic set $\Theta \subset V \times M$ consisting of pairs $(a, P) \in V \times M$ such that a is h-tangent at P. The second projection $\pi : \Theta \to M$ is onto and has fibers the spaces $\Theta_P^{(h)}M$. By [25, Chap. I.63 th.7] we have that $\dim_k \Theta_P^{(h)}M \geq s$ for all $P \in M$ and equality is attained at a non-empty open subset of M.

Definition 33. We will say that a point $P \in M$ is h-nonsingular if

$$\dim_k \Theta_P^{(h)} = \dim_k T_P^{(*h)}M = \dim M.$$

2.6.2. Differential between tangent spaces

Consider the projective varieties $V \subset \mathbb{P}^n$, $W \subset \mathbb{P}^m$ defined in terms of the homogeneous ideals $\langle f_1, ..., f_r \rangle \in k[x_0, ..., x_n]$ and $\langle g_1, ..., g_s \rangle \in k[y_0, ..., y_m]$ respectively. A map $F : V \to W$ is given by polynomials $F_0, ..., F_m \in k[x_0, ..., x_n]$ such that $y_i = F_i(x_0, ..., x_n)$ for $i = 0, ..., m$. Set

$$J_{0,h}(f_1, ..., f_r) = \left(\begin{array}{ccc|ccc} D_{x_0}^{(0)} f_1 & \cdots & D_{x_n}^{(0)} f_1 & D_{x_0}^{(h)} f_1 & \cdots & D_{x_n}^{(h)} f_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ D_{x_0}^{(0)} f_r & \cdots & D_{x_n}^{(0)} f_r & D_{x_0}^{(h)} f_r & \cdots & D_{x_n}^{(h)} f_r \end{array}\right) = (A|A')$$

and similarly

$$J_{0,h}(g_1, ..., g_s) = \left(\begin{array}{ccc|ccc} D_{y_0}^{(0)} g_1 & \cdots & D_{y_m}^{(0)} g_1 & D_{y_0}^{(h)} g_1 & \cdots & D_{y_m}^{(h)} g_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ D_{y_0}^{(0)} g_s & \cdots & D_{y_m}^{(0)} g_s & D_{y_0}^{(h)} g_s & \cdots & D_{y_m}^{(h)} g_s \end{array}\right) = (B|B').$$

The kernel of the matrix A at P (resp. B at $F(P)$) corresponds to the ordinary tangent space of V (resp. W) while the kernel of A' (resp. B') corresponds to the h-tangent space.

By substitution of $y_i = F(x_0, ..., x_n)$ for $0 \leq i \leq m$ in $g_1, ..., g_s$ we write each $g_1, ..., g_s$ as an element in the ideal $\langle f_1, ..., f_r \rangle$. Therefore elements in $T_P V$, resp. $T_P^{(h)} V$, given as elements in the kernel of A (resp. A') are sent to elements in T_{PW}, resp. $T_P^{(h)} W$.

Consider now the matrix (observe that $y_i^{p^h} = F_i(x_0, \ldots, x_n)^{p^h}$)

$$J_{0,h}(F_1, \ldots, F_n, F_1^q, \ldots, F_n^q) = \left(\begin{array}{ccc|ccc} D^{(0)}_{x_0}F_0 & \cdots & D^{(0)}_{x_n}F_0 & D^{(h)}_{x_0}F_0 & \cdots & D^{(h)}_{x_n}F_0 \\ \vdots & & \vdots & \vdots & & \vdots \\ D^{(0)}_{x_0}F_m & \cdots & D^{(0)}_{x_n}F_m & D^{(h)}_{x_0}F_m & \cdots & D^{(h)}_{x_n}F_m \\ \hline D^{(0)}_{x_0}F_0^{p^h} & \cdots & D^{(0)}_{x_n}F_0^{p^h} & D^{(h)}_{x_0}F_0^{p^h} & \cdots & D^{(h)}_{x_n}F_0^{p^h} \\ \vdots & & \vdots & \vdots & & \vdots \\ D^{(0)}_{x_0}F_m^{p^h} & \cdots & D^{(0)}_{x_n}F_m^{p^h} & D^{(h)}_{x_0}F_m^{p^h} & \cdots & D^{(h)}_{x_n}F_m^{p^h} \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} D^{(0)}_{x_0}F_0 & \cdots & D^{(0)}_{x_n}F_0 & D^{(h)}_{x_0}F_0 & \cdots & D^{(h)}_{x_n}F_0 \\ \vdots & & \vdots & \vdots & & \vdots \\ D^{(0)}_{x_0}F_m & \cdots & D^{(0)}_{x_n}F_m & D^{(h)}_{x_0}F_m & \cdots & D^{(h)}_{x_n}F_m \\ \hline 0 & \cdots & 0 & D^{(h)}_{x_0}F_0^{p^h} & \cdots & D^{(h)}_{x_n}F_0^{p^h} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & D^{(h)}_{x_0}F_m^{p^h} & \cdots & D^{(h)}_{x_n}F_m^{p^h} \end{array}\right)$$

$$= \left(\begin{array}{c|c} J & J' \\ \hline 0 & J^{p^h} \end{array}\right).$$

The chain rule implies

$$\left(\begin{array}{ccc|ccc} D^{(0)}_{x_0}g_1 & \cdots & D^{(0)}_{x_m}g_1 & D^{(h)}_{x_0}g_1 & \cdots & D^{(h)}_{x_m}g_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ D^{(0)}_{x_0}g_s & \cdots & D^{(0)}_{x_m}g_s & D^{(h)}_{x_0}g_s & \cdots & D^{(h)}_{x_m}g_s \end{array}\right) = (B|B')\left(\begin{array}{c|c} J & J' \\ \hline 0 & J^{p^h} \end{array}\right)$$
$$= (BJ|BJ' + B'J^{p^h}).$$

An element $\bar{a} = (a_0, \ldots, a_n)^t \in T_{F(P)}W$, $\bar{b} = (b_0, \ldots, b_n)^t \in T^{(h)}_{F(P)}W$ by definition of the tangent spaces satisfies

$$BJ\bar{a} = 0 \qquad (BJ' + B'J^{p^h})\bar{b} = 0.$$

On the other hand we have

$$\left(\begin{array}{c|c} J & J' \\ \hline 0 & J^{p^h} \end{array}\right)\begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = \begin{pmatrix} J\bar{a} + J'\bar{b} \\ J^{p^h}\bar{b} \end{pmatrix}$$

therefore $B(J\bar{a} + J'\bar{b}) = 0$ and $B'J^{p^h}\bar{b} = 0$. This allows us to write the differentials:

$$dF : T_P(V) \to T_{F(P)}W \text{ and } dF^{(h)} : T_P^{(h)}(V) \to T^{(h)}_{F(P)}W$$

as follows

$$dF(\bar{a}) = J\bar{a} + J'\bar{b} \text{ and } dF^{(h)}(\bar{b}) = J^{p^h}\bar{b}.$$

When $\bar{b} = 0$ is the zero h-tangent vector then dF is the classical map. The differential in the h-tangent space is independent on the choice of $\bar{a} \in T_P^{(0)}V$.

We have proved the following:

Proposition 34. *Let $F : V \to W$ be a map between polynomial varieties, expressed in terms of polynomials $F_0, \dots, F_s$. Then the p^h-power of the ordinary differential $T_P^{(0)}V \to T_{F(P)}^{(0)}W$ is the natural map $T_P^{(h)}V \to T_{F(P)}^{(h)}V$.*

2.7. Vector fields and differential forms

We will now define vector fields as differential operators in terms of Hasse-derivatives. The identification

$$\frac{\mathfrak{m}_P}{\mathfrak{m}_P^2} \xrightarrow{d} T_P^* M$$

proves that $dx_0, \dots, dx_n$ give a basis of the cotangent space, since $\mathfrak{m}_P/\mathfrak{m}_P^2$ is generated as vector space by the classes of $x_0, \dots, x_n$ modulo $\mathfrak{m}_P^2$. Also in the classical case the partial derivatives $\partial/\partial x_i$ give rise to naturally dual elements, i.e. elements in $T_P M$.

Let us assume that the variety M has a non-empty open set of h-nonsingular points. On this open set we will employ the identification $\frac{\mathfrak{m}_P^{(h)}}{(\mathfrak{m}_P^{(h)})^2} \xrightarrow{d^{(h)}} T_P^{(*h)}M$ of Lemma 29, which sends

$$\frac{\mathfrak{m}_P^{(h)}}{\left(\mathfrak{m}_P^{(h)}\right)^2} \ni m = \sum_{i=0}^{n} a_i x_i^{p^h} \mapsto d^{(h)}m = \sum_{i=0}^{n} a_i d^{(h)} x_i^q \in T_P^{(*h)}M.$$

Definition 35. A vector field X is a sum

$$X = \sum_{h=0}^{\infty} \sum_{i=0}^{n} a_{h,i}(X) D_{x_i}^{(h)}, \tag{2.14}$$

where all but finite coefficients $a_{h,i}(X)$ are zero. The elements $a_{h,i}(X)$ are coefficients in $\mathcal{O}_M$, depending linearly on X. Vector fields form $\mathcal{O}_M$-modules.

