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On abelian automorphism groups of Mumford curves

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Abstract

We use rigid analytic uniformization by Schottky groups to give a bound for the order of the abelian subgroups of the automorphism group of a Mumford curve in terms of its genus.

Introduction

Let X be a smooth irreducible projective algebraic curve of genus $g \geqslant 2$ over a field k. The automorphism group $\mathrm{Aut}(X)$ is always finite, and it is an interesting problem to determine its size with respect to the genus. When the ground field k has characteristic 0, it is known that the Hurwitz bound holds:

$$|\mathrm{Aut}(X)| \leqslant 84(g-1). \tag{1}$$

Moreover, this bound is best possible in the sense that there exist curves of genus g that admit $84(g-1)$ automorphisms for infinitely many different values of g.

When $\mathrm{char}(k) = p > 0$, then $|\mathrm{Aut}(X)|$ is bounded by a polynomial of degree 4 in g. In fact, it holds that

$$|\mathrm{Aut}(X)| \leqslant 16g^4$$

(see [**13**]), provided that X is not any of the Fermat curves $x^{q+1} + y^{q+1} = 1$, $q = p^n$, $n \geqslant 1$, which have even larger automorphism group [**8**].

For Mumford curves X over an algebraic extension of the p-adic field $\mathbb{Q}_p$, Herrlich [**4**] was able to improve Hurwitz's bound (1) by showing that actually

$$|\mathrm{Aut}(X)| \leqslant 12(g-1),$$

provided that $p \geqslant 7$.

Moreover, the first author, in a joint work with Cornelissen and Kato [**2**], proved that a bound of the form

$$|\mathrm{Aut}(X)| \leqslant \max\{12(g-1), 2\sqrt{g}(\sqrt{g}+1)^2\}$$

holds for Mumford curves defined over non-archimedean-valued fields of characteristic $p > 0$.

For ordinary curves X over an algebraically closed field of characteristic $p > 0$, Guralnik and Zieve (in a forthcoming paper, 'Automorphisms of ordinary curves', Talk in Leiden, Workshop on Automorphism of curves, 18 August 2004) announced that there exists a sharp bound of the order of $g^{8/5}$ for $|\mathrm{Aut}(X)|$.

In [**10**], Nakajima employs the Hasse–Arf theorem to prove that

$$|\mathrm{Aut}(X)| \leqslant 4g + 4$$

for any algebraic curve X with a group of automorphisms that is abelian.

The results of Herrlich compared to those of Hurwitz and those of [**2**] compared to Guralnik–Zieve's indicate that if we restrict ourselves to Mumford curves with abelian automorphism group, a stronger bound than the one of Nakajima should be expected.

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The aim of this note is to study the size of the abelian subgroups of the automorphism group $\mathrm{Aut}(X)$ of a Mumford curve over a complete field k with respect to a non-archimedean valuation. These curves are rigid, analytically uniformized by a Schottky group $\Gamma \subset \mathrm{PGL}_2(k)$, and their automorphism group is determined by the normalizer N of Γ in $\mathrm{PGL}_2(k)$.

Our results are based on the *Gauss–Bonnet* formula of Karass, Pietrowski and Solitar, which relates the rank of the free group Γ to the index $[N : \Gamma]$, and on the characterization of the possible abelian stabilizers $N_v \subset N$ of the vertices $v \in \mathcal{T}_k$ on the Bruhat–Tits tree of k acted upon by the group N.

1. *Abelian automorphism groups of Mumford Curves*

For a nice introduction to the theory of automorphism groups of Mumford curves we refer the reader to [**1**].

Let k be a complete field with respect to a non-archimedean valuation. Let $\bar{k}$ denote the residue field of k and write $p = \mathrm{char}(\bar{k})$ for its characteristic. Choose a separable closure K of k.

Let $\Gamma \subset \mathrm{PGL}_2(k)$ be a Schottky group, that is, a discrete finitely generated subgroup consisting entirely of hyperbolic elements acting on $\mathbb{P}^1_k$ with limit set $\mathcal{L}_\Gamma$ (cf. [**3**]). By a theorem of Ihara, Γ is a free group. The rigid analytic curve

$$\Gamma \setminus (\mathbb{P}^1_k - \mathcal{L}_\Gamma)$$

turns out to be the analytic counterpart of a smooth algebraic curve of genus $g = \mathrm{rank}(\Gamma) \geqslant 1$ over k, which we shall denote by X_Γ/k. In a fundamental work, Mumford [**9**] showed that the stable reduction of X_Γ is a $\bar{k}$-split degenerate curve: all its connected components are rational over $\bar{k}$, and they meet at ordinary double points rational over $\bar{k}$. Conversely, he showed that all such curves admit a rigid analytic uniformization by a Schottky subgroup of $\mathrm{PGL}_2(k)$.

