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Automorphisms of Fermat-like varieties

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Abstract. We study the automorphisms of some nice hypersurfaces and complete intersections in projective space by reducing the problem to the determination of the linear automorphisms of the ambient space that leave the algebraic set invariant.

1. Introduction

H. W. Leopoldt [12] (in characteristic p) and P. Tzermias [16] (in characteristic zero) studied the automorphism group of the Fermat curves, given as the zero locus of the homogeneous polynomial $x_0^n + x_1^n + x_2^n = 0$, where $n > 3$ and the characteristic p does not divide n. The author [11] generalized the above result by studying the group of automorphisms of a projective non-singular model of the affine curves $1 + x_1^m + x_2^n = 0$ for $n \neq m$.

The aim of this paper is to study the group of automorphisms of similar algebraic sets in higher dimensions. By the group of automorphisms $\mathrm{Aut}(X)$ of the projective variety $X \subset \mathbb{P}^r$ we mean the group of biregular transformations of X. All varieties are defined over an algebraically closed field k of characteristic $p \geq 0$.

For a complete intersection X in $\mathbb{P}^r$ of dimension ≥ 3, the study of the automorphism group is reduced to the study of the linear automorphisms $\mathrm{Lin}(X)$, *i.e.*, to automorphisms of the ambient space $\mathbb{P}^r$ that leave X invariant. This is a known theorem based on the generalization of the Lefschetz theorem, due to A. Grothendieck and P. Deligne.

Using this method, T. Shioda [15] was able to compute the group $\mathrm{Aut}(X)$ of automorphisms for the Fermat hypersurfaces (also for the Fermat curves), given by equations

$$x_0^n + x_1^n + \cdots + x_r^n = 0,$$

where the characteristic p does not divide n, and $n \geq 3$. It is clear that the group of automorphisms of the above equations contains the semidirect product $\mathbb{Z}_n^r \rtimes \mathrm{S}_{r+1}$ of the abelian group $\mathbb{Z}_n^r$ and the symmetric group S_{r+1}. This is the whole automorphism group if $n-1$ is not a power of the characteristic. If $n - 1 = p^h$ is

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a power of the characteristic then the automorphism group is the projective unitary group $\mathrm{PGU}(r+1, p^{2h})$.

In this paper we simplify the computations of Shioda and we generalize his results by studying the automorphism group of a hypersurface defined as the zero locus of a hermitian form

$$\sum_{\kappa,\lambda} x_\kappa a_{\kappa\lambda} x_\lambda^q = 0,$$

where q is a power of the characteristic and (a_{ij}) is an $(r+1)\times(r+1)$ matrix with elements in k. We also study the case of different exponents, *i.e.*, automorphisms of the projective closure of affine hypersurfaces of the form

$$\sum_{i=1}^{r} x_i^{n_i} + 1 = 0.$$

Namely we prove the following:

Theorem 1.1. *Let $n = n_{t_0} = \dots = n_{t_1-1} > n_{t_1} \geq \dots \geq n_r > 1$ be a decreasing sequence of integers, where $1 = t_0 < t_1 < \cdots < t_s$ such that n_i is constant for $t_k \leq i < t_{k+1}$, $r > t_1 - 1$ and the characteristic p does not divide n_i for all i. (Notice that in general $r \geq t_1 - 1$ and if $r = t_1 - 1$ then the hypersurface is Fermat). Let X be the projective hypersurface defined by the homogeneous irreducible polynomial*

$$\sum_{i=1}^{r} x_i^{n_i} x_0^{n-n_i} + x_0^n.$$

We have assumed that $n_r > 1$ because otherwise the defining polynomial will have as summand a linear polynomial, forcing the automorphism group to be infinite. Denote by $d_k = t_{k+1} - t_k$. The group of automorphisms G of X is given by a direct sum

$$G := \bigoplus_{k=0}^{s-1} G_k,$$

where

$$G_k \cong \begin{cases} 1 & \text{if } d_k = 1 \\ \mathbb{Z}_{n_k} & \text{if } d_k = 2 \\ \mathbb{Z}_{n_k} \rtimes \mathrm{S}_{d_k} & \text{if } d_k > 2 \text{ and } n_k - 1 \text{ is not a } p-\text{power} \\ \mathrm{PGU}(d_k, p^{2h_k}) & \text{if } d_k > 2 \text{ and } n_k - 1 = p^{h_k} \end{cases}$$