Definition 36. For every $i \in \{0, \dots, n\}$ we define the differential form $d^{(h)}x_i^q$, seen as a formal symbol. This definition can be given a functorial interpretation, by considering the module of p-graded Kähler differentials as a universal object representing the functor of Hasse derivations, see [6, chap. 16].

For a function $f \in \mathcal{O}_M(U)$ we define the differentials $d^{(h)}f$ (with respect to Hasse derivatives, see also Eq. (2.13)):

$$d^{(h)}(f) = \sum_{i=0}^{n} D_{x_i}^{(h)}(f) d^{(h)} x_i^{p^h}. \tag{2.15}$$

Recall the notation $q = p^h$ and note that from Eq. (2.15) we see that $d(x_i^q) = d^{(h)}x_i^q$ which can be seen as an element in $T_P^{(*h)}M$. Without the Hasse derivatives, the differential $d(x^q)$, when computed in terms of Eq. (2.15) is zero, but here it is a generator of the alternating algebra of differential forms.

Definition 37. For $q_i = p^i$ define the formal monomials $d^{(h_1)}x_{i_1}^{q_1} \wedge d^{(h_2)}x_{i_2}^{q_2} \wedge \cdots \wedge d^{(h_j)}x_{i_j}^{q_{i_j}}$ of degree j, where for monomials m, n of degrees k and l we have

$$m \wedge n = (-1)^{kl} n \wedge m.$$

A differential form of degree i is a formal linear combination of monomials of degree p, with coefficients from $\mathcal{O}_X(U)$.

We also require that for a function f we have

$$fdx_i \wedge d^{(h)} x_j^{p^h} = dx_i \wedge f^{p^h} d^{(h)} x_j^{p^h}. \tag{2.16}$$

The above requirement is natural since that alternating algebras are defined as quotients of the tensor algebra, see [6, Appendix 2] of an ordinary form by a p^h-form. We will use this definition in Lemma 46 in order to prove that the h-conormal space is Lagrangian.

A derivation of degree $s \in \mathbb{Z}$ on $\mathcal{O}_M(U)$ is a k-linear operator sending a form of degree j to a form of degree $j+s$ such that

$$D(\omega \wedge \tau) = D\omega \wedge \tau + (-1)^{sj} \omega \wedge D\tau.$$

We will need the following derivations.

(1) The derivations $d^{(h)}$ of degree $+\ 1$, such that $d^{(h)}f$ is given by Eq. (2.15) and $d^{(h')} d^{(h)} = 0$ for all $h, h' \in \mathbb{N}$.
(2) The derivation i_X of degree -1 corresponding to vector field X, given by $i_X(\mathcal{O}_X) = 0$ while for X given by Eq. (2.14) and ω given by

$$\omega = \sum_{h=0}^{\infty} \sum_{i=0}^{r} b_{h,i}(\omega) d^{(h)} x_i^q, \text{ for } b_{h,i}(\omega) \in \mathcal{O}_X(U) \text{ we have}$$

$$i_X(\omega) = \sum_{h=0}^{\infty} \sum_{i=0}^{r} \left(a_{h,i}(X)^{p^h} b_{h,i}^{p^h}(\omega) \right). \tag{2.17}$$

Remark 38. A vector field is a section of the tangent bundle, i.e. for every $P \in X$ if the functions $a_{h,i}$ are in $\mathcal{O}_X(U)$ for an open set U containing P, then the evaluation of $a_{h,i}$ at P gives us a tangent vector in $T_P M$,

$$X(P) = \sum_{h=0}^{\infty} \sum_{i=0}^{r} a_{h,i}(X)(P) D_{x_i}^{(h)}. \tag{2.18}$$

Indeed, using the i_X derivation we see that the vector field $D_{x_i}^{(h)}$ is the dual basis element to the differential form $d^{(h)} x_i^q$. Thus, the evaluated vector field gives rise to an element in the dual space of $T_P^* M$.

Assume now that the maximal ideal at $P \in M$ is generated by $t_1, \ldots, t_s$, and consider the differentials $dt_1, \ldots, dt_s$.

The classical cotangent vector bundle (see [26, p. 60]) is the vector bundle

$$T^* M = \bigoplus_{i=1}^{r} \mathcal{O}_M dt_i.$$

A classical differential form ξ is given by

$$\xi = \sum_{i=0}^{r} \xi_i dt_i, \ \xi_i \in \mathcal{O}_M.$$

Keep in mind that a vector bundle in algebraic geometry over an open set $U \subset M$ is described in terms of $\mathbb{A}_U^r = \mathrm{Spec} \mathcal{O}_M(U)[\xi_1, \ldots, \xi_r]$, see [11, ex. 5.18, p. 128].

In analogy to the classical case, an h-differential form is given by

$$\xi^h = \sum_{i=0}^{r} \xi_i dt_i^{(h)}, \ \ \xi_i \in \mathcal{O}_M.$$

3. The case of hypersurfaces

In this section we focus on the hypersurface case. When the variety is given as the zero set of a single polynomial we can use a form of implicit-inverse function theorem which allows us to express the coordinates x_i as functions of the dual coordinates. This method works if the h-Hessian is generically invertible. In characteristic zero we consider the hypersurface $V(f) \subset \mathbb{P}^n$ given by a polynomial f, if we set $\Xi_i = D_{x_i} f \in k[\underline{x}]$, we can find the ideal in $k[\underline{\Xi}]$ by eliminating the variables $\underline{x}$. Let us illustrate this method in characteristic zero by the following

Example 39. Consider the Fermat curve given as the zero locus of

$$x_0^5 + x_1^5 + x_2^5 = 0.$$

This in magma [2] can be done as follows: If $y_i = D_{x_i} f$, we fist define the ideal

$$I = \langle x_0^5 + x_1^5 + x_2^5, \ -5x_0^4 + y_0, \ -5x_1^4 + y_1, \ -5x_2^4 + y_2 \rangle \ \triangleleft \ k[x_0, \ldots, x_2, y_0, \ldots, y_2],$$

and then we eliminate the variables x_0, x_1, x_2 using the EliminationIdeal function:

$$J = \left\langle \begin{array}{l} y_0^{20} - 4y_0^{15}y_1^5 - 4y_0^{15}y_2^5 + 6y_0^{10}y_1^{10} - 124y_0^{10}y_1^5y_2^5 + 6y_0^{10}y_2^{10} - 4y_0^5y_1^{15} - 124y_0^5y_1^{10}y_2^5 \\ -124y_0^5y_1^5y_2^{10} - 4y_0^5y_2^{15} + y_1^{20} - 4y_1^{15}y_2^5 + 6y_1^{10}y_2^{10} - 4y_1^5y_2^{15} + y_2^{20} \end{array} \right\rangle.$$

We can now consider the same elimination process, arriving at the ideal J generated by the elements

$$\begin{aligned} g_1 &= y_0^{20} - 4y_0^{15}y_1^5 - 4y_0^{15}y_2^5 + 6y_0^{10}y_1^{10} - 124y_0^{10}y_1^5y_2^5 + 6y_0^{10}y_2^{10} - 4y_0^5y_1^{15} - 124y_0^5y_1^{10}y_2^5 \\ &\quad - 124y_0^5y_1^5y_2^{10} - 4y_0^5y_2^{15} + y_1^{20} - 4y_1^{15}y_2^5 + 6y_1^{10}y_2^{10} - 4y_1^5y_2^{15} + y_2^{20} \\ g_2 &= x_0 - 20y_0^{19} + 60y_0^{14}y_1^5 + 60y_0^{14}y_2^5 - 60y_0^9y_1^{10} + 1240y_0^9y_1^5y_2^5 \\ &\quad - 60y_0^9y_2^{10} + 20y_0^4y_1^{15} + 620y_0^4y_1^{10}y_2^5 + 620y_0^4y_1^5y_2^{10} + 20y_0^4y_2^{15} \\ g_3 &= x_1 + 20y_0^{15}y_1^4 - 60y_0^{10}y_1^9 + 620y_0^{10}y_1^4y_2^5 + 60y_0^5y_1^{14} + 1240y_0^5y_1^9y_2^5 + 620y_0^5y_1^4y_2^{10} \\ &\quad - 20y_1^{19} + 60y_1^{14}y_2^5 - 60y_1^9y_2^{10} + 20y_1^4y_2^{15} \\ g_4 &= x_2 + 20y_0^{15}y_2^4 + 620y_0^{10}y_1^5y_2^4 - 60y_0^{10}y_2^9 + 620y_0^5y_1^{10}y_2^4 + 1240y_0^5y_1^5y_2^9 + 60y_0^5y_2^{14} \\ &\quad + 20y_1^{15}y_2^4 - 60y_1^{10}y_2^9 + 60y_1^5y_2^{14} - 20y_2^{19}. \end{aligned}$$

Observe that the generators g_2, g_3, g_4 express x_0, x_1, x_2 as a function of $\underline{y}$, which follows by differentiating the defining equation g_1 of the dual hypersurface with respect to y_0, y_1, y_2, i.e., $x_i = D_{y_i} g_1$ for $i = 0, 1, 2$. After elimination in the ideal J of the variables $\underline{y}$ we arrive at the original equation as expected.

