Automorphisms and the canonical ideal

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Abstract. The automorphism group of a curve is studied from the viewpoint of the canonical embedding and Petri's theorem. A criterion for identifying the automorphism group as an algebraic subgroup the general linear group is given. Furthermore, the action of the automorphism group is extended to a linear action on the generators of the minimal free resolution of the canonical ring of the curve X.

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1. Introduction

Let X be a non-singular complete algebraic curve defined over an algebraically closed field of characteristic $p \geq 0$. If the genus g of the curve X is $g \geq 2$ then the automorphism group $G = \mathrm{Aut}(X)$ of the curve X is finite. The theory of automorphisms of curves is an interesting object of study, see the surveys [1], [5] and the references therein.

On the other hand the theory of syzygies which originates in the work of Hilbert and Sylvester has attracted a lot of researchers and it seems that a lot of geometric information can be found in the minimal free resolution of the ring of functions of an algebraic curve. For an introduction to this fascinating area we refer to [7].

In this article we aim to put together the theory of syzygies of the canonical embedding and the theory of automorphisms of curves. Throughout this article X is a non-hyperelliptic, non-trigonal and a non-singular quintic of genus 6 and we also assume $p \neq 2$. These conditions are needed for Petri's theorem to hold, while the $p \neq 2$ condition is needed to ensure the faithful action of the automorphism group on the space of holomorphic differentials $H^0(X, \Omega_X)$.

More precisely, in section 2.1 we use Petri's theorem in order to give a necessary and sufficient condition for an element in $\mathrm{GL}(H^0(X,\Omega_X))$ to act as an automorphism of our curve. In this way we can arrive to

Proposition 1. *The automorphism group of a curve X as a finite set can be seen as a subset of the $g^2(g+1)^2-1$-dimensional projective space and can be described by explicit quadratic equations.*

In section 3 we show that the automorphism group G of the curve acts linearly on a minimal free resolution $\mathbf{F}$ of the ring of regular functions S_X of the curve X canonically embedded in $\mathbb{P}^{g-1}$. Notice that an action of a group G on a graded module M gives rise to a series of linear representations $\rho_d : G \to M_d$ to all linear spaces M_d of degree d for $d \in \mathbb{Z}$. For the case of the free modules F_i of the minimal free resolution $\mathbf{F}$ we relate the actions of the group G in both F_i and in the dual F_{g-2-i} in terms of an inner automorphism of G.

This information is used in order to show that the action of the group G on generators of the modules F_i sends generators of degree d to linear combinations of generators of degree d. Let $S = \mathrm{Sym}(H^0(X,\Omega_X))$ be the symmetric algebra of $H^0(X,\Omega_X)$.

Proposition 2. *There is a well defined linear action of the automorphism group G on minimal generators of the free resolution, which sends a minimal generator of degree d of the free module F_i to a linear combination of other generators of degree d.*

The degree d-part of $\mathrm{Tor}_i^S(k,S_X)$ will be denoted by $\mathrm{Tor}_i^S(k,S_X)_d$, which is a vector space of dimension $\beta_{i,d}$. We can use our computation in order to show that all $\mathrm{Tor}_i^S(k,S_X)_d$ are acted on by the group G, but this also follows by Koszul cohomology, see [2]. Indeed, one starts with the vector space $V = H^0(X,\Omega_X)$, $\dim V = g$, $S = \mathrm{Sym}(V)$ and considers the exact Koszul complex

$$0 \to \wedge^g V \otimes S(-g) \to \wedge^{g-1} V \otimes S(-g+1) \to \cdots$$

$$\cdots \to \wedge^2 V \otimes S(-2) \to v \otimes S(-1) \to S \to k \to 0.$$

The symmetry property of the Tor functor implies that one can calculate $\mathrm{Tor}_i^S(k,S_X)$ by using the Koszul resolution of k instead of the Koszul resolution of S_X. Since the Koszul resolution of k is a complex of G-modules and all differentials are G-module morphisms the $\mathrm{Tor}_i^S(k,S_X)_d$ are naturally G-modules. On the other hand the passage to the action on generators is not explicit since the isomorphism between the graded components of the terms in the minimal resolution and Koszul cohomology spaces is not explicit, as as it comes from the spectral sequence that ensures the symmetry of Tor functor.

Finally, the representations to the d graded space of each F_i, $\rho_{i,d} : G \to \mathrm{GL}(F_{i,d})$ can be expressed as a direct sum of the G-modules $\mathrm{Tor}_i^S(k,S_X)_d$. We conclude by showing that the G-module structure of all F_i is determined by knowledge of the G-module structure of $H^0(X, \Omega_X)$ and the G-module structure of each $\mathrm{Tor}_i^S(k, S_X)$ for all $0 \leq i \leq g-2$.

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2. Automorphisms of curves and Petri's theorem

Consider a complete non-singular non-hyperelliptic curve of genus $g \geq 3$ over an algebraically closed field K. Let Ω_X denote the sheaf of holomorphic differentials on X.

Theorem 3 (Noether-Enriques-Petri). *There is a short exact sequence*

$$0 \to I_X \to \mathrm{Sym} H^0(X, \Omega_X) \to \bigoplus_{n=0}^{\infty} H^0(X, \Omega_X^{\otimes n}) \to 0,$$

where I_X is generated by elements of degree 2 and 3. Also if X is not a non-singular quintic of genus 6 or X is not a trigonal curve, then I_X is generated by elements of degree 2.

