REPRESENTATIONS OF CYCLIC GROUPS IN POSITIVE CHARACTERISTIC AND WEIERSTRASS SEMIGROUPS

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ABSTRACT. We study the $k[G]$-module structure of the space of holomorphic differentials of a curve defined over an algebraically closed field of positive characteristic, for a cyclic group G of order $p^\ell n$. We also study the relation to the Weierstrass semigroup for the case of Galois Weierstrass points.

1. INTRODUCTION

Let X be a projective nonsingular curve, defined over an algebraically closed field K of positive characteristic p. The study of the curve X is equivalent to the study of the corresponding function field F.

An open question in positive characteristic is the determination of the Galois module structure of the space of holomorphic differentials of X. This problem is still open and only some special cases are known [16],[17],[8],[12],[1] where restrictions are made either on the ramification type or on the group structure of G. R.Valentini and M. Madan in [19] computed the Galois module structure of the space of holomorphic differentials for the case of a cyclic group action G, where G was a cyclic p-group of order either prime to p or a power of p. One of the aims of this paper is to extend the result of Valentini Madan to the more general case of a cyclic group that has order $p^\ell n$, $(n, p) = 1$. We will characterize the indecomposable summands $V(\lambda, k)$ (see section 2 for a precise definition in terms of the Jordan indecomposable blocks of the generator) and we will decompose the space V of holomorphic differentials as:

$$V := \bigoplus_{\lambda=0}^{n-1} \bigoplus_{k=1}^{p^\ell} V(\lambda, k)^{d(\lambda,k)}. \tag{1}$$

The numbers $d(\lambda, k)$ will be described in terms of the ramification of the extension F/F^G in theorem 7.

The G–module structure is expressed in terms of the *Boseck* invariants. These are invariants introduced by Boseck [2] coming from the construction of bases of holomorphic differentials. The Boseck invariants have rich connections with other subjects in the literature: computation of Weierstrass points, [2], [3], [4]; the computation of the rank of the Hasse–Witt matrix, [9]; the classification of curves with certain rank of the Hasse–Witt matrix [11]; the study of the Artin–Schreier (sub)extensions of rational functions fields, [18],etc. Here we choose to focus only on the G module structure as well as on the structure of the Weierstrass semigroup that is attached to a ramified point.

The complicated notation needed in order to state the main results prevents us from presenting our main theorem here.

The paper is organized as follows: In section 2 we introduce a notation for the places that are ramified in extension $F/F^P/F^G$ and give a filtration of the module of holomorphic differentials used in the computations. Next section is devoted to dimension computations

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with the aid of Riemann-Roch formula. In the final section we see the relation to the Weierstrass semigroup. We tried to relate our results to known results in the literature. This way we discovered an inaccuracy in the work of Boseck [2] in the case of a $\mathbb{Z}/p\mathbb{Z}$-extension of the rational function field ramified above one point. Finally we extend results from characteristic zero relating the Galois module structure of the space of holomorphic differentials and the Weierstrass semigroup attached to a ramified point.

2. Notation

Let $G = \langle g \rangle$ be a cyclic subgroup of automorphisms acting on the space of holomorphic differentials $V := H^0(X, \Omega_X)$. The group G can be written as a direct product of a group $T = \langle g^{p^\ell} \rangle$ of order n and a cyclic p-group $P = \langle g^n \rangle$. We consider the tower of function fields $F/F^P/F^G$. Let $np^\ell = |G|$, $(n,p) = 1$ and consider a primitive n-th root of unity $\zeta_n \in K$. By Jordan decomposition theory we see that we can decompose V as a direct sum of $K[G]$-modules $V(\lambda, k)$. The modules $V(\lambda, k)$ are k-dimensional K-vector spaces with basis $\{v_1, \ldots, v_k\}$ and action given by

$$gv_i = \zeta_n^\lambda v_i + v_{i+1} \text{ for all } 1 \leq i \leq k-1 \tag{2}$$

and

$$gv_k = \zeta_n^\lambda v_k.$$

The action of the generator g on $V(\lambda, k)$ is given in terms of the matrix:

$$A := \begin{pmatrix} \zeta_n^\lambda & 1 & 0 & \cdots & 0 \\ 0 & \zeta_n^\lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta_n^\lambda & 1 \\ 0 & \cdots & 0 & 0 & \zeta_n^\lambda \end{pmatrix}.$$

Observe that for a cyclic group G of order np^ℓ generated by g the module $K[G]$ can be decomposed as $K[G] = \oplus_{\lambda=0}^{n-1} V(\lambda, p^\ell)$. Indeed, the characteristic polynomial of g acting on $K[G]$ is up to ± 1 equal to $x^{np^\ell} - 1 = (x^n - 1)^{p^\ell}$, and every root of unity in K appears as a character in $K[G]$.

Remark 1. The indecomposable $K[P]$-modules of a cyclic p-group of order p^ℓ and with generator σ are given by the quotients $K[P]/(\sigma - 1)^k$, where $k = 1, \ldots, p^\ell$ [19]. In our notation these are the modules $V(0, k)$ i.e. the indecomposable Jordan forms of dimension k.

Proposition 2. *The indecomposable $K[G]$-module $V(\lambda, k)$ seen as a $K[T]$-module is a direct sum of k characters of the form $\zeta \mapsto \zeta^{p^\ell n}$. The module $V(\lambda, k)$ seen as $K[P]$-module is indecomposable and isomorphic to the module $K[P]/(\sigma - 1)^k$.*

Proof. We will use the following idea: The action of G on the indecomposable summand $V(\lambda, k)$ is described by the action of the generator g of G. We would like to view the module $V(\lambda, k)$ as a P and T module respectively. A generator for the T group is given by g^{p^ℓ}. Write the matrix A as $A = \mathrm{diag}(\zeta^\lambda) + N$ where N is a nilpotent $k \times k$ matrix with $k \leq p^\ell$. Therefore, the generator g^{p^ℓ} of the T group is given by the matrix $A^{p^\ell} = \mathrm{diag}(\zeta^{\lambda p^\ell})$. This means that $V(\lambda, k)$ seen as a T module is decomposed as a direct sum of k characters of the form $\zeta \mapsto \zeta^{\lambda p^\ell}$. Since $(p^\ell, n) = 1$ raising an n-th root of unity to the p^ℓ-power is an automorphism of the group of n-th roots of one.

On the other hand the action of the generator g^n on the module $V(\lambda, k)$ is given by the n-th power of A. We observe first that that all eigenvalues of A^n are 1. We will prove that A^n is similar to the matrix $\mathrm{Id} + N$, i.e. a Jordan indecomposable block. Since all eigenvalues of A^n are 1 the characteristic polynomial of A^n is $(x-1)^k$. The minimal polynomial of A^n is $(x-1)^d$ for some integer $1 \leq d \leq k$. Since A is an indecomposable Jordan block the minimal polynomial of A is $(x-\zeta^\lambda)^k$. On the other hand, since $(A^n - 1)^d = 0$ we have that $(x-\zeta^\lambda)^k$ divides $(x^n-1)^d$ and this is possible only if $d = k$. This implies that A^n is similar to an indecomposable Jordan form of dimension k. $\square$

2.1. **Fields and ramification.** We will introduce some notation on the ramification places in the extensions F/F^P and F/F^G. Let us denote by $\bar{P}_1, \ldots, \bar{P}_s$ the places of F^P that are ramified in F/F^P. The places of F that are above $\bar{P}_i$ will be denoted by $P_{i,\nu}$, $1 \leq \nu \leq p^\ell/e_i$, where $e_i = p^{\epsilon_i}$ is the common ramification index $e(P_{i,\nu}/\bar{P}_i)$.

