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RESEARCH

On the Galois-module Structure of Polydifferentials of Subrao Curves.

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Abstract
We investigate the Galois module structure of polydifferentials for Subrao curves defined over an algebraically closed field of positive characteristic, and explicitly compute the decomposition of the space of holomorphic polydifferentials into indecomposable modules.

Keywords Automorphisms of curves · Differentials · Galois module structure · Modular representation theory.

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1 Introduction

Let X be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$, and let G be a subgroup of the group of automorphisms of X. The group G acts on X from the left, by our convention, and hence on the space of n-polydifferentials $H^0(X, \Omega_X^{\otimes n})$ from the right. The so-called *Galois-module structure problem* for X asks for the direct sum decomposition of $H^0(X, \Omega_X^{\otimes n})$ into G-indecomposable pieces. In characteristic zero, the $n = 1$ case is a classical result ([15]), which can be generalized for $n \geq 1$, [11].

In positive characteristic, the Galois-module structure is unknown in general. There are only some partial results known. Let us give a brief overview. If the cover $X \to G\backslash X$ is unramified or if $(|G|, p) = 1$, Tamagawa [32] determined the G-module structure of $H^0(X, \Omega_X)$. Valentini [34] generalized this result to unramified extensions with G being a p-group. In the p-group case, moreover, Salvador and Bautista [23] determined the semi-simple part of the representation with respect to the Cartier operator. For the cyclic-group case, Valentini

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and Madan [35] and S. Karanikolopoulos [16] determined the structure of $H^0(X, \Omega_X)$ in terms of indecomposable modules. A similar study has been done for the elementary abelian case by Calderón, Salvador and Madan [28]. Finally, N. Borne [6] developed a theory, using advanced techniques from both modular representation theory and K-theory, for computing in some cases the $k[G]$-module structure of the space of polydifferentials $H^0(X, \Omega_X^{\otimes n})$. On the other hand F. Bleher and T. Chinburg together with the first author in [4] proved that if the p-Sylow subgroup of a group G acting on a curve X is cyclic, then the Galois module structure of holomorphic differentials is determined by the ramification of the cover $X \to X/G$. This result was extended to the case of polydifferentials by F. Bleher and A. Wood in [5]. If the p-Sylow subgroup of the group G is not cyclic and $p > 2$, then the representation theory of G is much more difficult, for instance the problem of classifying the indecomposable representations is unsolvable, that is the representation type is wild, see [3, Theorem 1.2.1]. It is worth noticing that the case we study in this article has an elementary abelian p-Sylow subgroup $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and hence falls into the wild representation type, and in this article we provide one of few known examples of Galois module structure of polydifferentials for actions of groups with non trivial, non cyclic p-Sylow subgroup.

Let us point out that the determination of the Galois-module structure as above has several applications. For example, in [18, 19], the first author connected the $k[G]$-module structure of $H^0(X, \Omega_X^{\otimes 2})$ to the computation of the tangent space of the global deformation functor of curves.

In this paper, we consider the Galois-module structure problem for the so-called *Subrao curves*, that is nonsingular curves X whose function field $k(X)$ admits an equation of the form

$$k(X) : (x^p - x)(y^p - y) = c, \quad c \in k - \{0\}. \tag{1}$$

Subrao curves are double covers of the projective line and fit in the following diagram:

$$\begin{tikzcd} & X \arrow[dl, "A"'] \arrow[d, "A\times B"] \arrow[dr, "B"] & \\ \mathbb{P}^1_k \arrow[dr, "B"'] & & \mathbb{P}^1_k \arrow[dl, "A"] \\ & Y \cong \mathbb{P}^1_k & \end{tikzcd} \qquad \begin{tikzcd} & k(X) \arrow[dl, dash, "A"'] \arrow[d, dash, "A\times B"] \arrow[dr, dash, "B"] & \\ k(x) \arrow[dr, dash, "B"'] & & k(y) \arrow[dl, dash, "A"] \\ & k(T) & \end{tikzcd} \tag{2}$$

where in the right hand side we see the function field $k(X)$ as an Artin-Schreier extension of the rational function fields $k(x)$ and $k(y)$, and $A \cong B$ are both cyclic groups of order p.

Remark 1 If k is the algebraic closure of a discrete valuation field K and $|c| < 1$, then the above defined curves have split multiplicative reduction and are examples of Mumford curves [25]. These special case of Subrao curves are known in the literature as Artin-Schreie-Mumford curves. Mumford curves are algebraization of rigid analytic curves over K of the form $\Gamma\backslash(\mathbb{P}_K^{1,\mathrm{an}} - \mathcal{L}_\Gamma)$, where Γ is a finitely generated torsion free subgroup of $\mathrm{PGL}(2, K)$, called a Schotky group and $\mathcal{L}_\Gamma$ is the set of limit points.