Let $\mathcal{T}_k$ denote the Bruhat–Tits tree of k. The set of ends of $\mathcal{T}_k$ is in one-to-one correspondence with the projective line $\mathbb{P}^1(k)$; we thus identify $\mathbb{P}^1(k)$ with the boundary of $\mathcal{T}_k$.

Let N be a finitely generated discrete subgroup of $\mathrm{PGL}_2(k)$ that contains Γ as a normal subgroup of finite index. The group N naturally acts on $\mathcal{T}_k$. By taking an appropriate extension of k, we may assume that all fixed points of N in the boundary are rational. In turn, this implies that N acts on $\mathcal{T}_k$ without inversion.

Theorem 1.1 ([**2**; **3**, p. 216]). *The group $G = N/\Gamma$ is a subgroup of the automorphism group of the Mumford curve X_Γ. If N is the normalizer of Γ in $\mathrm{PGL}_2(k)$, then $G = \mathrm{Aut}(X_\Gamma)$.*

For every vertex v on $\mathcal{T}_k$ let N_v be the stabilizer of v in N, that is,

$$N_v = \{g \in N : g(v) = v\}.$$

Let $\mathrm{star}(v)$ denote the set of edges emanating from the vertex v. It is known that $\mathrm{star}(v)$ is in one-to-one correspondence with elements in $\mathbb{P}^1(\bar{k})$. Since N_v fixes v, it acts on $\mathrm{star}(v)$ and describes a natural map

$$\rho : N_v \longrightarrow \mathrm{PGL}_2(\bar{k}). \tag{2}$$

See [**2**, Lemma 2.7] for details. The kernel of ρ is trivial unless N_v is isomorphic to the semidirect product of a cyclic group with an elementary abelian group. In this case, $\ker \rho$ is an elementary abelian p-group (see [**2**, Lemma 2.10]).

Assume that $g \geqslant 2$. This implies that Γ has finite index in N, due to which both groups N and Γ share the same set of limit points $\mathcal{L}$. We shall denote by $\mathcal{T}_N$ the subtree of $\mathcal{T}_k$ with end points that are the limit points of $\mathcal{L}$.

The tree $\mathcal{T}_N$ is acted on by N, and we can consider the quotient graph $T_N := N \setminus \mathcal{T}_N$. The graph $N \setminus \mathcal{T}_N$ is the dual graph of the intersection graph of the special fibre of the quotient curve

$$X_N = G \setminus X_\Gamma = N \setminus (\mathbb{P}^1_k - \mathcal{L}).$$

Notice that T_N is a tree whenever X_N has genus 0.

The quotient graph T_N can be regarded as a graph of groups as follows: For every vertex v and edge e of T_N, consider a lift v' and e' in $\mathcal{T}_N$ and the corresponding stabilizer $N_{v'}$ and $N_{e'}$, respectively. We decorate the vertex v and edge e with the stabilizer $N_{v'}$ and $N_{e'}$, respectively.

Let T be a maximal tree of T_N, and let $T' \subset \mathcal{T}_N$ be a tree of representatives of $\mathcal{T}_N$ mod N, that is, a lift of T in $\mathcal{T}_N$. Consider the set Y of lifts of the remaining edges $T_N - T$ in $\mathcal{T}_N$ such that, for every $E \in Y$, the origin $o(E)$ lies in T'.

The set $Y = \{E_1, \ldots, E_r\}$ is finite. There exist elements $g_i \in N$ such that $g_i(t(E_i)) \in T'$, where $t(E_i)$ denotes the terminal vertex of the edge E_i of Y. Moreover, the elements g_i can be taken from the free group Γ.

The elements g_i act by conjugation on the groups $N_{t(E_i)}$ and impose the relations $g_i N_{t(E_i)} g_i^{-1} = N_{g_i(t(E_i))}$. Denote $N_{t(E_i)}$ by M_i and $N_{g_i(t(E_i))}$ by N_i.