We also study automorphisms of intersections of Fermat hypersurfaces and prove the following theorems:

Theorem 1.2. *Let n, m be integers not divisible by the characteristic p of the algebraic closed field k and let $r > 4$ or $n + m \neq 5$ for $r = 4$. The variety X defined by the homogeneous ideal $\langle \sum x_i^m, \sum x_i^n \rangle$ is reduced and irreducible. The group*

$\mathrm{Aut}(X)$ *of automorphisms of* X *is the intersection of the group of automorphisms of the hypersurfaces* $\sum x_i^m = 0$ *and* $\sum x_i^n = 0$, i.e.,

$$\mathrm{Aut}(X) = \mathrm{Aut}(V(\sum x_i^m)) \cap \mathrm{Aut}(V(\sum x_i^n)).$$

Theorem 1.3. *Let* X *be the algebraic set given as intersection of the weighted Fermat hypersurfaces*

$$X_1 = V(x_0^{q+1} + ... + x_r^{q+1}), X_2 = V(c_0 x_0^{q+1} + ... + c_r x_r^{q+1}),$$

where q *is a power of the characteristic and* $c_i \neq c_j$ *for* $i \neq j$. X *is reduced and irreducible. Let* G *be the automorphism group of* X *and* $A := \mathbb{Z}_{q+1}^r$ *be the subgroup of* G *given by* $\{x_i \mapsto \zeta^{a_i} x_i\}$ *where* ζ *is an* $q+1$ *root of unity and* a_i *runs over the set* $\{0, ..., q\}$. *The group* A *is a normal subgroup of* G, *and* G *is given as an extension of groups*

$$1 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 1.$$

The group H *acts on the set* $\{c_0, ..., c_r\}$ *as a group of linear fractional transformations and is one of the following groups* $\mathbb{Z}_s, \mathbb{Z}_p^t, \mathbb{Z}_{p_1}^{t_1} \rtimes \mathbb{Z}_n$ *where* $s \mid r+1$ *or* $s \mid r$, $pt = r$, $p_1 t_1 n \mid r+1$ *or* $p_1 t_1 n \mid r$. *Moreover* G *is a subgroup of* $\mathrm{Aut}(X_1) = \mathrm{PGU}(r+1, q^2)$.

2. Automorphisms of complete intersections

In this section we reduce the study of the automorphism group of a complete intersection $X \subset \mathbb{P}^r$ to the study of linear automorphisms of X, *i.e.*, automorphisms of the ambient space $\mathbb{P}^r$ that leave X invariant. This theorem is proved in the literature for hypersurfaces, using the Grothendieck-Deligne version of Lefschetz theorem [2, par. 16]. For the sake of completeness we present a proof for the case of complete intersections, which essentially follows the hypersurface proof.

Proposition 2.1. *Let* $i : X \to \mathbb{P}^r$ *be a closed subvariety of the projective space* $\mathbb{P}^r$, *such that the map* $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \xrightarrow{i^*} H^0(X, \mathcal{O}_X(1))$ *is an isomorphism. If* ϕ *is an automorphism of* X *preserving* $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r}(1)$, *then* ϕ *can be extended to an automorphism of* $\mathbb{P}^r$.

Proof. The homogeneous coordinate ring S of $\mathbb{P}^r$ is given as a direct sum [8, ex. 5.14a p.126]

$$S = \bigoplus_{n \geq 0} H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)),$$

and S is generated by $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ as a k-algebra (The k-vector space

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))$$

can be identified with the vector space of homogeneous polynomials of degree n). Every k linear automorphism of $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ can be extended to an automorphism of S. Every automorphism ϕ of X that preserves $\mathcal{O}_X(1)$ induces a linear map acting on $H^0(X, \mathcal{O}_X(1))$; since the map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \xrightarrow{i^*} H^0(X, \mathcal{O}_X(1)),$$

is an isomorphism, ϕ acts on $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ as well.