Subrao curves are special from quite a few points of view. For example, they have large automorphism group. Also in the case a Subrao curve is a Mumford curve, it has maximal automorphism group (and hence their Schottky groups are the analogue of classical Hurwitz groups), cf. [10] and [8]. They were first studied by D. Subrao [31], Valentini-Madan [35], and S. Nakajima [26]. M. Matignon has studied their equivariant liftability to characteristic zero [24], and these curves play a special role when studying the 'field of definition versus field of moduli' question for cyclic covers of the projective line (cf. [20]).

The main result of this article is the following:

Theorem 2 *Let k be an algebraically closed field of characteristic p, $p \neq 2$. For $n > 1$, let r $(0 \leq r < p)$ be the remainder of $2n-1$ modulo p.*

As a $k[G]$-module, the following decomposition holds:

(i) If $r \neq 0$,

$$H^0(X, \Omega_X^{\otimes n}) \cong k[G]^{\oplus\left(2n-1-2\left\lceil \frac{2n-1}{p} \right\rceil\right)} \oplus k[G]/(\epsilon_A - 1)^{p-r} \oplus k[G]/(\epsilon_B - 1)^{p-r}.$$

(ii) If $r = 0$,

$$H^0(X, \Omega_X^{\otimes n}) \cong k[G]^{\oplus\left(2n-1-2\left\lfloor \frac{2n-1}{p} \right\rfloor\right)} = k[G]^{\oplus\left((2n-1)(1-\frac{2}{p})\right)},$$

that is, $H^0(X, \Omega_X^{\otimes n})$ is a free $k[G]$-module.

Corollary 3 *The following decomposition for the structure of $H^0(X, \Omega_X^{\otimes n})$ as a $k[A]$ module can be obtained by restriction. If $r \neq 0$,*

$$H^0(X, \Omega_X^{\otimes n}) \cong k[A]^{\oplus\left((p-1)(2n-1)-p\left\lceil \frac{2n-1}{p} \right\rceil\right)} \oplus \left(k[A]/(\epsilon_A - 1)^{p-r}\right)^{\oplus p},$$

while if $r = 0$,

$$H^0(X, \Omega_X^{\otimes n}) \cong k[A]^{\oplus\left((p-1)(2n-1)-p\left\lfloor \frac{2n-1}{p} \right\rfloor\right)} = k[A]^{\oplus((p-2)(2n-1))},$$

that is $H^0(X, \Omega_X^{\otimes n})$ is a free $k[A]$-module. A similar result holds for the group B.

In Section 2.2 we will obtain this result for the $k[A]$ module structure using the method of Nakajima [26], who described the Galois module structure of cyclic p-groups of order p. This result can be also proved by a combinatorial method using the theory of Mumford curves, see [21]. More precisely, when X is a Mumford curve, that is when k is the algebraic closure of a discrete valuation field K and $|c| < 1$ as in Remark 1, a theorem of Teitelbaum [33, Theorem 1] provides a natural A-equivariant isomorphism between $H^0(X, \Omega_X^{\otimes n})$ and the group cohomology space $H^1(A, P_n)$, where P_n denotes the space of polynomials of one variable of degree at most $2n-2$ on which A acts by fractional linear transformations. On this cohomological model the action of A can be described explicitly, by means of cocycles attached to the generators of A with respect to an explicit basis of P_n, and this is precisely the point of view used in the combinatorial proof given in [21]. We note that there is recently an increased interest for Artin-Schreier-Mumford curves and the Galois actions on them, see [12, 13, 22].

The curve X is known to be ordinary by the work of Subrao [31], and the cover $X \to X/G$ is only weakly ramified [27, Theorem 2]. This is exactly the setting in which the theory of B. Köck [17] applies: admitting controlled poles of order $(2n-1)R_{\mathrm{red}}$ at the ramification points turns $\Omega_X^{\otimes n}$ into a sheaf whose space of global sections is a free $k[G]$-module of the expected rank $2n-1$ (Theorem 5). Comparing this free module with $H^0(X, \Omega_X^{\otimes n})$ via the short exact sequence (5), the difference is identified with the space of global sections of the skyscraper sheaf Σ supported at the (two) ramification points of $X \to X/G$. We compute the $k[G(P)]$-module structure of each stalk Σ_P explicitly, in terms of Jordan blocks for the transformation $\sigma(1/t) = 1/t + 1$, and induce these local pieces up to G. Comparing the resulting module with a projective cover of $H^0(X,\Sigma)$ and applying the snake lemma then yields the decomposition of Theorem 2.

