FRAMED THOMPSON GROUPS

ARISTIDES KONTOGEORGIS AND SOFIA LAMBROPOULOU

Abstract. We introduce the notion of the framed Thompson group, which can be seen as a categorification of the ordinary Thompson group, and we show how framed links can be obtained from elements of the framed Thompson group.

In memory of Vaughan F. R. Jones

1. Introduction

The Thompson group was introduced by R. Thompson in 1965 and is a widely studied group in the literature, see [4] and references therein. In a series of articles V.F.R. Jones developed a method in order to produce all knots and links from elements of the Thompson group F, see [7, 8, 9]. (Moreover, Aiello [1] proved that all oriented knots and links arise from the oriented subgroup $\overrightarrow{F}$, which was also introduced by Jones [7].) The reader is also referred to Figures 8,12. In this article we will follow Jones' categorical approach in order to define the notion of the *framed Thompson group*, $\mathscr{G}_{V,k}$. This group is interesting in itself and can be seen as a categorification of the ordinary Thompson group. We also show that every framed link can be obtained by an element of the $\mathscr{G}_{V,k}$.

More precisely, in Section 2 we introduce the category of forests decorated by unitary operators on a fixed vector space V. Following Jones, the trees in this category are seen as a direct system and the framed Thompson group is the direct limit of this system. In Section 3 the category of decorated tangles $\mathfrak{C}_V$, together with a functor from the category of decorated forests to $\mathfrak{C}_V$. Then we define the ring R of formal linear combinations over $\mathbb{Z}$ of isotopy classes of decorated links, with multiplication the disjoint union. We also define the R-module generated by all $(1, 2k+1)$ decorated tangles, where the scalar product is again the disjoint union. This module is equipped with a hermitian inner product with values in R. Jones' result is modified and the existence of a special tangle Ω is shown, so that decorated links can be obtained in terms of the introduced inner product, as $\langle g\Omega, \Omega\rangle$ for some element g in the framed Thompson group $\mathscr{G}_{V,k}$. If the selected vector space V is one-dimensional and the unitary operator is a root of unity, then we can recover in this way the notion of classical framing. Furthermore unitary operators with eigenvalues n-th roots of unity correspond to a multitude of framings for the same link component. These are discussed in Section 4.

It is well-known that every closed, connected, oriented (c.c.o.) 3-manifold can be obtained by surgery along a framed link in S^3 and that the framed links of two homeomorphic 3-manifolds are related by isotopy and the so-called Kirby moves. Now, framing can be represented by the blackboard framing, where in the equivalence the first Reidemeister move is not permitted. Also by integral or rational framing assigned to each component of the link, where in the equivalence all three Reidemeister moves are allowed.

Relating now the discussion to the Thompson groups, through our construction, all ambient isotopy classes of framed link diagrams arise from elements of the framed Thompson groups (in analogy to the connection of framed links to elements of the framed braid groups). Note that in [15] Raghavan and Sweeney show that not all regular isotopy classes of link diagrams arise from elements of some Thompson group. Therefore, our approach has the potential of relating the theory of Thompson groups to the study of 3-manifolds. In fact, through our construction each component of a given link can be assigned a multitude of framings and, thus, a multitude of associated 3-manifolds. Then, an interesting question would be the topological connection of the 3-manifolds assigned to one framed Thompson group element. Another question is the algebraic relation between different (framed) Thompson group elements that give rise to ambient isotopic (framed) links (the analogues of the classical Markov theorem), and then to homeomorphic 3-manifolds.

Acknowledgment: We would like to thank the referee for all their remarks, corrections and improvements.

Date: August 13, 2022.

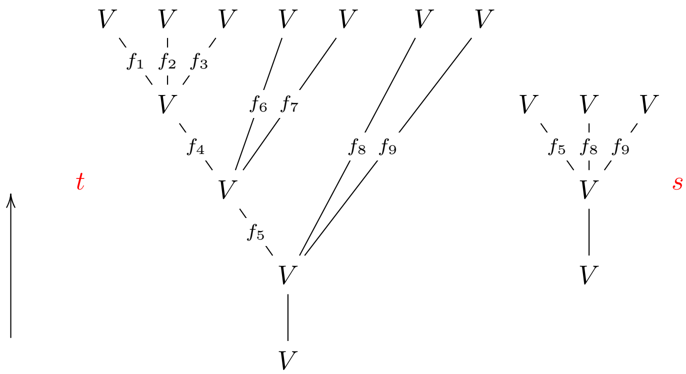


FIGURE 1. Two decorated trees; the upward arrow indicates the direction of the maps

2. THE FRAMED THOMPSON GROUP

In a series of articles [7, 8, 9] V.F.R. Jones studied the Thompson group F, which he views as a group of fractions of an appropriate category, and he developed a method for producing all knots and links from elements of the group F. Notice that there are other known examples of this construction (besides F): the Brown-Thomson group F_k [8, sec. 2.3], the braided Thompson group [8, sec. 2.4], the braid groups [8, p.6], the oriented Thompson group $\overrightarrow{F}$ [2]. In this section we will extend the construction of Jones [8] in order to introduce a framing on the classical Thomson group.

2.1. **A categorical approach to the framed Thompson group.** Let V be a fixed vector space of dimension d over $\mathbb{C}$. Select a fixed basis $(e_1, \ldots, e_d)$ of the vector space V.

Definition 1. Define the category of *decorated forests*, $\mathscr{F}_{V,k}$, which has as objects elements of the form $V^{\otimes n} = V \otimes \cdots \otimes V$ (n-factors), where n is a natural number. The morphisms of $\mathscr{F}_{V,k}$ from V to $V^{\otimes n}$ are equivalence classes of decorated trees of the form given in Figure 1. That is each inner vertex is adjacent to $k+1$ edges, one incoming and k outgoing. Each edge is decorated by a unitary linear map $f : V \to V$, namely $f \in U(V)$, that is $ff^* = \mathrm{Id}_V$. In this way we have linear maps

$$\begin{aligned} V &\longrightarrow V^{\otimes k} = V \otimes \cdots \otimes V \qquad k \text{ factors} \\ e_i &\longmapsto f_1(e_i) \otimes f_2(e_i) \otimes \cdots \otimes f_k(e_i) = \sum_{\nu_1=1}^{d} a^{(1)}_{\nu_1,i} \sum_{\nu_2=1}^{d} a^{(2)}_{\nu_2,i} \cdots \sum_{\nu_k=1}^{d} a^{(k)}_{\nu_k,i}\, e_{\nu_1} \otimes e_{\nu_2} \otimes \cdots \otimes e_{\nu_k}, \end{aligned}$$

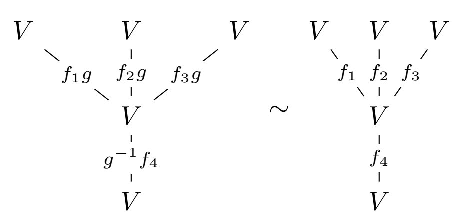
where we assume that

$$f_\mu(e_i) = \sum_{\nu_\mu=1}^{d} a^{(\mu)}_{\nu_\mu,i}\, e_{\nu_\mu}, \quad a^{(\mu)}_{\nu_\mu,i} \in \mathbb{C},\ \mu = 1, \ldots, k.$$

Two decorated trees t_1, t_2 will be considered *equivalent* if:

- The corresponding trees t_1, t_2, if we forget the decoration, are equal
- At each inner vertex we can change the linear maps as follows

(1)



That is, we can compose the functions decorating the bottom edge and the middle edge on top by a unitary map g and its inverse.

The set of morphisms of the form $V \to V^{\otimes n}$, $n \in \mathbb{N}$, will be denoted by $\mathscr{T}_{V,k}$ and elements of $\mathscr{T}_{V,k}$ will be called *decorated trees*. An arrangement of decorated trees next to each other will be called a *forest*. A decorated forest consisting of vertical segments placed next to each other, will be called *trivial*, even with non-trivial decorations. A decorated forest gives rise to a morphism $V^{\otimes n} \to V^{\otimes m}$ in the category $\mathscr{F}_{V,k}$ by placing n decorated trees of the form $(1, m_i)$ next to each other for $m_1 + \cdots + m_n = m$, that is we consider the tensor product of the trees' morphisms. One can compose two forests $V^{\otimes n} \to V^{\otimes m}$, $V^{\otimes m} \to V^{\otimes \ell}$ by placing the second forest on top of the first.

Remark 2. Using the above equivalence one can move all (unitary) maps on the leaves of the tree so that all other edges are decorated by identity maps. The topological interpretation of this definition will be clear, when we will define the functor from forests to tangles.

Remark 3. Notice that ternary trees, that is $k = 3$, will lead to (framed) knots and links.