According to [**12**, Lemma 4, p. 34], the group N can be recovered as the group generated by

$$N := \langle N_v, g_i \rangle = \langle g_1, \ldots, g_r, K \mid \operatorname{rel} K, g_1 M_1 g_1^{-1} = N_1, \ldots, g_r M_r g_r^{-1} = N_r \rangle,$$

where K is the tree product of T'.

Assume that the tree T' of representatives has κ edges and $\kappa + 1$ vertices. Let v_i denote the order of the stabilizer of the ith vertex and e_i the order of the stabilizer of the ith edge. If $f_i = |M_i|$, then we define the *volume* of the fundamental domain as

$$\mu(T_N) := \left(\sum_{i=1}^{r} \frac{1}{f_i} + \sum_{i=1}^{\kappa} \frac{1}{e_i} - \sum_{i=1}^{\kappa+1} \frac{1}{v_i} \right).$$

Notice that when $r = 0$, that is, the quotient graph T_N is a tree, this definition coincides with that given in [**2**].

Karrass, Pietrowski and Solitar proved in [**7**] the following *discrete Gauss–Bonnet* theorem.

Proposition 1.2. *Let N, T_N, g be as above. The following equality holds:*

$$|N/\Gamma| \cdot \mu(T_N) = g - 1.$$

In order to obtain an upper bound for the group of automorphisms with respect to the genus, we aim for a lower bound for $\mu(T_N)$. Observe that if we restrict the above sum to the maximal tree T of T_N, we deduce the inequality

$$\mu(T) := \sum_{i=1}^{\kappa} \frac{1}{e_i} - \sum_{i=1}^{\kappa+1} \frac{1}{v_i} \leqslant \mu(T_N),$$

where equality is achieved if and only if T_N is a tree, that is, the genus of X_N is 0.

In what follows we pursue lower bounds for $\mu(T)$, where T is a maximal tree. These should be lower bounds for $\mu(T_N)$ as well.

Lemma 1.3. *Let G be a finite abelian subgroup of $\mathrm{PGL}_2(\mathbb{F}_{p^n})$ acting on $\mathbb{P}^1(\mathbb{F}_{p^n})$. Let S be the subset of $\mathbb{P}^1(\bar{\mathbb{F}}_p)$ of ramified points of the cover*

$$\mathbb{P}^1 \longrightarrow G \setminus \mathbb{P}^1.$$

Then either

(i) $G \simeq \mathbb{Z}/n\mathbb{Z}$, *where* $(n,p)=1$, $S=\{P_1,P_2\}$ *and the ramification indices are* $e(P_1)=e(P_2)=n$, *or*
(ii) $G \simeq D_2 = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, $p \neq 2$ and $S=\{P_1,P_2,P_3\}$ *with ramification indices* $e(P_1)=e(P_2)=e(P_3)=2$, *or*
(iii) $G \simeq E(r) = \mathbb{Z}/p\mathbb{Z}\times \overset{(r)}{\cdots} \times\mathbb{Z}/p\mathbb{Z}$ *for some* $r \geqslant 0$ *and* $S=\{P\}$, *with ramification index* $e(P)=p^r$.

Proof. The finite subgroups of $\mathrm{PGL}_2(\mathbb{F}_{p^n})$ were classified by Dickson (cf. [**2**, Theorem 2.9; **5**, II.8.27; **15**]). The list of abelian groups follows by selecting the abelian groups among the possible finite subgroups of $\mathrm{PGL}_2(\mathbb{F}_{p^n})$. Notice that the case $E(r)\rtimes\mathbb{Z}/n\mathbb{Z}$, where $(n,p)=1$ and $n \mid p-1$, is never abelian. Indeed, this is due to the fact that $\mathbb{Z}/n\mathbb{Z}$ acts on $E(r)$ by means of a primitive nth root of unity [**11**, Corollary 1, p. 67]. The description of the ramification locus S in each case is given in [**15**, Theorem 1]. □

LEMMA 1.4. *Let* v *be a vertex of* T_N. *If the finite group* N/Γ *is abelian, then* N_v *is abelian. Moreover, the map* $\rho : N_v \to \mathrm{PGL}_2(\bar{k})$ *is injective unless* $N_v = E(r_1)$. *In this case,* $\ker(\rho) \simeq E(r_2)$ *for some* $r_2 \leqslant r_1$.