2 Proofs

Let X be a smooth projective curve over an algebraically closed field k of characteristic $p>0$ as in the introduction, let G be a p-group acting on X, and consider the cover $\pi: X\to X/G =: Y$. For every point P of X we consider a local uniformizer t at P, the stabilizer $G(P)$ of P and assign a sequence of ramification groups

$$G_i(P)=\{\sigma\in G(P): v_P(\sigma(t)-t)\geq i+1\}.$$

Notice that $G_0(P)=G(P)$ for p-groups, see [29, Chapter IV]. Let $e_i(P)$ denote the order of $G_i(P)$. We use the notation X_{ram} for the set of ramification points. We will say that the cover $X\to X/G$ is weakly ramified if all $e_i(P)$ vanish for $i\geq 2$.

In this section we will employ the results of B. Köck on the projectivity of the cohomology groups of certain sheaves in the weakly ramified case. Notice that the curves X are ordinary [31, Corollary 3.3, th. 4.1], and in the cover $X\to X/G$ only weak ramification can happen, for example by [27, Theorem 2].

We denote by Ω_X the sheaf of differentials on X and by $\Omega_X(D)$ the sheaf $\Omega_X\otimes_{\mathcal{O}_X}\mathcal{O}_X(D)$. For a divisor $D=\sum_{P\in X} n_P P$ we denote by $D_{\mathrm{red}}=\sum_{P\in X: n_P\neq 0} P$ the associated reduced divisor. We will also denote by

$$L(D)=H^0(X,\mathcal{O}_X(D))=\{D+(f)>0: f\in k(X)\}\cup\{0\},$$

where $k(X)$ is the function field of the curve X. The ramification divisor equals $R=\sum_{P\in X}\sum_{i=0}^{\infty}\big(e_i(P)-1\big)$. Finally, Σ denotes the skyscraper sheaf defined by the short exact sequence:

$$0\to\Omega_X^{\otimes n}\to\Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\to\Sigma\to 0. \tag{3}$$

From now on we will restrict ourselves to the case of Subrao curves as defined in Eq. 1.

Lemma 4 *For $n>1$ the cohomology group $H^1(X,\Omega_X^{\otimes n})=0$.*

Proof We have $\deg(\Omega_X^{\otimes n})=2n(g-1)$, [14, Example 1.3.3]. The result follows by [14, Example 1.3.4]. □

Now we apply the functor of global sections to the short exact sequence in Eq. 3 and obtain the following short exact sequence:

$$0\to H^0(X,\Omega_X^{\otimes n})\to H^0\big(X,\Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big)\to H^0(X,\Sigma)\to H^1(X,\Omega_X^{\otimes n})=0. \tag{4}$$

Theorem 5 *The $k[G]$-module $H^0\big(X,\Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big)$ is a free $k[G]$-module of rank $2n-1$.*

Proof Since G is a p-group a module is free if and only if it is projective. So we have to show that $H^0\big(X,\Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big)$ is projective. B. Köck proved [17, Theorem 2.1] that if $D=\sum_{P\in X} n_P P$ is a G-equivariant divisor, the map $\pi: X\to Y:=X/G$ is weakly ramified, $n_P\equiv -1 \bmod e_P$ for all $P\in X$ and $\deg(D)\geq 2g_X-2$, then the module $H^0(X,\mathcal{O}_X(D))$ is projective.

We have to check the conditions for the divisor $D = nK_X + (2n-1)R_{\mathrm{red}}$, where K_X is a canonical divisor on X. By [14, Proposition IV.2.3], $K_X = \pi^* K_Y + R$ and $R = \sum_{P\in X} 2\big(e_0(P)-1\big)$, therefore

$$D = n\pi^* K_Y + \sum_{P\in X: e_0(P)>1} (2ne_0(P) - 2n + 2n - 1)\, P.$$

Therefore, the condition $n_P \equiv -1 \bmod e_0(P)$ is satisfied, notice that $e_0 \mid n$.

We will now compute the dimension of $H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big)$ using Riemann–Roch theorem, keep in mind that $H^1\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big) = 0$. Let r_0 denote the cardinality of $X^G_{\mathrm{ram}} = \{P \in X/G : e(P) > 1\}$, and g_Y denotes the genus of the quotient curve $Y = X/G$.

$$\begin{aligned}\dim_K H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big) &= n(2g_X - 2) + (2n-1)|X_{\mathrm{ram}}| + 1 - g_X\\ &= (2n-1)(g_X - 1 + |X_{\mathrm{ram}}|)\\ &= |G|(2n-1)(g_Y - 1 + r_0),\end{aligned}$$

where in the last equality we have used the Riemann–Hurwitz formula [14, 7, Corollary IV 2.4]

$$g_X - 1 = |G|(g_Y - 1) + \sum_{P\in X_{\mathrm{ram}}} (e_0(P) - 1).$$

Since $g_Y = 0$ and $r_0 = 2$ the desired result follows. □

Remark 6 This method was applied by the first author and B. Köck in [18] for the $n = 2$ case in order to compute the dimension of the tangent space to the deformation functor of curves with automorphisms. Deformations of curves with automorphisms for Mumford curves were also studied by G. Cornelissen and F. Kato in [7].