For a proof of this theorem we refer to [16], [9]. The ideal I_X is called *the canonical ideal* and it is the homogeneous ideal of the embedded curve $X \to \mathbb{P}_k^{g-1}$. The automorphism group of the ambient space $\mathbb{P}^{g-1}$ is known to be $\mathrm{PGL}_g(k)$, [10, example 7.1.1 p. 151]. On the other hand every automorphism of X is known to act on $H^0(X, \Omega_X)$ giving rise to a representation

$$\rho : G \to \mathrm{GL}(H^0(X, \Omega_X)),$$

which is known to be faithful, when X is not hyperelliptic and $p \neq 2$, see [11]. The representation ρ in turn gives rise to a series of representations

$$\rho_d : G \to \mathrm{GL}(S_d),$$

where S_d is the vector space of degree d polynomials in the ring $S := k[\omega_1, \ldots, \omega_g]$.

Let $X \subset \mathbb{P}^r$ be a projective algebraic set. Is it true that every automorphism $\sigma : X \to X$ comes as the restriction of an automorphism of the ambient projective space, that is by an element of $\mathrm{PGL}_k(r)$? For instance such a criterion for complete intersections is explained in [13, sec. 2]. In the case of canonically embedded curves $X \subset \mathbb{P}^{g-1}$ it is clear that any automorphism $\sigma \in \mathrm{Aut}(X)$ acts also on $\mathbb{P}^{g-1} = \mathrm{Proj} H^0(X, \Omega_X)$. In this way we arrive at the following:

Lemma 4. *Every automorphism $\sigma \in \mathrm{Aut}(X)$ corresponds to an element in $\mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$ and every element in $\mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$ gives rise to an automorphism of X.*

In the next section we will describe the elements $\sigma \in \mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$.

2.1. Algebraic equations of automorphisms

For now on we will assume that the canonical ideal I_X is generated by polynomials in $k[\omega_1, \ldots, \omega_g] = \mathrm{Sym} H^0(X, \Omega_X)$ of degree 2, that is the requirements for Petri's theorem hold. Consider such a set of quadratic polynomials $\tilde{A}_1, \ldots, \tilde{A}_r$ generating I_X.

A polynomial $\tilde{A}_i$ of degree two can be encoded in terms of a symmetric $g \times g$ matrix $A_i = (a_{\nu,\mu})$ as follows. Set $\bar{\omega} = (\omega_1, \ldots, \omega_g)^t$. We have

$$\tilde{A}_i(\bar{\omega}) = \bar{\omega}^t A_i \bar{\omega}.$$

The polynomial $\sigma(\tilde{A}_i)$ is still a polynomial of degree two so we write $\sigma(A_i)$ for the symmetric $g \times g$ matrix such that $\sigma(\tilde{A}_i) = \bar{\omega}^t \sigma(A)_i \bar{\omega}$. It is clear that for an element $\sigma \in \mathrm{GL}_g(k)$, $\sigma(I_X) \subset I_X$ holds if and only if for all $1 \leq i \leq r$, $\sigma(A_i) \in \mathrm{span}_k\{A_1, \ldots, A_r\}$. This means that

$$(\sigma_{\mu,\nu})^t A_i (\sigma_{\mu,\nu}) = \sum_{j=1}^{r} \lambda(\sigma)_{ji} A_j \qquad \text{for every } 1 \leq i \leq j. \tag{1}$$

2.2. The automorphism group as an algebraic set.

Let $A_1, \ldots, A_r$ be a set of linear independent $g \times g$ matrices such that the $w^t A_i w$ $1 \leq i \leq r$ generate the canonical ideal, and $w^t = (w_1, \ldots, w_g)$ is a basis of the space of holomorphic differentials. By choosing an ordered basis of the vector space of symmetric $g \times g$ matrices we can represent any symmetric $g \times g$ matrix A as an element $\bar{A} \in k^{\frac{g(g+1)}{2}}$, that is

$$\begin{aligned} \bar{\cdot} : \text{Symmetric } g \times g \text{ matrices} &\longrightarrow k^{\frac{g(g+1)}{2}} \\ A &\longmapsto \bar{A} \end{aligned}$$

We can now put together the r elements $\bar{A}_i$ as a $g(g+1)/2 \times r$ matrix $\left(\bar{A}_1 | \cdots | \bar{A}_r\right)$, which has full rank r, since $\{A_1, \ldots, A_r\}$ are assumed to be linear independent.

Proposition 5. *An element $\sigma = (\sigma_{ij}) \in \mathrm{GL}_g(k)$ induces an action on the curve X, if and only if the $g(g+1)/2 \times 2r$ matrix*

$$B(\sigma) = \left[\bar{A}_1, \ldots, \bar{A}_r, \overline{\sigma^t A_1 \sigma}, \ldots, \overline{\sigma^t A_r \sigma}\right]$$

has rank r.

We have that σ is an automorphism if the $g(g+1)/2 \times 2r$-matrix $B(\sigma)$ has rank r, which means that $(r+1) \times (r+1)$-minors of $B(\sigma)$ are zero. This provides us with a description of the automorphism group as a determinantal variety given by explicit equations of degree $(r+1)^2$.