2.2. **Direct limit constructions.** The set $\mathscr{T}_{V,k}$ is a directed set by setting, for $s, t \in \mathscr{T}_{V,k}$, $s \leq t$, if and only if there is a morphism f in the category $\mathscr{F}_{V,k}$ such that $t = fs$. For example if s is the element on the right of Figure 1 and t is the element on the left, then $s \leq t$.

Lemma 4. *For each two elements $s_1, s_2 \in \mathscr{T}_{V,k}$ there is a common element s, such that $s_1, s_2 \leq s$, that is*

$$s = f_1 s_1 = f_2 s_2, \text{ for some morphisms } f_1, f_2 \text{ in the category } \mathscr{F}_{V,k}.$$

Proof. Using [8] we can find forests f_1', f_2, with edges decorated by the identity map so that $f_1' s_1 = f_2 s_2$ as non-decorated trees. Furthermore, it is clear that using the equivalence relation of decorated trees, given in Eq. (1), we can move all decorations on the top edges (leaves). We can then compose one of the trees with a trivial forest f, whose segments are decorated so that $f f_1' s_1 = f_2 s_2$. The desired forest f_1 is given by $f_1 = f f_1'$. □

For a morphism $f : V^{\otimes n} \to V^{\otimes m}$ we will denote by $\mathrm{t}(f) = V^{\otimes m}$ and $\mathrm{o}(f) = V^{\otimes n}$, the target and origin of f respectivelly.

Given a functor

$$\Phi : \mathscr{F}_{V,k} \to \mathscr{C} \tag{2}$$

we define the direct system S_Φ indexed by $\mathscr{T}_{V,k}$, associating to each morphism $s \in \mathscr{T}_{V,k}$ the object $\Phi(\mathrm{t}(s)) = \Phi(V^{\otimes n})$, where n is the number of leaves of s, that is the number of terminal edges of the underlying tree. Then, for $s \leq t$, that is $t = fs$, we associate the direct system morphisms i_s^t given by

$$i_s^t := \Phi(f) \in \mathrm{Hom}_{\mathscr{C}}\big(\Phi(\mathrm{t}(s)), \Phi(\mathrm{t}(t))\big).$$

Remark 5. The categories denoted by $\mathscr{C}$ in this article will have elements of the form $V^{\otimes n}$ as objects, for a fixed vector space V. The functors the form $\Phi : \mathscr{F}_{V,k} \to \mathscr{C}$ will satisfy $\Phi(V^{\otimes n}) = V^{\otimes n}$.

Given now an fixed object, $\omega \in \mathrm{Ob}(\mathscr{C})$, we define the category $\mathscr{C}^\omega$, which has as objects the sets $\mathrm{Hom}_{\mathscr{C}}(\omega, \mathrm{obj})$ for every object $\mathrm{obj} \in \mathrm{Ob}(\mathscr{C})$, while the morphisms f_* of $\mathscr{C}^\omega$ are given by compositions with morphisms $f : \mathrm{obj}_1 \to \mathrm{obj}_2$ of $\mathscr{C}$ with morphisms $\phi \in \mathrm{Hom}_{\mathscr{C}}(\omega, \mathrm{obj})$ as follows:

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(\omega, \mathrm{obj}_1) \arrow[d, "f_*"] \\ \mathrm{Hom}_{\mathscr{C}}(\omega, \mathrm{obj}_2) \end{tikzcd} \qquad \begin{tikzcd} \omega \arrow[r, "\phi"] \arrow[dr, "f\circ\phi"'] & \mathrm{obj}_1 \arrow[d, "f"] \\ & \mathrm{obj}_2 \end{tikzcd}$$

Example 6. If $\mathscr{C} = \mathscr{F}_{V,k}$ and $\omega = V$, then $\mathrm{Hom}_{\mathscr{F}_{V,k}}(V, \mathrm{obj})$ is the set of $s \in \mathscr{T}_{V,k}$ with $\mathrm{t}(s) = \mathrm{obj}$. That is an object in the category $\mathscr{F}_{V,k}^\omega = \mathscr{F}_{V,k}^V$ is the set of elements s in $\mathscr{T}_{V,k}$ with $\mathrm{t}(s) = \mathrm{obj}$ for a fixed object $\mathrm{obj} = V^{\otimes n} \in \mathscr{F}_{V,k}$. Denote by $\mathscr{T}_{V,k}(\mathrm{obj}) := \mathrm{Hom}_{\mathscr{F}_{V,k}}(V, \mathrm{obj})$ such a set. The push-forward construction above gives rise to a morphism

$$\begin{aligned} \mathscr{T}_{V,k}(\mathrm{obj}_1) = \{s \in \mathscr{T}_{V,k} : \mathrm{t}(s) = \mathrm{obj}_1\} &\xrightarrow{f_*} \{s' \in \mathscr{T}_{V,k} : \mathrm{t}(s') = \mathrm{obj}_2\} = \mathscr{T}_{V,k}(\mathrm{obj}_2) \\ s &\longmapsto fs \end{aligned}$$

whenever there is a morphism $f : \mathrm{obj}_1 \to \mathrm{obj}_2$.

Given a functor Φ as in eq. (2) we can define a functor

$$\Phi^\omega : \mathscr{F}_{V,k} \longrightarrow \mathscr{C}^\omega$$
$$\Phi^\omega(\mathrm{obj}) = \mathrm{Hom}_{\mathscr{C}}(\omega, \Phi(\mathrm{obj}))$$
$$\Phi^\omega(f : \mathrm{obj}_1 \to \mathrm{obj}_2) = \Phi(f)_* : \mathrm{Hom}_{\mathscr{C}}(\omega, \Phi(\mathrm{obj}_1)) \to \mathrm{Hom}_{\mathscr{C}}(\omega, \Phi(\mathrm{obj}_2))$$

From the above $\Phi(f)_*(\phi) = \Phi(f) \circ \phi$.

As before we define a direct system S_{Φ^ω}, indexed by $\mathscr{T}_{V,k}$, associating to each morphism $s \in \mathscr{T}_{V,k}$ the object

$$\Phi^\omega(\mathrm{t}(s)) = \mathrm{Hom}_{\mathscr{C}}\big(\omega, \Phi(\mathrm{t}(s))\big).$$

In the Φ^ω setting the maps i_s^t of the direct system S_{Φ^ω} indexed by $\mathscr{T}_{V,k}$, are given by mapping $\phi \in \mathrm{Hom}\big(\omega, \Phi(\mathrm{t}(s))\big) \in \mathrm{Ob}(\mathscr{C}^\omega)$ to

$$i_s^t(\phi) = \Phi^\omega(f)(\phi) = \Phi(f)_*(\phi) = \Phi(f) \circ \phi \in \mathrm{Hom}_{\mathscr{C}}\big(\omega, \Phi(\mathrm{t}(t))\big).$$

In the above formula recall that f is selected by the equality $t = fs$. We can now define the direct limit

$$\lim_{\to} S_{\Phi^\omega} = \{(t, x) \text{ with } t \in \mathscr{T}_{V,k},\ x \in \Phi^\omega(\mathrm{t}(t))\} / \sim \tag{3}$$

where $(t, x) \sim (s, y)$ if there is an r such that $t, s \le r$ with

$$r = ft = gs \text{ and } \Phi^\omega(f)(x) = \Phi^\omega(g)(y) = z$$

for some $z \in \Phi^\omega(\mathrm{t}(r))$. In particular,

$$(t, x) \sim (ft, \Phi^\omega(f)x) = (ft, \Phi(f)_*(x)) = (ft, \Phi(f) \circ x).$$

We will denote the equivalence class of (t, x) by t/x.

2.3. **Definition of the framed Thompson group.** Suppose now that Φ is the identity functor from $\mathscr{F}_{V,k} \to \mathscr{F}_{V,k}$ and $\bar{\omega}$ is the tree with one leaf with V both at the top and bottom and the identity map $V \to V$ at the edge. Set $\omega = V = \Phi(\mathrm{t}(\bar{\omega}))$.