Proof. The composition

$$N_v \subset N \longrightarrow N/\Gamma$$

is injective, since it is not possible for an element of finite order to be cancelled out by factoring out the group Γ. Hence N_v is a abelian. The possible kernels of ϱ are collected in [**2**, Lemma 2.10]. □

Let v be a vertex of T_N decorated by the group N_v and assume that there exist $s \geqslant 1$ edges in its star, decorated by groups $N^v_{e_\nu} \subset N_v$, $\nu = 1,\dots,s$. We define the *curvature* $c(v)$ of v as

$$c(v) := \frac{1}{2}\sum_{i=1}^{s}\frac{1}{|N^v_{e_\nu}|} - \frac{1}{|N_v|}.$$

It is obvious that the following formula holds:

$$\mu(T) = \sum_{v\in \mathrm{Vert}(T)} c(v).$$

In what follows, we shall provide lower bounds for the curvature of each vertex.

We shall call a tree of groups *reduced* if $|N_v| > |N_{e_\nu}|$ for all vertices v and edges $e_\nu \in \mathrm{star}(v)$. Notice that, if $N_v = N_e$ for a vertex v and an edge $e \in \mathrm{star}(v)$, then the opposite vertex v' of e is decorated by a group $N_{v'} \supseteq N_e$. The contribution of e to the tree product is the amalgam $N_v *_{N_e} N_{v'} = N_{v'}$. This means that e can be contracted without altering the tree product. From now on we shall assume that *the tree* T *is reduced.*

For an element $\gamma \in N$, define the *mirror* of γ to be the smallest subtree $M(\gamma)$ of $\mathcal{T}_k$ generated by the point-wise fixed vertices of $\mathcal{T}$ by γ.

Let $\gamma \in N$ be an elliptic element (that is, an element of N of finite order with two distinct eigenvalues of the same valuation). Then γ has two fixed points in $\mathbb{P}^1(k)$, and $M(\gamma)$ is the geodesic connecting them.

If $\gamma \in N$ is a parabolic element (that is, an element in N having a single eigenvalue), then it has a unique fixed point z on the boundary $\mathbb{P}^1(k)$.

LEMMA 1.5. *Let P_1, P_2, Q_1, Q_2 be four distinct points on the boundary of $\mathcal{T}_k$. Let $g(P_1, P_2)$, $g(Q_1, Q_2)$ be the corresponding geodesic on $\mathcal{T}_k$ connecting P_1, P_2 and Q_1, Q_2, respectively. For the intersection of the geodesics $g(P_1, P_2)$ and $g(Q_1, Q_2)$ there are the following possibilities:*

(i) *$g(P_1, P_2)$, $g(Q_1, Q_2)$ have an empty intersection;*
(ii) *$g(P_1, P_2)$, $g(Q_1, Q_2)$ intersect at only one vertex of $\mathcal{T}_k$;*
(iii) *$g(P_1, P_2)$, $g(Q_1, Q_2)$ have a common interval as intersection.*

Proof. It immediately follows from the fact that $\mathcal{T}_k$ is simply connected. □

We refer to [6, Proposition 3.5.1] for a detailed description on the arrangement of the geodesics with respect to the valuations of the cross-ratio of the points P_1, P_2, Q_1, Q_2.

LEMMA 1.6. *Two non-trivial elliptic elements $\gamma, \gamma' \in \mathrm{PGL}_2(k)$ have the same set of fixed points in $\mathbb{P}^1(k)$ if and only if $\langle \gamma, \gamma' \rangle$ is a cyclic group.*

Proof. If γ and γ' generate a cyclic group, then there exists an element σ such that $\sigma^i = \gamma$ and $\sigma^{i'} = \gamma'$ for some $i, i' \geqslant 1$. Since any non-trivial elliptic element has exactly two fixed points, it is immediate that γ, γ' and σ have the same set of fixed points.