But we can do better. Using Gauss elimination we can find a $\frac{g(g+1)}{2} \times \frac{g(g+1)}{2}$ invertible matrix Q which puts the matrix $\left(\bar{A}_1|\cdots|\bar{A}_r\right)$ in echelon form, that is

$$Q\left(\bar{A}_1|\cdots|\bar{A}_r\right) = \left(\begin{array}{c} \mathbb{I}_r \\ \hline \mathbb{O}_{\left(\frac{g(g+1)}{2}-r\right)\times r} \end{array}\right).$$

But then for each $1 \leq i \leq r$ eq. (1) is satisfied if and only if the lower $(\frac{g(g+1)}{2} - r) \times r$ bottom block matrix of the matrix

$$Q\left(\overline{\sigma^t A_1 \sigma}|\cdots|\overline{\sigma^t A_r \sigma}\right) \tag{2}$$

is zero, while the top $r \times r$ block matrix gives rise to the representation

$$\rho_1 : G \to \mathrm{GL}_r(k),$$

defined by equation (1). Assuming that the lower $(\frac{g(g+1)}{2} - r) \times r$ bottom block matrix gives us $r(\frac{g(g+1)}{2} - r)$ equations where the entries $\sigma = (\sigma_{ij})$ are seen as indeterminates. In this way we can write down elements of the automorphism group as a zero dimensional algebraic set, satisfying certain quadratic equations.

3. Syzygies

3.1. Extending group actions

Recall that $S = k[\omega_1, \ldots, \omega_g]$ is the polynomial ring in g variables. Let M be a graded S-module acted on by the group G, generated by the elements $m_1, \ldots, m_r$ of corresponding degrees $a_1, \ldots, a_r$. We consider the free S-module $F_0 = \bigoplus_{j=1}^r S(-a_j)$ together with the onto map

$$F_0 = \bigoplus_j S(-a_j) \xrightarrow{\pi} M. \tag{3}$$

Let us denote by $M_1, \ldots, M_r$ elements of F_0, such that $\pi(M_i) = m_i$, assuming also that $\deg(M_i) = \deg(m_i)$, for $1 \leq i \leq r$. The action on the generators m_i is given by

$$\sigma(m_i) = \sum_{\nu=1}^r a_{\nu,i} m_i, \text{ for some } a_{\nu,i} \in S. \tag{4}$$

Remark 6. We would like to point out here that unlike the theory of vector spaces, an element $x \in F_0$ might admit two different decompositions

$$x = \sum_{i=1}^r a_i m_i = \sum_{i=1}^r b_i m_i, \text{ that is } \sum_{i=1}^r (a_i - b_i) m_i = 0,$$

and if $a_{i_0} - b_{i_0} \neq 0$ we cannot assume that $a_{i_0} - b_{i_0}$ is invertible, so we can't express m_{i_0} as an S-linear combination of the other elements m_i, for $i_0 \neq i, 1 \leq i \leq r$ in order to contradict minimality. We can only deduce that $\{a_i - b_i\}_{i=1,\ldots,r}$ form a syzygy.

Therefore one might ask if the matrix $(a_{\nu,i})$ given in eq. (4) is unique. In proposition 9 we will prove that the elements $a_{\nu,i}$ which appear as coefficients in eq. (4) are in the field k and therefore the expression is indeed unique.

The natural action of $\mathrm{Aut}(X)$ on $H^0(X, \Omega_X)$ can be extended to an action on the ring $S = \mathrm{Sym} H^0(X, \Omega_X)$, so that $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in S$. Therefore if $M = I_X$ then for all $s \in S$, $m \in I_X = M$ we have $\sigma(sm) = \sigma(s)\sigma(m)$. All the actions in the modules we will consider will have this property.

For a free module $F = \bigoplus_{j=1}^{s} S(-a_j)$, generated by the elements M_i, $1 \leq i \leq r$, $\deg(M_i) = a_i$ and a map $\pi : F \to M$ we define the action of G by

$$\sigma\left(\sum_{j=1}^{r} s_j M_j\right) = \sum_{j=1}^{r} \sigma(s_j) \sum_{\nu=1}^{r} a_{\nu,j}(\sigma) M_\nu = \sum_{\nu=1}^{r}\left(\sum_{j=1}^{r} a_{\nu,j}(\sigma)\sigma(s_j)\right) M_\nu,$$

where $\deg_S a_{\nu,j} + a_\nu = \deg_S m_j$. This means that under the action of $\sigma \in G$ the r-tuple $(s_1, \ldots, s_r)^t$ is sent to

$$\begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix} \xmapsto{\sigma} \begin{pmatrix} a_{1,1}(\sigma) & a_{1,2}(\sigma) & \cdots & a_{1,r}(\sigma) \\ \vdots & \vdots & & \vdots \\ a_{r,1}(\sigma) & a_{r,2}(\sigma) & \cdots & a_{r,r}(\sigma) \end{pmatrix} \begin{pmatrix} \sigma(s_1) \\ \vdots \\ \sigma(s_r) \end{pmatrix}.$$

If $A(\sigma) = \big(a_{i,j}(\sigma)\big)$ is the matrix corresponding to σ then for $\sigma, \tau \in G$ the following cocycle condition holds:

$$A(\sigma\tau) = A(\sigma)A(\tau)^\sigma.$$

If we can assume that G acts trivially on the matrix $A(\tau)$ for every $\tau \in G$ (for instance when $A(\tau)$ is a matrix with entries in k for every $\tau \in G$), then the above cocycle condition becomes a homomorphism condition.

Also if $A(\sigma)$ is a principal derivation, that is there is an $r \times r$ matrix Q, such that

$$A(\sigma) = \sigma(Q) \cdot Q^{-1}$$

then after a basis change of the generators we can show that the action on the coordinates is just given by

$$(s_1, \cdots, s_r)^t \xmapsto{\sigma} (\sigma(s_1), \cdots, \sigma(s_r))^t,$$

that is the matrix $A(\sigma)$ is the identity. We will call the action on the free resolution $\mathbf{F}$ obtained by extending the action on M the *standard action*.