A complete intersection X in $\mathbb{P}^r$ is a projective algebraic variety, *i.e.*, reduced and irreducible, whose homogeneous ideal I is generated by $c = \operatorname{codim}(X, \mathbb{P}^r)$ polynomials.

Theorem 2.1. *Let X be a complete intersection in $\mathbb{P}^r$, of dimension ≥ 2. Then the group $\operatorname{Pic}(X)$ is torsion free. If moreover $\dim(X) \geq 3$, then the group $\operatorname{Pic}(X)$ is the free group generated by the class of $\mathcal{O}_X(1)$.*

Proof. [1, Th. II.1.8] and [7, Exp. XII Corollary 3.7].

Theorem 2.1 can also be expressed in the following form:

Corollary 2.1. *Let X be a complete intersection in $\mathbb{P}^r$ of dimension ≥ 3. Every Cartier divisor on X comes from an intersection of X by a hypersurface of $\mathbb{P}^r$. In particular, the semigroup of effective divisors on X is generated by the linear systems of intersections by hyperplanes.*

Proof. Let $i : X \to \mathbb{P}^r$ denote the inclusion map. By definition $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r}(1)$. Since $\operatorname{Pic}(X)$ is generated by $\mathcal{O}_X(1)$ the result follows.

Corollary 2.2. *Let X be a complete intersection of dimension $X \geq 3$, or a non-singular complete intersection of dimension 2, such that ω_X is not the identity in $\operatorname{Pic}(X)$. Then for every automorphism ϕ we have that $\phi^*(\mathcal{O}_X(1)) = \mathcal{O}_X(1)$. In particular, if the surface X is non-singular, given by the intersection of two Fermat hypersurfaces $x_0^n + ... + x_4^n = 0$ and $x_0^m + ... + x_4^m = 0$ such that $n + m - 5 \neq 0$, then $\phi^*(\mathcal{O}_X(1)) = \mathcal{O}_X(1)$ for every automorphism of X.*

Proof. If $\dim X \geq 3$ then by corollary 2.1, we have that the linear system L_1 of intersections with hyperplanes of $\mathbb{P}^r$ is complete and is the unique base of the additive semigroup of equivalent positive divisors. Therefore L_1 is invariant under the action of automorphisms. (See also [13].) In the second case, since X is a nonsingular intersection in $\mathbb{P}^4$, one can compute [8, Exer. 8.4 chap. II], that the canonical invertible sheaf $\omega_X := \wedge^{\dim X}\Omega_{X/k} = \mathcal{O}_X(n + m - 5)$. The desired result follows since ϕ^* preserves ω_X and $\operatorname{Pic}(X)$ is torsion free.

We have thus proved the following:

Proposition 2.2. *If the variety X given by the intersection of two Fermat hypersurfaces $x_0^n + ... + x_r^n = 0$ and $x_0^m + ... + x_r^m = 0$ is a complete intersection and $\dim X \geq 3$ then $\operatorname{Aut}(X) = \operatorname{Lin}(X)$. The same result holds for $\dim(X) = 2$,* i.e., *when $r = 4$, under the additional hypotheses that X is non-singular and that $n - m - 5 \neq 0$.*

3. Linear automorphisms

Let $X \subseteq \mathbb{P}^r$ be a projective algebraic variety which is the zero locus of a homogeneous ideal I, such that there is a homogeneous basis $\{f_a\}$ of the ideal I consisting of polynomials f_a of degree prime to the characteristic p. In this section we will study the linear automorphisms of X, *i.e.*, automorphisms of the form

$$\sigma(x_i) = \sum_j a_{ij} x_j, \quad (a_{ij}) \in \mathrm{PGL}(r+1, k)$$

such that

$$\forall f \in I, \quad f(\sigma(x_0), ..., \sigma(x_r)) \in I.$$

Computing the automorphism group this way is in general quite difficult, but in several cases can be carried out in a straightforward manner. For example T. Shioda [15] uses this method to study the automorphism group of Fermat hypersurfaces given as the zero locus of polynomials of the form $\sum_i x_i^n$.