By Lemma 4 and Theorem 5 the sequence in Eq. 4 leads to the following short exact sequence of modules:

$$0 \to H^0(X, \Omega_X^{\otimes n}) \to k[G]^{(2n-1)} \to H^0(X, \Sigma) \to 0. \tag{5}$$

Since Σ is a skyscraper sheaf the space $H^0(X, \Sigma)$ is the direct sum of the stalks of Σ

$$H^0(X, \Sigma) = \bigoplus_{P\in X_{\mathrm{ram}}} \Sigma_P \cong \bigoplus_{j=1}^{r_0} \mathrm{Ind}^G_{G(P_j)}(\Sigma_{P_j}),$$

where, for a subgroup H of G, $\mathrm{Ind}_H^G V$ denotes the induced representation of an $k[H]$-module V to a $k[G]$-module, i.e., $\mathrm{Ind}_H^G V = V \otimes_{k[H]} k[G]$.

2.1 Proof of Theorem 2

Lemma 7 *The module $H^0(X, \Sigma)$ is $k[G]$-isomorphic to*

$$k[G]^{\oplus 2\left\lfloor \frac{2n-1}{p} \right\rfloor} \bigoplus k[G]/(e_A - 1)^r \bigoplus k[G]/(e_B - 1)^r.$$

Proof It follows from the ramification of the function field of Subrao curve, seen as a double Artin-Schreier extension of the rational function field, where $r_0 = 2$, i.e., only two points p_1, p_2 of $X/(A \times B)$ are branched in the cover $X \to X/(A \times B)$.

Let $e_A : y \mapsto y+1, x \mapsto x$ be the generator of A and $e_B : x \mapsto x+1, y \mapsto y$ be the generators of the groups A, B. The branched points for the Artin-Schreier extension $k(X)/k(x)$ are exaclty the roots of $x^p - x$. These roots are identified in the quotient curve $X/(A \times B)$ under the action of B, and correspond to one of the two branched points p_1. A similar construction on the other curve gives rise to the second branch point p_2.

Select a point P_1 of X which lies above p_1 and a point P_2 of X which lies above p_2. Let $G(P_1) = A$ and $G(P_2) = B$.

We will use an approach similar to [18] in order to study the Galois module structure of the stalk Σ_{P_j} as a $k[G_{P_j}]$-module. Let $P = P_j$ for $j = 1$ or $j = 2$.

Recall from the proof of Theorem 5 that, by [14, Proposition IV.2.3], for a ramified cover of curves with canonical divisors K_X, K_Y and ramification divisor R we have $K_X = \pi^* K_Y + R$; here X is still our Subrao curve, and we apply this formula to the cover $\pi : X \to X/G(P) =: Y$ associated to the stabilizer $G(P)$ of P (rather than to the full quotient $X \to X/G$ considered before). This implies $nK_X = n\pi^* K_Y + nR$, so if the multiplicity of the divisor K_Y at $\pi(P)$ is m, then the multiplicity of nK_X at P is $mnp + 2n(p-1)$ and the multiplicity of $nK_X + (2n-1)R_{\mathrm{red}}$ at P is $mnp + 2n(p-1) + (2n-1)$. The space Σ_P is thus the space of Laurent series (in a local parameter at P) of pole order at most $(mn+2n)p - 1$, modulo the subspace of those of pole order at most $(mn+2n)p - 2n$. So if t is a local uniformizer at P and s is a local uniformizer at $\pi(P)$ we can identify Σ_P with the finite-dimensional space

$$\Sigma_P = \left\langle \frac{t}{s^{mn+2n}}, \frac{t^2}{s^{mn+2n}}, \ldots, \frac{t^{2n-1}}{s^{mn+2n}} \right\rangle_k .$$