The different $\mathrm{Diff}(F/F^P)$ is supported at the places $P_{i,\nu}$ while the discriminant $D(F/F^P)$ is supported at the places $\bar{P}_1, \ldots, \bar{P}_s$. Let us denote the different exponent at each ramified place P_j by δ_j. The discriminant is then computed:

$$D(F/F^P) = \sum_{j=1}^{s} p^{\ell-\epsilon_j} \delta_j \bar{P}_j,$$

while the different is given by

$$\mathrm{Diff}(F/F^P) = \sum_{j=1}^{s} \delta_j \sum_{\nu} P_{j,\nu}.$$

The cyclic group extension F^P/F^G is a Kummer extension with Galois group T and it is defined by an equation of the form:

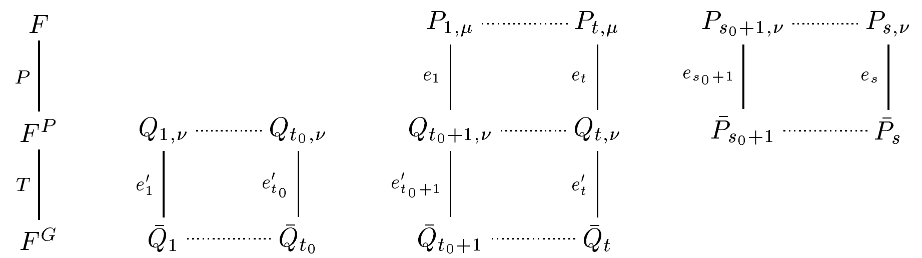
$$F^P = F^G(y), \quad y^n = b, \quad b \in F^G. \tag{3}$$

Let $\bar{Q}_1, \ldots, \bar{Q}_t$ be the places of F^G that are ramified in extension F^P/F^G. We define $Q_{i,\nu}$, $1 \leq \nu \leq n/e'_i$ to be the places of F^P which are above $\bar{Q}_i$, where e'_i denotes the common ramification index, $e'_i = e(Q_{i,\nu}/\bar{Q}_i)$.

Assume that the set of places $\{\bar{Q}_1, \ldots, \bar{Q}_{t_0}\}$ extend to places $Q_{i,\nu}$ of F^P that do not ramify on F/F^P and that each place $\bar{Q}_i$ of the places $\{\bar{Q}_{t_0+1}, \ldots, \bar{Q}_t\}$ extends to places $Q_{i,\nu}$ that ramify in F/F^P. The total number of places of the form $Q_{i,\nu}$ $t_0 + 1 \leq i \leq t$ equals

$$s_0 := \sum_{i=t_0+1}^{t} \frac{n}{e'_i} = |\{Q_{i,\nu} : t_0 + 1 \leq i \leq t, 1 \leq \nu \leq n/e'_i\}|.$$

We enumerate the places $\bar{P}_i$ such that $\{\bar{P}_{s_0+1}, \ldots, \bar{P}_s\}$ do not ramify in F^P/F^G and $\{P_1, \ldots, P_{s_0}\} = \{Q_{i,\nu} : t_0 + 1 \leq i \leq t, 1 \leq \nu \leq n/e'_i\}$.



We can select b in eq. (3) such that [19, sec. 2]

$$\mathrm{div}_{F^G}(b) = nA + \sum_{i=1}^{t} \phi_i \bar{Q}_i, \tag{4}$$

where $0 < \phi_i < n$, A is a divisor of F^G. The ramification indices are given by $e'_i = n/(n,\phi_i)$, and the discriminant is given by

$$D(F^P/F^G) = \sum_{i=1}^{t} \left(n - \frac{n}{e'_i} \right) \bar{Q}_i. \tag{5}$$

We also define $\Phi_i = \phi_i/(n,\phi_i)$.

2.2. **Modules.** Let us now focus on the cyclic p-group extension F/F^P. The G-module structure on holomorphic differentials on a cyclic p-group is studied by R. Valentini and M. Madan in [19]. Let $\sigma := g^n$ be a generator of the cyclic group P. Recall that V denotes the set of holomorphic differentials. Following the article of Valentini Madan we consider the set of subspaces $V^i \subset V$ defined by

$$V^i := \{\omega \in V : (\sigma-1)^i \omega = 0\} \text{ for } i = 0, \ldots, p^\ell.$$

We compute that (the set $\{v_1, \ldots, v_m\}$ is the basis of $V(\mu, m)$ given in eq. (2)).

$$V^{i+1} \cap V(\lambda, m) = \begin{cases} V(\lambda, m) & \text{if } m \leq i+1 \\ \langle v_{m-i}, \ldots, v_m \rangle & \text{if } m > i+1 \end{cases}$$

Since G is a commutative group there is a well defined action of $T = \langle g^{p^\ell} \rangle$ on the quotient space V^{i+1}/V^i and the natural map

$$V^{i+1} \rightarrow V^{i+1}/V^i,$$

is T-equivariant. The images of the spaces $V^{i+1} \cap V(\lambda, m)$ under this map are 0 for $m \leq i$ and are one dimensional if $m > i$.

The space V^{i+1}/V^i is decomposed into characters of the group T. Let $d(\lambda, k)$ be the number of $V(\lambda, k)$ blocks in V. Let $c(\lambda, i)$, $0 \leq i \leq p^\ell - 1$ be the number of characters of the form $g\omega = \zeta^\lambda \omega$ in V^{i+1}/V^i.

We have that

$$c(\lambda, i) = \sum_{k \geq i+1} d(\lambda, k).$$

Therefore

$$\begin{aligned} d(\lambda, p^\ell) &= c(\lambda, p^\ell - 1) \\ d(\lambda, k) &= c(\lambda, k-1) - c(\lambda, k). \end{aligned} \tag{6}$$

Lemma 3. *There is a basis $\{w_0, \ldots, w_{p^\ell - 1}\}$ of F over F^P such that:*

(1) *For $0 \leq k \leq p^\ell - 1$ with p-adic expansion $k = a_1^k + a_2^k p + \cdots + a_\ell^k p^{\ell-1}$, we have*

$$(\sigma - 1)^k w_k = a_1^k! a_2^k! \cdots a_n^k! w_k.$$

(2) *Every $\omega \in V$ can be written as*

$$\omega = \sum_{\nu=0}^{p^\ell - 1} c_\nu w_\nu dx$$

with $x, c_\nu \in F^P$ and with the additional property that

$$\omega \in V^i \Leftrightarrow c_i = c_{i+1} = \cdots = c_{p^\ell - 1} = 0.$$

(3) *There are numbers $\Phi(\mu, j)$ prime to p such that*

$$v_{P_{\mu,\nu}}(w_k) = -\sum_{j=1}^{\ell} a_j^k \Phi(\mu, j) p^{\ell - j}.$$

Proof. The definition of the basis is given in [19, p.108] while the second assertion is proved in the same article in the proof of theorem 1. The existence of the numbers $\Phi(\mu, j)$ follows by the construction of the extension F/F^P in terms of successive Artin-Schreier extensions (see [19, sec. 1]). ∎

Define the integers:

$$\nu_{\mu,k} := \left\lfloor \frac{\delta_\mu + v_{P_{\mu,\nu}}(w_k)}{e_\mu} \right\rfloor . \tag{7}$$

Notice that the valuation $v_{P_{\mu,\nu}}(w_k)$ does not depend on the selection of the place $P_{\mu,\nu}$ over $\bar{P}_\mu$.