In this case, i.e. when Φ is the identity functor, an equivalence class t/x in the limit is given by

$$t \in \mathscr{T}_{V,k} \quad \text{ and } \quad x \in \Phi^\omega(\mathrm{t}(t)) = \mathrm{Hom}_{\mathscr{F}_{V,k}}\big(V, \mathrm{t}(t)\big) = \mathrm{Hom}_{\mathscr{F}_{V,k}}\big(V, V^{\otimes n}\big).$$

This means that $x : V \to \mathrm{t}(t) = V^{\otimes n}$, therefore $x \in \mathscr{T}_{V,k}$. Therefore, t/x is the equivalence class of (t, x) in the direct limit $\lim\limits_{\to} S_{\Phi^\omega}$, for Φ the identity functor and $\omega = V$. Denote this direct limit in this case by

$$\mathscr{G}_{V,k} = \{(t, s), t, s \in \mathscr{T}_{V,k}\} / \sim$$

Two classes $(t_1, x_1), (t_2, x_2) \in \mathscr{G}_{V,k}$ can by multiplied in the following way: We can select elements f_1, f_2 in the morphisms of the category $\mathscr{F}_{V,k}$ such that

$$f_1 x_1 = f_2 t_2.$$

We have that $(t_1, x_1) \sim (f_1 t_1, f_1 x_1)$ and $(t_2, x_2) \sim (f_2 t_2, f_2 x_2)$. We define

$$\frac{t_1}{x_1} \cdot \frac{t_2}{x_2} = \frac{f_1 t_1}{f_1 x_1} \cdot \frac{f_2 t_2}{f_2 x_2} := \frac{f_1 t_1}{f_2 x_2}. \tag{4}$$

The geometric interpretation of the above cancellation can be explained in terms of elementary cobordisms. Here we generalize the multiplication algorithm of Belk-Guba-Sapir [3], [5] to the framed Thompson group. Suppose that we would like to multiply the elements α, β in $\mathscr{G}_{V,k}$. We place β on top of α. In the resulting diagram, each local region of the form shown in the left hand-side of Figure 2, is replaced by three parallel lines, each one decorated by $f_1 g_1^*, f_2 g_2^*, f_3 g_3^*$. In order to see why this procedure gives the same result as the one obtained in Eq. (4) we argue as follows. The branch shown in left hand-side of Figure 2 is decorated by f_1, f_2, f_3 is in the denominator of β while the branch bellow, decorated by g_1, g_2, g_3, is in the numerator of α. We change the decorations f_1, f_2, f_3 by $f_1 g_1^* g_1$, $f_2 g_2^* g_2$, $f_3 g_3^* g_3$ so that we can cancel the denominator by the numerator using the definition of the direct limit. The resulting element is the one with three parallel lines as shown in the right part of Figure 2.

In this way the set $\mathscr{G}_{V,k}$ becomes a group. The identity element is the equivalence class of $(\bar{\omega}, \bar{\omega}) \sim (s, s)$, for any $s \in \mathscr{T}_{V,k}$, while the inverse element of the equivalence class of (x, t) is given by the equivalence class of (t, x), that is:

$$(x, t)^{-1} = (t, x).$$

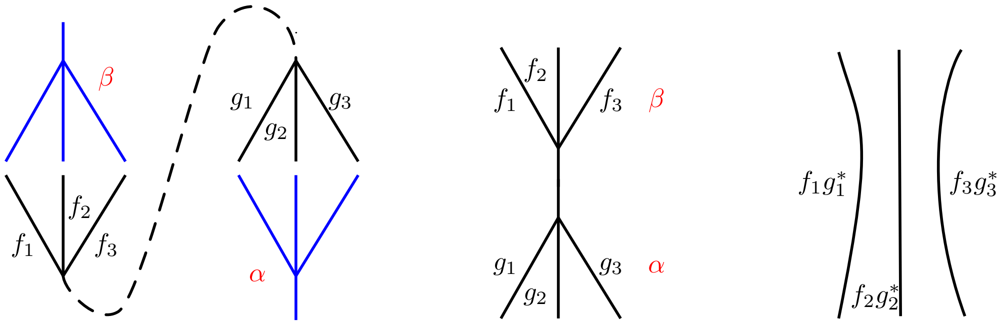


FIGURE 2. Multiplying the Thompson group elements α, β

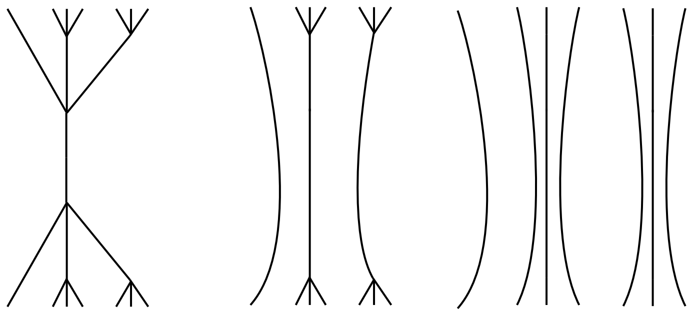


FIGURE 3. A series of simplifications in the group multiplication

Definition 7. The group $\mathscr{G}_{V,k}$ is called the *framed Thompson group*.

Remark 8. If $V = \mathbb{C}$ and if we consider the subcategory of $\mathscr{F}_{\mathbb{C},k}$. where all edge maps are the identity, then we recover the classical Thompson group [9].

3. The category of decorated tangles

Definition 9. Let $n, m \in \mathbb{Z}$, $n, m \geq 0$. A *decorated* (n,m)*-tangle* is an isotopy class of rectangles with m "top" and n "bottom" boundary points, containing immersions of circles and intervals with only double points, the crossings, which are assigned with 'over' or 'under' information, and which meet the boundary transversally in the $m+n$ boundary points, see Fig. 4. The special case of $(0,2)$-tangle comprising a (possibly knotted) arc that connects two top boundary points, as illustrated on the left-hand side of Fig. 5, will be called a *cup*, while a $(2,0)$-tangle comprising a (possibly knotted) arc connecting two bottom boundary points, as illustrated on the right-hand side of Fig. 5, will be called a *cap*. Furthermore, to each boundary point we attach a fixed vector space V. The tangle arcs, whether connecting boundary points or forming closed loops in the interior of the framed tangle, have the following extra decoration information:

- To each arc connecting a bottom to a top boundary point we attach a unitary map $f : V \to V$ between the two boundary vector spaces V.
- To each closed arc we attach a complex number.
- To each cup and to each cap we attach functions $\eta(f_1, f_2)$ and $\epsilon(f_1, f_2)$ respectively, as explained in Eqs. (5), (6), where $f_1, f_2 : V \to V$ are given unitary maps, see also Fig. 5. More precisely cups and caps give rise to linear maps: $\mathbb{C} \to V \otimes V$ and $V \otimes V \to \mathbb{C}$ respectively, defined as follows: For given unitary maps $f_1, f_2 : V \to V$, the "cup" tangle gives rise to the linear form:

$$\begin{aligned} \eta(f_1, f_2) &: \mathbb{C} \longrightarrow V \otimes V \\ 1 &\longmapsto \sum_{i=1}^{d} f_1(e_i) \otimes f_2^*(e_i) \end{aligned} \tag{5}$$

while the "cap" tangle gives rise to the map

$$\begin{aligned} \epsilon(f_1,f_2): V\otimes V &\longrightarrow \mathbb{C} \\ e_i\otimes e_j &\longmapsto \epsilon(f_1(e_i)\otimes f_2^*(e_j)), \end{aligned} \tag{6}$$

where ϵ is the linear map defined by $\epsilon(e_i\otimes e_j)=\delta_{ij}$. In particular, if $A^{(i)}=(a^{(i)}_{\nu,\mu})$ is the matrix describing f_i, then we can compute the matrices of $\eta(f_1,f_2)$ and $\epsilon(f_1,f_2)$ as follows:

$$\eta(f_1,f_2)(1)=\sum_{i=1}^d\sum_{\nu_1=1}^d\sum_{\nu_2=1}^d a^{(1)}_{\nu_1,i}\bar{a}^{(2)}_{i,\nu_2}e_{\nu_1}\otimes e_{\nu_2}$$

Similarly

$$\epsilon(f_1,f_2)(e_i\otimes e_j)=\epsilon\left(\sum_{\nu_1=1}^d\sum_{\nu_2=1}^d a^{(1)}_{\nu_1,i}\bar{a}^{(2)}_{j,\nu_2}e_{\nu_1}\otimes e_{\nu_2}\right)=\sum_{\nu=1}^d \bar{a}^{(2)}_{j,\nu}a^{(1)}_{\nu,i}.$$

