Deformation of Curves with Automorphisms

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Deformation of Curves with Automorphisms

Let X be a projective nonsingular curve defined over an algebraically closed field of characteristic $p \geq 0$. Let $G \subset \mathrm{Aut}(X)$ be a fixed subgroup of the automorphism group of the curve X. We will denote by (X, G) the couple of the curve X together with the group G.

A deformation of the couple (X, G) over the local Artin ring A is a proper, smooth family of curves

$$\mathcal{X} \to \mathrm{Spec}(A)$$

parametrized by the base scheme $\mathrm{Spec}(A)$,together with a group homomorphism $G \to \mathrm{Aut}_A(\mathcal{X})$ such that there is a G-equivariant isomorphism ϕ from the fibre over the closed point of A to the original curve X:

$$\phi : \mathcal{X} \otimes_{\mathrm{Spec}(A)} \mathrm{Spec}(k) \to X.$$

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Deformation of Curves with Automorphisms

Two deformations $\mathcal{X}_1, \mathcal{X}_2$ are considered to be equivalent if there is a G-equivariant isomorphism ψ, making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\psi} & \mathcal{X}_2 \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec} A & \end{array}$$

The deformation functor of curves with automorphisms is defined:

$$D_{\mathrm{gl}} : \mathcal{C} \to \mathrm{Sets}, \mathrm{A} \mapsto \left\{ \begin{array}{l} \text{Equivalence classes} \\ \text{of deformations of} \\ \text{couples } (X, G) \text{ over } A \end{array} \right\}$$

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An Example

The Fermat curve defines a curve over $\mathbb{Z}$.

$$C : X^n + Y^n + Z^n = 0$$

This can be seen as a family of Curves over elements of $\mathrm{Spec}\mathbb{Z}$.
Indeed for every prime p the fibre over p is the curve C_p defined over $\mathbb{F}_p$.

$$\mathrm{Aut}\, C_p = S_3 \rtimes (\mu(n) \times \mu(n)) \text{ if } n-1 \text{ not a power of } p$$

$$\mathrm{Aut}\, C_p = PGU(3, \mathbb{F}_{p^{2h}}) \text{ if } n-1 = p^h.$$

The case C_p for $n-1 = p^h$ is very special. It has maximal number of $\mathbb{F}_p$ with respect to the Hasse bound and there is unique.

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Artin-Schreier Curves

Consider the curves:

$$C : y^p - y = x^\ell$$

These curves define covers $C \to \mathbb{P}^1_k$ ramified only above ∞. These curves can not exist for $k = \mathbb{C}$, $\mathbb{P}^1 - \infty$ is simply connected.

$$y^p - y = \sum_{i=1,\ldots,\ell-1,(i,p)=1} a_i x^i + x^\ell$$

is a family of non-isomorphic curves over $k[a_i]_{i=1,\ldots,\ell-1,(i,p)=1}$ all of them have $\mathbb{Z}/p\mathbb{Z}$-action and is a deformation of $(C, \mathbb{Z})$.

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- Look at the tangent space of the deformation functor $D_{gl}(k[\epsilon])$, $\epsilon^2 = 0$.
- $D_{gl}(k[\epsilon]) = H^1(X, G, \mathcal{T}_X)$, where $H^1(X, G, \mathcal{T}_X)$ is Grothendieck's equivariant cohomology.
- How can we compute $H^1(X, G, \mathcal{T}_X)$?

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Equivariant Čech Cohomology

Let $\{U_i\}_{i\in I}$ be an open affine covering of the curve X consisting of G-stable open sets U_i.

Let ζ_i^σ be a family of G-derivations *i.e.*, elements in $\Gamma(U_i, \mathcal{T}_X)$ and let δ_{ij} be Čech-cocycles, in $\Gamma(U_i \cap U_j, \mathcal{T}_X)$.