Similarly, the implicit-inverse function method will allow us to solve “locally” and express Ξ_i as functions of $k[x_0, \ldots, x_n]$. The problem with this method is that Zariski topology does not have fine enough open sets for the implicit (or the equivalent inverse) function theorem to hold. Actually this was one of the reasons for inventing etalé topology [23, p. 11]. The approach of Wallace is based on defining algebraic functions in order for the implicit function theorem to work. We will follow the ideas of Wallace [29, sec. 4.1]. Let $X_1, \ldots, X_n$ be a set of indeterminates of the field k. A separable algebraic function ϕ over $k(X_1, \ldots, X_n)$ will be called a k-function of $X_1, \ldots, X_n$. If $x_1, \ldots, x_n$ is any set of elements of k and y is a specialization of ϕ over the specialization $(X_1, ..., X_n) \mapsto (x_1, ..., x_n)$, then y will be called a value of ϕ at $(x_1, ..., x_n)$, and will be written $y = \phi(x_1, ..., x_n)$. The partial derivative $\partial\phi/\partial X_i$ for each i, is a rational function of $X_1, ..., X_n$ and ϕ. If this rational function is defined at $(x_1, ..., x_n, y)$ (i.e. has non zero denominator), then the k-function ϕ will be called differentiable at $(x_1, ..., x_n, y)$.

Remark 40. If we allow k-functions then the duality theorems have a simpler form. For example for $(a, p) = 1$ the dual curve of the Fermat curve $x_0^a + x_1^a + x_2^a = 0$ is the dual curve $x_0^b + x_1^b + x_2^b = 0$ such that $\frac{1}{a} + \frac{1}{b} = 1$, see [9, Example 2.3, p. 20].

Theorem 41 (Implicit function theorem). *If $x_1, ..., x_{2n}$ satisfy the k-functions $\phi_i(x_1, ..., x_{2n}) = 0$ for $i = 1, ..., n$, differentiable at $(x_1, ..., x_{2n}, 0)$ and the Jacobian $n \times n$-matrix $(\partial\phi_i/\partial x_j)$ is invertible, then there are k-functions $f_1, ..., f_n$ of $y_1, ..., y_n$ such that $x_i = f_i(x_{n+1}, ..., x_{2n})$ for all $1 \leq i \leq n$.*

Proof. Theorem 6 in [29]. □

The above theorem in practice allows us to work with hypersurfaces as follows: Let $V(f)$ be a projective hypersurface. We put coordinates $(x_0, ..., x_n)$ on the space $\mathbb{P}^n$ and $y_0, ..., y_n$ on $\mathbb{P}^{*n}$. We have the equations:

$$y_i = \partial f/\partial x_i = \phi_i(x_0, ..., x_n). \tag{3.1}$$

If the Hessian matrix $(\partial\phi_j/\partial x_i) = (\partial^2/f\partial x_i\partial x_j)$ is not singular, then the implicit function theorem allows us to express x_i as k-functions of $y_0, ..., y_n$.

For example, in characteristic zero (or if $p \nmid a - 1$, the hypersurface defined by $f = \sum_{i=0}^n x_i^a$ has $y_i = \partial f/\partial x_i = a x_i^{a-1}$, therefore $x_i = (y_i/a)^{\frac{1}{a-1}}$. The last expression is in accordance to Theorem 41, since the Hessian matrix equals $a(a-1) \cdot \operatorname{diag}(x_0^{a-2}, ..., x_n^{a-2})$, which is generically invertible. We can arrive to the dual hypersurface by replacing x_i in the defining equation of f, i.e.

$$\sum_{i=0}^n x_i(y_0, ..., y_n)^a = a(a-1)\sum_{i=0}^n y_i^{\frac{a}{a-1}}.$$

Notice that $b = \frac{a}{a-1}$ satisfies the symmetric equation $1/a + 1/b = 1$.

If $p \mid a - 1$, then the equation $y_i = a x_i^{a-1}$ does not allow us to express x_i in terms of y_i. Keep in mind that the rational function field is not perfect, and we are not allowed to take p-roots of polynomials.

Let $V(f)$ be a hypersurface corresponding to the irreducible homogeneous and h-homogeneous polynomial f of degree prime to the characteristic. By Eq. (2.10) we have that if the Gauss map is not separable then $y_i = \partial f/\partial x_i = g_i^{p^h}(\underline{x})$. Moreover by Euler's theorem we have

$$f = \deg(f) \cdot \sum_{i=0}^n x_i g_i(\underline{x})^{p^h}.$$

In our approach we propose to consider instead of Eq. (3.1) the equations

$$y_i = D_{x_i}^{(h)}(f).$$

Then under the assumption that the "Hessian" $D_{x_j}^{(0)} D_{x_i}^{(h)} f$ is invertible we can express

$$x_i = g_i(y_0, ..., y_n),$$

where g_i is a k-function.

Remark 42. Even in characteristic zero, the Hessian might be singular. Consider for example a hyperplane $V(\sum a_i x_i)$. The first derivatives are constants and the Hessian is zero. This situation is related to the case of singular Gauss map. For a detailed study of this case in terms of classical differential geometry see [1].

A similar example can be given in positive characteristic, for example for $h \geq 1$ the polynomial $f = x_0^{p^h}$ has $D_{x_i}^{(h)} f = 1$, which has degree 0, and the Hessian $D_{x_j}^{(0)} D_{x_i}^{(h)}(f)$ is zero.

These two cases will be excluded in next lemma where we will assume that the degree of the derivatives are prime to the characteristic.

In some cases we can prove that the Hessian is invertible.

Lemma 43. *Let f be a homogeneous polynomial so that so that one at least of its derivatives $D_{x_i}^{(h)} f$, $0 \leq i \leq n$ is not zero, and all non-zero $D_{x_i}^{(h)} f$ derivatives have degree d_i prime to the characteristic p. Then the $(n+1) \times (n+1)$ matrix $D_{x_j}^{(0)} D_{x_i}^{(h)}(f)$ is generically invertible.*

Proof. Assume that the above mentioned map is not invertible, then one column, say the first one, is a linear combination of the other columns, that is

$$\begin{pmatrix} D_{x_0}^{(0)} D_{x_0}^{(h)}(f) \\ \vdots \\ D_{x_n}^{(0)} D_{x_0}^{(h)}(f) \end{pmatrix} = \sum_{\mu=1}^{n} \lambda_\mu \begin{pmatrix} D_{x_0}^{(0)} D_{x_\mu}^{(h)}(f) \\ \vdots \\ D_{x_n}^{(0)} D_{x_\mu}^{(h)}(f) \end{pmatrix} \tag{3.2}$$

Notice that if $\lambda_1 = \cdots = \lambda_n = 0$, then $D_{x_i}^{(0)} D_{x_0}^{(h)}(f) = 0$ for all $0 \leq i \leq n$, this means that

$$D_{x_0}^{(h)} \in k\left[x_0^q, \ldots, x_n^q\right],$$

and so it has degree divisible by the characteristic. Summing along each column of Eq. (3.2) after multiplying by x_ν and using Euler's theorem we have (set $d_\mu = \deg D_{x_\mu}^{(h)} f$)

$$d_0 D_{x_0}^{(h)}(f) = \sum_{\mu=1}^{n} \lambda_\mu d_\mu D_{x_\mu}^{(h)}(f). \tag{3.3}$$

Let δ_0 be the degree of the polynomial f in the variable x_0^q, $q = p^h$. The above Eq. (3.3) is impossible for $\delta_0 > 0$ by considering the degrees of both sides in the variable x_0^q, since the degree of x_0^q on the left hand side of Eq. (3.3) is less than the degree of x_0^q of the right hand side. This forces $\delta = 0$ and in this case $D_{x_0}^{(h)} f = 0$. This forces the right hand side of Eq. (3.3) to be zero, which allows us to repeat the above argument (recall that in Eq. (3.3) there is at least one more $\lambda_i \neq 0$) for another variable x_i, until we prove inductively that all derivatives $D_{x_i}^{(h)}$ are zero, a contradiction. □