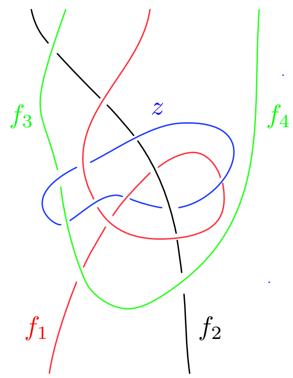


FIGURE 4. A decorated (n,m)-tangle

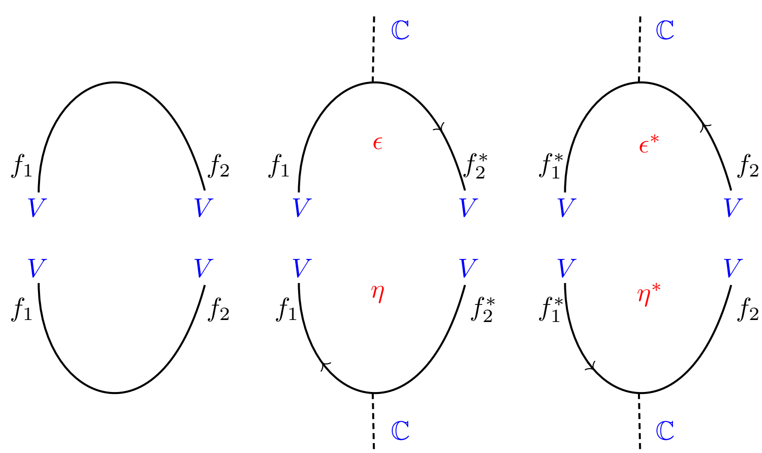


FIGURE 5. The cup and cap maps

Remark 10. The definitions of the maps $\eta(f_1,f_2)$ and $\epsilon(f_1,f_2)$ induce "orientations" on the cup and cap respectively, in the sence that we introduce a convention on how to compose the decorating functions on each arc. Indeed, taking the $*$ on the second function of the definition can be considered as inverting the orientation of the corresponding arc. Therefore the cup connecting two top boundary points has an orientation from right to left, while the cap connecting two bottom boundary points has an orientation from left to the right, see Fig. 5. We can also consider the maps $\eta^*(f_1, f_2)$ and $\epsilon^*(f_1, f_2)$ with the inverse orientation defined as

$$\eta^*(f_1, f_2) := \eta(f_1^*, f_2^*) \qquad \epsilon^*(f_1, f_2) := \epsilon(f_1^*, f_2^*).$$

Definition 11. We will call the orientations defined by the maps η and ϵ on the cup and cap the *standard orientations.*

Remark 12. The notion of a decorated tangle is well-defined, since any isotopy between two tangles does not affect the decorations.

3.1. **Composition of tangles.** Topologically, if we have an (n, m)-tangle and an (m, ℓ)-tangle we can compose them by placing the second on top of the first and concatenating the corresponding endpoints, resulting in an (n, ℓ)-tangle. Regarding decorations, observe that the convention of the direction of the decorating maps from bottom to top, induces an upwards orientation on all tangle arcs with respect to the height function. So for the composition we distinguish the following cases:

• Arcs in the composed tangle connecting a bottom point to a top point have an orientation from bottom to top. We realize this by composing the decorating maps along the arc consecutively, changing, if needed, intermediate maps f to f^*, for achieving compatible orientations. For example in Fig. 6 the arc on the left is decorated by the map $f_6 f_3 f_2^* f_5^* f_4 f_1$ while the arc on the right is decorated by the map $f_4 f_1 f_2^* f_5^* f_6 f_3$.

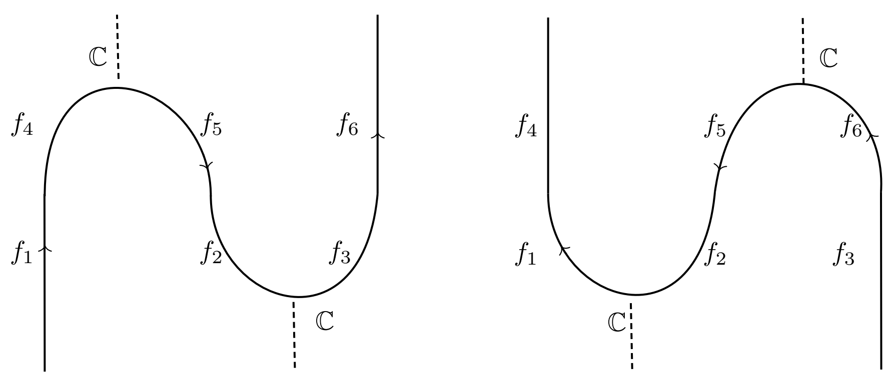


FIGURE 6. Composition of maps in a bottom-to-top arc

We can arrive at the same result as a composition of maps η and ϵ by following the path from bottom to top and inverting the maps η and ϵ to η^* and ϵ^*, whenever the cups and caps are travelled by the non-standard orientation. For example for the left hand side of Fig. 6 we consider the composition of the functions

$$V \cong V \otimes \mathbb{C} \xrightarrow{f_1 \otimes \eta(f_2^*, f_3^*)} V \otimes V \otimes V \xrightarrow{\epsilon(f_4, f_5) \otimes f_6} \mathbb{C} \times V \cong V \ .$$

The basis element $e_i \otimes 1$ of $V \otimes \mathbb{C}$ goes by $f_1 \otimes \eta(f_2^*, f_3^*)$ to

$$\sum_{\nu=1}^{d} f_1(e_1) \otimes f_2^*(e_\nu) \otimes f_3(e_\nu) = \sum_{\nu,\mu,\lambda,\kappa=1}^{d} a^{(1)}_{\mu,i} \bar{a}^{(2)}_{\nu,\lambda} a^{(3)}_{\kappa,\nu} \quad e_\mu \otimes e_\lambda \otimes e_\kappa,$$

which in turn goes by the function $\epsilon(f_4, f_5) \otimes f_6$ to

$$\sum_{\nu,\mu,\lambda,\kappa,s,t=1}^{d} a^{(6)}_{sk} a^{(3)}_{\nu,\kappa} \bar{a}^{(2)}_{\lambda,\nu} \bar{a}^{(5)}_{\lambda,t} a^{(4)}_{t,\mu} a^{(1)}_{\mu,i} e_s,$$

which is the matrix corresponding to the composition $f_6 f_3 f_2^* f_5^* f_4 f_1$.

• Arcs in the composed tangle connecting two bottom or two top points are given the standard orientation. In terms of the decoration maps this means that we traverse the arcs, composing the consecutive decorating maps and switching, if needed, maps f to f^*. Also we can obtain the same result by composing maps ϵ and η, taking care of the consistency of their orientations, as explained in the previous case.

• For composing a cup to a cap as shown, in Figure 7 we proceed as follows:

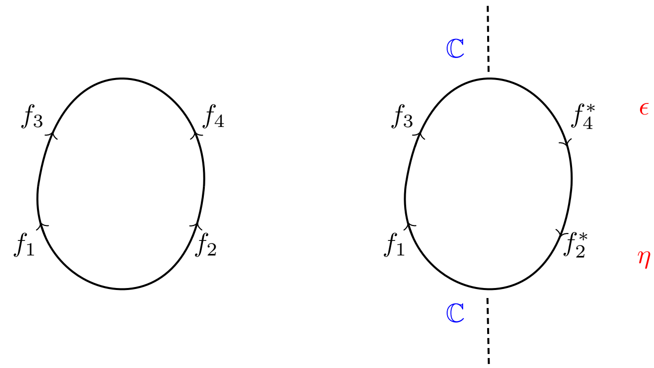


FIGURE 7. Composing cup to cap

After glueing together the cup with the cup we have a map

$$\begin{array}{ccccc} \mathbb{C} & \xrightarrow{\eta(f_1,f_2)} & V\otimes V & \xrightarrow{\qquad\epsilon(f_3,f_4)\qquad} & \mathbb{C} \\ 1 & \longmapsto & \sum_{i=1}^d f_1(e_i)\otimes f_2^*(e_i) & \longmapsto & \epsilon\left(\sum_{i=1}^d f_3 f_1(e_i)\otimes f_4^* f_2^*(e_i)\right) \end{array}$$

If f_i is represented by the matrix $A^{(i)} = (a^{(i)}_{\nu,\mu})$ then this composition is given by

$$\begin{aligned} \epsilon(f_3,f_4)\left(\sum_{i,\nu_1,\nu_2=1}^d a^{(1)}_{\nu_1,i}\bar{a}^{(2)}_{i,\nu_2} e_{\nu_1}\otimes e_{\nu_2}\right) &= \sum_{i,\nu_1,\nu_2=1}^d a^{(1)}_{\nu_1,i}\bar{a}^{(2)}_{i,\nu_2}\sum_{\mu=1}^d \bar{a}^{(4)}_{\nu_2,\mu} a^{(3)}_{\mu,\nu_1} \\ &= \mathrm{tr}(A_1\cdot A_2^*\cdot A_4^*\cdot A_3) = \mathrm{tr}(A_2^*\cdot A_4^*\cdot A_3\cdot A_1). \end{aligned}$$

Note that the above given expression can be also derived as follows: we start from a given point on the closed arc. Then, we move along the closed arc clockwise (according to the standard orrientation) and we compose the linear maps, switching from f to f^*, if needed. The expression $\mathrm{tr}(A_2^*\cdot A_4^*\cdot A_3\cdot A_1) = \mathrm{tr}(f_2^*\cdot f_4^*\cdot f_3\cdot f_1)$ Notice also that since $\mathrm{tr}(AB)=\mathrm{tr}(BA)$, the starting point does not matter.