Then the equivariant cohomology is given by

$$H^1(X, G, \mathcal{T}_X) = \frac{\{\{\zeta_i^\sigma\}, \{\delta_{ij}\}\}}{\{\{\sigma\gamma_i - \gamma_i\}, \{\gamma_j - \gamma_i\}\}},$$

where $\sigma\gamma_i - \gamma_i$ is a family of principal G-derivations and $\gamma_j - \gamma_i$ is a family of 1-Čech coboundaries, and moreover

$$\zeta_j^\sigma - \zeta_i^\sigma = \sigma(\delta_{ij}) - \delta_{ij}.$$

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Spectral Sequences

First Approach:

Artin-Schreier curves are manageable. By the local-global principle the computation is reduced to a computation of deformations at wild ramified points. At a wild ramified points we have the following ramification filtration:

$$G_0(P) \subset G_1(P) \subset G_2(P) \subset \cdots \subset G_n(P),$$

so that

$$G_0(P)/G_1(P) \text{ is cyclic prime to } p$$

and

$$G_i(P)/G_{i+1}(P) \text{ is elementary abelian.}$$

Use the Lyndon-Hochshield-Serre spectral sequence at the wild ramified points. The above technique involves the difficult computation of the kernel of the transgression map.

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Galois Module stucture of holomorphic differentials

Second Approach: Restrict to p-groups. Using the computation of $H^1(X, G, \mathcal{T}_X)$ is terms of Čech cohomology we see that:

$$H^1(X, G, \mathcal{T}_X) = H^1(X, \mathcal{T}_X)^G \subset H^1(X, \mathcal{T}_X).$$

Use Serre duality:

$$H^1(X, \mathcal{T}_X) = H^0(X, \Omega_X^{\otimes 2})^*$$

The computation of invariants reduces to a computation of covariants.

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The space $H^0(X, \Omega^{\otimes 2})$ is a $k[G]$-module. Describe if possible $H^0(X, \Omega^{\otimes 2})$ as a sum of simple $k[G]$-modules.

If the characteristic $p = 0$ then the solution of this problem is known (Hurwitz 1900).

If the characteristic $p > 0$ the representations involved are modular and the problem is unsolved.

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The cyclic p-case

In the case $G = \mathbb{Z}/p\mathbb{Z}$, the problem is solved by Shoichi Nakajima.

Denote by V the $k[G]$-module with k-basis $\{e_1, \ldots, e_p\}$ and action given by $\sigma e_\ell = e_\ell + e_{\ell-1}$, $e_0 = 0$.

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Using the theory of Jordan normal form of matrices we can show that every $k[G]$-module is isomorphic to a direct sum of V_j.

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We observe that $\dim_k (V_i)_G = 1$, therefore

$$\dim_k H^0(X, \Omega_X^{\otimes 2})_G = \sum_{j=1}^{p} m_j.$$

Describe the m_j:

$$m_p := 3g_Y - 3 + \sum_{i=1}^{p} \left[\frac{n_i - (p-1)N_i}{p} \right],$$

and for $j = 1, \ldots, p-1$,

$$m_j = \sum_{i=1}^{r} \left\{ - \left[\frac{n_i - jN_i}{p} \right] + \left[\frac{n_i - (j-1)N_i}{p} \right] \right\}.$$

$$n_i := v_{P_i}(\mathrm{div}(f^*\omega)) = v_{P_i}(2R) = 2(N_i + 1)(p-1).$$

The cyclic p-case

We observe that $\dim_k(V_i)_G = 1$, therefore

$$\dim_k H^0(X, \Omega_X^{\otimes 2})_G = \sum_{j=1}^{p} m_j.$$

Describe the m_j:

$$m_p := 3g_Y - 3 + \sum_{i=1}^{p} \left[\frac{n_i - (p-1)N_i}{p} \right],$$

and for $j = 1, \ldots, p-1$,

$$m_j = \sum_{i=1}^{r} \left\{ - \left[\frac{n_i - jN_i}{p} \right] + \left[\frac{n_i - (j-1)N_i}{p} \right] \right\}.$$

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In what cases can this method applyied to?

- Weekly ramified covers, *i.e.* covers where at all wild ramified points P $G_2(P)=\{1\}$. B. Köck. Ordinary curves are weekly ramified. Mumford curves are ordinary.
- Cyclic $\mathbb{Z}/p^n\mathbb{Z}$-covers. N. Borne.
- N. Stadler Theory