Since s and t^p differ by a unit, multiplying through by s^{mn+2n} turns these Laurent series into power series representatives, giving a $G(P)$-equivariant isomorphism of Σ_P with the k-vector space of power series whose constant term vanishes, modulo the subspace of those of t-valuation greater than or equal to $2n$:

$$\Sigma_P = \left\langle t, t^2, \ldots, t^{2n-1} \right\rangle_k . \tag{6}$$

The action of $G(P)$ on Σ_P is given by the transformation $\sigma(1/t) = 1/t + 1$ for a generator σ of the cyclic group $G(P)$, or equivalently $\sigma(t) = \frac{t}{1+t}$, see [9, prop. 1.18]. Notice, that the element $t^{-p} - t^{-1} = \frac{1-t^{p-1}}{t^p}$ is invariant and so is its inverse $t^p(1-t^{p-1})^{-1}$. Here the unit $(1-t^{p-1})^{-1}$, can be seen as a polynomial modulo t^{2n}, if we expand it in terms of a geometric series and truncate the terms of degree $\geq 2n$. Now we analyse the $G(P)$-module structure of Σ_P using Jordan blocks. Observe that for $0 \leq \mu \leq p-1$:

$$\sigma \binom{1/t}{\mu} = \binom{1/t}{\mu} + \binom{1/t}{\mu - 1},$$

where

$$\binom{1/t}{\mu} = \frac{1}{\mu!} \prod_{\nu=0}^{\mu-1} \left(\frac{1}{t} - \nu \right) = \frac{1}{\mu! t^\mu} \prod_{\nu=0}^{\mu-1} (1 - t\nu).$$

Notice that $\binom{1/t}{-1}$, which appears above for $\mu = 0$, is equal to 0. Furthermore, $\binom{1/t}{\mu}$ is a rational function of t, whose denominator is $\mu! t^\mu$. Therefore, multiplying $\binom{1/t}{\mu}$ by the invariant element $t^p(1-t^{p-1})^{-1}$ we obtain a truncated Laurent polynomial in the basis of Σ_P given in Eq. 6 of t-valuation $p - \mu$:

$$t^p(1-t^{p-1})^{-1} \binom{1/t}{\mu} = \frac{t^{p-\mu}}{\mu!} + \text{higher order terms}$$

The following elements of Σ_P

$$\left(\frac{t^p}{1-t^{p-1}}\right)^i \binom{1/t}{\mu}, \quad \begin{array}{l} \text{where } 1 \leq i \leq \left\lfloor \frac{2n-1}{p} \right\rfloor \text{ and } 0 \leq \mu \leq p-1 \\ \text{or } i = \left\lfloor \frac{2n-1}{p} \right\rfloor + 1 \text{ and } p-r \leq \mu \leq p-1 \end{array},$$

seen as linear combinations of elements in the basis of Σ_P given in Eq. 6 (after expanding the denominators using the geometric series expansion) have t-valuations equal to $pi - \mu$. The set of their t-valuations start from 1 (obtained for $i = 1, \mu = p-1$) to $2n-1$ (obtained for $i = \left\lfloor \frac{2n-1}{p} \right\rfloor + 1, \mu = p-r$). Since these $2n-1$ elements realize all the distinct t-valuations $1, 2, \ldots, 2n-1$, they form another basis of Σ_P, and we now compute the action of σ on this basis.

For fixed i, $i = 1, \ldots, \left\lfloor \frac{2n-1}{p} \right\rfloor$, and by allowing μ to vary from $0 \leq \mu \leq p-1$, we obtain a representation given by Jordan block J_p. The remaining block $i = \left\lfloor \frac{2n-1}{p} \right\rfloor + 1$, $p-r \leq \mu \leq p-1$ gives rise to a representation corresponding to the Jordan block J_r.

So the structure of Σ_P is given by

$$\Sigma_P = J_p^{\left\lfloor \frac{2n-1}{p} \right\rfloor} \bigoplus J_r. \tag{7}$$

We now have

$$\begin{aligned} \operatorname{Ind}_{G(P_j)}^G(J_r) &\cong \operatorname{Ind}_{G(P_j)}^G\left(\frac{k[A]}{(e_A-1)^r}\right) \cong \left(\frac{k[A]}{(e_A-1)^r}\right) \otimes_{k[A]} k[A] \otimes k[B] \\ &\cong k[G]/(e_A-1)^r. \end{aligned}$$

Similarly

$$\operatorname{Ind}_{G(P_2)}^G(J_r) = \frac{k[G]}{(\epsilon_B-1)^r}$$

and both of the above $k[G]$-modules are indecomposable by Green's indecomposability criterion [1, th. 8 p. 62] stating that if N is a normal subgroup of G, G/N is a p-group, V is an indecomposable $k[N]$-module then $\operatorname{Ind}_N^G(V)$ is also indecomposable. □

We will now give the proof of Theorem 2.