Conversely if γ, γ' have the same set of fixed points $0, \infty$ say, then a simple computation shows that γ, γ' are of the form

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad \text{and} \quad \gamma' = \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix},$$

where a/d and a'/d' are roots of unity. Hence there exists a $\sigma \in \mathrm{PGL}_2(k)$ such that $\sigma^i = \gamma$ and $\sigma^{i'} = \gamma'$. □

LEMMA 1.7. *Assume that N/Γ is an abelian group, and let $\gamma, \gamma' \in N$, $\gamma \neq \gamma'$, be elements of prime-to-p finite order. If $M(\gamma) \cap M(\gamma') \neq \emptyset$, then $\langle \gamma, \gamma' \rangle$ is isomorphic to either D_2 or a cyclic group.*

Proof. By Lemma 1.4, the stabilizers N_v of those vertices v such that $(|N_v|, p) = 1$ are abelian subgroups of $\mathrm{PGL}_2(\bar{k})$. Let F_γ and $F_{\gamma'}$ denote the sets of the fixed points of γ and γ' in $\mathbb{P}^1(k)$, respectively.

If $M(\gamma) = M(\gamma')$ then $F_\gamma = F_{\gamma'}$, and it follows from Lemma 1.6 that $\langle \gamma, \gamma' \rangle$ is a cyclic group.

On the other hand, if $M(\gamma) \neq M(\gamma')$, then $F_\gamma \neq F_{\gamma'}$ and $\langle \gamma, \gamma' \rangle$ cannot be cyclic, again by Lemma 1.6. In this case, any vertex $v \in M(\gamma) \cap M(\gamma')$ is fixed by $\langle \gamma, \gamma' \rangle$, which must be isomorphic to D_2 by Lemma 1.3. □

LEMMA 1.8. *Assume that N/Γ is an abelian group. If N_v is a p-group for some vertex v in T, then $N_e = \{1\}$ for all $e \in \mathrm{star}(v)$.*

Proof. Assume that $v \in \mathrm{Vert}(T)$ is lifted to $v' \in \mathrm{Vert}(\mathcal{T}_N)$ and that $N_{v'}$ is an elementary abelian group. Recall the map $\rho : N_{v'} \to \mathrm{PGL}_2(\bar{k})$, which describes the action of $N_{v'}$ on $\mathrm{star}(v')$.

If $\mathrm{Im}(\rho) = \{\mathrm{Id}_{\mathrm{PGL}_2(\bar{k})}\}$, then every edge $e' \in \mathrm{star}(v')$ would be fixed by $N_{v'} = \ker(\rho)$. If one of the edges $e' \in \mathrm{star}(v')$ were reduced in T to an edge e with non-trivial stabilizer then it would follow that $N_v = N_e$ and the tree would not be reduced.

Suppose now that $\mathrm{Im}(\rho) \neq \{\mathrm{Id}_{\mathrm{PGL}_2(\bar{k})}\}$. By Lemma 1.3 there exists exactly one edge e' in the star of v' which is fixed by the whole group $\mathrm{Im}(\rho)$ and all other edges emanating from v' are not fixed by $\mathrm{Im}(\rho)$. Therefore the edge e' is fixed by the whole group $N_{v'}$. If the edge e' were reduced in T to an edge e with non-trivial stabilizer, then it would follow that $N_v = N_e$ and the tree would not be reduced.

Assume now that there exist two vertices v_1, v_2 on T joined by an edge e such that N_{v_1}, N_{v_2} are elementary abelian groups and N_e is a non-trivial proper subgroup both of N_{v_1} and N_{v_2}. Let us show that this cannot happen.

Let σ, τ be two commuting parabolic elements of $\mathrm{PGL}_2(\bar{k})$. They fix a common point in the boundary of $\mathbb{P}^1(k)$. Indeed, every parabolic element fixes a single point in the boundary. If P is the unique fixed point of σ, then

$$\sigma(\tau P) = \tau(\sigma P) = \tau P,$$

and $\tau(P)$ is also fixed by σ. Since the fixed point of σ in the boundary is unique, we have $\tau(P) = P$.

Let v_1', v_2' be two lifts of v_1, v_2 on the Bruhat–Tits tree. The apartment $[v_1', v_2']$ is contracted to the edge e and it is fixed by N_e, but not by a larger subgroup.