Let $\tau = g^{p^\ell}$ be a generator of the cyclic group T. Assume that $\tau y = \zeta_n^r y$. For each $\lambda = 0, \ldots, n-1$ we select $0 \leq \alpha_\lambda \leq n-1$ such that

$$r\alpha_\lambda \equiv \lambda \bmod n. \tag{8}$$

Definition 4. Define the Boseck invariants:

$$\Gamma_{k,\lambda} := \sum_{i=1}^{t} \left\langle -\frac{\alpha_\lambda \Phi_i}{e'_i} \right\rangle + \sum_{j=t_0+1}^{t} \left\lfloor \left\langle \frac{\alpha_\lambda \Phi_j - 1}{e'_j} \right\rangle + \frac{\nu_{j,k}}{n} \right\rfloor + \sum_{\mu=1}^{s-t+t_0} \left\lfloor \frac{\nu_{\mu,k}}{n} \right\rfloor$$

.

Remark 5. If $n = 1$, then

$$\Gamma_{k,\lambda} = \Gamma_k = \sum_{\mu=1}^{s} \nu_{\mu,k}.$$

This is the Boseck invariant for the p cyclic case, see [2] and [6].

If $p^\ell = 1$ then

$$\Gamma_{k,\lambda} = \Gamma_\lambda = \sum_{j=1}^{t} \left\langle -\frac{\alpha_\lambda \Phi_j}{e'_j} \right\rangle .$$

This is the Boseck invariant for the the cyclic tame case. These invariants coincide with the ones introduced by [2], and used by [6], after letting $r = 1$, to eq. (8) (this can be done without loss of the generality).

In next section we will prove the following:

Proposition 6. *Recall that ϵ_i are integers such that $p^{\epsilon_i} = e(P_{i,\nu}/\bar{P}_i)$. Consider the integer $r = \ell - \max \epsilon_i$. For $0 \leq k < p^\ell - p^r$, we have*

$$c(\lambda, k) = g_{F^G} - 1 + \Gamma_{k,\lambda} + \Lambda_{k,\lambda}.$$

The integer $\Lambda_{k,\lambda}$ is given by the following rule: If $\Gamma_{k,\lambda} = 0$ then $\Lambda_{k,\lambda} = 1$. In all other cases $\Lambda_{k,\lambda} = 0$.

For $p^\ell - p^r \leq k \leq p^\ell - 1$ we have

$$c(\lambda, k) = \begin{cases} \frac{1}{p^r}\left(g_{E_r^T} - 1 + \Gamma_{k.\lambda}\right) & \text{if } k \geq p^\ell - p^r + 1 \text{ or } \lambda \neq 0 \\ g_{F^G} & \text{if } k = p^\ell - p^r \text{ and } \lambda = 0. \end{cases}$$

This will allow us to see:

Theorem 7. *If $r = 0$ then*

$$d(\lambda, p^\ell) = g_{F^G} - 1 + \Gamma_{p^\ell,\lambda} + \Lambda_{p^\ell,\lambda}. \tag{9}$$

For all the values of r and for $k < p^\ell - p^r$ we have:

$$d(\lambda, k) = \Gamma_{k-1,\lambda} - \Gamma_{k,\lambda} + M_{k,\lambda},$$

where $\{-1, 0, 1\} \ni M_{k,\lambda} := \Lambda_{k-1,\lambda} - \Lambda_{k,\lambda}$.

If $k = p^\ell - p^r$ then $k - 1 = p^\ell - p^r - 1$ and

$$\begin{aligned} d(\lambda, p^\ell - p^r) &= g_{F^G} - 1 + \Gamma_{k-1,\lambda} + \Lambda_{k-1,\lambda} - c(\lambda, p^\ell - p^r) \\ &= \begin{cases} \Gamma_{k-1,\lambda} - \frac{1}{p^r}\Gamma_{k,\lambda} + \Lambda_{k-1,\lambda}, & \text{if } \lambda \neq 0 \\ \Gamma_{k-1,0} + \Lambda_{k-1,0} - 1, & \text{if } \lambda = 0 \end{cases} \end{aligned}$$

For $r \neq 0$ *and* $p^\ell - p^r \leq k \leq p^\ell - 1$*, we have:*

$$d(\lambda,k) = \begin{cases} 1 & \text{if } k = p^\ell - p^r + 1, \lambda = 0 \\ \frac{1}{p^r}\left(g_{E_r^T} - 1 + \Gamma_{k,\lambda}\right) & \text{if } k = p^\ell \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Proof. The proof is a simple application of proposition 6. □

2.3. **Computation of $\mathbf{c}(\lambda,\mathbf{k})$.** This section is devoted to the proof of proposition 6.

Lemma 8. *Let G be a group of order $p^\ell n$ acting on the curve X, with only tame ramification, i.e. every point that is ramified has decomposition group $G(P) \subset \langle g^{p^\ell}\rangle$. Let $T = \langle g^{p^\ell}\rangle$ be the tame cyclic part of the group G. Consider the integers $\phi_i, \Phi_i, \alpha_\lambda, e'_i$ describing the Kummer extension F/F^T and let g_{F^T} denote the genus of F^T. Then the decomposition of the space V of holomorphic differentials is given by*

$$V := \bigoplus_{k=1}^{p^\ell}\bigoplus_{\lambda=0}^{n-1} V(\lambda,k)^{d^*(\lambda,k)},$$

where $d^*(0,1) = 1$,

$$d^*(\lambda,p^\ell) = \frac{1}{p^\ell}\left(g_{F^T} - 1 + \sum_{i=1}^t \left\langle \frac{-\alpha_\lambda \Phi_i}{e'_i}\right\rangle\right)$$

and $d^*(\lambda,k) = 0$ *in all other cases.*

Proof. Group actions on curves without branched points on spaces of holomorphic differentials were studied by T. Tamagawa [16]. Tamagawa proved that the space of holomorphic differentials is decomposed as

$$V := K \oplus K[P]^{g_{X/P}-1},$$

where $g_{X/P}$ is the genus of the quotient curve X/P.

Actions with tame ramification where studied by E. Kani [5]. Kani proved that:

$$V := K \oplus K[G]^{g_{X/G}-1} \oplus \tilde{R}^*_G,$$

where $\tilde{R}^*_G$ is a $k[G]$-module such that $n\tilde{R}^*_G = R^*_G$ and R^*_G is a the contragredient module of the tame ramification module (for precise definition see [5, sec. 1]).

The result of Tamagawa for the action of the p-group $P = \langle g^n\rangle$ gives that

$$V = K \oplus \bigoplus_{\lambda=0}^{n-1} V(\lambda,p^\ell)^{d^*(\lambda,p^\ell)}. \tag{11}$$

The integers $d^*(\lambda,p^\ell)$ can be computed by a careful look at the definition of the tame ramification module. We will instead compute them using the results of Valentini-Madan for the extension F/F^T, $T = \langle g^{p^\ell}\rangle$.

The extension F/F^T is a cyclic Kummer extension with Galois group generated by $\sigma = g^{p^\ell}$ and it is characterized by the integers $\phi, \Phi, e'_i, \alpha_\lambda$ introduced in section 2.1. For the module of holomorphic differentials the multiplicities m_λ of the character λ given by the action $\sigma^j(v) = \zeta^{\lambda j} v$ are equal to

$$m_\lambda = g_{F^T} - 1 + \sum_{i=1}^t \left\langle \frac{-\alpha_\lambda \Phi_i}{e'_i}\right\rangle, \text{ if } \lambda \neq 0$$

and

$$\begin{aligned} m_0 &= g_{F^T} + \sum_{i=1}^t \left\langle \frac{-\alpha_0 \Phi_i}{e'_i}\right\rangle \\ &= g_{F^T}, \text{ if } \lambda = 0. \end{aligned}$$

For the last equality it is enough to notice that as $(\Phi_i, e_i') = 1$ then $\sum_{i=1}^{t} \left\langle \frac{-\alpha_0 \Phi_i}{e_i'} \right\rangle = 0$, since the condition $e_i' \mid \alpha_0$ is equivalent to $e_i' \mid 0$ for all i's (see [19, page 115]).