3.2. Group actions on free resolutions

Recall that $S = k[\omega_1, \ldots, \omega_g]$ is the polynomial ring in g variables. Let M be a graded S-module generated by the elements $m_1, \ldots, m_r$ of corresponding degrees $a_1, \ldots, a_r$. Consider the minimal free resolution

$$0 \longrightarrow F_g \xrightarrow{\phi_g} \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \ , \tag{5}$$

where $\operatorname{coker}(\phi_1) = F_0/\mathrm{Im}\phi_1 = F_0/\ker\pi \cong M$. Let $\mathfrak{m}$ be the maximal ideal of S generated by $\langle \omega_1, \ldots, \omega_g \rangle$. Each free module in the resolution can be written as

$$F_i = \bigoplus_j S(-j)^{\beta_{i,j}},$$

where the integers $\beta_{i,j}$ are the Betti numbers of the resolution. The Betti numbers satisfy

$$\beta_{i,j} = \beta_{g-2-i,g+1-j}. \tag{6}$$

as one can see by using the self duality of the above resolution by twisting by $S(-g)$ see [15, prop. 4.1.1], [7, prop. 9.5] or by using Koszul cohomology, see [8, prop. 4.1].

Assume that M and each F_i is acted on by a group G and that the maps δ_i are G-equivariant. We will now study the action of the group G on the generators of F_i. First of all we have that

$$F_i = \bigoplus_{\nu=1}^{r_i} \bigoplus_{\mu=1}^{\beta_{i,\nu}} e_{i,\nu,\mu} S \cong \bigoplus_{\nu=1}^{r_i} S(-d_{i,\nu})^{\beta_{i,\nu}}.$$

In the above formula we assumed that F_i is generated by elements $e_{i,\nu,\mu}$ such that the degree of $e_{i,\nu,\mu} = d_{i,\nu}$ for all $1 \leq \mu \leq \beta_{i,\nu}$. We also assume that

$$d_{i,1} < d_{i,2} < \cdots < d_{i,r_i}.$$

The action of σ is respecting the degrees, so an element of minimal degree $d_{i,1}$ is sent to a linear combination of elements of minimal degree $d_{i,1}$. In this way we obtain a representation

$$\rho_{i,1} : G \to \mathrm{GL}(\beta_{i,1}, k).$$

In a similar way an element $e_{i,2,\mu}$ of degree $d_{i,2}$ is sent to an element of degree $d_{i,2}$ and we have that

$$\sigma(e_{i,2,\mu}) = \sum_{j_1=1}^{\beta_{i,2}} \lambda_{i,2,\mu,j_1} e_{i,2,j_1} + \sum_{j_2=1}^{\beta_{i,1}} \lambda'_{i,2,\mu,j_1} e_{i,1,j_2},$$

where all $\lambda_{i,2,\mu,j_1} \in k$ and all $\lambda'_{i,1,\mu,j_2} \in \mathfrak{m}^{d_{i,2}-d_{i,1}}$. In this case we have a representation with entries in an ring instead of a field, which has the form:

$$\rho_{i,2} : G \to \mathrm{GL}(\beta_{i,1} + \beta_{i,2}, \mathfrak{m}^{d_{i,2}-d_{i,1}}),$$

$$\sigma \mapsto \begin{pmatrix} A_1(\sigma) & A_{1,2}(\sigma) \\ 0 & A_2(\sigma) \end{pmatrix},$$

where $A_1(\sigma) \in \mathrm{GL}(\beta_{i,1}, k)$ and $A_2(\sigma) \in \mathfrak{m}^{d_{i,2}-di,1}\mathrm{GL}(\beta_{i,2}, k)$.

By induction the situation in the general setting gives rise to a series of representations:

$$\rho_{i,j} : G \to \mathrm{GL}(\beta_{i,1} + \beta_{i,2}, \mathfrak{m}^{d_{i,j}-d_{i,1}})$$

$$\sigma \mapsto A(\sigma) = \begin{pmatrix} A_1(\sigma) & A_{1,2}(\sigma) & \cdots & A_{1,j}(\sigma) \\ 0 & A_2(\sigma) & & A_{2,j}(\sigma) \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 0 & A_j(\sigma) \end{pmatrix} \tag{7}$$

where $A_\nu(\sigma) \in \mathrm{GL}(\beta_{i,\nu}, k)$ and $A_{\kappa,\lambda}(\sigma)$ is an $\beta_{i,\kappa} \times \beta_{i,\lambda}$ matrix with coefficients in $\mathfrak{m}^{\beta_{i,\lambda}-\beta_{i,\kappa}}$. The representation ρ_{i,r_i} taken modulo $\mathfrak{m}$ reduces to $\mathrm{Tor}_i^S(k, M)$, seen as a $k[G]$-module.

3.3. Unique actions

Let us consider two actions of the automorphisms group G on $H^0(X, \Omega_X)$, which can naturally be extended on the symmetric algebra $\mathrm{Sym}H^0(X, \Omega_X)$. We will denote the first action by $g \star v$ and the second action by $g \circ v$, where $g \in G$, $v \in \mathrm{Sym}H^0(X, \Omega_X)$.