Lemma 44. *Consider a function f as given in Eq. (2.10) in proposition 18. Then this function satisfies the invertible Hessian criterion of Lemma 43 and the dual variety given by equation*

$$G(y_0, \ldots, y_n) = f(x_0(y_0, \ldots, y_n), \ldots, x_n(y_0, \ldots, y_n)) = 0.$$

Let us now consider the Hasse derivatives $D_{y_\nu}^{(h)}$ for $q = p^h$ of $G(\bar{y}) = \sum_{i=0}^{n} x_i^q y_i$, where x_i are considered as functions of y_i

$$z_\nu := D_{y_\nu}^{(h)} \left(\sum_{i=0}^{n} x_i^q y_i \right) = \sum_{i=0}^{n} \left(D_{y_\nu}^{(0)} x_i \right)^q y_i.$$

We compute

$$\sum_{\nu=0}^{n} z_\nu y_\nu^q = \sum_{\nu=0}^{n}\sum_{i=0}^{n}\left(D_{y_\nu}^{(0)} x_i\right)^q y_i y_\nu^q. \tag{3.4}$$

Since $x_i(y_0, ..., y_n)$ is homogenous in the variables $y_0, ..., y_n$ as well the classical Euler identity gives us that

$$\sum_{\nu=0}^{n} y_\nu D_{y_\nu}^{(0)} x_i = cx_i \text{ for some } c \in k,$$

so Eq. (3.4) gives us

$$\sum_{\nu=0}^{n} z_\nu y_\nu^q = c\sum_{i=0}^{n} x_i^q y_i = 0.$$

This means that the point $\bar{x} = (x_0, ..., x_n) \in V(f) \subset \mathbb{P}^n$ has the q-hyperplane

$$(X_0 : \cdots : X_n) \in \mathbb{P}_k^n \text{ such that } \sum_{\nu=0}^{n} y_i X_i^q = 0$$

with coordinates $(y_0, ..., y_n)$ as h-tangent and the point $(y_0, ..., y_n) \in V(G) \subset \mathbb{P}^{h*}$ has the q-hyperplane

$$(Y_0 : \cdots : Y_n) \in \mathbb{P}_k^n \text{ such that } \sum_{\nu=0}^{n} z_\nu Y_\nu^q = 0$$

with coordinates $(z_0, ..., z_n)$ as h-tangent. Reflexivity essentially means that the map

$$\begin{aligned} V \times V^{*h} &\rightarrow V^{*h} \times \left(V^{*h}\right)^{*h} \\ (x, y) &\mapsto (y, F(x)) \end{aligned}$$

induces an isomorphism from $\mathrm{Con}^{(h)}(X)$ to $\mathrm{Con}^{(h)}(Y)$, where F is the isomorphism $F : V \longrightarrow (V^{*h})^{*h}$ introduced in Theorem 12. For proving this we will require the notion of Lagrangian variety for algebraic sets defined over the field of complex numbers.

3.1. Example: A class of Fermat hypersurfaces

Let $p \neq 2$ be a prime. Consider the hypersurface

$$\sum_{i=0}^{n} x_i^{2p+1} = 0.$$

We set also $y_i = D_{x_i}^{(h)} f = 2x_i^{p+1}$. We can express x_i in terms of y_i, that is

$$x_i = \left(\frac{1}{2} y_i\right)^{\frac{1}{p+1}}.$$

The dual variety is then described as the zero set of the k-function

$$G(y_0, ..., y_n) = \sum_{i=0}^{n} y_i^{\frac{2p+1}{p+1}} = 0.$$

We now compute the derivatives $z_i = D_{y_i}^{(h)} G = cx_i^{p-p^2+1}$ for some $c \in k$. We now expand

$$\begin{aligned}0=\left(\sum_{i=0}^{n}x_i^{2p+1}\right)^{1+p-p^2} &= \left(\sum_{i=0}^{n}x_i^{2p+1}\right)\left(\sum_{j=0}^{n}x_j^{p(2p+1)}\right)\left(\sum_{k=0}^{n}x_k^{-p^2(2p+1)}\right)\\ &= \sum_{i=0}^{n}x_i^{(2p+1)(1+p-p^2)}+\sum_{i=0}^{n}x_i^{2p+1}\sum_{\substack{j=0\\ j\neq i}}^{n}x_j^{p(2p+1)}\sum_{\substack{k=0\\ k\neq i}}^{n}x_k^{-p^2(2p+1)}\\ &= \sum_{i=0}^{n}x_i^{(2p+1)(1+p-p^2)}=c^{-1}\sum_{i=0}^{n}z_i^{2p+1}.\end{aligned}$$

This proves that $(z_0,\dots,z_n)$ are in $V(f)$.

4. Lagrangian varieties

4.1. h-Cotangent bundle and h-Lagrangian subvarieties

The space $V\times V^{*h}$ can be identified to the h-cotangent bundle $T^{(*h)}(V)$ of V. Let $x_0,\dots,x_n$ be a set of coordinates on V and $\xi_0,\dots,\xi_n$ be a set of coordinates on V^{*h}. Notice that a vector field (here we use vector fields which have non-zero coefficients only at a certain value of h)

$$X=\sum_{j\in\{0,h\}}\sum_{\nu=0}^{n}a_\nu(X)D_{x_\nu}^{(j)}+\sum_{j\in\{0,h\}}\sum_{\nu=0}^{n}b_\nu(X)D_{\xi_\nu}^{(j)} \tag{4.1}$$

by Eq. (2.17) acts on differential forms in terms of the derivation i_X by the rule:

$$\begin{aligned} i_X\left(d^{(h)}x_i^q\right)&=a_i(X)^{p^h}, & i_X(d^{(0)}\xi_i)&=b_i(X),\\ i_X\left(d^{(0)}x_i\right)&=a_i(X), & i_X(d^{(h)}\xi_i^q)&=b_i(X)^{p^h}.\end{aligned}$$

Consider $M\subset\mathbb{P}(V)$ a projective variety and consider the cone $M'\subset V$ seen as an affine variety in V. Assume that the homogeneous ideal of M' is generated by the homogeneous polynomials $f_1,\dots,f_r$, and the set of h-nonsingular points of M is non-empty. Consider the $n+1$-upple

$$\nabla^{(h)}f_i=\left(D_0^{(h)}f_i(P),D_1^{(h)}f_i(P),\dots,D_n^{(h)}f_i(P)\right).$$

Each f_i defines an h-linear form given by

$$L_i^{(h)}:=\sum_{\nu=0}^{n}D_\nu^{(h)}f_i(P)x_\nu^{p^h}. \tag{4.2}$$

The h-tangent space at P is the variety defined by the equations $L_i^{(h)}=0$. Recall the definition of $\mathrm{Lag}^{(h)}(M)\subset V\times V^{*h}$,

$$\mathrm{Lag}^{(h)}(M)=\overline{\left\{(P,H):P\in\mathrm{Cone}(M^h_{\mathrm{sm}}),H\text{ is a }p^h-\text{linear form which vanishes on }T_P^{(h)}M\right\}}.$$

Let $\pi_1:V\times V^{*h}\to V$ be the first projection. We have seen in the introduction, that for every $P\in M^h_{\mathrm{sm}}$ the set $\pi_1^{-1}\cap\mathrm{Lag}^{(h)}(M)$ can be identified to the space of p^h-linear forms on the h-normal space $N_P^{(h)}(M)$ defined as

$$N_P^{(h)}(M)=T_P^{(h)}V/T_P^{(h)}(M)\cong V/T_P^{(h)}(M).$$

Also by the definition of $T_P^{(h)}M$ the fiber of the h-conormal space at the point P for a projective variety defined by the elements $f_1,\dots,f_r$ is the vector subspace of V^{*h} spanned by $L_i^{(h)}$ given in Eq. (4.2):

$$\mathrm{Lag}_P^{(h)}(M)=\langle L_i^{(h)}:1\le i\le r\rangle_k.$$

4.2. The symplectic structure on $V \times V^{*h}$

Definition 45. Let x_i, ξ_i be coordinates on the vector spaces V, V^{*h} respectively.

A subvariety Λ of $V \times V^{*h}$ with non empty h-nonsingular locus will be called conical h-Lagrangian if

(1) The form $\omega = \sum_{j=0}^{n} d^{(h)} x_j^q \wedge d\xi_j + \sum_{j=0}^{n} d^{(h)} \xi_j^q \wedge dx_j$ is zero on Λ.
(2) $\dim\Lambda = n$
(3) If $(P, H) = (x_0, \ldots, x_n, \xi_0, \ldots, \xi_n) \in \Lambda$ then $(\mu P, \lambda H) = (\mu x_0, \ldots, \mu x_n, \lambda\xi_0, \ldots, \lambda\xi_n) \in \Lambda$ for every, $\mu, \lambda \in k^*$.