Proof Consider Eq. 4, and let $P \xrightarrow{\phi} H^0(X, \Sigma)$ be the projective cover of $H^0(X, \Sigma)$. Recall from Lemma 7 that $H^0(X, \Sigma) \cong k[G]^{\oplus 2\left\lfloor \frac{2n-1}{p} \right\rfloor} \oplus k[G]/(e_A-1)^r \oplus k[G]/(e_B-1)^r$; its free summand is already projective, while (assuming $r \neq 0$) each of the two remaining cyclic summands needs one further copy of $k[G]$ to be covered. Hence, using $\left\lceil \frac{2n-1}{p} \right\rceil = \left\lfloor \frac{2n-1}{p} \right\rfloor + 1$ for $r \neq 0$, the projective cover is

$$P = k[G]^{\oplus 2\left\lceil \frac{2n-1}{p} \right\rceil}.$$

Assuming $r \neq 0$, we compute that $\ker \phi = k[G]/(e_A-1)^{p-r} \oplus k[G]/(e_B-1)^{p-r}$. (If $r = 0$, then by Lemma 7 the module $H^0(X, \Sigma)$ is already free, so $P = H^0(X, \Sigma)$, $\phi = \mathrm{Id}$, and

$\ker\phi = 0$; we treat this case separately below.) By [2, Lemma 17.17] we have the following decomposition $k[G]^{2n-1} = F \oplus P$, fitting in the following diagram:

$$\begin{tikzcd}
& 0 \arrow[d] & & 0 \arrow[d] & \\
& \ker f \arrow[d] & 0 \arrow[d] & \ker\phi \arrow[d] & \\
0 \arrow[r] & F \arrow[r] \arrow[d, "f"] & k[G]^{2n-1} \arrow[r] \arrow[d, equal, "\mathrm{Id}"] & P \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & H^0(X, \Omega_X^{\otimes n}) \arrow[r] \arrow[d] & k[G]^{2n-1} \arrow[r] \arrow[d] & k[G]^{\oplus 2\left\lfloor \frac{2n-1}{p} \right\rfloor} \oplus \frac{k[G]}{(e_A-1)^r} \oplus \frac{k[G]}{(e_B-1)^r} \arrow[r] \arrow[d] & 0 \\
& \operatorname{coker} f & 0 & 0 &
\end{tikzcd}$$

The Snake lemma implies that the following sequence is exact

$$0 \to \ker f \to 0 = \ker(\mathrm{Id}) \to \ker\phi \to \operatorname{coker} f \to 0 = \operatorname{coker}(\mathrm{Id})$$

therefore $\ker f = 0$ and $\operatorname{coker}(f) \cong \ker\phi$.

This gives us the short vertical exact sequence

$$0 \to F \to H^0(X, \Omega_X^{\otimes n}) \to \ker\phi \to 0.$$

Since F is projective and in our setting injective as well the above sequence splits, therefore

$$H^0(X, \Omega_X^{\otimes n}) \cong k[G]^{2n-1-2\left\lceil \frac{2n-1}{p} \right\rceil} \oplus k[G]/(e_A - 1)^{p-r} \oplus k[G]/(e_B - 1)^{p-r},$$

which proves part (i) of the theorem.

If $r = 0$, then by Lemma 7 the module $H^0(X, \Sigma) \cong k[G]^{\oplus 2\left\lfloor \frac{2n-1}{p} \right\rfloor}$ is already free, so $P = H^0(X, \Sigma)$, $\phi = \mathrm{Id}$, and $\ker\phi = 0$. The same Snake lemma argument then gives $\operatorname{coker}(f) \cong \ker\phi = 0$ (in addition to $\ker f = 0$), so $f : F \to H^0(X, \Omega_X^{\otimes n})$ is an isomorphism. Since F has rank $2n - 1 - 2\left\lceil \frac{2n-1}{p} \right\rceil = 2n - 1 - 2\left\lfloor \frac{2n-1}{p} \right\rfloor$ (as the ceiling and floor of $(2n-1)/p$ coincide when $r = 0$), this proves part (ii).

The Proof of Theorem 2 is now complete. □

2.2 Comparison to the Work of S. Nakajima

The goal of this subsection is to give an independent derivation of the $k[A]$-module structure of $H^0(X, \Omega_X^{\otimes n})$ stated in the Corollary above, using the classical theorem of S. Nakajima [26] on the Galois module structure of polydifferentials for covers with cyclic p-group of order p. This provides a sanity check for our main results, independent of the projective-cover argument used in the proof of Theorem 2.

Recall that the curves we are studying, are given by the following algebraic model

$$X : (x^p - x)(y^p - y) = c,$$

The group $A = \mathbb{Z}/p\mathbb{Z}$ is a subgroup of the automorphism group and acts for instance on the curve X_c by letting the generator τ of $\mathbb{Z}/p\mathbb{Z}$ to act on the curve in terms of the map $(x, y) \mapsto (x, y+1)$. We call Y the quotient curve $X_c/\langle\tau\rangle$. Note that Y is isomorphic to $\mathbb{P}^1$, and hence the genus g_Y is zero, see also the diagram in Eq. 2.