Proposition 7. *If the curve X satisfies the conditions of faithful action of $G = \mathrm{Aut}(X)$ on $H^0(X, \Omega_X)$, that is X is not hyperelliptic and $p > 2$,* [11, th. 3.2] *and moreover both actions $\star, \circ$ restrict to actions on the canonical ideal I_X, then there is an automorphism $i : G \to G$, such that $g \star v = i(g) \circ v$.*

Proof. Both actions of G on $H^0(X, \Omega_X)$ introduce automorphisms of the curve X. That is since $G \star I_X = I_X$ and $G \circ I_X = I_X$, the group G is mapped into $\mathrm{Aut}(X) = G$. This means that for every element $g \in G$ there is an element $g^* \in \mathrm{Aut}(X) = G$ such that $g \star v = g^* v$, where the action on the right is the standard action of the automorphism group on holomorphic differentials. By the definition of the group action for every $g_1, g_2 \in G$ we have $(g_1 g_2)^* v = g_1^* g_2^* v$ for all $v \in H^0(X, \omega_X)$ and the faithful action of the automorphism group provides us with $(g_1 g_2)^* = g_1^* g_2^*$, i.e. the map $i_* : g \mapsto g^*$ is a homomorphism. Similarly the map corresponding to the $\circ$-action, $i_\circ : g \mapsto g^\circ$ is a homomorphism and the desired homomorphism i is the composition of $i_* i_\circ^{-1}$. □

The map $\mathrm{Hom}_S(F_i, S(-g))$ induces a symmetry of the free resolution $\mathbf{F}$ by sending F_i to F_{g-2-i}. Each free module F_i of the resolution $\mathbf{F}$ is equipped by the extension of the action on holomorphic differentials, according to the construction of section 3.2. On the other hand since $S(-g)$ is a G-module we have that $F_{g-2-i} \cong \mathrm{Hom}_S(F_i, S(-g))$ is equipped by a second action namely every $\phi : F_i \to S(-g)$ is acted naturally by G in terms of $\phi \mapsto \phi^\sigma = \sigma^{-1}\phi\sigma$. How are the two actions related?

Lemma 8. *Denote by $\star$ the action of G on F_i induced by taking the $S(-g)$-dual. The standard and the $\star$-actions are connected in terms of an automorphism ψ_i of G, that is for all $v \in F_i$ $g \star v = \psi_i(g)v$.*

Proof. Assume that $i \leq g-2-i$. Consider the standard action of G on the free resolution $\mathbf{F}$. The module F_{g-2-i} obtains a new action $g \star v$ for $g \in G, v \in F_i$. By 3.2 this $\star$ action is transferred to an action on all F_j for $j \geq g-2-i$, including the final term F_{g-2} which is isomorphic to $S(-1)$. This gives us two actions on $H^0(X, \Omega_X)$ which satisfy the requirements of proposition 7. The desired result follows, since the action can be pulled back to all syzygies using either $\mathbf{F}$ or $\mathbf{F}^*$. $\square$

Proposition 9. *Under the faithful action requirement we have that all automorphisms $\sigma \in G$ send the direct summand $S(-j)^{\beta_{i,j}}$ of F_i to itself, that is the representation matrix in eq. (7) is block diagonal.*

Proof. Consider $F_i = \bigoplus_{\nu=1}^{r_i} M_{i,\nu} S$, where $M_{i,1}, \ldots, M_{i,r_i}$ are assumed to be minimal generators of F_i with descending degrees $a_{i,\nu} = \deg(m_{i,\nu})$, $1 \leq \nu \leq r_i$. The action of an element σ is given in terms of the matrix $A(\sigma)$ given in equation (7). The element $\phi \in \mathrm{Hom}_S(F_i, S(-g))$ is sent to

$$\begin{aligned} h : \mathrm{Hom}_S(F_i, S(-g)) &\xrightarrow{\cong} F_{g-2-i} \\ \phi &\longmapsto \big(\phi(M_{i,1}), \ldots, \phi(M_{i,r_i})\big) \end{aligned} \tag{8}$$

Each $\phi(M_{i,\nu})$ can be considered as an element in $S(-g-1+\deg(m_{i,\nu}))$ inside F_{g-2-i}. Observe that the element $\phi \in \mathrm{Hom}_S(F_i, S(-g))$ is known if we know all $\phi(M_{i,\nu})$ for $1 \leq \nu \leq r_i$. From now on we will identify such an element ϕ as a r_i-tuple $\big(\phi(M_{i,\nu})\big)_{1 \leq \nu \leq r_i}$.

Recall that if A, B are G-modules, then there is an natural action on $\mathrm{Hom}(A, B)$, sending $\phi \in \mathrm{Hom}(A, B)$ to ${}^{\sigma}\phi$, which is the map

$$ {}^{\sigma}\phi : A \ni a \mapsto \sigma\phi(\sigma^{-1}a). $$

We have also a second action on the module F_{g-2-i}. We compute ${}^{\sigma}\phi(M_{i,\nu})$ for all base elements $M_{i,\nu}$ in order to describe ${}^{\sigma}\phi$:

$$\begin{aligned} \sigma\left(\phi(\sigma^{-1}M_{i,\nu})\right)_{1 \leq \nu \leq \kappa} &= \left(\sum_{\mu=1}^{r_i} \sigma\big(\alpha_{\mu,\nu}(\sigma^{-1})\big)\sigma\phi(M_{i,\mu})\right)_{1 \leq \nu \leq r_i} \\ &= \left(\sum_{\mu=1}^{r_i} \sigma\big(\alpha_{\mu,\nu}(\sigma^{-1})\big)\chi(\sigma)\phi(M_{i,\mu})\right)_{1 \leq \nu \leq r_i} \end{aligned}$$

where in the last equation we have used the fact that $\phi(M_i)$ are in the rank one G-module $S(-g) \cong \wedge^{g-1}\Omega^1_X$ hence the action of $\sigma \in G$ is given by multiplication by $\chi(\sigma)$, where $\chi(\sigma)$ is an invertible element is S.