In order to simplify the presentation we will assume that the ideal I is generated by $c = 1, 2$ homogeneous polynomials. We have

$$\sigma(f_\nu) = \sum_\mu g_{\nu\mu}(\sigma) f_\mu$$

where $g_{\nu\mu}(\sigma)$ are homogeneous polynomials of degree $n - n_k$ and n_k is the degree of f_k.

3.1. The case of hypersurfaces $(c = 1)$

In this case the ideal I is generated by a single homogeneous polynomial f of degree n, prime to the characteristic p. For a linear automorphism σ, represented by a matrix A, we have

$$\sigma(f) = \chi(\sigma) f, \tag{3.1}$$

where χ is a character of the automorphism group.

Assume for a moment that X is a nonsingular hypersurface. We consider the dual variety X^* of the hypersurface. For nonsingular hypersurfaces, that are not hyperplanes, it is known that X^* is also a hypersurface [5, 7.2, p.58 Zak Theorem][9], and every automorphism σ of X induces an automorphism σ^* of X^*. Since X^* is a hypersurface of the same dimension as X, σ^* is also linear. We will give a direct proof for the following proposition, not based on projective duality, and by not assuming that X is non singular.

Proposition 3.1. *Let X be an irreducible projective hypersurface of dimension ≥ 3, which might be singular. Let $f(x_0, ..., x_r)$ be the defining polynomial of X. Every automorphism of X induces a linear automorphism on $Y_i := \partial f/\partial x_i$,* i.e. *$\partial f/\partial x_i(\sigma(P)) = \sum_\nu \lambda_\nu \partial f/\partial x_i(P)$.*

Proof. Denote by ∇f the vector $(\partial f/\partial x_0, ..., \partial f/\partial x_r)$. Let $A = (a_{ij})$ be a matrix representing the automorphism σ of Y. By differentiation of (3.1) we obtain

$$\nabla f|_{\sigma(P)} \cdot A = \nabla(fA)|_P = \chi(\sigma)\nabla f|_P,$$

for every $P = (x_0 : ...x_r)$. So

$$\nabla f|_{\sigma(P)} = \nabla(fA)|_P = \chi(\sigma)\nabla f|_P \cdot A^{-1}$$

and the action of σ on the partial derivatives is linear as well.

In order to compute the automorphism group of some interesting examples we will need the following

Lemma 3.1. *The binomial coefficient $\binom{n}{k}$ is not divisible by the characteristic p, if and only if $k_i \leq n_i$ for all i, where $n = \sum n_i p^i$, $k = \sum k_i p^i$ are the p-adic expansions of n and m.*

Proof. [4, p. 352]

Example 1. In this example we simplify Shioda's [15] calculation of the group of automorphisms of the Fermat hypersurfaces $f := \sum_{i=0}^{r} x_i^n$, defined over an algebraically closed field k, of characteristic $p \geq 0$, $p \nmid n$. The partial derivatives of f are $Y_i := nx_i^{n-1}$. If $r \geq 4$, then every automorphism of the Fermat hypersurfaces is linear. Let $A = (a_{ij}) \in \mathrm{PGL}(r+1, k)$ be such an automorphism.

For a point $P = (x_0 : ... : x_r)$ we compute the coefficients of $Y(A(P))$:

$$Y_i(A(P)) := n(\sum_j a_{ij}x_j)^{n-1} = n\sum_j a_{ij}^{n-1}Y_j(x_0, ..., x_r)+$$

$$+n \sum_{\substack{\nu_0+...+\nu_r = n-1 \\ \nu_i < n-1}} \binom{n-1}{\nu_0, ..., \nu_r} a_{i0}^{\nu_0} \cdots a_{ir}^{\nu_r} x_0^{\nu_0} \cdots x_r^{\nu_r}. \tag{3.2}$$