Denote the function field of X by F. The extension $F/k(x)$ is a cyclic extension of the rational function field $k(x)$. In this extension the p places $P_i = (x-i)$, $0 \leq i \leq p-1$ of $k(x)$ are weakly ramified, since the polynomial $x^p - x$ has simple roots, see [30, prop. 3.7.8]. The ramification divisor of the extension $F/k(x)$ is given by:

$$R_{F/k(x)} = \sum_{i=0}^{p-1} 2(p-1)P_i.$$

We will employ the results of S. Nakajima [26]. We have the following decomposition in terms of indecomposable modules

$$H^0(X, \Omega_X^{\otimes n}) = \bigoplus_{i=1}^{p} J_i^{\oplus m_i},$$

and the coefficients are given by [26, Theorem 1] (we have used the equivalent expression given at the final displayed equation of the proof of this theorem):

$$m_p = (2n-1)(g_Y - 1) + \sum_{i=1}^{p} \left\lfloor \frac{n_i - (p-1)N_i}{p} \right\rfloor,$$

where $N_i = 1$ (ordinary curves) and $n_i = 2n(p-1)$, see [19, Section 4].

Since $g_Y = 0$ we compute:

$$\begin{aligned} m_p &= (2n-1)(g_Y - 1) + p\left\lfloor \frac{2n(p-1) - (p-1)}{p} \right\rfloor \\ &= (2n-1)(g_Y - 1) + p(2n-1) - p\left\lceil \frac{2n-1}{p} \right\rceil \\ &\overset{g_Y=0}{=} (p-1)(2n-1) - p\left\lceil \frac{2n-1}{p} \right\rceil. \end{aligned}$$

For $1 \leq j \leq p-1$ the coefficients m_j are given by the following formulas [26, Theorem 1]:

$$\begin{aligned} \frac{m_j}{p} &= -\left\lfloor \frac{n_i - jN_i}{p} \right\rfloor + \left\lfloor \frac{n_i - (j-1)N_i}{p} \right\rfloor \\ &= -\left\lfloor \frac{2n(p-1) - j}{p} \right\rfloor + \left\lfloor \frac{2n(p-1) - (j-1)}{p} \right\rfloor \\ &= -\left\lfloor \frac{-2n - j}{p} \right\rfloor + \left\lfloor \frac{-2n - (j-1)}{p} \right\rfloor \\ &= \left\lceil \frac{2n+j}{p} \right\rceil - \left\lceil \frac{2n+j-1}{p} \right\rceil. \end{aligned}$$

We now notice that for $0 \leq j \leq p-1$ the above expression is zero unless $p \mid 2n + j - 1$.

We write $2n - 1 = \left\lfloor \frac{2n-1}{p} \right\rfloor p + r$. If $r \neq 0$, we arrive at

$$H^0(X, \Omega_X^{\otimes n}) = k[A]^{\oplus\left((p-1)(2n-1) - p\left\lceil \frac{2n-1}{p} \right\rceil\right)} \bigoplus J_{p-r}^{\oplus p}. \tag{8}$$

If $r = 0$, the multiplicity m_p computed above already accounts for the whole of $H^0(X, \Omega_X^{\otimes n})$, that is

$$H^0(X, \Omega_X^{\otimes n}) = k[A]^{\oplus\left((p-1)(2n-1)-p\left\lfloor \frac{2n-1}{p} \right\rfloor\right)}, \tag{9}$$

with no further $J_p^{\oplus p}$ summand: since $J_p \cong k[A]$, adding $J_p^{\oplus p} = k[A]^{\oplus p}$ on top of m_p would over-count, as the p copies of J_p coming from the two ramification points are already included in m_p in this case. In both cases, this confirms the decomposition of $H^0(X, \Omega_X^{\otimes n})$ as a $k[A]$-module already stated in Corollary 3.

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Declarations

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References

1. Alperin, J.L.: Local representation theory, volume 11 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge. Modular representations as an introduction to the local representation theory of finite groups (1986)
2. Anderson, F.W., Fuller, K.R.: Rings and categories of modules, volume 13 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, (1992)
3. Benson, D.J.: Representations of elementary abelian p-groups and vector bundles. Cambridge Tracts in Mathematics, vol. 208. Cambridge University Press, Cambridge (2017)
4. Bleher, F.M., Chinburg, T., Kontogeorgis, A.: Galois structure of the holomorphic differentials of curves. J. Number Theory **216**, 1–68 (2020)
5. Bleher, F.M., Wood, A.: The Galois module structure of holomorphic poly-differentials and Riemann-Roch spaces. J. Algebra **631**, 756–803 (2023)
6. Borne, N.: Cohomology of G-sheaves in positive characteristic. Adv. Math. **201**(2), 454–515 (2006)
7. Cornelissen, G., Kato, F.: Equivariant deformation of Mumford curves and of ordinary curves in positive characteristic. Duke Math. J. **116**(3), 431–470 (2003)
8. Cornelissen, G., Kato, F.: Mumford curves with maximal automorphism group. Proc. Amer. Math. Soc. **132**(7), 1937–1941 (2004)
9. Cornelissen, G., Kato, F.: Zur Entartung schwach verzweigter Gruppenoperationen auf Kurven. J. Reine Angew. Math. **589**, 201–236 (2005)
10. Cornelissen, G., Kato, F., Kontogeorgis, A.: Discontinuous groups in positive characteristic and automorphisms of Mumford curves. Math. Ann. **320**(1), 55–85 (2001)
11. Ellingsrud, G., Lønsted, K.: An equivariant Lefschetz formula for finite reductive groups. Math. Ann. **251**(3), 253–261 (1980)