Since N_e is contained in both abelian groups $N_{v_1'}, N_{v_2'}$, all parabolic elements in $N_{v_1'}, N_{v_2'}$ have the same fixed point P in the boundary $\mathbb{P}^1(k)$. Therefore, the apartments $[v_1', P[$ and $[v_2', P[$ are fixed by $N_{v_1'}$ and $N_{v_2'}$, respectively. Moreover, the apartments $[v_1', P[, [v_2', P[$ have a non-empty intersection. Since the Bruhat–Tits tree is simply connected, the apartment $[v_1', v_2']$ intersects $[v_2', P[\cap [v_1', P[$ at a bifurcation point Q:

$$[v_1', v_2'] \cap \big([v_2', P[\cap [v_1', P[\big) = \{Q\}.$$

The point Q is then fixed by $N_{v_1'}$ and $N_{v_2'}$ and it is on the apartment $[v_1', v_2']$, which is a contradiction. □

LEMMA 1.9. *Let v be a vertex in T. If $c(v) > 0$, then $c(v) \geqslant \frac{1}{6}$. Let $s = \#\mathrm{star}(v)$, and let $N^v_{e_\nu}$ denote the stabilizers of the edges in the star of v for $\nu = 1, \ldots, s$. It holds that $c(v) = 0$ if and only if:*

(i) *$N_v = D_2$, $s = 1$, $N^v_{e_1} = \mathbb{Z}_2$, or*
(ii) *$N_v = \mathbb{Z}_2$, $s = 1$, $|N^v_{e_1}| = 1$.*

We have

- *$c(v) = \frac{1}{6}$ if and only if $N_v = \mathbb{Z}_3$ and $s = 1$;*
- *$c(v) = \frac{1}{4}$ if and only if $N_v = D_2$ with $s = 2$ and $|N^v_{e_1}| = |N^v_{e_2}| = |2|$, or $N_v = D_2$ with $s = 1$ and $|N^v_{e_1}| = 1$.*

In the remaining cases we have $c(v) \geqslant \frac{1}{3}$.

Proof. • Assume that $N_v = D_2$. Then $c(v) \geqslant 0$. The equality $c(v) = 0$ holds only when $s = 1$ and the only edge leaving v is decorated by a group of order 2. If we assume that $c(v) > 0$, then

$$\tfrac{1}{4} \leqslant c(v),$$

and equality is achieved if $s = 2$ and $|N_{e_1}| = |N_{e_2}| = 2$, or if $s = 1$ and $|N^v_{e_1}| = 1$.

• Assume that $N_v = \mathbb{Z}_n$. Then

$$c(v) = \sum_{i=\nu}^{s} \frac{1}{2|N_{e_\nu}|} - \frac{1}{n}.$$

By Lemma 1.7, the stabilizer of each edge in the star of v is trivial. Indeed, if there were an edge $e \in \mathrm{star(v)}$ with $N_e > \{1\}$, then e would be fixed by a cyclic group $\mathbb{Z}_m$, where $m \mid n$. Let σ be a generator of $\mathbb{Z}_n$, and let σ^κ be the generator of $\mathbb{Z}_m$. The elements σ, σ^κ have the same fixed points. Hence a lift of the edge e in $\mathcal{T}_N$ would lie on the mirror of σ. But then $N_e = N_v$, and this is not possible by the reducibility assumption. See also [**4**, Lemma 1].

If $n > 2$, then

$$\frac{1}{6} \leqslant \frac{n-2}{2n} \leqslant \frac{sn-2}{2n} = s\frac{1}{2} - \frac{1}{n} \leqslant c(v),$$

and equality holds only if $s = 1, n = 3$. If $n = 2$ and $c(v) > 0$ then $s \geqslant 2$, and $c(v) = s\frac{1}{2} - \frac{1}{2} \geqslant \frac{1}{2}$.

• Assume that $N_v = E(r)$. Then $s = 1$, and it follows from Lemma 1.8 that $|N_{e_1}| = 1$. If $p^r = 2$ then $c(v) = 0$. Hence if $c(v) > 0$ then $p^r > 2$ and

$$\frac{1}{6} \leqslant c(v) = \frac{1}{2} - \frac{1}{p^r} = \frac{p^r - 2}{2p^r},$$

and equality holds only if $p^r = 3$. □

THEOREM 1.10. *Assume that N/Γ is abelian. If N is isomorphic to neither $\mathbb{Z}_2 * \mathbb{Z}_3$ nor $D_2 * \mathbb{Z}_3$, then*

$$|N/\Gamma| \leqslant 4(g-1).$$

If we exclude the groups of Table 1 then

$$|N/\Gamma| \leqslant 3(g-1).$$

The case $N = \mathbb{Z}_2 * \mathbb{Z}_3$ gives rise to a curve of genus 2 with an automorphism group that is a cyclic group of order 6. The case $N = D_2 * \mathbb{Z}_3$ gives rise to a curve of genus 3 with automorphism group $D_2 \times \mathbb{Z}_3$.