From Tamagawa result we have that $d^*(0,1) = 1$,while for the remaining $m_0 - 1 = g_{F^T} - 1$ representations give us that in both cases ($\lambda = 0$ and $\lambda \neq 0$) we have

$$d^*(\lambda, p^\ell) = \frac{1}{p^\ell}\left(g_{F^T} - 1 + \sum_{i=1}^{t} \left\langle \frac{-\alpha_\lambda \Phi_i}{e_i'} \right\rangle \right)$$

and $d^*(\lambda, k) = 0$ in all other cases. Notice that the eigenvalue ζ^λ appears p^ℓ times in every component $V(\lambda, k)$. □

Remark 9. Applying Riemann-Hurwitz formula, we obtain:

$$d^*(\lambda, p^\ell) = g_{F^G} - 1 + \frac{1}{p^\ell} \sum_{i=1}^{t} \left\langle \frac{-\alpha_\lambda \Phi_i}{e_i'} \right\rangle$$

- If F/F^T is unramified, i.e. when $e_i' = 1$ for all i, then this coincides with the result of Tamagawa, [16].
- If $p^\ell = 1$, i.e. $F^G = F^T$, then this coincides with the result of Hurwitz [10, Theorem 3.5, p. 600], after letting $r = 1$, to eq. (8).

Is there a place P of F that is fully ramified in extension $F/F^{\langle g^n \rangle}$? If not then we consider the place P with maximal ramification index. Set $r = \ell - \max\{\epsilon_i\}$. The wild decomposition group $\langle g^n \rangle(P)$ at this place is cyclic and we will denote the corresponding fixed field by E_r. Call E the fixed field of the wild part $\langle g^n \rangle$. Then we will have a tower of fields $F/E_r/E$ such that in extension E_r/E there is no ramification at all. Notice that $r = 0$ and $E_r = E$ if and only if there is a place P fully ramified in extension $F/F^{\langle g^n \rangle}$.

For the study of the spaces V^{k+1}/V^k, with $k = 0, \ldots, p^\ell - 1$, we will distinguish two cases:
Case 1. $k < p^\ell - p^r$.

Lemma 10. *Assume that $k < p^\ell - p^r$. If the differential $\omega = \sum_{\nu=0}^{k} c_\nu w_\nu dx \in V^{k+1}$, representing a class in V^{k+1}/V^k, is holomorphic then*

$$c_k \in L_{F^P}\left(\mathrm{div}_{F^P}(dx) + \sum_{\mu=1}^{s} \nu_{\mu,k} \bar{P}_\mu \right).$$

The space V^{k+1}/V^k is of dimension $g_{F^P} - 1 + \sum_{\mu=1}^{s} \nu_{\mu,k}$.

Proof. See the proof of theorem 1 and page 112 in [19]. □

In order to study the $k[T]$-module structure of the space V we will apply the previous argument with f in place of the c_k and we focus our study to the space of differentials which have poles at $\sum_{\mu=1}^{s} \nu_{\mu,k} P_\mu$, i.e. differentials of the form:

$$\omega = f dx \text{ such that } \mathrm{div}_{F^P}(f dx) \geq - \sum_{\mu=1}^{s} \nu_{\mu,k} \bar{P}_\mu. \tag{12}$$

We may choose the function $x \in F^P$ to be a function in F^G. Let $\tau = g^{p^\ell}$ be a generator of the cyclic group T. Recall that we assumed that $\tau y = \zeta_n^r y$ and we have selected α_λ such that $r\alpha_\lambda = \lambda \bmod n$. Assume that

$$\tau(f dx) = \zeta^\lambda f dx.$$

By eq. (8) we have

$$\tau(f/y^{\alpha_\lambda}) = f/y^{\alpha_\lambda},$$

so $f = hy^{\alpha_\lambda}$ with $h \in F^G$. Therefore, eq. (12) is satisfied if and only if

$$N_{F^P/F^G}(\mathrm{div}(fdx)) \geq N_{F^P/F^G}\left(-\sum_{\mu=1}^{s}\nu_{\mu,k}\bar{P}_\mu\right). \tag{13}$$

We compute:

$$N_{F^P/F^G}(\mathrm{div}(fdx)) = n\mathrm{div}_{F^G}h + \alpha_k\mathrm{div}_{F^G}(b) + n\mathrm{div}_{F^G}(dx) + \mathrm{D}(F^P/F^G).$$

Remark 11. Whenever we write down a reduced divisor $A = \sum \alpha_i P_i$ (i.e. $P_i \neq P_j$) with $q_i \in \mathbb{Q}$ we mean the divisor $\sum \lfloor q_i \rfloor P_i$. Notice that if $A = \sum \alpha_i P_i$ is a divisor (with possible rational coefficients) and $B = \sum \beta_j P_j$ is a divisor with integer coefficients, then since for $\alpha \in \mathbb{Q}, \beta \in \mathbb{Z}$ $\lfloor \alpha + \beta \rfloor = \lfloor \alpha \rfloor + \beta$ we have that

$$A + B = \sum \lfloor \alpha_i \rfloor P_i + \sum \beta_j P_j,$$

i.e. we don't have to write down $A + B$ in reduced form, before taking the integral part of its coefficients.

Using eq. (4),(5) we see that eq. (13) is equivalent to

$$\mathrm{div}(h) \geq -\mathrm{div}_{F^G}(dx) - \alpha_\lambda A - \sum_{j=1}^{t}\left(\frac{\alpha_\lambda\phi_j}{n} + 1 - \frac{1}{e'_j}\right)\bar{Q}_j - \frac{1}{n}N_{F^P/F^G}\left(\sum_{\mu=1}^{s}\nu_{\mu,k}\bar{P}_\mu\right),$$

i.e. $h \in L(W + E_{k,\lambda})$. Notice that the norm $N_{F^P/F^G}(P_\mu)$ is just the place of F^G lying below P_μ. We proved the following

Lemma 12. *The subspace of V^{k+1}/V^k of elements where g acts by multiplication by ζ^λ is isomorphic to the space $L_{F^G}(W + E_{k,\lambda})$, where W is a canonical divisor on F^G and*

$$E_{k,\lambda} := \alpha_\lambda A + \sum_{j=1}^{t}\left(\frac{\alpha_\lambda\phi_j}{n} + 1 - \frac{1}{e'_j}\right)\bar{Q}_j + \frac{1}{n}N_{F^P/F^G}\left(\sum_{\mu=1}^{s}\nu_{\mu,k}P_\mu\right)$$

is an effective divisor.

We will now write $E_{k,\lambda}$ as a sum of an integral divisor and of a divisor in reduced form. We can assume that $\{\bar{Q}_1, \ldots, \bar{Q}_{t_0}\}$ is the set of ramified places such that their extensions in F^P do not ramified further in F/F^P. We will denote by $\{\bar{Q}_{t_0+1}, \ldots, \bar{Q}_t\}$ the rest of the ramified places. For the $s - (t - t_0)$ places of F^P that are not ramified in F^P/F^G we will denote by Π_μ the places $\sum_{j=1}^{n}\tau P_\mu$.