Definition 13. We define the *decorated tangle category* $\mathfrak{C}_V$ to be the category with objects $V^{\otimes n}$, $n\in\mathbb{Z}, n\geq 0$ and morphisms the decorated tangles as defined above. By convention $V^{\otimes 0}=\mathbb{C}$.

Remark 14. The category of classical tangles can be seen as a subcategory of the category $\mathfrak{C}_V$, if we assign on any classical tangle the identity operator on open arcs and the dimension of V on closed arcs.

3.2. **A functor from the forest to the tangle category.** We will now define a functor

$$\Phi : \mathscr{F}_{V,3} \longrightarrow \mathfrak{C}_V$$

as follows: The object $V^{\otimes n}$ of $\mathscr{F}_{V,3}$ goes to the same object of $\mathfrak{C}_V$. A morphism f goes to $\Phi(f)$ which is the tangle obtained by isotoping the forest to be in a rectangle with roots on the bottom edge and leaves on the top edge. Each vertex of the forest is replaced with a crossing, where the middle leaf at each vertex connects with the root coming in and passes under the crossing, see Figure 8.

We can also define the set $\tilde{\mathfrak{C}}_V$.

Definition 15. The elements of the set $\tilde{\mathfrak{C}}_V$ are given by equivalence classes of pairs (t,T), where $t\in\mathscr{T}_{V,3}$ with $\mathrm{t}(t)=V^{\otimes n}$ and T is an $(1,n)$ tangle in $\mathfrak{C}$, where $(t_1,T_1)\sim(t_2,T_2)$ if there are elements $g_1,g_2\in\mathscr{F}_{V,3}$, such that $g_1t_1=g_2t_2$ and $\Phi(g_1)T_1=\Phi(g_2)T_2$. In particular $(gt,\Phi(g)T)\sim(t,T)$. This is the direct limit for the functor $\Phi:\mathfrak{F}_{V,k}\to\mathfrak{C}$ as defined in Eq. (3).

3.3. **The inner product.** If S,T are two $(1,2k+1)$ tangles in $\mathfrak{C}_V$, then we can take the horizontal mirror image of the second one and we change all maps f_i on arcs to f_i^*. The complex number attached to each closed arc is changed to its complex conjugate.

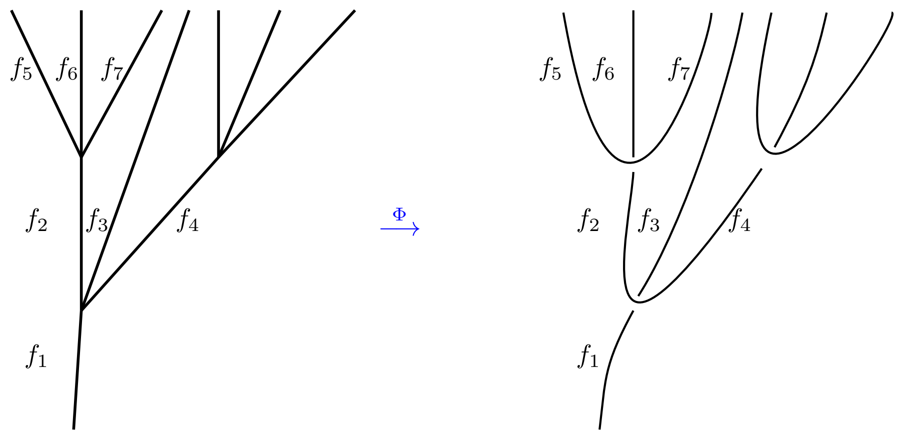


FIGURE 8. Relating a forest to a tangle via the functor Φ

We then place the altered tangle on top of the first one and we compose them as morphisms in the tangle category $\mathfrak{C}_V$ obtaining a decorated $(1,1)$-tangle, as shown in Figure 9. This $(1,1)$-tangle contains an arc connecting the bottom to the top boundary point. We will now explain how we can close this open arc in order to arrive at a closed link.

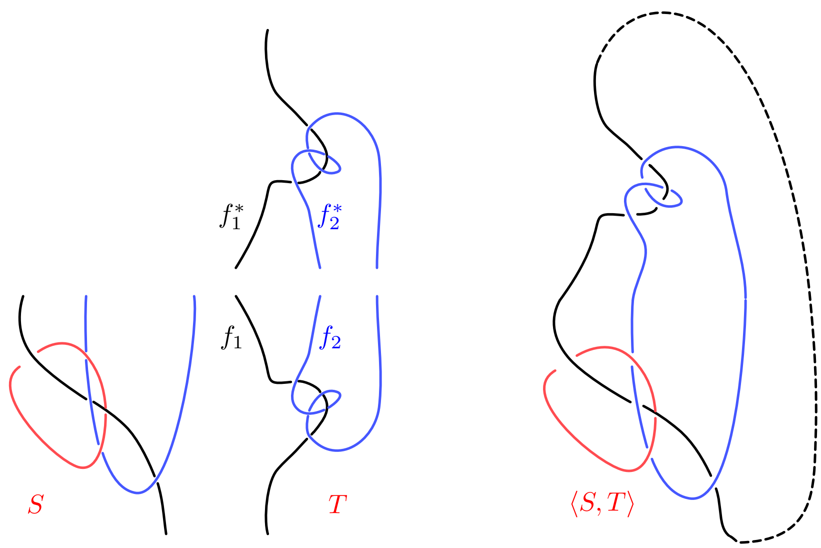


FIGURE 9. The inner product of two decorated tangles

Closing the bottom-top arc

The open arc is decorated by a linear map $f : V \to V$. In order to close the arc we draw an arc decorated by the identity map parallel to the arc carrying f and we close the bottom and top by the maps $\eta : \mathbb{C} \to V \otimes V$ and $\epsilon : V \otimes V \to \mathbb{C}$. We finally arrive to a map $\mathbb{C} \to \mathbb{C}$ given as follows:

$$1 \mapsto \epsilon\eta(1) = \epsilon\left(\sum_{i=1}^{d} f(e_i) \otimes e_i\right) = \epsilon\left(\sum_{i=1}^{d}\sum_{\nu=1}^{d} a_{\nu,i} e_\nu \otimes \epsilon_i\right) = \mathrm{tr}(f).$$

Definition 16. The decorated link resulting by the above described procedure is the *inner product of* S, T and it will be denoted by $\langle S, T\rangle$.

We expand the above definition in the following sense:

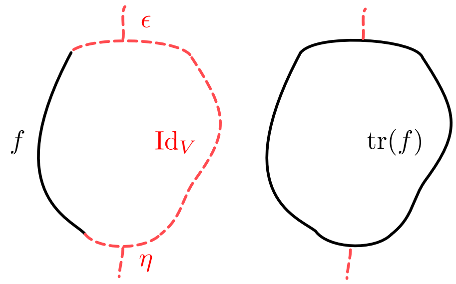


Figure 10. Closing the bottom-top arc

Definition 17. Let R be the ring of formal linear combinations over $\mathbb{Z}$ of isotopy classes of decorated unoriented links, where multiplication is the distant union. Let $R\,\mathfrak{C}_{1,k}$ denote the free R-module having the set of all $(1, 2k+1)$ decorated tangles as a basis. The above construction gives rise to an inner product:

$$\begin{aligned} R\,\mathfrak{C}_{1,k} \times R\,\mathfrak{C}_{1,k} &\longrightarrow R \\ (A, B) &\longmapsto \langle A, B\rangle. \end{aligned}$$

This inner product is Hermitian, that is $\langle A, B\rangle = \overline{\langle B, A\rangle}$, where for a decorated link L, the $\overline{L}$ is the mirror image of L with complex conjugates applied to all decorations of L.

3.4. **Normalising the inner product.** This inner product is not compatible with the connecting maps defining the direct limit $\tilde{\mathfrak{C}}$, as given in Definition 15. This means that for a forest $f \in \mathscr{F}_{V,3}$ we might have $\langle S, T\rangle \neq \langle \Phi(f)S, \Phi(f)T\rangle$. The composition with the tangles $\Phi(f)$ on both entries of the inner product will produce several extra compositions of unknotted cups and caps as shown in Figure 11.