By proposition 3.1 we have that modulo the defining polynomial f, which is of degree n, the right hand side of (3.2) is linear in Y_i, and since no polynomial of degree $n-1$ can be equal to a polynomial of degree $\geq n$, we finally arrive at

$$(\sum_j a_{ij}x_j)^{n-1} = \sum_j a_{ij}^{n-1}x_j^{n-1}, \tag{3.3}$$

so if there are more than two $a_{ij} \neq 0$ in some column, then by lemma (3.1) $n-1$ is a power of the characteristic. If A is the matrix (a_{ij}), then by $A^{(q)}$ we denote the matrix (a_{ij}^q). Let $q := n-1 = p^h$ then equation (3.3) implies $A^{(q)}A^t = \mathrm{Id}$, hence the automorphism group is $\mathrm{PGU}(r+1, p^{2h})$.

If, on the other hand, there is only one non-zero element in every column of (a_{ij}), then after a permutation (a_{ij}) is diagonal, so the automorphism group is isomorphic to $\mathbb{Z}_n^r \rtimes \mathrm{S}_{r+1}$.

Example 2. Let $A = (a_{ij})$ be an $m \times m$ matrix with elements in the algebraically closed field k. We consider the hypersurface defined by the equation

$$A_q[X] := \sum_{\kappa,\lambda} x_\kappa a_{\kappa\lambda} x_\lambda^q = 0, \tag{3.4}$$

where q is a power of the characteristic. By the theory of Jordan forms, the matrix A can be decomposed after a linear change of coordinates in block diagonal matrices of the form:

$$A \sim \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_s \end{pmatrix},$$

where A_i are square matrices of the form $A_i = \mathrm{diag}(\lambda_1, ..., \lambda_t)$ or

$$A_i = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

This proves that after a linear change of coordinates, the hypersurface defined in equation (3.4) can be decomposed as a product of hypersurfaces defined by Fermat polynomials (A_i diagonal), and polynomials of the form

$$\sum_{\kappa=0}^{r-1} x_k(\lambda x_\kappa^q + x_{\kappa+1}^q) + \lambda x_r^{q+1} = 0. \tag{3.5}$$

Hence, $\mathrm{Aut}(V(A_q[X])) = \oplus_i \mathrm{Aut}(V(A_{i,q}[X]))$. The automorphism group of Fermat hypersurfaces is studied in the previous example. We are left with the study of non diagonal case. Let J be the non diagonal $r \times r$ matrix $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$, and let f be the polynomial defined by equation (3.5). The partial derivatives are given by

$$\frac{\partial f}{\partial x_i} = \begin{cases} \lambda x_i^q + x_{i+1}^q & \text{if i} < r \\ \lambda x_i^q & \text{if } i = r \end{cases}.$$

We observe that after every linear change of coordinates given by an $r \times r$ matrix $B = (b_{ij})$, the partial derivatives change linearly. Moreover, B is an automorphism of $V(f)$ if and only if :

$$(x_0, ..., x_r)B^t(B(\frac{\partial f}{\partial x_0}), ..., B(\frac{\partial f}{\partial x_r})) = (x_0, ..., x_r)B^t J B^{(q)}(x_0^q, ..., x_r^q)^t = J, \tag{3.6}$$

where $B^{(q)}$ is the matrix (b_{ij}^q). Let B^* be the matrix defined by $B^*J = JB^{(q)}$, *i.e.* $B^* = JB^{(q)}J^{-1}$. By (3.6) the matrix B satisfies

$$B^t B^* = B^t J B^{(q)} J^{-1} = I \bmod Z,$$

where Z denotes the centre of $\mathrm{GL}(r+1,k)$. Therefore $\mathrm{Aut}(V(f))$ is the subgroup of $\mathrm{PGL}(r+1,k)$ consisting of matrices of the form