Notice that if

$$X = \sum_{i\in\{0,h\}} \sum_{\nu=0}^{n} a_{i,\nu}(X) D_{x_\nu}^{(i)} + \sum_{i\in\{0,h\}} \sum_{\nu=0}^{n} b_{i,\nu}(X) D_{\xi_\nu}^{(i)},$$

then

$$\begin{aligned} &\omega(X, Y) := i_Y i_X \omega = \\ &= \sum_{\nu=0}^{n} \left(a_{h,\nu}(X)^{p^h} b_{0,\nu}(Y) - a_{h,\nu}(Y)^{p^h} b_{0,\nu}(X) - a_{0,\nu}(X) b_{h,\nu}(Y)^{p^h} + a_{0,\nu}(Y) b_{h,\nu}(X)^{p^h} \right). \end{aligned} \tag{4.3}$$

If one restricts on $(h, 0)$-tangent vectors, i.e. $a_{0,i}(X) = a_{0,i}(Y) = b_{h,i}(X) = b_{h,i}(Y) = 0$ for all i, then the above computation is compatible with the definition given in Sec. 2.3 since in this case

$$\omega(X, Y) := \sum_{\nu=0}^{n} \left(a_{h,\nu}(X)^{p^h} b_{0,\nu}(Y) - a_{h,\nu}(Y)^{p^h} b_{0,\nu}(X) \right).$$

Lemma 46. *Assume that h is selected such that $\pi_2 : \mathrm{Lag}^{(h)} M \to \mathrm{Im}\pi_2 = Z$ is separable. If $M_{\mathrm{sm}}^h \neq \emptyset$, then the conormal bundle $\mathrm{Lag}^{(h)}(M)$ is a Lagrangian manifold of $V \times V^{*h}$.*

Proof. Assume that M is the zero locus of the homogeneous polynomials $F_1, \ldots, F_r$. When we restrict ourselves to $\mathrm{Lag}^{(h)}(M)$ we have that

$$\xi_j = \sum_{i=1}^{r} \lambda_i \left(D_j^{(h)} |_P F_i \right) \qquad \lambda_i \in k,$$

and

$$d\xi_j = \sum_{i=1}^{r} \lambda_i d D_j^{(h)} \Big|_P F_i = \sum_{i=1}^{r} \lambda_i \sum_{\nu=0}^{n} D_\nu^{(0)} \Big|_P D_j^{(h)} \Big|_P F_i dx_\nu.$$

This means that the first summand of ω restricted to $\mathrm{Lag}^{(h)}(X)$ has the form

$$\sum_{j=0}^{n} d^{(h)} x_j^q \wedge d\xi_j = \sum_{i=1}^{r} \lambda_i \sum_{j=0}^{n} \sum_{\nu=0}^{n} D_\nu^{(0)} \Big|_P D_j^{(h)} \Big|_P F_i \; d^{(h)} x_j^q \wedge dx_\nu.$$

In a similar way we have, using Eq. (2.16)

$$\begin{aligned} \sum_{j=0}^{n} dx_j \wedge d^{(h)} \xi_j^q &= \sum_{j=0}^{n} dx_j \wedge \left(\sum_{i=1}^{r} \lambda_i \sum_{\nu=0}^{n} D_\nu^{(0)} \Big|_P D_j^{(h)} \Big|_P F_i \right)^q d^{(h)} x_\nu^q \\ &= \sum_{j=0}^{n} \sum_{i=1}^{r} \lambda_i \sum_{\nu=0}^{n} D_\nu^{(0)} \Big|_P D_j^{(h)} \Big|_P F_i \; dx_j \wedge d^{(h)} x_\nu^q \end{aligned}$$

Therefore the form

$$\omega = \sum_{j=0}^{n} d^{(h)} x_j^q \wedge d\xi_j + \sum_{j=0}^{n} d^{(h)} \xi_j^q \wedge dx_j$$

is zero on Λ.

We now compute the dimension of $\mathrm{Lag}^{(h)}(M)$. If P is an h-nonsingular point, then the dimension of the h-tangent space equals $\dim M$, therefore the dimension of the conormal space is $n - r$ and the dimension of $\mathrm{Lag}^{(h)}(M) = \dim(M) + n - r = n$.

Finally, if $(x_0, ..., x_n, \xi_0, ..., \xi_n) \in \mathrm{Lag}^{(h)}(M)$ then it is obvious that for $\mu, \lambda \in k^*$ the element $(\mu x_0, ..., \mu x_n, \lambda\xi_0, ..., \lambda\xi_n)$ is an element of $\mathrm{Lag}^{(h)}(M)$ as well. $\square$

Definition 47. A map $f : X \to Y$ between varieties will be called generically smooth if the induced map $f_* : T_P^{(0)} X \to T_{f(P)}^{(0)} Y$ is surjective for an open dense subset $U \subset X$.

Similarly we will call a map $f : X \to Y$ h-generically smooth if the induced map $f_* : T_P^{(h)} X \to T_{f(P)}^{(h)} Y$ is surjective for an open dense subset $U \subset X$ such that $f(U)$ is an open dense subset of Y.

Remark 48. Proposition 34 implies that if f is generically smooth then it is h-generically smooth. Also if the function field extension $k(X)/k(Y)$ is separable, then there is an open set U so that f_* is smooth for all points in U [22, p. 169], [15, p. 68].

Remark 49. We consider the identification $F : V \to (V^{*h})^{*h}$ given in Theorem 12. Define the map $\Psi : V \times V^{*h} \to V^{*h} \times (V^{*h})^{*h}$ given by sending $\Psi : (v, w) \mapsto (w, F(v))$. Notice that if $\bar{c}, \bar{b}$ are the coordinates of V^{*h}, $(V^{*h})^{*h}$, then the coordinates in $V \times V^{*h}$ are given by $(\bar{b}, \bar{c})$, see also the diagram in Eq. (2.5).

The following is essential for proving reflexivity.

Proposition 50. *Let $\pi_2 : V \times V^{*h} \to V^{*h}$ be the second projection. A conical Lagrangian variety $\Lambda \subset V \times V^{*h}$ has projection $Z = \pi_2(\Lambda) \subset V^{*h}$. If the set of h-smooth points of Z forms an non-empty dense open subset of Z and the map $\pi_2 : \Lambda \to Z \subset V^{*h}$ is h-generically smooth, then the conormal variety $\mathrm{Lag}^{(h)}(Z) \subset V^{*h} \times V$ coincides with Λ under the natural identification $\Psi : V \times V^{*h} \to V^{*h} \times V$.*

Proof. This proof follows [9, prop. 3.1], see also [27]. The set of h-smooth points of Λ is non empty by definition 45, so it is an open dense set of the irreducible variety Λ. The projection $\pi_2(\Lambda) = Z$ is irreducible since Λ is irreducible. By assumption the map $\pi_2 : \Lambda \to Z$ is h-generically smooth, so we can find an open dense set $\Lambda_0 \subset \Lambda$, consisted of h-smooth points with the additional property that $\pi_2(\Lambda_0) = Z_0$ consists also of h-smooth points and moreover the induced map $\pi_{2,*}$ forms a surjective map

$$T_P^{(h)} \Lambda \to T_{\pi_2(P)}^{(h)} Z.$$

Since π_2 is h-generically smooth we can see every element $v \in T_P^{(h)} Z$ written as

$$v = \sum_{i=0}^{n} a_i D_{\xi_i}^{(h)},$$

as the image of an element $v' \in T_Q^{(h)} \Lambda$ which can be written as

$$v' = \sum_{i=0}^{n} 0 \cdot D_{x_i}^{(0)} + \sum_{i=0}^{n} a_i{}' \cdot D_{x_i}^{(h)} + \sum_{i=0}^{n} 0 \cdot D_{\xi_i}^{(0)} + \sum_{i=0}^{n} a_i D_{\xi_i}^{(h)} \tag{4.4}$$

for a selection of values $a_i' \in k$. In selecting Λ_0 and Z_0 it is essential that open non-empty sets in irreducible varieties are dense. We have the following diagram:

$$\begin{tikzcd}
V \times V^{*h} \arrow[d, "\pi_2"] \arrow[rrrr, bend left, "\cong \Psi"] & \Lambda \arrow[l, hook'] \arrow[d, "\pi_2"] \arrow[rr, bend left, "\cong"] & \Lambda_0 \arrow[l, hook'] \arrow[d, "\pi_2"] & \mathrm{Lag}^{(h)}(Z) \arrow[r, hook] \arrow[d] & V^{*h} \times V \arrow[dl, "\pi"] \\
V^{*h} & Z \arrow[l, hook'] & Z_0 \arrow[l, hook'] \arrow[r, equal] & Z_0 &
\end{tikzcd}$$

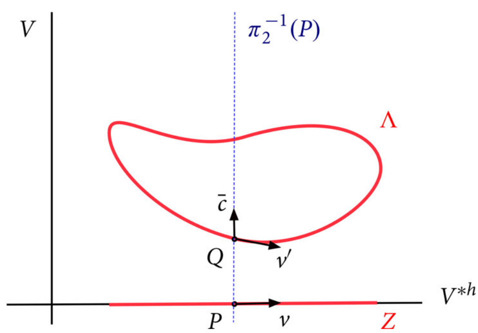
The map Ψ is the map sending $(x, y) \in V \times V^{*h}$ to $(y, x) \in V^{*h} \times V$.