Now $E_{k,\lambda}$ can be written:

$$\begin{aligned} E_{k,\lambda} \quad := \quad & \alpha_\lambda A + \sum_{j=1}^{t_0}\left(\frac{\alpha_\lambda\phi_j}{n} + \frac{e'_j - 1}{e'_j}\right)\bar{Q}_j + \sum_{j=t_0+1}^{t}\left(\frac{\alpha_\lambda\phi_j + \nu_{j,k}}{n} + \frac{e'_j - 1}{e'_j}\right)\bar{Q}_j \\ & + \sum_{\mu=1}^{s-t+t_0}\frac{\nu_{\mu,k}}{n}\Pi_\mu. \end{aligned}$$

The divisor $E_{k,\lambda}$ as it is written above is not necessarily in reduced form. We don't know whether the divisor A is prime to $\bar{Q}_i$ or Π_μ. But since it has integer coefficients and since all the divisors with possibly rational coefficients are prime to each other, we arrive at

$$\begin{aligned} E_{k,\lambda} \quad := \quad & \alpha_\lambda A + \sum_{j=1}^{t_0}\left\lfloor\frac{\alpha_\lambda\phi_j}{n} + \frac{e'_j - 1}{e'_j}\right\rfloor\bar{Q}_j + \sum_{j=t_0+1}^{t}\left\lfloor\frac{\alpha_\lambda\phi_j + \nu_{j,k}}{n} + \frac{e'_j - 1}{e'_j}\right\rfloor\bar{Q}_j \\ & + \sum_{\mu=1}^{s-t+t_0}\left\lfloor\frac{\nu_{\mu,k}}{n}\right\rfloor\Pi_\mu. \end{aligned} \tag{14}$$

Lemma 13. *The degree of $E_{k,\lambda}$ equals to*

$$\deg(E_{k,\lambda}) := \sum_{i=1}^{t} \left\langle -\frac{\alpha_\lambda \Phi_i}{e'_i} \right\rangle + \sum_{j=t_0+1}^{t} \left\lfloor \left\langle \frac{\alpha_\lambda \Phi_j - 1}{e'_j} \right\rangle + \frac{\nu_{j,k}}{n} \right\rfloor + \sum_{\mu=1}^{s-t+t_0} \left\lfloor \frac{\nu_{\mu,k}}{n} \right\rfloor .$$

Proof. Following Valentini Madan we see that that

$$\deg(A) = -\sum_{j=1}^{t} \frac{\phi_j}{n} = -\sum_{j=1}^{t} \frac{\Phi_j}{e'_j} \tag{15}$$

(recall that $\Phi_i = \phi_i/(\phi_i, n)$).

$$\begin{aligned}\deg E_{k,\lambda} &= \sum_{j=1}^{t_0} \left\langle -\frac{\alpha_\lambda \Phi_j}{e'_j} \right\rangle + \sum_{j=t_0+1}^{t} \left(\left\lfloor \frac{\alpha_\lambda \phi_j + \nu_{j,k}}{n} + \frac{e'_j - 1}{e'_j} \right\rfloor - \frac{\alpha_\lambda \Phi_j}{e'_j} \right) \\ &\quad + \sum_{\mu=1}^{s-t+t_0} \left\lfloor \frac{\nu_{\mu,k}}{n} \right\rfloor .\end{aligned}$$

We will use

$$\frac{\alpha_\lambda \Phi_j}{e'_j} = \left\lfloor \frac{\alpha_\lambda \Phi_j}{e'_j} \right\rfloor + \left\langle \frac{\alpha_\lambda \Phi_j}{e'_j} \right\rangle ,$$

We have

$$\left\lfloor \frac{\alpha_\lambda \phi_j + \nu_{j,k}}{n} + \frac{e'_j - 1}{e'_j} \right\rfloor - \frac{\alpha_\lambda \Phi_j}{e'_j} =$$

$$= \left\langle -\frac{\alpha_\lambda \Phi_j}{e'_j} \right\rangle + \left\lfloor \frac{\alpha_\lambda \Phi_j + e'_j - 1}{e'_j} + \frac{\nu_{j,k}}{n} \right\rfloor + \left\lfloor -\frac{\alpha_\lambda \Phi_j}{e'_j} \right\rfloor =$$

$$= \left\langle -\frac{\alpha_\lambda \Phi_j}{e'_j} \right\rangle + \left\lfloor \left\lfloor \frac{\alpha_\lambda \Phi_j + e'_j - 1}{e'_j} \right\rfloor + \left\langle \frac{\alpha_\lambda \Phi_j + e'_j - 1}{e'_j} \right\rangle + \frac{\nu_{j,k}}{n} \right\rfloor + \left\lfloor -\frac{\alpha_\lambda \Phi_j}{e'_j} \right\rfloor =$$

$$= \left\langle -\frac{\alpha_\lambda \Phi_j}{e'_j} \right\rangle + \left\lceil \frac{\alpha_\lambda \Phi_j}{e'_j} \right\rceil + \left\lfloor \left\langle \frac{\alpha_\lambda \Phi_j + e'_j - 1}{e'_j} \right\rangle + \frac{\nu_{j,k}}{n} \right\rfloor + \left\lfloor -\frac{\alpha_\lambda \Phi_j}{e'_j} \right\rfloor =$$

$$= \left\langle -\frac{\alpha_\lambda \Phi_j}{e'_j} \right\rangle + \left\lfloor \left\langle \frac{\alpha_\lambda \Phi_j - 1}{e'_j} \right\rangle + \frac{\nu_{j,k}}{n} \right\rfloor .$$

□

Proposition 14. *If $k < p^\ell - p^r$, we have*

$$c(\lambda, k) = \dim L(W + E_{k,\lambda}) = g_{F^G} - 1 + \deg(E_{k,\lambda}) + \Lambda_{k,\lambda}$$

Moreover, if

$$\deg(E_{k,\lambda}) = 0$$

then $\Lambda_{k,\lambda} = 1$. In all other cases $\Lambda_{k,\lambda} = 0$.

Proof. By Riemann-Roch theorem and lemma 13 we see that

$$\dim L(W + E_{k,\lambda}) = g_{F^G} - 1 + \deg E_{k,\lambda} + \dim L(-E_{k,\lambda}).$$

If the divisor $\deg(E_{k,\lambda}) > 0$ then $\dim L(-E_{k,\lambda}) = 0$ and the result follows.

Assume now that $\deg(E_{k,\lambda}) = 0$. Since $E_{k,\lambda}$ is effective this means that $E_{k,\lambda} = 0$ and in this case $\Lambda_{k,\lambda} = \ell(0) = 1$. □

Case 2. $p^\ell - p^r \leq k \leq p^\ell - 1$. In this case we will apply the same procedure as we did in Case 1 and then we will apply lemma 8 for the extension E_r/F^G. Write $k = k_1 + tp^r$ with $0 \leq k_1 < p^r$ and $t = p^{\ell-r} - 1$. Set $k_0 = tp^r = p^\ell - p^r$. Let $\sigma = g^n$ be a generator for the p-cyclic part of G. Let $V^k_{E_r}$ be the space of holomorphic differentials of E_r that are annihilated by $(\sigma-1)^k$. Valentini-Madan [19, p.111-112] proved that

$$(\sigma - 1)^{k_0} : V^{k+1}/V^k \to V^{k_1+1}_{E_r}/V^{k_1}_{E_r} \tag{16}$$

is an isomorphism. We will now consider the extension E_r/F^G and we will apply lemma 8 in order to compute the decomposition into indecomposable $G/\langle g^{p^r}\rangle$-modules. Let $c^*(\lambda, k_1)$ be the number of characters $\zeta \mapsto \zeta^\lambda$ in the module $V^{k_1+1}_{E_r}/V^{k_1}_{E_r}$. We compute that $c(\lambda, k)$ equals to:

$$\begin{aligned} c(\lambda, k_1 + p^\ell - p^r) &= c^*(\lambda, k_1) \\ &= \sum_{\mu \geq k_1+1} d^*(\lambda,\mu) = \begin{cases} d^*(\lambda, p^\ell), & \text{if } k_1 \geq 1 \text{ or } \lambda \neq 0 \\ d^*(0,p^\ell) + 1 = g_{F^G}, & \text{if } k_1 = 0, \text{ and } \lambda = 0 \end{cases} \end{aligned}$$