$$B = \left(J\left(B^{(q)}\right)^{-1} J^{-1}\right)^t.$$

Example 3. We consider now the hypersurface X given as the zero locus of the irreducible polynomial $f := \sum_{i=1}^r x_i^{n_i} x_0^{n-n_i} + x_0^n$, defined over an algebraically closed field of characteristic $p \geq 0$. We assume that n_i form a decreasing sequence, and we consider the set of indices $\{1 = t_0 < t_1 =: t < \cdots < t_s\}$, such that n_i is constant for $t_k \leq i < t_{k+1}$, *i.e.*

$$n = n_{t_0} = \ldots = n_{t_1 - 1} > n_{t_1} \geq \ldots \geq n_{t_s} > 1.$$

We want f to be irreducible, so at least one n_i is equal to n. Moreover, we assume that $p \nmid n_i$ for all i. For technical reasons we also assume that $r > t$. In order to use proposition 3.1, we also assume $r \geq 4$.

The partial derivatives $Y_i = \partial_i f$ of f are:

$$Y_0 = \sum_{i=t_1}^{r} (n - n_i) x_i^{n_i} x_0^{n-n_i-1} + n x_0^{n-1}$$

$$Y_i = \begin{cases} n_i x_i^{n_i - 1} & \text{if } \ 0 < i < t_1 \\ n_i x_i^{n_i-1} x_0^{n-n_i} & \text{if } \quad t_1 \leq i \end{cases}.$$

The action of the automorphism $\sigma = (a_{ij})$ is, by proposition, 3.1 linear on the partial derivatives Y_i. For $0 < i < t$ we compute:

$$\sigma(Y_i) = n_i \left(\sum_{j=0}^{r} a_{ij} x_j\right)^{n_i - 1} =$$

$$n_i \sum_{j=1}^{r} a_{ij}^{n_i-1} x_j^{n_i-1} + n_i \sum_{\substack{\nu_0 + \ldots + \nu_r = n_i - 1 \\ 0 \leq \nu_i < n_i - 1}} \binom{n_i - 1}{\nu_0, \ldots., \nu_r} a_{i0}^{\nu_0} \cdots a_{ir}^{\nu_r} x_0^{\nu_0} \cdots x_r^{\nu_r}$$

$$= \sum \lambda_i Y_i \bmod f.$$

As in the previous example, every summand of the above sum that does not fit in a linear combination of Y_i is zero, since no no-zero polynomial of degree $n-1$

belongs to the principal ideal generated by f. By comparing coefficients we have $a_{ij} = 0$ for $j \geq t$ and $0 < i < t$.

For $t \leq i$ we compute:

$$\sigma(Y_i) = n_i \left(\sum_{j=0}^{r} a_{ij} x_j \right)^{n_i - 1} \left(\sum_{k=0}^{r} a_{0k} x_k \right)^{n - n_i}. \tag{3.7}$$

The term $n_i a_{ij}^{n_i-1} a_{0k}^{n-n_i} x_j^{n_i-1} x_k^{n-n_i}$ that appears in the above equation can not be canceled out by linear combinations of Y_i, for $k \neq j, k \neq 0, j \neq 0$. This proves that $a_{ij} a_{0k} = 0$ for $k \neq j, k \neq 0, j \neq 0$. If there is a $k \neq 0$ such that $a_{0k} \neq 0$, then $a_{ij} = 0$ for $t \leq i$ and $j \neq 0, k$ for $i \neq 0$. Therefore for all $i \geq t$ we have

$$\sigma(x_i) = a_{i0} x_0 + a_{ik} x_k. \tag{3.8}$$

Assume that $a_{ik} = 0$ for all $i \geq t_1$. Since $r - t_1 > 0$ there are at least two x_{i_1}, x_{i_2} with $i_1, i_2 \geq t_1$ such that