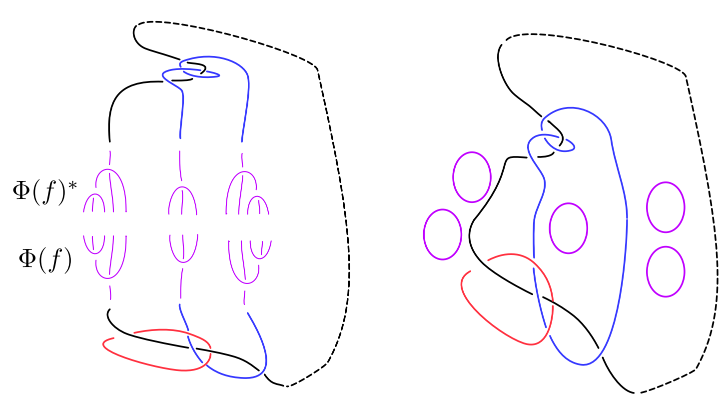
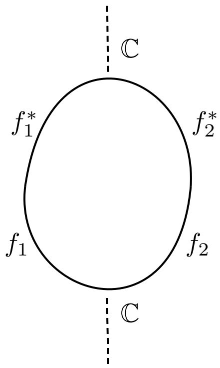


Figure 11. Canceling carets after composing by the image $\Phi(f)$ of a forest f with 5 nodes

This will lead to several unknotted circles, corresponding to cancelling carets (as many as the number of vertices of the forest $f \in \mathscr{F}_{V,3}$), with framing the dimension $\dim V$ of the fixed vector space V, which is independent of the decoration of $\Phi(f)$. Indeed,



We have the map

$$\mathbb{C} \xrightarrow{\eta(f_1,f_2)} V \otimes V \xrightarrow{\qquad\epsilon(f_1^*,f_2^*)\qquad} \mathbb{C}$$

$$1 \longmapsto \sum_{i=1}^{d} f_1(e_i) \otimes f_2(e_i) \longmapsto \epsilon\left(\sum_{i=1}^{d} f_1^* f_1(e_i) \otimes f_2^* f_2(e_i)\right) = \epsilon\left(\sum_{i=1}^{d} e_i \otimes e_i\right) = d.$$

We will denote by δ the element of R, which topologically is the unknotted circle and is decorated by the complex number $d = \dim V$. We consider the extension of R, given by $R[\sqrt{\delta}, 1/\sqrt{\delta}]$. We define the normalization $\bar{\Phi}$ of the functor Φ, which is given by multiplying the $R[\sqrt{\delta}, 1/\sqrt{\delta}]$-linear map induced by Φ by $\left(\frac{1}{\sqrt{\delta}}\right)^p$, where p is the number of vertices of the forest in $\mathscr{F}_{V,3}$. Notice that we don't count the leaves and root as vertices. Thus

$$\bar{\Phi}(s) = \frac{1}{\sqrt{\delta}^p}\Phi(s). \tag{7}$$

Notice, that when we move from (s, r), $s, r \in \mathscr{T}_{V,3}$, to the equivalent (fs, fr), by multiplying both s, r by some forest $f \in \mathscr{F}_{V,3}$ with $\mathrm{o}(f) = \mathrm{t}(r) = \mathrm{t}(s)$, then

$$\langle \bar{\Phi}(fs), \bar{\Phi}(fr)\rangle = \langle \bar{\Phi}(s), \bar{\Phi}(r)\rangle. \tag{8}$$

Indeed, each extra vertex introduced by f gives rise to an extra δ. Let p be the number of such vertices. On the other hand fs and fr each gives rise to $1/\sqrt{d}^p$ so

$$\langle \bar{\Phi}(fs), \bar{\Phi}(fr)\rangle = d^{-p}d^p\langle \bar{\Phi}(s), \bar{\Phi}(r)\rangle = \langle \bar{\Phi}(s), \bar{\Phi}(r)\rangle.$$

Therefore the normalization of $\bar{\Phi}$ given in Eq. (7) induces an inner product compatible with the connecting maps of the direct limit $\tilde{\mathfrak{C}}$.

Example 18. In Figure 12 the three branches at position A are labelled by f_5, f_6, f_7, at position B they are labelled by f_8, f_9, f_{10}, at position C they are labelled by g_5, g_6, g_7, and at position D by g_8, g_9, g_{10}. Hence, the framing on the resulting trefoil knot is given by

$$\mathrm{tr}(g_5^*\, g_7\, f_6\, f_2\, f_4\, g_4^*\, g_2^*\, g_9^*\, f_{10}^*\, g_3\, g_1^*\, f_5^*\, f_7^*\, g_8^*\, g_{10}\, f_5^*\, f_3^*\, f_1).$$

3.5. **An action of the framed Thompson group $\mathscr{G}_{V,k}$.** Let $\Phi : \mathscr{F}_{V,k} \to \mathscr{C}$ be a functor, and let $\alpha = s/t \in \mathscr{G}_{V,k}$ and $m = t'/x \in \varinjlim S_{\Phi^\omega}$, that is s/t is the class of the pair $(s,t) \in \mathscr{T}_{V,k} \times \mathscr{T}_{V,k}$, while t'/x is the class of (t', x), $t' \in \mathscr{T}_{V,k}$ and $x \in \Phi^\omega(\mathrm{t}(t')) = \mathrm{Hom}_{\mathscr{C}}(\omega, \Phi(\mathrm{t}(t')))$.

The interesting case in this article is when $\mathscr{C}$ stands for $\mathfrak{C}$. We have

$$\alpha = \frac{s}{t} = \frac{f_1 s}{f_1 t} \quad \text{and} \quad m = \frac{t'}{x} = \frac{f_2 t'}{\Phi(f_2)x}$$

and after selecting $f_1, f_2 \in \mathscr{F}_{V,k}$, such that $f_1 t = f_2 t'$, we can multiply

$$\alpha \cdot m = \frac{f_1 s}{f_1 t} \cdot \frac{f_2 t'}{\Phi(f_2)x} = \frac{f_1 s}{\Phi(f_2)x} \in \varinjlim S_{\Phi^\omega}.$$

In this way the R-module $M = \varinjlim S_{\Phi^\omega}$ is acted on by the group $\mathscr{G}_{V,k}$. Checking that this is a group action is left to the reader.

Definition 19. For two elements $m_1 = [(t, T)] = \frac{t}{T}$, $m_2 = [(s, S)] = \frac{s}{S}$ in M, we consider elements $f_1, f_2 \in \mathscr{F}_{V,3}$ such that $f_1 t = f_2 s$. We then define

$$\langle m_1, m_2\rangle = \left\langle \frac{f_1 t}{\bar{\Phi}(f_1)T}, \frac{f_2 s}{\bar{\Phi}(f_2)S}\right\rangle_M := \left\langle \bar{\Phi}(f_1)T, \bar{\Phi}(f_2)S\right\rangle.$$

By construction of $\bar{\Phi}$ given in Eq. (7), the inner product on classes is well-defined.

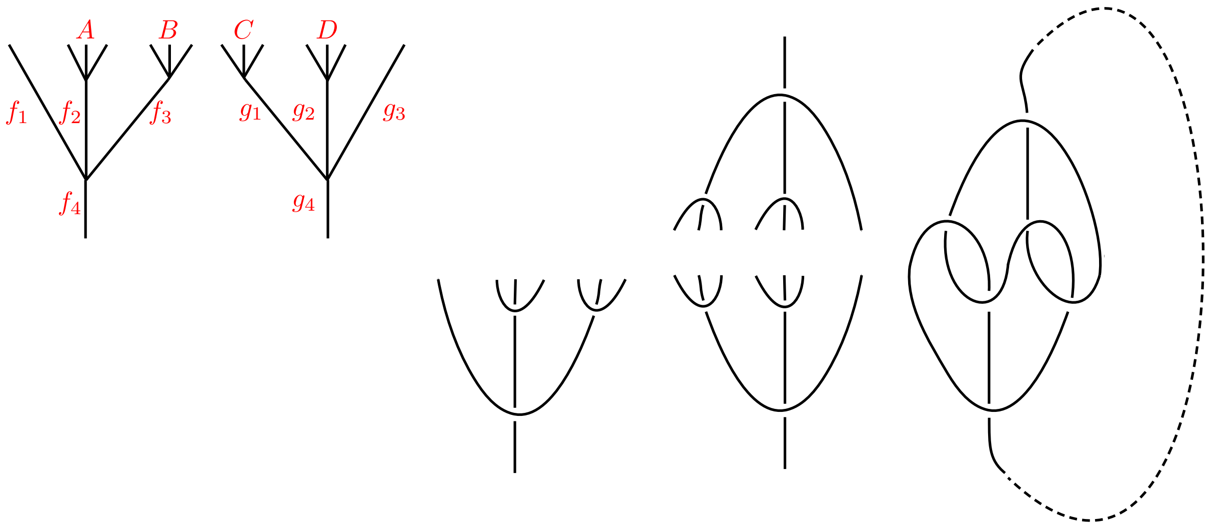


FIGURE 12. Computing the inner product of tangles coming from trees.