In what follows we consider $V^{*h} \times V \to V^{*h}$ as a (trivial) vector bundle, which is identified, using Theorem 12, to the h-cotangent bundle $T^{(*h)}(V)$ of V and to the h-cotangent bundle of V^{*h},

$$T^{(*h)}(V^{*h}) = V^{*h} \times (V^{*h})^{*h} \cong V^{*h} \times V \xrightarrow{\pi} V^{*h}. \tag{4.5}$$

Keep in mind that the h-cotangent bundle of a vector space V is $V \times V^{*h}$. The h-cotangent bundle of V^{*h} is $V^{*h} \times (V^{*h})^{*h}$.

The space $V^{*h} \cong V^{*h} \times \{0\}$ is considered as the zero section of the bundle $T^{*h}(V^{*h}) = V^{*h} \times V$. Each point $z \in V^{*h}$ has the vector space V as fiber.



We will prove first that for any h-smooth point $P \in Z_0 \neq \emptyset$, $\Psi(\pi_2^{-1}(P) \cap \Lambda) \subset \mathrm{Lag}^{(h)}(Z)$. Let $\bar{b} = (b_0, b_1, \ldots, b_n)$ be the coordinates of $P \in Z_0 \subset V^{*h}$. A point $Q = \pi^{-1}(P) \cap \Lambda$ has coordinates $(\bar{b}, \bar{c})$. The second part of coordinates of Q given by $\bar{c} = (c_0, \ldots, c_n)$ corresponds to an element in $V \cong (V^{*h})^{*h}$ seen as the vertical fiber $\pi^{-1}(P)$ of the vector bundle in Eq. (4.5). Since the fiber $\pi^{-1}(P)$ is the vector space V, we can identify V with its tangent space and write

$$\bar{c} = \sum_{i=0}^{n} c_i D_{x_i}^{(0)}.$$

This element is considered as an element in the tangent space $T^{(0)}_{(\bar{b},0)} T^{(*h)}(V^{*h})$ of the point $(\bar{b}, 0)$ of the zero section of π. Since Λ is conical the element $\bar{c}$ can be also considered as a tangent vector of Λ at $Q = (\bar{b}, \bar{c})$.

Since ω is zero on Λ, for v' defined in Eq. (4.4) and $\bar{c}$ given as follows

$$\bar{c} = \sum_{i=0}^{n} c_i \cdot D_{x_i}^{(0)} + \sum_{i=0}^{n} 0 \cdot D_{x_i}^{(h)} + \sum_{i=0}^{n} 0 \cdot D_{\xi_i}^{(0)} + \sum_{i=0}^{n} 0 D_{\xi_i}^{(h)}$$

we have by Eq. (4.3)

$$0 = \omega(\bar{c}, v') = \sum_{i=0}^{n} a_i^q c_i,$$

that is the q-linear form

$$(x_0, ..., x_n) \mapsto \sum_{i=0}^{n} x_i^q c_i,$$

vanishes on the element of the tangent space with coordinates $\bar{a}$. The last equation implies that we can see $\Psi(\xi) = (\bar{b}(P), \bar{c}(P)) \in V^{*h} \times V$ as an element in $\mathrm{Lag}^{(h)} Z_0 \subset V^{*h} \times V$, so $\Psi(\pi^{-1}(P) \cap \Lambda) \subset \mathrm{Lag}^{(h)}(Z)$, notice that the coordinates $\bar{b}$ of P satisfy the defining equations of Z, while the coordinates of $\bar{c}$, satisfy the defining equations of $\pi_1(\Lambda)$.

We thus arrive to the desired inclusion $\Psi(\pi^{-1}(P) \cap \Lambda_0) \subset \mathrm{Con}^{(h)}(Z)$ and $\Psi(\Lambda_0)$ is a dense subset of the same dimension of the irreducible variety $\mathrm{Lag}^{(h)}(Z)$, therefore $\Psi(\Lambda) = \mathrm{Lag}^{(h)}(Z)$. □

Theorem 51 (Reflexivity). *Let $M \in \mathbb{P}(V)$ be an irreducible, reduced projective variety generated by h-homogeneous elements, which also has a non-empty h-nonsingular locus. Assume that $Z := \pi_2(\mathrm{Lag}^{(h)}(M))$ has a nonempty open set of h-nonsingular points and that the map*

$$\pi_2 : V \times V^{*h} \supset \mathrm{Lag}^{(h)}(M) \to \pi_2(\mathrm{Lag}^{(h)}(M)) := Z \subset V^{*h}$$

is generically smooth. Then

$$\Psi(\mathrm{Lag}^{(h)}(M)) = \mathrm{Con}^{(h)}(Z) \subset V^{*h} \times (V^{*h})^{*h} = V^{*h} \times V.$$

Proof. The conormal variety $\mathrm{Lag}^{(h)}(M)$ which is originally defined as a subset of $V \times V^{*h}$ can be also seen through Ψ as a subset of $V^{*h} \times V \cong V^{*h} \times (V^{*h})^{*h}$ and by symmetry it is still Lagrangian of dimension n.

Let us now prove that the map π_2 is h-generically smooth. Let $f : X \to Y$ be a map and suppose that P is a smooth point of X and $f(P)$ is a smooth point of Y and $f_* T_P X \to T_{f(P)} Y$ is surjective. If X, Y are irreducible then such a point P exists since the space of smooth points is a non-empty open dense subset and f is generically smooth.

Set $\Lambda = \mathrm{Lag}^{(h)}(M)$. The map $\pi_2 : V \times V^{*h} \supset \mathrm{Lag}^{(h)}(M) \to \pi_2(M) := Z \subset V^{*h}$ is assumed to be generically smooth and by proposition 34 the natural map $T_P^{(h)} \Lambda \to T_{\pi_2(P)}^{(h)} Z$ is the p^h-power of the classical differential $d\pi_2 : T_P^{(0)} \Lambda \to T_{\pi_2(P)}^{(0)} Z$.

There is an open dense set U of Λ such that for every $P \in U$ we have $n = \dim \Lambda = T_P^{(0)} \Lambda$ (is a classical nonsingular point) and $\dim \mathrm{Im}(d\pi_2) = \dim T_{\pi_2(P)} Z = \dim Z$ (surjective differential and $\pi_2(P)$ is classical nonsingular) and moreover that $\pi_2(P)$ is h-nonsingular point of Z. Notice that non-empty sets of irreducible varieties are dense and hence have nonempty intersection. But $\dim \ker(d\pi_2(P)) = \dim \ker(d\pi_2^{p^h}(P))$ hence we obtain

$$\begin{aligned} n &= \dim \ker(d\pi_2(P)) + \dim \mathrm{Im}(d\pi_2(P)) \\ &= \dim \ker(d\pi_2(P))^{p^h} + \dim \mathrm{Im}(d\pi_2(P))^{p^h}. \end{aligned}$$

The assumption that $\pi_2(P)$ is h-nonsingular gives us that $\dim T^{(h)}_{\pi_2(P)} Z = \dim Z$. On the other hand $\dim \mathrm{Im}(d\pi_2(P))^{p^h} = \dim Z$ and since $\mathrm{Im}(d\pi_2(P)^{p^h}) \subset T^{(h)}_{\pi_2(P)} Z$ if $\pi_2(P)$ is h-nonsingular, that is $\dim T^{(h)}_{\pi_2(P)} Z = \dim Z$ we finally have that $d\pi_2(P)^{p^h}$ is surjective. Since we proved that π_2 is generically smooth reflexivity follows by Theorem 50. $\square$