In order to simplify the notation consider i fixed, and denote $M_\nu = M_{i,\nu}$, $r = r_i$, $a_{i,j} = a_j$. We can consider as a basis of $\mathrm{Hom}(F_i, S(-g))$ the morphisms ϕ_μ given by

$$ \phi_\mu(M_j) = \delta_{\mu,j} \cdot E, \tag{9}$$

where E is a basis element of degree g of the rank 1 module $S(-g) \cong S \cdot E$. This is a different basis than the basis $M_{g-2-i,\nu}$, $1 \leq n \leq r_{g-2-i}$ of F_{g-2-i} we have already introduced.

According to eq. (6) if M_j has degree a_j then the element ϕ_j has degree $g+1-a_j$. Assume that M_r has maximal degree a_r. Then, ϕ_r has minimal degree. Moreover, in order to describe ${}^{\sigma}\phi_r$ we have to consider the tuple $({}^{\sigma}\phi_r(M_1),\ldots,{}^{\sigma}\phi_r(M_r))$. We have

$$\begin{aligned}\left({}^{\sigma}\phi_r(M_\nu)\right)_{1\leq\nu\leq r} &= \left(\sum_{\mu=1}^{r}\sigma\big(\alpha^{(i)}_{\mu,\nu}(\sigma^{-1})\big)\chi(\sigma)\phi_r(M_\mu)\right)_{1\leq\nu\leq r}\\ &\overset{(9)}{=\!=} \left(\sigma\big(\alpha^{(i)}_{r,\nu}(\sigma^{-1})\big)\chi(\sigma)E\right)_{1\leq\nu\leq r}\end{aligned}$$

and we finally conclude that

$$ {}^{\sigma}\phi_r = \sum_{\nu=1}^{r}\sigma^{-1}\big(\alpha^{(i)}_{r,\nu}(\sigma^{-1})\big)\chi(\sigma)\phi_\nu .$$

In this way every element $x\in F_{g-2-i}$ is acted on by σ in terms of the action

$$\sigma\star x = h\big({}^{\sigma}h^{-1}(x)\big),$$

where h is the map given in eq. (8). On the other hand the elements $h(\phi_r)$ are in F_{g-2-i} and by lemma 8 there is an element $\sigma'\in G$ such that

$$\sigma' h(\phi_r) = \sum_{\nu=1}^{r}\alpha^{(g-2-i)}_{\nu,r}(\sigma')h(\phi_\nu).$$

Since the element ϕ_ν has maximal degree among generators of F_i the element $h(\phi_r)$ has minimal degree. This means that all coefficients

$$\alpha^{(g-2-i)}_{\nu,r}(\sigma') = \sigma\big(\alpha^{(i)}_{r,\nu}(\sigma^{-1})\big)\chi(\sigma)$$

are zero for all ν such that $\deg m_\nu < \deg \mu_r$. Therefore all coefficients $a^{(i)}_{\nu,r}(\sigma)$ for ν such that $\deg m_\nu < \deg m_r$ are zero. This holds for all $\sigma\in G$. By considering in this way all elements $\phi_{r-1},\phi_{r-2},\ldots,\phi_1$, which might have greater degree than the degree of ϕ_r the result follows. $\square$

4. Representations on the free resolution

Each S-module F_i in the minimal free resolution can be seen as a series of representations of the group G. Indeed, the modules F_i are graded and there is an action of G on each graded part $F_{i,d}$, given by representations

$$\rho_{i,d}: G\rightarrow \mathrm{GL}\big(F_{i,d}\big),$$

where $F_{i,d}$ is the degree d part of the S-module F_i. The space $\mathrm{Tor}_i^S(k,S_X))$ is clearly a G-module, and by proposition 9 there is a decomposition of G-modules

$$\mathrm{Tor}_i^S(k,S_X) = \bigoplus_{j\in\mathbb{Z}}\mathrm{Tor}_i^S(k,S_X)_j,$$

where $\mathrm{Tor}_i^S(k,S_X)_j$ is the k-vector space generated by generators of F_i that have degree j. This is a vector space of dimension $\beta_{i,j}$.

Denote by $\mathrm{Ind}(G)$ the set of isomorphism classes of indecomposable $k[G]$-modules. If k is of characteristic $p > 0$ and G has no-cyclic p-Sylow subgroup then the set $\mathrm{Ind}(G)$ is infinite, see [3, p.26]. Suppose that each $\mathrm{Tor}_i^S(k, S_X)_j$ admits the following decomposition in terms of $U \in \mathrm{Ind}(G)$:

$$\mathrm{Tor}_i^S(k, S_X)_j = \bigoplus_{U \in \mathrm{Ind}(G)} a_{i,j,U} U \text{ where } a_{i,j,U} \in \mathbb{Z}.$$

We obviously have that

$$\beta_{i,j} = \sum_{U \in \mathrm{Ind}(G)} a_{i,j,U} \dim_k U.$$

The G-structure of F_i is given by

$$\mathrm{Tor}_i^S(k, S_X) \otimes S,$$

that is the G-module structure of $F_{i,d}$ is given by

$$F_{i,d} = \bigoplus_{d \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} \mathrm{Tor}_i^S(k, S_X)_{d-j} \otimes S_j.$$