$$\sigma(x_{i_1}) = a \cdot \sigma(x_{i_2}),$$

where $a \in k$. Taking the inverse of σ of both hand sides we arrive at $x_{i_1} = a \cdot x_{i_2}$, a contradiction. So there is an $i \geq t_1$ such that $a_{ik} \neq 0$. The term $a_{00}^{n-n_i} a_{ik}^{n_i-1} x_k^{n_i-1} x_0^{n-n_i}$ appears in (3.7), therefore $a_{00} = 0$ or $k \geq t$. In the first case, *i.e.* when $a_{00} = 0$, $\sigma(Y_i)$ in (3.7) could not be linear sum of Y_i, unless $r = t - 1$, *i.e.* we are studying the case of a Fermat hypersurface. Hence, $k \geq t$. On the other hand, $n_i a_{ik}^{n_i-1} a_{0k}^{n-n_i} x_k^{n-1}$ is also a term in $\sigma(Y_i)$, and since $k \geq t_1$, it cannot be canceled out by a linear sum of Y_i. This proves that the original assumption $a_{0k} \neq 0$, for some $k \neq 0$ is false.

We have proved so far that

$$\sigma(x_0) = a_{00} x_0.$$

Equation (3.7) is now transformed to

$$\sigma(Y_i) = n_i \left(\sum_{j=0}^{r} a_{ij} x_j \right)^{n_i - 1} a_{00}^{n-n_i} x_0^{n-n_i},$$

and by comparing coefficients, we arrive at $a_{ij} = 0$, for $i \geq t_1, j < t_1$. Let us write the polynomial f as a sum of two polynomials $f = f_1 + f_2$, where $f_1 := \sum_{i=1}^{t_1-1} x_i^n$ and $f_2 := x_0^{n-n_{t_1}} \left(\sum_{i=t_1}^{t_2-1} x_i^{n_{t_1}} + \sum_{i=t_2}^{r} x_i^{n_i} x_0^{n_{t_1}-n_i} + x_0^{n_{t_1}} \right)$. For the arbitrary automorphism σ we have

$$\sigma(f) = \chi(\sigma) f \Rightarrow \sigma(f_1) - \chi(\sigma) f_1 = \chi(\sigma) f_2 - \sigma(f_2). \tag{3.9}$$

Since $a_{ij} = 0$ for $i < t_1$ and $j \geq t_1$ the polynomial

$$\sigma(f_1) - \chi(\sigma) f_1 \in k[x_1, ..., x_{t_1-1}].$$

Moreover, since $a_{ij} = 0$ for $i \geq t_1$, $j < t_1$ and $\sigma(x_0) = a_{00}x_0$, we have that $\chi(\sigma)f_2 - \sigma(f_2) \in k[x_0, x_{t_1}, ..., x_r]$, and finally we arrive at

$$\sigma(f_1) = \chi(\sigma)f_1 \text{ and } \sigma(f_2) = \chi(\sigma)f_2.$$

The polynomial $f_2/x_0^{n-n_{t_1}}$ is of the same form as the original polynomial f_1, so proceeding inductively we have that the matrix representation (a_{ij}) of σ is a block diagonal matrix of the form:

$$\begin{pmatrix} a_{00} & 0 & \cdots & 0 \\ 0 & A_1 & 0 & \\ \vdots & & A_i & \\ 0 & & & \ddots \end{pmatrix},$$

where A_k are $(t_{k+1} - t_k) \times (t_{k+1} - t_k)$ square invertible matrices. Moreover, the study of automorphisms of Fermat hypersurfaces, gives us that if in a column of the block matrix A_k there are more than one non zero elements, then $n_k - 1$ is a power of the characteristic. Let $d_k := t_{k+1} - t_k$. There is a direct sum decomposition of the group of automorphisms of X :

$$\mathrm{Aut}(X) := \bigoplus_{k=0}^{s-1} G_k,$$

where

$$G_k \cong \begin{cases} 1 & \text{if } d_k = 1 \\ \mathbb{Z}_{n_k} & \text{if } d_k = 2 \\ \mathbb{Z}_{n_k} \rtimes S_{d_k} & \text{if } d_k > 2 \text{ and } n_k - 1 \text{ is not a } p-\text{power} \\ \mathrm{PGU}(d_k, p^{2h_k}) & \text{if } d_k > 2 \text{ and } n_k - 1 = p^{h_k} \end{cases}$$