Therefore, for $k = p^\ell$ we compute:

$$d(\lambda, p^\ell) = c(\lambda, p^\ell - 1) = d^*(\lambda, p^\ell) = \frac{1}{p^r}\left(g_{E_r^T} - 1 + \sum_{i=1}^{t} \left\langle \frac{-\alpha_\lambda \Phi_i}{e'_i} \right\rangle \right)$$

by lemma 8. Moreover for $p^\ell - p^r \leq k \leq p^\ell - 1$ and from eq. (6) and the isomorphism given in eq. (16) we obtain:

$$\begin{aligned} d(\lambda, k) &= c(\lambda, k-1) - c(\lambda, k), \\ &= c^*(\lambda, k - 1 - (p^\ell - p^r)) - c^*(\lambda, k - (p^\ell - p^r)) \\ &= d^*(\lambda, k - (p^\ell - p^r)) = 0 \end{aligned}$$

unless $k = p^\ell - p^r + 1$ and $\lambda = 0$. In this case $d^*(0,1) = 1 = d(0, p^\ell - p^r + 1)$.

Remark 15. Notice that when $k \geq p^\ell - p^r$, then $\nu_{\mu,k} = 0$ (see also [19, p. 110]). Thus Boseck invariants (Definition 4) take now the form

$$\Gamma_{k,\lambda} = \Gamma_\lambda = \sum_{i=1}^{t} \left\langle \frac{-\alpha_\lambda \Phi_i}{e'_i} \right\rangle .$$

With this in mind, we take that

$$d(\lambda, p^\ell) = \frac{1}{p^r}\left(g_{E_r^T} - 1 + \Gamma_{k,\lambda} \right).$$

This completes the proof of theorem 7.

Example 16. Suppose that $F^G = F^P$, i.e. $n = 1$ then, from eq. (9) and (10) respectively, we get that the regular representation of G occurs

$$d(\lambda, p^\ell) = d(p^\ell) = \begin{cases} g_{F^G} - 1, & \text{if } r \neq 0 \\ g_{F^G}, & \text{otherwise} \end{cases}$$

times in the representation of G in V. This result coincide with the results obtained in [19].

3. Relation to the theory of Weierstrass semigroups

Aim of this section is to find a relation between the Galois module structure of the space of holomorphic differentials and the Weierstrass semigroup attached to a ramified point. In characteristic zero there are results [10] relating the structure of the Weierstrass semigroup at P to the subgroup $G(P)$. For example there is a theorem due to J. Lewittes [7, Th. 5] which relates the structure of the semigroup to the module structure of holomorphic differentials. Also I. Morrison and H. Pinkham [10] considered the case of Galois Weierstrass points, i.e. covers of the form $X \to \mathbb{P}^1$ with cyclic cover group in characteristic 0.

Let us start with a convenient description of a semigroup: Let $\Sigma \subset \mathbb{N}$ be a semigroup and let d be the least positive number in Σ. For $1 \leq i \leq d-1$ we denote by b_i the smallest element in Σ congruent to $i \bmod d$, and define ν_i by the equation:

(17) $$b_i = \nu_i d + i,$$

i.e. $\nu_i = \left\lfloor \frac{d}{b_i} \right\rfloor$. The numbers ν_i equals to the number of gaps a_k for which $a_k \equiv i \bmod d$, and the semigroup Σ is characterized by them.

From now on Σ will be the Weierstrass semigroup attached to a point P. Let f be a function on X such that $(f)_\infty = dP$. This gives rise to a map $f : X \mapsto \mathbb{P}^1$ and we assume that this map is a Galois cover with Galois group $G(P)$. In characteristic 0 the group $G(P)$ is always cyclic and the space of holomorphic differentials is described by a theorem due to Lewittes and Hurwitz [10, th. 1.3, th 3.5].

In this paper we assume that $G(P)$ is a cyclic group of order np^ℓ. The following theorem is a generalization of the theorems of Lewittes and Hurwitz written in the language of Brauer characters [13].

Proposition 17. *Let T be the tame cyclic part of $G(P)$. Let L be a complete local ring that contains the n-th roots of unity, let $\mathscr{O}_L$ be its valuation ring and let m_L be the maximal ideal of $\mathscr{O}_L$ such that $\mathscr{O}_L/m_L = K$. For example we can take $L = W(K)[\zeta_n]$. The modular character of $\mu : G_{\mathrm{reg}} \to \mathscr{O}_L$ induced by the $K[G]$-module of holomorphic differentials can be written as*

$$\mu = \sum_{i=1}^{d-1} \mu_i \chi^i,$$

where χ is a generator of the character group $\hat{G}_{\mathrm{reg}}$ of the cyclic group $G_{\mathrm{reg}} = \mathbb{Z}/n\mathbb{Z}$ and μ_i are equal to the number of gaps at P that are equivalent to $i \bmod n$.

Proof. The proof we will write is a modification of the characteristic zero proof given in [10, th. 1.3].

By construction for every $\sigma \in T$ we have $\sigma(T) = T$. Let τ be the generator of T. By the lemma of Hensel we might assume that there is a local uniformizer t at P such that $\sigma(t) = \zeta t$ where ζ is a primitive n-th root of unity.

Every gap a_k corresponds by Riemann-Roch theorem to a holomorphic differential ω_k with a root of order $a_k - 1$ at P. Observe that the flag of vector spaces $\langle \omega_g, \ldots, \omega_k \rangle$ are invariant under the action of G and by a trigonal change of coordinates we might assume that ω_k can be selected in such a way so that $\omega_k = t^{a_k - 1} dt$. For this selection of ω_k we have that $\tau\omega_k = \zeta^{a_k}\omega_k$ and the result follows.

□

This proposition does not describe completely the relation of the semigroup and the $K[G]$-module structure since it gives information of the number gaps modulo n and not modulo np^ℓ as required. Notice that by construction $d := np^\ell$ is the smallest non zero pole number.

Definition 18. For every i, $0 \leq i < np^\ell$ we consider the reductions of of i modulo p^ℓ and n respectively, namely: $i_0 = i \bmod n$ and $i_1 = i \bmod p^\ell$. We will denote by

$$\bar{c}(i_0, i_1) = \text{ the number of gaps at } P \text{ that are equivalent to } i \bmod np^\ell.$$

Of course these quantities are related to the μ_i defined in proposition 17. For an i_0 with $0 \leq i_0 < n$ we have

$$\mu_{i_0} = \sum_{i_1=0}^{p^\ell - 1} \bar{c}(i_0, i_1).$$

We will give an independent and complete description in terms of the decomposition given in eq. (1).

Proposition 19. *Let $r_{\mu,k}$ be the remainder of the division of $\delta_\mu + v_{P_{\mu,\nu}}(w_k)$ by p^ℓ, i.e.*

$$r_{\mu,k} = \delta_\mu + v_{P_{\mu,\nu}}(w_k) - p^\ell \nu_{\mu,k}. \tag{18}$$

The holomorphic differentials in V^{k+1} have roots at $P_{\mu,\nu}$ of orders

$$r_{\mu,k} + p^\ell \xi, \text{ with } \xi \in \{0, 1, \ldots, \sum_{\mu=1}^{s} \nu_{\mu,k} - 2\} \cup B_{\mu,k}, \tag{19}$$

where $B_{\mu,k}$ is a subset of natural numbers with g_{F^P} elements, all greater than $\sum_{\mu=1}^{s} \nu_{\mu,k}$. The dimension of the space of holomorphic differentials in V^{k+1} that have roots of order x such that:

$$\begin{aligned} x &\equiv r_{\mu,k} \mod p^\ell, \\ x &\equiv \alpha_\lambda \mod n \end{aligned} \tag{20}$$

is equal to $c(\lambda, k) = \bar{c}(a_\lambda + 1, r_{\mu,k} + 1)$.