Consider the tangle $\bar{\omega}$, consisting of a single straight line connecting two boundary points of a rectangle, and set for any $t' \in \mathscr{T}_{V,k}$

$$\Omega = [(t', \Phi(t')\bar{\omega})] = [(1, \bar{\omega})] = \frac{1}{\bar{\omega}} \in M.$$

Let us now consider for every $g \in \mathscr{G}_{V,3}$ the element

$$\langle g\Omega, \Omega \rangle \in R,$$

which is called the "vacuum expectation of the element g". Let $g = \frac{s}{t} \in \mathscr{G}_{V,3}$. Then since

$$\frac{1}{\bar{\omega}} = \frac{t}{\bar{\Phi}(t)\bar{\omega}} = \frac{s}{\bar{\Phi}(s)\bar{\omega}}$$

we have

$$\left\langle \frac{s}{t}\Omega, \Omega \right\rangle = \left\langle \frac{s}{t} \frac{t}{\bar{\Phi}(t)\bar{\omega}}, \frac{1}{\bar{\omega}} \right\rangle = \left\langle \frac{s}{\bar{\Phi}(t)\bar{\omega}}, \frac{1}{\bar{\omega}} \right\rangle = \left\langle \frac{s}{\bar{\Phi}(t)\bar{\omega}}, \frac{s}{\bar{\Phi}(s)\bar{\omega}} \right\rangle = \left\langle \bar{\Phi}(t), \bar{\Phi}(s) \right\rangle.$$

Jones proved [7, 9] that topologically every link L can be obtained as $\langle \bar{\Phi}(t), \bar{\Phi}(s) \rangle$, that is as $\langle g\Omega, \Omega \rangle$ for some g in the ordinary Thompson group.

Proposition 20. *Every decorated link, where each component is decorated by a complex number, which is the trace of a unitary operator on V, can arise as $\langle g\Omega, \Omega \rangle$ for some $g \in \mathscr{G}_{V,3}$.*

Proof. By Jones' theorem [7, 9] each non-decorated link L can arise as $\langle g\Omega, \Omega \rangle$. Write this g as $g = s/t$, where s, t are non decorated ternary trees. Take now a component of L decorated by $\mathrm{tr}(f)$. We decorate a leaf of t contributing to this component by f and all other edges of s and t contributing to this component are decorated by the identity map. □

4. FRAMED LINKS

4.1. **Obtaining classical framings.** Assume now that the fixed vector space V is $\mathbb{C}$. The set of unitary operators $\mathbb{C} \to \mathbb{C}$ is identified with complex numbers z, such that $|z| = 1$. If we consider the category $\mathscr{F}_{\mathbb{C},3,n}$, where we consider only unitary maps which are n-th roots of unity, then we end up with links L having knot components, each of them is decorated by an n-th root of unity. This is equivalent to the notion of a $\mathbb{Z}/n\mathbb{Z}$-framing. We can interpret a knot component decorated by an n-th root of unity, as a twist of the thickened knot along its core curve. Indeed, let $\zeta_n = e^{\frac{2i\pi}{n}}$, every n-th root of unity is expressed by ζ_n^a, where $a \mod n$ is a class in $\mathbb{Z}/n\mathbb{Z}$. Multiplication by ζ_n^a gives rise to rotation in $D = \{z \in \mathbb{C}, |z| < 1\}$. Subdivide the core curve of a link component into n-segments and suppose that in $D \times [0, 1]$, the elements in $D \times \{k/n\}$

for $0 \leq k \leq n$ are multiplied by ζ_n^{ak}, then we obtain a twisting of the thickened knot component, which is rotated a times.

The continuous version of this can be given in the following way: each rational number $\alpha = a/n$ gives rise to the function

$$\begin{aligned} D \times I &\longrightarrow D \times I \\ (z,t) &\longmapsto (e^{2\pi i \alpha} z, t) = (e^{\frac{2\pi i a}{n}} z, t). \end{aligned}$$

We can obtain an integer framing, by considering an infinite number of compatible framings, one for each $n \in \mathbb{N}$ and then form $\hat{\mathbb{Z}} = \lim\limits_{\leftarrow} \mathbb{Z}/n\mathbb{Z}$, which contains $\mathbb{Z}$ as a dense subgroup. Instead of $\hat{\mathbb{Z}}$ we can also restrict ourselves to framings in p^ℓ-roots unity, with $\ell \in \mathbb{N}$ and forming the $\mathbb{Z}_p$ rings. For a topological interpretation of p-adic integers as infinite cablings in the case of the Braid groups, see the articles of J. Juyumaya and the second author in [10], [11].

The interpretation of a twist by an element z of $|z| = 1$ which is not a root of unity is still interesting. Such rotations lead to non-closed orbits which are dense in the boundary torus of the thickened knot.

4.2. **Obtaining multiple framings.** We will now return to the general case where $d = \dim_{\mathbb{C}} V > 1$. Suppose that we have a unitary operators with eigenvalues n-th roots of unity. The conjugation class of such an operator U can be described by the exponents $(a_1, \ldots, a_d) \in (\mathbb{Z}/n\mathbb{Z})^d$, that is there exist $Q \in \mathrm{GL}_d(\mathbb{C})$ such that $QUQ^{-1} = \mathrm{diag}(\zeta_n^{a_1}, \ldots, \zeta_n^{a_d})$. Closed components lead to taking the trace of U as decoration, i.e. we get the quantity $\zeta_n^{a_1} + \cdots + \zeta_n^{a_d}$. Next lemma shows that under some mild assumptions the exponents $a_1, \ldots, a_d$ can be recovered from the trace of the operator.

Lemma 21. *Let $n = p_1^{\lambda_1} \cdots p_r^{\lambda_r}$ be the decomposition of the integer into prime factors $p_1, \ldots, p_r$, such that $d < p_1 < p_2 < \cdots < p_r$. Assume that $2 \leq d \leq n-2$. If*

$$\zeta_n^{a_1} + \cdots + \zeta_n^{a_d} = \zeta_n^{b_1} + \cdots + \zeta_n^{b_d} \tag{9}$$

and $0 \leq a_i, b_i < n$ then there is a permutation $\sigma \in S_d$, such that $a_i = b_{\sigma(i)}$ for all $1 \leq i \leq d$.

Proof. We will modify the proof of [6] given in Mathoverflow in order to include the case of having the same exponents.

Notice first the case $n = p$ is easy. Indeed, if we consider an equality as given in Eq. (9), then we can form the polynomial

$$f(x) = \sum_{i=1}^{d} \left(x^{a_i} - x^{b_i} \right) \in \mathbb{Z}[x],$$

which has ζ_p as a root and degree $\deg f(x) \leq p-1$. So for the polynomial $\Phi_p(x) = 1 + x + \cdots + x^{p-1}$ we should have $\Phi_p(x) \mid f(x)$, and if $f(x)$ is not identically zero then $f(x) = c\,\Phi_p(x)$, for some $c \in \mathbb{Z}$ and in this case $d = p-1$ contradicting the assumption $d \leq n-2 = p-2$.

For the next step we will use Theorem 4.1 in [12], which we now describe. Let $G = \langle g \rangle$ be a cyclic group of order $n = p_1 p_2 \cdots p_r$, where $p_1 < p_2 < \cdots < p_r$ are the primes in the decomposition of n and $r \geq 2$. Let ϕ be the natural map

$$\begin{aligned} \phi : \mathbb{Z}[G] &\longrightarrow \mathbb{Z}[\zeta_n] \subset \mathbb{C} \\ \sum_{i=0}^{n-1} c_i g^i &\longmapsto \sum_{i=0}^{n-1} c_i \zeta_n^i \end{aligned}$$

and $\epsilon_0 : \mathbb{Z}[G] \to \mathbb{Z}$ be the augmentation map, namely $\epsilon_0(\sum_{i=0}^{n-1} c_i \zeta_n^i) = \sum_{i=0}^{n-1} c_i$. An expression of the form $\zeta_1^{a_1} + \cdots + \zeta_d^{a_d} \in \mathbb{Z}[\zeta_n]$, is obtained as image of an element $\sum_{i=0}^{n-1} c_i g^i \in \mathbb{Z}[G]$, where c_i is the multiplicity of the root $\zeta_n^{a_\nu}$ for $i = a_\nu$. That is if we have equal exponents a_i, then we write

$$\zeta_n^{a_1} + \cdots + \zeta_n^{a_d} = c_{a_{j_1}} \zeta_n^{a_{j_1}} + \cdots + c_{a_{j_t}} \zeta_n^{a_{j_t}} = \phi \left(c_{a_{j_1}} g^{a_{j_1}} + \cdots + c_{a_{j_t}} g^{a_{j_t}} \right),$$

where all exponents a_{j_s} are different and $c_{a_{j_\nu}}$ are positive integers. We also have $e_0(c_{a_{j_1}} g^{a_{j_1}} + \cdots + c_{a_{j_t}} g^{a_{j_t}}) = d$.