Proof. Since $g \geqslant 2$ and therefore $\mu(T_N) > 0$, we see by Proposition 1.2 that

$$|\mathrm{Aut}(X)| = \frac{1}{\mu(T_N)}(g-1) \leqslant \frac{1}{\mu(T)}(g-1) \leqslant \frac{6}{\lambda}(g-1),$$

where

$$\lambda = \#\{v \in \mathrm{Vert}(T_N) : c(v) > 0\}.$$

If $\lambda \geqslant 2$ the result follows. Assume that there is only one vertex v such that $c(v) > 0$. Since $g \geqslant 2$, there exist other vertices v' on the tree T_N but their contribution is $c(v') = 0$. Notice that if we contract a tree along an edge connecting the vertices v_1, v_2 forming a new vertex v then $c(v_1) + c(v_2) = c(v)$. Therefore, one can check that $c(v) \geqslant 0$, using Lemma 1.3.

Case 1: $c(v) = \frac{1}{6}$. Then $N_v = \mathbb{Z}_3$ and there exists a single edge e at the star of v. Let v' denote the terminal vertex of e. Since $c(v') = 0$ if and only if there exists a single edge leaving

N	N/Γ	g
$D_2 * \mathbb{Z}_4$	$D_2 \times \mathbb{Z}_4$	2
$\mathbb{Z}_2 * \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	2
$D_2 * D_2$	$\mathbb{Z}_2^4$	4
$\mathbb{Z}_2 * D_2$	$\mathbb{Z}_2^3$	2
$D_2 *_{\mathbb{Z}_2} D_2 *_{\mathbb{Z}_2} D_2$	$\mathbb{Z}_2^4$	2

TABLE 1.

v', the only possibilities for N are $N = D_2 * \mathbb{Z}_3$ and $N = \mathbb{Z}_2 * \mathbb{Z}_3$. Since N/Γ is abelian, the groups $\Gamma_1 := [D_2, \mathbb{Z}_3]$ and $\Gamma_1 := [\mathbb{Z}_2, \mathbb{Z}_3]$ are contained in Γ. According to [**2**, Lemma 6.6], Γ_1 is a maximal free subgroup of N and thus $\Gamma = \Gamma_1$. The rank of Γ is $(4-1)(2-1) = 3$ in the first case and $(3-1)(2-1) = 2$ in the second. Therefore the first amalgam gives rise to a curve of genus 3 with automorphism group $D_2 \times \mathbb{Z}_3$, and the second gives rise to a curve of genus 2 with automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$.

Case 2: $c(v) = \frac{1}{4}$. This occurs only when $N_v = \mathbb{Z}_4, D_2$ and $s = 1$ or $N_v = D_2$, $s = 2$, $|N_{e_1}| = |N_{e_2}| = \frac{1}{2}$. The possible groups are given in Table 1.

In this case we have the following bound:

$$|\mathrm{Aut}(X)| \leqslant \frac{1}{\mu(T)}(g-1) \leqslant 4(g-1).$$

Case 3: $c(v) \geqslant \frac{1}{3}$. Similarly as above we obtain that

$$|\mathrm{Aut}(X)| \leqslant \frac{1}{\mu(T)}(g-1) \leqslant \frac{3}{\lambda}(g-1) \leqslant 3(g-1). \qquad \square$$

EXAMPLE (Subrao curves). Let $(k, |\cdot|)$ be a complete field of characteristic p with respect to a non-archimedean norm $|\cdot|$. Assume that $\mathbb{F}_q \subset k$, for some $q = p^r$, $r \geqslant 1$. Define the curve

$$(y^q - y)(x^q - x) = c,$$

with $|c| < 1$. This curve was introduced by Subrao in [**14**], and it has a large automorphism group compared to the genus. This curve is a Mumford curve [**2**, p. 9] and has *chessboard* reduction [**2**, par. 9]. It is a curve of genus $(q-1)^2$ and admits the group $G := \mathbb{Z}_p^r \times \mathbb{Z}_p^r$ as a subgroup of the automorphism group. The group G consists of the automorphisms $\sigma_{a,b}(x, y) = (x + a, y + b)$, where $(a, b) \in \mathbb{F}_q \times \mathbb{F}_q$. The discrete group N' corresponding to G is given by $\mathbb{Z}_{p^r} * \mathbb{Z}_{p^r}$ and the free subgroup Γ giving the Mumford uniformization is given by the commutator $[\mathbb{Z}_{p^r}, \mathbb{Z}_{p^r}]$, which is of rank $(q-1)^2$ (see [**12**]).