12. Fukasawa, S.: Galois lines for the Artin-Schreier-Mumford curve. Finite Fields Appl. **75**, 10 (2021)
13. Fukasawa, S.: Automorphism group, Galois points and lines of the generalized Artin-Schreier-Mumford curve. Geom. Dedicata. **216**(2), 9 (2022)
14. Hartshorne, R.: Algebraic Geometry. Springer-Verlag, New York. Graduate Texts in Mathematics, No. 52 (1977)
15. Hurwitz, A.: Über algebraische Gebilde mit eindeutigen Tranformationen in sich. Math. Ann. **41**, 403–442 (1893)
16. Karanikolopoulos, S., Kontogeorgis, A.: Representation of cyclic groups in positive characteristic and Weierstrass semigroups. J. Number Theory **133**(1), 158–175 (2013)
17. Köck, B.: Galois structure of Zariski cohomology for weakly ramified covers of curves. Amer. J. Math. **126**(5), 1085–1107 (2004)
18. Köck, B., Kontogeorgis, A.: Quadratic differentials and equivariant deformation theory of curves. Ann. Inst. Fourier (Grenoble) **62**(3), 1015–1043 (2012)
19. Kontogeorgis, A.: Polydifferentials and the deformation functor of curves with automorphisms. J. Pure Appl. Algebra **210**(2), 551–558 (2007)
20. Kontogeorgis, A.: Field of moduli versus field of definition for cyclic covers of the projective line. J. Théor. Nombres Bordeaux **21**(3), 679–692 (2009)
21. Kontogeorgis, A., Stergiopoulou, D.-D.: On the galois-module structure of polydifferentials of artin-schreier-mumford curves, modular and integral representation theory, (2025)
22. Korchmáros, G., Montanucci, M.: The geometry of the Artin-Schreier-Mumford curves over an algebraically closed field. Acta Sci. Math. (Szeged) **83**(3–4), 673–681 (2017)
23. Pedro Ricardo López-Bautista and Gabriel Daniel Villa-Salvador: On the Galois module structure of semisimple holomorphic differentials. Israel J. Math. **116**, 345–365 (2000)
24. Matignon, M.: Lifting the curve $(x^p - x)(y^p - y) = 1$ and its automorphism group. Note non destinée à publication (2003). http://www.math.u-bordeaux1.fr/~matignon
25. Mumford, D.: An analytic construction of degenerating curves over complete local rings. Compositio Math. **24**, 129–174 (1972)
26. Nakajima, S.: Action of an automorphism of order p on cohomology groups of an algebraic curve. J. Pure Appl. Algebra **42**(1), 85–94 (1986)
27. Nakajima, S.: p-ranks and automorphism groups of algebraic curves. Trans. Amer. Math. Soc. **303**(2), 595–607 (1987)
28. Rzedowski-Calderón, M., Villa-Salvador, G., Madan, M.L.: Galois module structure of holomorphic differentials in characteristic p. Arch. Math. (Basel) **66**(2), 150–156 (1996)
29. Serre, J.-P. :Local fields. Springer-Verlag, New York. Translated from the French by Marvin Jay Greenberg (1979)
30. Stichtenoth, H.: Algebraic function fields and codes. Springer-Verlag, Berlin (1993)
31. Subrao, D.: The p-rank of Artin-Schreier curves. Manuscripta Math. **16**(2), 169–193 (1975)
32. Tamagawa, T.: On unramified extensions of algebraic function fields. Proc. Japan Acad. **27**, 548–551 (1951)
33. Teitelbaum, J.T.: Values of p-adic L-functions and a p-adic Poisson kernel. Invent. Math. **101**(2), 395–410 (1990)
34. Valentini, R.C.: Representations of automorphisms on differentials of function fields of characteristic p. J. Reine Angew. Math. **335**, 164–179 (1982)
35. Valentini, R.C., Madan, M.L.: Automorphisms and holomorphic differentials in characteristic p. J. Number Theory **13**(1), 106–115 (1981)