Let $M \subset \mathbb{P}(V)$ be an irreducible, reduced projective variety. We can form the connical h-Lagrangian $\mathrm{Lag}^{(h)} M \subset V \times V^{*h}$ which has a nonempty open set of h-nonsingular points and also form the h-dual variety $Z = \pi_2(\Lambda)$, where $\pi_2 : V \times V^{*h} \to V^{*h}$ is the second projection. The set Z is irreducible but determining whether the set of h-nonsingular points is non-empty is a subtle problem. Irreducible algebraic sets are known to have open dense sets of classical nonsingular points. For proving a reflexivity theorem we need the set of h-nonsingular points of Z to be non-empty, hence dense subset of Z. When M is a hypersurface we have given conditions in Lemma 43 so that Z has non-empty set of h-nonsingular points. The condition of h-nonsingular points requires a computation of the dimension of the algebraic set. Understanding the dimension of the dual variety Z is a subtle task, see [1, 5, 8, 17, 21, sec. 2.5]. Let us treat here the following case

Proposition 52. *Let M be a complete intersection described as the zero locus of r polynomials $F_1, \ldots, F_r$ and $\dim M = n - r$, such that all Hasse derivatives $D^{(h)}_{x_j} F_i$ have degree prime to the characteristic. Then the dual variety is a hypersurface. If moreover all Hasse derivatives $D^{(h)}_{x_j} F_i$ have zero h-derivatives for all $i = 0, \ldots, n$ and $1 \le j \le r$ then the dual hypersurface has non-empty h-singular locus.*

Proof. In this case we can prove that Z has dimension $n-1$ since the coordinates $(\xi_0, \ldots, \xi_n)$ are given by

$$\begin{pmatrix} \xi_0 \\ \vdots \\ \xi_n \end{pmatrix} = \sum_{i=1}^{r} \lambda_i \begin{pmatrix} D^{(h)}_{x_0} F_i \\ \vdots \\ D^{(h)}_{x_n} F_i \end{pmatrix}. \tag{4.6}$$

We now compute the $(n+1) \times (n+1)$-matrix

$$\begin{pmatrix} D^{(0)}_{x_0} \xi_0 & \cdots & D^{(0)}_{x_0} \xi_n \\ \vdots & & \vdots \\ D^{(0)}_{x_n} \xi_0 & \cdots & D^{(0)}_{x_n} \xi_n \end{pmatrix} = \sum_{i=1}^{r} \lambda_i \begin{pmatrix} D^{(0)}_{x_0} D^{(h)}_{x_0} F_i & \cdots & D^{(0)}_{x_0} D^{(h)}_{x_n} F_i \\ \vdots & & \vdots \\ D^{(0)}_{x_n} D^{(h)}_{x_0} F_i & \cdots & D^{(0)}_{x_n} D^{(h)}_{x_n} F_i \end{pmatrix}. \tag{4.7}$$

If the elements F_i have at least a derivative $D^{(h)}_{x_\mu}(F_i)$ which is not zero, and degrees $d_{\mu,i}$ which are prime to p, then by Lemma 43 we obtain that each matrix summand in the right hand side of Eq. (4.7) is generically invertible. Without loss of generality we can assume that for $\lambda_1 = 1, \lambda_2 = \cdots = \lambda_r = 0$ the matrix in Eq. (4.7) is invertible (change projective coordinates in the projective space $\mathbb{P}^r$ if not.) In this case the subvariety Z_λ of the projection Z cut out by equations $\lambda_2 = \cdots = \lambda_r$ is locally isomorphic to our original variety M by using Wallace inverse function construction, which allows as to express $(x_0, \ldots, x_n)$ in terms of $(\xi_0, \ldots, \xi_n)$. The dual variety is then ruled in projective spaces with base Z_λ and has dimension equal to

$$\dim Z_\lambda + r - 1 = \dim M + r - 1 = n - r + r - 1 = n - 1.$$

This means that Z is a hypersurface defined as the zero locus of the polynomial $G(\xi_0, \ldots, \xi_n)$. If Z has empty set of h-nonsingular points, then all Hasse derivatives $D^{(h)}_{\xi_\nu} G = 0$ for $0 \le \nu \le n$

and this means that G has degree smaller than q. Using Eq. (4.6) we can write G as a function of $x_0, ..., x_n$, depending on $\lambda_1, ..., \lambda_r$. Then G is zero on M and this means that $G(\xi_0(x_0, ..., x_n), ..., \xi_n(x_0, ..., x_n))$ is in the ideal generated by $F_1, ..., F_r$. Let us write

$$G(\xi_0(\bar{x}), ..., \xi_n(\bar{x})) = \sum_{i=1}^{r} g_i(\bar{x})F_i(\bar{x}). \tag{4.8}$$

The chain rule gives us that (recall we assumed that $D^{(h)}_{\xi_\nu}G = 0$ for $0 \le \nu \le n$)

$$\left(D^{(0)}_{x_0}G, ..., D^{(0)}_{x_n}G, D^{(h)}_{x_0}G, ..., D^{(h)}_{x_n}G\right) = \left(D^{(0)}_{\xi_0}G, ..., D^{(0)}_{\xi_n}G, 0, ..., 0\right)\left(\begin{array}{c|c} J & J' \\ \hline 0 & J^{p^h}\end{array}\right)$$

where

$$J = \begin{pmatrix} D^{(0)}_{x_0}\xi_0 & \cdots & D^{(0)}_{x_n}\xi_0 \\ \vdots & & \vdots \\ D^{(0)}_{x_0}\xi_n & \cdots & D^{(0)}_{x_n}\xi_n \end{pmatrix} \text{ and } J' = \begin{pmatrix} D^{(h)}_{x_0}\xi_0 & \cdots & D^{(h)}_{x_n}\xi_0 \\ \vdots & & \vdots \\ D^{(h)}_{x_0}\xi_n & \cdots & D^{(h)}_{x_n}\xi_n \end{pmatrix}$$

Therefore if $D^{(h)}_{x_i}\xi_j = 0$ for all $0 \le i, j \le n$ then

$$(D^{(h)}_{x_0}G, ..., D^{(h)}_{x_n}G) = (0, ..., 0).$$

In this case we have by Eq. (4.8)

$$\left(D^{(h)}_{x_0}G(P), ..., D^{(h)}_{x_n}G(P)\right) = \sum_{i=1}^{r} g_i(P)\left(D^{(h)}_{x_0}F_i(P), ..., D^{(h)}_{x_n}F_i(P)\right).$$

But the vectors

$$\nabla^{(h)}F_i = \left(D^{(h)}_{x_0}F_i(P), ..., D^{(h)}_{x_n}F_i(P)\right)$$

are linear independent for every point P in the non-empty set U containing all h-nonsingular points. This means that for all $P \in U$ $g_i(P) = 0$ for $1 \le i \le r$, which in turn implies that g_i are zero polynomials and G is also zero, a contradiction. □

4.3. Examples

Consider the complete intersection in $\mathbb{P}^n_k$ given by ($\bar{\lambda} = (\lambda_1, ..., \lambda_{n-2})$)

$$C^k(\bar{\lambda}) := \left\{ \begin{array}{ccc} x_0^k + x_1^k + x_2^k & = & 0 \\ \lambda_1 x_0^k + x_1^k + x_3^k & = & 0 \\ \vdots & \vdots & \vdots \\ \lambda_{n-2}x_0^k + x_1^k + x_n^k & = & 0 \end{array} \right\} \subset \mathbb{P}^n_k. \tag{4.9}$$

These curves are called "generalized Fermat curves," see [16]. We consider the matrix of ∇f_i written as rows,

$$\begin{pmatrix} kx_0^{k-1} & kx_1^{k-1} & kx_2^{k-1} & 0 & \dots & 0 \\ \lambda_1 kx_0^{k-1} & kx_1^{k-1} & 0 & kx_3^{k-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \lambda_{n-2}kx_0^{k-1} & kx_1^{k-1} & 0 & \dots & 0 & kx_n^{k-1} \end{pmatrix}. \tag{4.10}$$

The conormal space is the subspace in V^* of linear forms spanned by the linear forms

$$L_i = \sum_{\nu=0}^{n} D_{x_i}^{(0)} f_i X_i.$$

Consider an arbitrary element in the span of L_i, $\mu_0, \dots, \mu_{n-2} \in k$:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = k \begin{pmatrix} \sum_{\nu=0}^{n-2} \mu_\nu \lambda_\nu x_0^{k-1} \\ \sum_{\nu=0}^{n-2} \mu_\nu x_1^{k-1} \\ \mu_0 x_2^{k-1} \\ \vdots \\ \mu_{n-2} x_n^{k-1} \end{pmatrix}. \tag{4.11}$$

The ordinary Lagrangian space is given by

$$\mathrm{Lag}(C^k(\bar{\lambda})) = \left\{ (x_0, \dots, x_n, y_0, \dots, y_n) : \begin{array}{l} \text{where } x_0, \dots, x_n \text{ satisfy eq. (4.9)} \\ \text{and } y_0, \dots, y_n \text{ eq. (4.11)} \end{array} \right\}.$$

The image of the projection π_2 is a codimensional 1 subvariety, hence a hypersurface given by a single polynomial $F(y_0, \dots, y_n) = 0$. Finding this polynomial F explicitly is a complicated task in this case. If $p|k-1$ it is clear by Eq. (4.11) that y_i are given as polynomials of x_i^p and the map π_2 cannot be separable, hence reflexivity fails.