Proof. By lemma 10 the differential $\sum_{\nu=0}^{k} c_\nu w_\nu dx$ is holomorphic if the elements c_k are in $\mathscr{L} := L(\mathrm{div}_{F^P}(dx) + \sum_{\mu=1}^{s} \nu_{\mu,k} \bar{P}_\mu)$. What are the possible valuations of such elements at a fixed $\bar{P}_{\mu_0}$? Fix μ_0 and consider the divisors:

$$A_j := \mathrm{div}_{F^P}(dx) + \sum_{\mu=1, \mu \neq \mu_0}^{s} \nu_{\mu,k} \bar{P}_\mu + j \bar{P}_{\mu_0},$$

for $j < \nu_{\mu_0,k}$. We have that $L(A_j) \subset L(A_{\nu_{\mu_0,k}})$. There is an element c_k with $v_{\bar{P}_{\mu_0}}(c_k) = -v_{\bar{P}_{\mu_0}}(\mathrm{div}_{F^P}(dx)) - j$ if and only if $\ell(A_j) - \ell(A_{j-1}) = 1$: Indeed, by using Riemann-Roch theorem we see that

$$\ell(A_j) = g_{F^P} - 1 + \sum_{\mu=1, \mu \neq \mu_0}^{s} \nu_{\mu,k} + j + \ell\left(- \sum_{\mu=1, \mu \neq \mu_0}^{s} \nu_{\mu,k} \bar{P}_\mu - j \bar{P}_{\mu_0} \right).$$

Therefore,

$$\text{if } \sum_{\mu=1, \mu \neq \mu_0}^{s} \nu_{\mu,k} + j - 1 \geq 0 \text{ then } \ell\left(- \sum_{\mu=1, \mu \neq \mu_0}^{s} \nu_{\mu,k} \bar{P}_\mu - j \bar{P}_{\mu_0} \right) = 0$$

and $\ell(A_j) - \ell(A_{j-1}) = 1$ and there is an element c_k with valuation at $\bar{P}_{\mu_0}$ equal to $-v_{\bar{P}_{\mu_0}}(\mathrm{div}_{F^P}(dx)) - j$. This proves that possible valuations $v = -v_{\bar{P}_{\mu_0}}(\mathrm{div}_{F^P}(dx)) - j$ of elements in $\mathscr{L}$ at $\bar{P}_{\mu_0}$ satisfy

$$-\nu_{\mu_0,k} - v_{\bar{P}_{\mu_0}}(\mathrm{div}_{F^P}(dx)) \leq v \leq \sum_{\mu=1, \mu \neq \mu_0}^{s} \nu_{\mu,k} - 1 - v_{\bar{P}_{\mu_0}}(\mathrm{div}_{F^P}(dx)),$$

i.e.

$$0 \leq \nu_{\mu_0,k} - j \leq \sum_{\mu=1}^{s} \nu_{\mu,k} - 1. \tag{21}$$

The valuation at P_{μ_0} of the differential $c_k w_k dx$ of F equals:

$$p^\ell v_{\bar{P}_{\mu_0}}(c_k dx) + \delta_{\mu_0} + v_{P_{\mu_0,\nu}}(w_k). \tag{22}$$

Recall that $\delta_{\mu_0} + v_{P_{\mu_0,\nu}}(w_k) = r_{\mu_0,k} + p^\ell \nu_{\mu_0,k}$ by eq. (18), so (22) becomes

$$p^\ell v_{\bar{P}_{\mu_0}}(c_k dx) + r_{\mu_0,k} + p^\ell \nu_{\mu_0,k}$$

which in turn by using (21) implies that the possible valuations of differentials in V^{k+1} contain the set

$$r_{\mu_0,k}+p^{\ell}\xi, 0\leq \xi < \sum_{\mu=1}^{s}\nu_{\mu,k}-1.$$

If $g_{F^P}=0$ then there are no other possible valuations for the elements c_k since the above valuations are different, the corresponding functions are linear independent and have the correct dimension given in lemma 10. If $g_{F^P}>0$ then there are g_{F^P} more possible valuations, but their exact values can not be easily described. Indeed, notice that always $\ell(A_j)-\ell(A_{j-1})\leq 1$ by [15, I.4.8].

Using lemma 12 and proposition 14 we compute that the dimension of the differentials satisfying the conditions given in (20), is equal to $c(k,\lambda)$.

It is a well known application of the Riemann-Roch theorem that the existence of a differential with root of order $a-1$ at P implies that a is a gap at P. Therefore if we add 1 to the natural numbers appearing in eq. (19) then we obtain all the gaps at P coming from holomorphic differentials in V^{k+1}. All of them are equivalent to $r_{\mu,k}+1$ modulo p^{ℓ}. Moreover, if a is a gap at P then there is a onedimensional subspace of V such that the action of the tame part is given by $\zeta\mapsto\zeta^a$ [10, th. 1.3]. This proves the equality $c(\lambda,k)=\bar{c}(a_\lambda+1,r_{\mu,k}+1)$. □

Remark 20. Notice, that now we are able to describe completely Σ at $P_{\mu,\nu}$ by the method introduced by Morrison and H. Pinkham [10] and explained in eq. (17), when $P_{\mu,\nu}$ ramifies completely. Indeed :

(1) The numbers $c(\lambda,k)=\bar{c}(a_\lambda+1,r_{\mu,k}+1)$ equal to the number of gaps $x+1$ for which $x+1\equiv i \bmod d$ and thus from the Chinese remainder theorem are equivalent to $\alpha_\lambda+1$ (or equivalently, see (2), to $\lambda+1$) and $r_{\mu,k}+1$ modulo n and p^{ℓ} respectively.

(2) $r_{\mu,k}$ forms a complete system modulo p^{ℓ} as k takes all the values $0,\ldots,p^{\ell}-1$, and thus takes all the values from 1 to $d-1$. Moreover, let $r=1$ to eq. (8) (we use this argument widely through this paper). Then, in the same way we see that α_λ forms a complete system modulo n, as λ runs through $0,\ldots,n-1$.

4. THE CASE OF A CYCLIC p-GROUP

We will now focus on the case of cyclic extensions of the rational function field of order p^{ℓ}. We will also assume that every ramified place is ramified completely. In this case we construct explicitly a basis of holomorphic differentials as follows:

We denote the ramified places of $K(x)$, by $\bar{Q}_i=(x-\alpha_i)$, $1\leq i\leq s$, since in a rational function field every ramified place corresponds to an irreducible polynomial, which is linear since the field K is algebraically closed. We set

$$g_k(x)=\prod_{i=1}^{s}(x-\alpha_i)^{\nu_{ik}}.$$

Definition 21. For $k=0,1,\ldots,p^{\ell}-1$, we define

$$\Gamma_k:=\sum_{i=1}^{s}\nu_{ik}.$$

Proposition 22. *Let X be a cyclic extension of degree p^{ℓ} of the rational function field. The set*

$$\left\{\omega_{k\nu}^{(\alpha_i)}=(x-\alpha_i)^{\nu^{(k)}}g_k(x)^{-1}w_k dx : 0\leq \nu^{(k)}\leq \Gamma_k-2, 0\leq k\leq p^n-2\right\}$$

forms a basis for the set of holomorphic differentials for a cyclic extension of the rational function field of order p^{ℓ}.