Our bound is given by

$$q^2 = |G| \leqslant 2(g-1) = 2(q^2 - 2q).$$

Notice that the group N' is a proper subgroup of the normalizer of Γ in $\mathrm{PGL}_2(k)$, since the full automorphism group of the curve is isomorphic to $\mathbb{Z}_p^{2r} \rtimes D_{p^r-1}$ (see [**2**]).

1.1. *Elementary abelian groups*

PROPOSITION 1.11. *Let ℓ be a prime number, and let X_Γ/k be a Mumford curve over a non-archimedean local field k such that $p = \mathrm{char}(\bar{k}) \neq \ell$. Let $A \subset \mathrm{Aut}(X_\Gamma)$ be a subgroup of the group of automorphisms of X_Γ such that $A \simeq \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z} \times \ldots \times \mathbb{Z}/\ell\mathbb{Z}$.*

If $\ell = 2$ then all stabilizers of vertices and edges of the quotient graph T_N are subgroups of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mu(T_N) = a/4$ for some $a \in \mathbb{Z}$. If $\ell > 2$ then all stabilizers of vertices and edges of T_N are subgroups of $\mathbb{Z}/\ell\mathbb{Z}$ and $\mu(T_N) = a/\ell$ for some $a \in \mathbb{Z}$.

Proof. Let $A \subset \mathrm{Aut}(X_\Gamma)$. There is a discrete finitely generated subgroup $N' \subset N$ such that $\Gamma N'$ and $N'/\Gamma = A$. Let N_v be the stabilizer of a vertex in T_N, and let $N'_v = N_v \cap N'$. The composition

$$N'_v \subset N_v \subset N \longrightarrow N/\Gamma$$

is injective, since it is not possible for an element of finite order to be cancelled out by factoring out the group Γ.

The map $\rho : N'_v \to \mathrm{PGL}_2(\bar{k}) = \mathrm{PGL}_2(\mathbb{F}_{p^m})$ of Lemma 1.4 is injective since $(|N'_v|, p) = 1$, and hence we can regard N'_v as a finite subgroup of $\mathrm{PGL}_2(\bar{k}) = \mathrm{PGL}_2(\mathbb{F}_{p^m})$.

Assume first that $\ell = 2$. Then by Lemma 1.3 the only abelian finite subgroups of $\mathrm{PGL}_2(\bar{k})$ for $p \neq 2$ are $\mathbb{Z}/2\mathbb{Z}$ and the dihedral group of order 4. Hence N'_v is a subgroup of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Since the group N acts without inversions, the stabilizer of a vertex is the intersection of the stabilizers of the limiting edges. It again follows that $N_e \subseteq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Finally, we obtain from its very definition that $\mu(T_N) = a/4$ for some $a \in \mathbb{Z}$.

For the case $\ell > 2$ we observe that $\mathbb{Z}/\ell\mathbb{Z}$ is the only abelian subgroup of $\mathrm{PGL}_2(\mathbb{F}_{p^m})$, and it follows similarly that $\mu(T_N) = a/\ell$ for some $a \in \mathbb{Z}$. □

As an immediate corollary of Proposition 1.11 we obtain the following formula for the ℓ-elementary subgroups of the group of automorphisms of Mumford curves.

Notice that the result below actually holds for arbitrary algebraic curves, as it can be proved by applying the Riemann–Hurwitz formula to the covering $X \to X/A$.

Corollary 1.12. *Let $\ell \neq \mathrm{char}(\bar{k})$ be a prime number, and let X/k be a Mumford curve of genus $g \geqslant 2$ over a non-archimedean local field k. Let $A \subseteq \mathrm{Aut}(X)$ be a subgroup of the group of automorphisms of X such that $A \simeq \bigoplus_{i=1}^{s} \mathbb{Z}/\ell\mathbb{Z}$ for some $s \geqslant 2$.*

(i) *If $\ell \neq 2$ then $\ell^{s-1} \mid g - 1$.*
(ii) *If $\ell = 2$ then $2^{s-2} \mid g - 1$.*

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