Let us study the conormal space of the dual variety $Z = \pi_2(\mathrm{Con}(C^k \bar{\lambda}))$. We see equations (4.11) as parametric equations with parameters $\mu_0, \dots, \mu_{n-2}$. In this case we have that the tangent space is generated by the vectors

$$V_i := \left(\frac{\partial y_i}{\partial \mu_0}, \frac{\partial y_i}{\partial \mu_1}, \dots, \frac{\partial y_i}{\partial \mu_{n-2}} \right) = \left(\lambda_i x_0^{k-1}, x_1^{k-1}, 0, \dots, 0, x_i^{k-1}, 0, \dots, 0 \right) \text{ for } 0 \le i \le n-2,$$

which are subject to the additional condition

$$\nabla F \perp V_i \mathrm{i.e.} \langle \nabla \mathrm{F}, \mathrm{V_i} \rangle = 0. \tag{4.12}$$

In order to study further Eq. (4.12) we consider the following cases:

- If $(k-1, p) = 1$ then we obtain:

$$\begin{aligned} x_0 &= \left(\frac{y_0}{k \sum_{\nu=0}^{n-2} \mu_\nu \lambda_\nu} \right)^{\frac{1}{k-1}} \\ x_1 &= \left(\frac{y_1}{k \sum_{\nu=0}^{n-2} \mu_\nu} \right)^{\frac{1}{k-1}} \\ x_i &= \left(\frac{y_i}{k \mu_{i-2}} \right)^{\frac{1}{k-1}} \qquad \text{for } 2 \le i \le n-2. \end{aligned} \tag{4.13}$$

This way we obtain a relative curve $X \to \mathbb{P}_k^{n-1}$, where $[\mu_0 : \cdots : \mu_{n-2}]$ serve as projective coordinates of $\mathbb{P}_k^{n-1}$. The precise equations in terms of algebraic functions are given by:

$$G_i = \lambda_i \left(\frac{y_0}{k\sum_{\nu=0}^{n-2} \mu_\nu \lambda_\nu} \right)^{\frac{k}{k-1}} + \left(\frac{y_1}{k\sum_{\nu=0}^{n-2} \mu_\nu} \right)^{\frac{k}{k-1}} + \left(\frac{y_{i+2}}{k\mu_i} \right)^{\frac{k}{k-1}} = 0 \qquad \text{for } 0 \le i \le n-2.$$

The polynomial F can be computed by eliminating $\mu_0, \dots, \mu_{n-2}$ from the system of the G_i. We compute (over the open set $\mu_0\mu_1 \cdots \mu_{n-2} \neq 0$)

$$\nabla G_i = \frac{k}{k-1} \left(\lambda_i \left(\frac{y_0}{k\sum_{\nu=0}^{n-2} \mu_\nu \lambda_\nu} \right)^{\frac{1}{k-1}}, \left(\frac{y_1}{k\sum_{\nu=0}^{n-2} \mu_\nu} \right)^{\frac{1}{k-1}}, \dots, \left(\frac{y_{i+2}}{k\mu_i} \right)^{\frac{1}{k-1}}, \dots, 0 \right)$$
$$= \frac{k}{k-1} (\lambda_i x_0, x_1, 0 \dots, 0, x_i, 0, \dots, 0).$$

Therefore, the compatibility condition given in Eq. (4.12) can be replaced by the conditions:

$$V_i \perp \nabla G_j \text{ i.e. } \langle V_i, \nabla G_j \rangle = 0 \text{ for all } 0 \le i, j \le n-2. \tag{4.14}$$

We can now confirm that the conditions given in (4.14) are equivalent to the original defining equations for our curve. It is clear now that the vector $(x_0, x_1, \dots, x_n)$ is normal to every generator of the tangent space of the dual variety Z hence

$$\mathrm{Con}(Z) = \{(y_0, \dots, y_n, x_0, \dots, x_n) : F(y_0, \dots, y_n) = 0\} = \mathrm{Con}(C^k(\bar{\lambda})).$$

In our computation it was essential that we were able to express x_i for $0 \le i \le n-2$ in terms of y_i for $0 \le i \le n-2$ in Eq. (4.13). This could not be done if $p \mid k-1$. We now proceed to the extreme case $k-1$ is a power of p.

- Assume that $k = q+1$ for $q = p^h$. Then instead of the matrix given in Eq. (4.10) we consider the matrix of $\nabla^{(h)} f_i$ given as

$$\begin{pmatrix} x_0 & x_1 & x_2 & 0 & \dots & 0 \\ \lambda_1 x_0 & x_1 & 0 & x_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \lambda_{n-2} x_0 & x_1 & 0 & \dots & 0 & x_n \end{pmatrix}. \tag{4.15}$$

And now

$$\begin{pmatrix} y_0^{(h)} \\ y_1^{(h)} \\ y_2^{(h)} \\ \vdots \\ y_n^{(h)} \end{pmatrix} = k \begin{pmatrix} \sum_{\nu=0}^{n-2} \mu_\nu \lambda_\nu x_0 \\ \sum_{\nu=0}^{n-2} \mu_\nu x_1 \\ \mu_0 x_2 \\ \vdots \\ \mu_{n-2} x_n \end{pmatrix}. \tag{4.16}$$

The relations among elements $y_0^{(h)}, \dots, y_n^{(h)}$ are given by:

$$G_i^{(h)} = \lambda_i \left(\frac{y_0^{(h)}}{\sum_{\nu=0}^{n-2} \mu_\nu \lambda_\nu} \right)^{q+1} + \left(\frac{y_1^{(h)}}{\sum_{\nu=0}^{n-2} \mu_\nu} \right)^{q+1} + \left(\frac{y_{i+2}^{(h)}}{\mu_i} \right)^{q+1} = 0 \qquad \text{for } 0 \le i \le n-2.$$

The h-conormal space is given by

$$\mathrm{Lag}^{(h)}(C^k(\bar{\lambda})) = \left\{ (x_0, \ldots, x_n, y_0^{(h)}, \ldots, y_n^{(h)}) : \text{ where } \begin{array}{l} x_0, \ldots, x_n \text{ satisfy eq. } (4.9) \\ \text{and } y_0^{(h)}, \ldots, y_n^{(h)} \text{ eq. } (4.15) \end{array} \right\}.$$

The variety $Z^{(h)} = \pi_2(\mathrm{Lag}^{(h)}(C^k(\bar{\lambda})))$ is given by a hypersurface $F^{(h)}(y_0^{(h)}, \ldots, y_n^{(h)}) = 0$, which can be computed by eliminating $\mu_0, \ldots, \mu_{n-2}$ from the system of $G_i^{(h)}$. Similarly we can compute

$$\begin{aligned} \nabla^{(h)} G_i^{(h)} &= \left(\lambda_i \frac{y_0^{(h)}}{k \sum_{\nu=0}^{n-2} \mu_\nu \lambda_\nu}, \frac{y_1^{(h)}}{k \sum_{\nu=0}^{n-2} \mu_\nu}, \ldots, \frac{y_{i+2}^{(h)}}{k \mu_i}, \ldots, 0 \right) \\ &= (\lambda_i x_0, x_1, 0 \ldots, 0, x_i, 0, \ldots, 0). \end{aligned}$$

Again we see Eq. (4.16) as parametric equations with parameters $\mu_0, \ldots, \mu_{n-2}$. The tangent space is generated by the vectors

$$V_i^{(h)} := \left(\frac{\partial y_i}{\partial \mu_0}, \frac{\partial y_i}{\partial \mu_1}, \ldots, \frac{\partial y_i}{\partial \mu_{n-2}} \right) = \left(\lambda_i x_0^{k-1}, x_1^{k-1}, 0, \ldots, 0, x_i^{k-1}, 0, \ldots, 0 \right) \text{ for } 0 \le i \le n-2,$$

which are subject to the additional condition

$$V_i^{(h)} \perp \nabla G_j^{(h)} \text{ i.e. } \langle V_i^{(h)}, \nabla G_j^{(h)} \rangle = 0 \text{ for all } 0 \le i, j \le n-2. \tag{4.17}$$

As in the zero characteristic case the last conditions are equivalent to the defining equations of the curve.

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