Proof. We take the basis of [6, Lemma 10], set $m = 1$ and modify it in order to evaluate holomorphic differentials in the ramified primes of the extension. The same construction is given by Garcia in [4, Theorem 2, Claim] where the elementary abelian, totally ramified case is studied. The proof is identical to the one given there. □

Keep in mind that the natural number i is a gap at P if and only if there is a holomorphic differential ω with root at P of order $i-1$.

Lemma 23. *The remainders $r_{\mu,i}$ for different values of i are different modulo p^ℓ and form a full set of representatives modulo p^ℓ.*

Proof. Observe first that the valuations of the functions w_k as k runs over $0,\ldots,p^\ell-2$ are all different, since

$$v_{P_{\mu,\nu}}(w_k) = -\sum_{j=1}^{\ell} a_j^k \Phi(\mu,j)p^{\ell-j}.$$

Therefore the values $\delta_\mu + v_{P_{\mu,\nu}}(w_k) = -\sum_{j=1}^{\ell} a_j^k \Phi(\mu,j)p^{\ell-j}$ take all possible values modulo p^ℓ. □

Definition 24. For every natural number $0 \leq a < p^\ell$ define by $\psi(a)$ the natural number such that

$$r_{\psi(a),\mu} = a.$$

Such a number exists by lemma 23.

Remark 25. Recall that $r_{\mu,k}$ was defined in eq. (18) to denote the remainder of the division of $\delta_\mu + v_{P_{\mu,\nu}}(w_k)$ by p^ℓ. Boseck in his seminal paper [2, Satz 18], where the $G = \mathbb{Z}/p\mathbb{Z}$ case is studied, states that as as k takes all the values $0 \leq k \leq p-2$ the remainder of the Boseck's basis construction $r_{\mu,k}$ takes all the values $0 \leq r_{\mu,k} \leq p-2$ and thus all the numbers $1,\ldots,p-1$ are gaps. This is not entirely correct as we will show in example 28. The problem appears if there is exactly one ramified place in the Galois extension.

Lemma 26. *If all $\Gamma_k \geq 2$ then all numbers $1,\ldots,p^\ell-1$ are gaps. If there exist Boseck invariants $\Gamma_k = 1$, then the set of gaps smaller than p^ℓ is exactly the set $\{r_{\mu,k} : 0 \leq k \leq p^\ell-2, \Gamma_k \geq 2\}$.*

Proof. As k runs in $0 \leq k \leq p^\ell-2$ the $r_{\mu,k}$ run in $0,\ldots,p^\ell-2$. But the Γ_k that are equal to 1 have to be excluded since they give not rise to a holomorphic differentials in proposition 22, see [6, Eq. (21)] and example 28. □

Remark 27. Notice that elements $\Gamma_k = 1$ can appear only for primes $p \geq \Phi(\mu,j)$ and only if there is only one ramified place.

Example 28. We consider the now the case of an Artin-Schreier extension of the function field $k(x)$, of the form $y^p - y = 1/x^m$. In this extension only the place $(x-0)$ is ramified with different exponent $\delta_1 = (m+1)(p-1)$. The Boseck invariants in this case are

$$\Gamma_k = \left\lfloor \frac{(m+1)(p-1)-km}{p} \right\rfloor \quad \text{for } k = 0,\ldots,p-2.$$

The Weierstrass semigroup is known [14] to be $m\mathbb{Z}_+ + p\mathbb{Z}_+$. Let us now find the small gaps by using lemma 26. If $p < m$ then all numbers $1,\ldots,p-1$ are gaps. If $p > m$ then m is a pole number smaller than p. Indeed, $\Gamma_{p-2} = 1$ and the remainder of the division of $(m+1)(p-1)-(p-2)m$ by p is $r_{p-2} = m-1$. But then $r_{p-2}+1 = m$ is not a gap.

REFERENCES

[1] Niels Borne, *Cohomology of G-sheaves in positive characteristic*, Adv. Math. **201** (2006), no. 2, 454–515. MR MR2211535

[2] Helmut Boseck, *Zur Theorie der Weierstrasspunkte*, Math. Nachr. **19** (1958), 29–63. MR MR0106221 (21 #4955)

[3] Arnaldo García, *On Weierstrass points on Artin-Schreier extensions of $k(x)$*, Math. Nachr. **144** (1989), 233–239. MR MR1037171 (91f:14021)

[4] ———, *On Weierstrass points on certain elementary abelian extensions of $k(x)$*, Comm. Algebra **17** (1989), no. 12, 3025–3032. MR MR1030607 (90m:14020)

[5] Ernst Kani, *The Galois-module structure of the space of holomorphic differentials of a curve*, J. Reine Angew. Math. **367** (1986), 187–206. MR 839131 (88f:14024)

[6] Sotiris Karanikolopoulos, *On holomorphic polydifferentials in positive characteristic*, arXiv:0905.1196v2 (2010).

[7] Joseph Lewittes, *Automorphisms of compact Riemann surfaces*, Amer. J. Math. **85** (1963), 734–752. MR MR0160893 (28 #4102)

[8] Pedro Ricardo López-Bautista and Gabriel Daniel Villa-Salvador, *On the Galois module structure of semisimple holomorphic differentials*, Israel J. Math. **116** (2000), 345–365. MR MR1759412 (2001f:12007)

[9] Daniel J. Madden, *Arithmetic in generalized Artin-Schreier extensions of $k(x)$*, J. Number Theory **10** (1978), no. 3, 303–323. MR MR506641 (80d:12009)

[10] Ian Morrison and Henry Pinkham, *Galois Weierstrass points and Hurwitz characters*, Ann. of Math. (2) **124** (1986), no. 3, 591–625. MR MR866710 (88a:14033)

[11] Martha Rzedowski-Calderón and Gabriel Villa-Salvador, *Function field extensions with null Hasse-Witt map*, Int. Math. J. **2** (2002), no. 4, 361–371. MR 1891121 (2003d:11172)

[12] Martha Rzedowski-Calderón, Gabriel Villa-Salvador, and Manohar L. Madan, *Galois module structure of holomorphic differentials in characteristic p*, Arch. Math. (Basel) **66** (1996), no. 2, 150–156. MR MR1367157 (97e:11142)

[13] Jean-Pierre Serre, *Linear representations of finite groups*, Springer-Verlag, New York, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42. MR MR0450380 (56 #8675)

[14] Henning Stichtenoth, *Über die Automorphismengruppe eines algebraischen Funktionenkörpers von Primzahlcharakteristik. II. Ein spezieller Typ von Funktionenkörpern*, Arch. Math. (Basel) **24** (1973), 615–631. MR 53 #8068

[15] ———, *Algebraic function fields and codes*, Springer-Verlag, Berlin, 1993. MR 94k:14016

[16] Tuneo Tamagawa, *On unramified extensions of algebraic function fields*, Proc. Japan Acad. **27** (1951), 548–551. MR MR0047705 (13,918a)

[17] Robert C. Valentini, *Representations of automorphisms on differentials of function fields of characteristic p*, J. Reine Angew. Math. **335** (1982), 164–179. MR MR667465 (84j:12013)

[18] Robert C. Valentini and Manohar L. Madan, *A hauptsatz of L. E. Dickson and Artin-Schreier extensions*, J. Reine Angew. Math. **318** (1980), 156–177. MR 82e:12030

[19] ———, *Automorphisms and holomorphic differentials in characteristic p*, J. Number Theory **13** (1981), no. 1, 106–115. MR MR602451 (83d:14011)

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