5. An example: the Fermat curve

Consider the projective non singular curve given by equation

$$F_n : x_1^n + x_2^n + x_0^n = 0$$

This curve has genus $g = \frac{(n-2)(n-1)}{2}$. Set $x = x_1/x_0$, $y = x_2/x_0$. For $\omega = \frac{dx}{y^{n-1}} = -\frac{dy}{x^{n-1}}$ we have that the set

$$x^i y^j \omega \text{ for } 0 \leq i + j \leq n - 3 \tag{10}$$

forms a basis for holomorphic differentials, [12], [17], [18]. These g differentials are ordered lexicographically according to (i, j), that is

$$\omega_{0,0} < \omega_{0,1} < \cdots < \omega_{0,n-3} < \omega_{1,0} < \omega_{1,1} < \cdots < \omega_{1,n-4} < \cdots < \omega_{n-3,0}.$$

The case $n = 2$ is a rational curve, the case $n = 3$ is an elliptic curve, the case $n = 4$ has genus 3 and gonality 3, the case $n = 5$ has genus 6 and is quintic so the first Fermat curve which has canonical ideal generated by quadratic polynomial is the case $n = 6$ which has genus 10.

Proposition 10. *The canonical ideal of the Fermat curve F_n for $n \geq 6$ consists of two sets of relations*

$$G_1 = \{\omega_{i_1,j_1}\omega_{i_2,j_2} - \omega_{i_3,j_3}\omega_{i_4,j_4} : i_1 + i_2 = i_3 + i_4, j_1 + j_2 = j_3 + j_4\}, \tag{11}$$

and

$$G_2 = \left\{ \omega_{i_1,j_1}\omega_{i_2,j_2} + \omega_{i_3,j_3}\omega_{i_4,j_4} + \omega_{i_5,j_5}\omega_{i_6,j_6} = 0 : \begin{smallmatrix} i_1+i_2=n+a, & j_1+j_2=b \\ i_3+i_4=a, & j_3+j_4=n+b \\ i_5+i_6=a, & j_5+j_6=b \end{smallmatrix} \right\} \tag{12}$$

where $0 \leq a, b$ *are selected such that* $0 \leq a + b \leq n - 3$.

We will now prove proposition 10 for $n \geq 6$, following the method developed in [6]. Observe that the holomorphic differentials given in eq. (10) are in 1-1 correspondence with the elements of the set $\mathbf{A} = \{(i,j) : 0 \leq i+j \leq n-3\} \subset \mathbb{N}^2$. First we introduce the following term order on the polynomial algebra $S := \mathrm{Sym} H^0(X, \Omega_X)$.

Definition 11. Choose any term order $\prec_t$ for the variables $\{\omega_{N,\mu} : (N,\mu) \in A\}$ and define the term order $\prec$ on the monomials of S as follows:

$$\omega_{N_1,\mu_1}\omega_{N_2,\mu_2}\cdots\omega_{N_d,\mu_d} \prec \omega_{N'_1,\mu'_1}\omega_{N'_2,\mu'_2}\cdots\omega_{N'_s,\mu'_s} \text{ if and only if} \tag{13}$$

- $d < s$ or
- $d = s$ and $\sum \mu_i > \sum \mu'_i$ or
- $d = s$ and $\sum \mu_i = \sum \mu'_i$ and $\sum N_i < \sum N'_i$
- $d = s$ and $\sum \mu_i = \sum \mu'_i$ and $\sum N_i = \sum N'_i$ and

$$\omega_{N_1,\mu_1}\omega_{N_2,\mu_2}\cdots\omega_{N_d,\mu_d} \prec_t \omega_{N'_1,\mu'_1}\omega_{N'_2,\mu'_2}\cdots\omega_{N'_s,\mu'_s}.$$

By evaluating $\sum_{i=0}^{E}\sum_{j=0}^{E-i} 1$ we can see that

$$\#\{(i,j) \in \mathbb{N}^2 : 0 \leq i+j \leq E\} = (E+1)(E+2)/2 \tag{14}$$

We will use the following lemma, for a proof see [6].

Lemma 12. *Let J be the ideal generated by the elements G_1, G_2 and let I be the canonical ideal. Assume that the cannonical ideal is generated by elements of degree 2. If $\dim_L (S/\mathrm{in}_\prec(J))_2 \leq 3(g-1)$, then $I = J$.*

We extend the correspondence between the variables $\omega_{i,j}$ and the points of $\mathbf{A}$ to a correspondence between monomials in S of standard degree 2 and points of the Minkowski sum of $\mathbf{A}$ with itself, defined as

$$\mathbf{A} + \mathbf{A} = \{(i+i', j+j') \mid (i,j),(i',j') \in \mathbf{A}\} \subseteq \mathbb{N}^2. \tag{15}$$

Proposition 13. *Let $\mathbf{A}$ be the set of exponents of the basis of holomorphic differentials, and let $\mathbf{A}+\mathbf{A}$ denote the Minkowski sum of $\mathbf{A}$ with itself, as defined in (15). Then*

$$(\rho, T) \in \mathbf{A}+\mathbf{A} \Leftrightarrow \exists\, \omega_{i,j}\omega_{i',j'} \in S \text{ such that } \mathrm{mdeg}(\omega_{\mathrm{i,j}}\omega_{\mathrm{i',j'}}) = (2, \rho, \mathrm{T}).$$

For each $n \in \mathbb{N}$ we write $\mathbb{T}^n$ for the set of monomials of degree n in S and proceed with the characterization of monomials that do not appear as leading terms of binomials in $G_1 \subseteq J$.