Theorem 4.1 in [12] implies that if $x, y \in \mathbb{N}[G]$ with $\phi(x) = \phi(y)$ and $\epsilon_0(x) \leq \epsilon_0(y)$, then $x \leq y$, meaning that the coefficient of x corresponding to g^i is less than or equal to the coefficient of y corresponding to the same g^i. But in our case if x, y are selected so that $\phi(x), \phi(y)$ are the left and right hand of Eq. (9)

respectively, then $e_0(x) = e_0(y) = d$, forcing $x \leq y$. The same argument shows that $y \leq x$ and the equality of the coefficients follows.

The above argument holds only for squarefree integers n. We will now use theorem 1 of [14] in order to reduce the problem to the squarefree case. This theorem states that whenever we have a selection $\zeta_1, \ldots, \zeta_r$ of roots of unity such that

$$\sum_{i=1}^{r} \alpha_i \zeta_i = 0, \alpha_i \in \mathbb{Z}, \alpha_i \neq 0$$

but there is no nonempty proper subset $S \subset \{1, 2, \ldots, r\}$ for which

$$\sum_{i \in S} \alpha_i \zeta_i = 0,$$

then for each i, j we have $(\zeta_i/\zeta_j)^m = 1$, where m is the product of primes which are less than r. We rewrite Eq. (9) as follows:

$$\nu_1 \zeta_n^{a_1} + \cdots + \nu_t \zeta_n^{a_t} = \mu_1 \zeta_n^{b_1} + \cdots + \mu_{t'} \zeta_n^{b_{t'}}, \tag{10}$$

where $1 \leq \nu_i, \mu_i \leq d$ are the multiplicities of $\zeta_n^{a_i}, \zeta_n^{b_i}$ in case of multiple eigenvalues. We subtract the left from the right hand side in order to obtain a relation

$$\sum_{i=1}^{r} \alpha_i \zeta_i = 0, \tag{11}$$

where $r \leq 2d$ and each $-d \leq \alpha_i \leq d$, and ζ_i are distinct n-th roots of unity. If there exist a nonempty proper subset $S \subset \{1, 2, \ldots, r\}$ for which

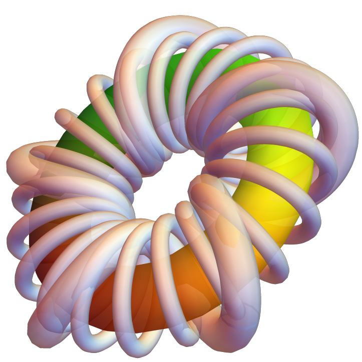
$$\sum_{i \in S} \alpha_i \zeta_i = 0$$

then the smallest such S has size at most $r/2 \leq d$. Indeed, for every S with the zero sum property the complement has also the zero sum property. Mann's theorem for S gives us that $(\zeta_i/\zeta_j)^m = 1$ where m is the product of primes $\leq d$. For the prime factors p_i of n we have assumed that $d < p_i$ therefore we have that $(n, m) = 1$ and the equation $(\zeta_i/\zeta_j)^n = 1$ we finally arrive to $\zeta_i = \zeta_j$, which is a contradiction. So no such set exists. We multiply now Eq. (11) by a reciprocal of an existing root of unity in order to obtain that some $\zeta_i = 1$. Again we use Mann's theorem in order to obtain a squarefree positive integer m such that for all $1 \leq i, j \leq r$ we have $(\zeta_i/\zeta_j)^m = 1$ and since some $\zeta_j = 1$ we finally arrive to $\zeta_i^m = 1$, for all $1 \leq i \leq r$. Therefore in Eq. (10) we have that all roots $\zeta_n^{a_i}, \zeta_n^{b_i}$ have squarefree order, meaning that $N = p_1^{\lambda_1 - 1} \cdots p_r^{\lambda_r - 1}$ divides all exponents a_i, b_i, that is $a_i = a_i' N$, $b_i = b_i' N$. But then setting $\zeta_{n/N} = \zeta_n^N$ we have that

$$\zeta_{n/N}^{a_1'} + \cdots + \zeta_{n/N}^{a_d'} = \zeta_{n/N}^{b_1'} + \cdots + \zeta_{n/N}^{b_d'}$$

and since n/N is square free we have the desired equality for a_i', b_i', which in turn implies the truth of the lemma for a_i, b_i.

□



We can restrict ourselves to the category $\mathscr{F}_{V,3,n}$, where V is a vector space of small dimension d, according to the previous lemma and the maps are forests decorated by unitary operators with eigenvalues n-th roots of unity. Then lemma 21 shows that the trace of U holds the information of d-exponents and gives rise to a $(\mathbb{Z}/n\mathbb{Z})^d$-framing of each link component. Using the previous interpretation of framing as twisting along the core curve we can model this way d-cables rotating along the core curve.

5. Relating to the construction of 3-manifolds

By classical results of Lickorish and Wallace [13], [16] it is well-known that every closed, connected, oriented (c.c.o.) 3-manifold can be obtained by the surgery technique along a framed link in S^3. Then two framed links give rise to homeomorphic 3-manifolds if and only if they are equivalent via isotopy and the so-called Kirby moves. There are various ways of representing framing. One of them is to use the blackboard framing, so in this case the isotopy part of the equivalence is restricted to regular isotopy (the first Reidemeister move is not allowed). Another one is to consider the framing as a decoration (integral or rational) assigned to each component of the link. In this approach, the isotopy part of the equivalence is ambient isotopy (all three Reidemeister moves). Relating now the discussion to the Thompson group, Jones showed in [7] that every link diagram is equivalent to a link diagram representing a Thompson group element. (Moreover, Aiello [1] proved an analogous result for oriented links). However, in [15] Raghavan and Sweeney show that not all regular isotopy classes of link diagrams arise from elements of the Thompson group. Through our construction, all ambient isotopy classes of framed link diagrams arise from elements of the framed Thompson groups (in analogy to the connection of framed links to elements of the framed braid groups). Therefore, our approach has the potential of relating the theory of Thompson groups to 3-manifolds.

In fact, from our construction in Section 4.2, each component of a given link can be assigned in a natural manner a multitude of framings and, thus, a multidute of associated 3-manifolds. The above constructions lead to several interesting questions, as for example the topological connection of the 3-manifolds assigned to one framed Thompson group element. Another question is the algebraic relation between different (framed) Thompson group elements that give rise to ambient isotopic (framed) links (the analogues of the classical Markov theorem), and then to homeomorphic 3-manifolds.

References

[1] V. Aiello. On the Alexander theorem for the oriented Thompson group $\overrightarrow{F}$. *Algebr. Geom. Topol.*, 20(1):429–438, 2020.
[2] V. Aiello, R. Conti, and V. F. R. Jones. The Homflypt polynomial and the oriented Thompson group. *Quantum Topol.*, 9(3):461–472, 2018.
[3] J. M. Belk. *Thompsons' group F*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–Cornell University.
[4] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.
[5] V. Guba and M. Sapir. Diagram groups. *Mem. Amer. Math. Soc.*, 130(620):viii+117, 1997.
[6] itotcg (https://mathoverflow.net/users/95358/itotcg). Uniqueness of sums of roots of unity. MathOverflow. URL:https://mathoverflow.net/q/244876 (version: 2017-04-13).
[7] V. F. R. Jones. Some unitary representations of Thompson's groups F and T. *J. Comb. Algebra*, 1(1):1–44, 2017.
[8] V. F. R. Jones. A no-go theorem for the continuum limit of a periodic quantum spin chain. *Comm. Math. Phys.*, 357(1):295–317, 2018.
[9] V. F. R. Jones. On the construction of knots and links from Thompson's groups. In *Knots, low-dimensional topology and applications*, volume 284 of *Springer Proc. Math. Stat.*, pages 43–66. Springer, Cham, 2019.
[10] J. Juyumaya and S. Lambropoulou. p-adic framed braids. *Topology Appl.*, 154(8):1804–1826, 2007.
[11] J. Juyumaya and S. Lambropoulou. p-adic framed braids II. *Adv. Math.*, 234:149–191, 2013. With an appendix by Paul Gérardin.
[12] T. Y. Lam and K. H. Leung. On vanishing sums of roots of unity. *J. Algebra*, 224(1):91–109, 2000.
[13] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. of Math. (2)*, 76:531–540, 1962.
[14] H. B. Mann. On linear relations between roots of unity. *Mathematika*, 12:107–117, 1965.
[15] R. Raghavan and D. Sweeney. Regular isotopy classes of link diagrams from thompson's groups, 2020.
[16] A. H. Wallace. Modifications and cobounding manifolds. *Canadian J. Math.*, 12:503–528, 1960.

Department of Mathematics, National and Kapodistrian University of Athens Panepistimioupolis, 15784 Athens, Greece
Email address: kontogar@math.uoa.gr

National Technical University Department of Mathematics Zografou Campus GR-15780 Athens GREECE
Email address: sofia@math.ntua.gr