Proposition 14. *Let σ be the map of sets*

$$\begin{aligned} \sigma : \mathbf{A}+\mathbf{A} &\rightarrow \mathbb{T}^2 \\ (\rho, T) &\mapsto \min_{\prec}\{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid (\rho,T) = (i+i', j+j')\} \end{aligned}$$

Then

$$\sigma(\mathbf{A}+\mathbf{A}) = \{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid \omega_{i,j}\cdot\omega_{i',j'} \neq \mathrm{in}_\prec(f),\ \forall\, f \in G_1\}$$

The above proposition gives a characterization of the monomials that do not appear as initial terms of elements of G_1, therefore they survive in the quotient $(S/\mathrm{in}_{\prec}(J))_2$. Indeed, the minimal of the set $\{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid (\rho, T) = (i+i', j+j')\}$ will never appear as the initial term of an element in G_1. Therefore $\mathbf{A}+\mathbf{A}$ is bijective with a basis of the vector space $(S/\mathrm{in}_{\prec}G_1)_2$. However, some of these monomials appear as initial terms of polynomials in G_2 and these have to be subtracted in order to compute $\dim_L (S/\mathrm{in}_{\prec}(J))_2$

Proposition 15. *Let*

$$C = \{(\rho, b) \in \mathbf{A}+\mathbf{A} \mid \rho = n+a, 0 \le a+b \le n-6, a, b \in \mathbb{N}\}$$

Then

$$\sigma(C) \subseteq \{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid \exists\, g \in G_2 \text{ such that } \omega_{i,j}\omega_{i',j'} = \mathrm{in}_{\prec}(g)\}$$

Moreover $\#C = \#\sigma(C) = (n-5)(n-4)/2$.

Proof. Observe that elements in G_2 are mapped into elements of the form $x^a y^b (x^n + y^n + 1)\omega^2 \in H^0(X, \Omega_X^{\otimes 2})$. By the form of the initial term of such an element of G_2 we have for $i_1 + i_2 = n + a = \rho, j_1 + j_2 = b$. Therefore

$$i_3 + i_4 = a = \rho - n, j_3 + j_4 = n + b, i_5 + i_6 = a = \rho - n, j_5 + j_6 = b = T$$

We should have $0 \le a+b \le n-6$ and by eq. (14) we have that the cardinality of C equals $(n-5)(n-4)/2$. $\square$

We now observe that

$$\mathbf{A}+\mathbf{A} \subset \{i, j \in \mathbb{N} : i+j \le 2n-6\}$$

so $\#(\mathbf{A}+\mathbf{A}) \le (2n-5)(2n-4)/2$ and

$$\begin{aligned}\dim_L (S/\mathrm{in}_{\prec}(J))_2 &= \#\big((\mathbf{A}+\mathbf{A})\backslash C\big) = \#(\mathbf{A}+\mathbf{A}) - \#C \\ &\le \frac{(2n-5)(2n-4)}{2} - \frac{(n-5)(n-4)}{2} = 3(g-1).\end{aligned}$$

so by lemma 12 we have that $I = J$.

5.1. Automorphisms of the Fermat curve

The group of automorphisms of the Fermat curve is given by [19], [14]

$$G = \begin{cases} \mathrm{PGU}(3, p^h), & \text{if } n = 1 + p^h \\ (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \rtimes S_3, & \text{otherwise} \end{cases}$$

The action of the automorphism group is given in terms of a 3×3 matrix A sending

$$x = (x_1/x_0) \mapsto \frac{\sum_{i=0}^2 a_{1,i}x_i}{\sum_{i=0}^2 a_{0,i}x_i} \qquad y = (x_2/x_0) \mapsto \frac{\sum_{i=0}^2 a_{2,i}x_i}{\sum_{i=0}^2 a_{0,i}x_i},$$

In characteristic 0, the matrix A is a monomial matrix, that is, it has only one non-zero element in each row and column and this element is an n-th root of unity. Two matrices A_1, A_2 give rise to the same automorphism if and only if they differ by an element in the group $\{\lambda \mathbb{I}_3 : \lambda \in k\}$. In any case the group G is naturally a subgroup of $\mathrm{PGL}_3(k)$. Finding the representation matrix of G as an element in $\mathrm{PGL}_{g-1}(k)$ is easy when $n \neq 1 + p^h$ and more complicated in $n = 1 + p^h$ case. We have two different embeddings of the Fermat curve F_n in projective space

$$\mathbb{P}_k^{g-1} \longleftarrow F_n \longrightarrow \mathbb{P}_k^2.$$

In both cases the automorphism group is given as restriction of the automorphism group of the ambient space.

The computation of the automorphism group in terms of the vanishing of the polynomials given in equation (2) is quite complicated.

We have performed this computation in magma [4], and it turns out the automorphism group for the $n = 6$ case is described as an algebraic set described by $g^2 = 100$ variables and 756 equations.

```
    FermatCurve(6, Rationals());
    x_{7,8}*x_{10,10} - 2*x_{9,8}*x_{9,10} + x_{10,8}*x_{7,10},
..................756 equations....................
    >x_{7,9}*x_{10,10} - 2*x_{9,9}*x_{9,10} + x_{10,9}*x_{7,10}
```

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