Remark 3.1. Let X be a variety in $\mathbb{P}^{n+1}$, let X^{sm} be the smooth locus of X and let $\mathbb{P}^{n+1^*}$ be the dual projective space. The conormal variety CX of X is defined as the closure of $CX^{\mathrm{sm}} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{n+1^*}$, where CX^{sm} is the set $(P, H) \in \mathbb{P}^{n+1} \times \mathbb{P}^{n+1^*}$ for all $P \in X^{\mathrm{sm}}$, and $H \in \mathbb{P}^{n+1^*}$, with $T_P X \subset H$. The dual variety X^* is the image of CX in $\mathbb{P}^{n+1^*}$. If $X = V(f)$ is a hypersurface, then the Gauss map γ, is defined as the map sending every point P in the smooth locus of X, to the tangent hyperplane $T_P X \in \mathbb{P}^{n+1^*}$, *i.e.*, in terms of the defining equation f of the hypersurface, the Gauss map is given by

$$X \ni P \to (\frac{\partial f}{\partial x_0}(P), ..., \frac{\partial f}{\partial x_n}(P)) \in X^*.$$

Let CX^* be the conormal variety of the dual variety. It is natural to expect that $CX = CX^*$ and as a matter of fact this is true in characteristic zero. In positive characteristic the Monge-Segre-Wallace criterion [10] asserts that $CX = CX^*$ if and only if the extension of function fields, that is induced by the map $CX \to X^*$, is separable.

In our examples, it is nice to point out that there are extra automorphisms in characteristic p exactly when $CX \neq CX^*$. For example when X is the Fermat hypersurface, then the Gauss map is the Frobenius map and the extension $k(CX)/k(X^*)$ is purely inseparable. For more information about projective duality we refer to the literature [6],[9],[10].

3.2. *The case $c = 2$*

Let $X \subseteq \mathbb{P}^r$ be a complete intersection corresponding to the ideal I, which is generated by two homogeneous polynomials f_1 and f_2 of degrees m and n respectively. Every linear automorphism σ defines polynomials $g_{ij}(\sigma)$ of degrees $\deg f_i - \deg f_j$, such that

$$\sigma(f_i) = \sum_{j=1}^{2} g_{ij}(\sigma) f_j$$

(by assumption polynomials of negative degree are zero). Let $\mathrm{PGL}(2, k[x])$ be the group of invertible matrices modulo diagonal matrices with coefficients in k. There is a group morphism

$$\rho : \mathrm{Aut}(X) \longrightarrow \mathrm{PGL}(2, k[x])$$

$$\sigma \longmapsto \big(g_{ij}(\sigma)\big)\,.$$

The automorphism σ can be extended to an automorphism of the hypersurfaces $V(f_1)$ and $V(f_2)$ if and only if $(g_{ij}(\sigma))$ is a diagonal matrix. In case $n = m$, the morphism ρ defines a representation of $\mathrm{Aut}(X)$ in $\mathrm{PGL}(2, k)$, and if $n > m$ then $\rho(\sigma)$ is a lower triangular matrix of the form

$$\rho(\sigma) = \begin{pmatrix} g_{11}(\sigma) & 0 \\ g_{21}(\sigma) & g_{22}(\sigma) \end{pmatrix}, \tag{3.10}$$

where $g_{ii}(\sigma) \in k$ and $g_{21}(\sigma)$ is a polynomial of degree $n - m$. Moreover, in this case every automorphism can be extended to an automorphism of the hypersurface $V(f_1)$.

As in the case of hypersurfaces, the automorphism σ preserves the normal bundle $N_P X = T_P\mathbb{P}^r / T_P X$, and acts on the base of $N_P X = \langle \nabla f_1|_P, \nabla f_2|_P\rangle_k$ as follows:

$$\nabla f_2|_{\sigma(P)} \cdot \sigma = \nabla g_{21}(\sigma) f_1(P) + g_{22}(\sigma)\nabla f_2(P) + g_{21}(\sigma)\nabla f_1(P). \tag{3.11}$$

Remark 3.2. In case $n > m$ every automorphism of X is the restriction of an automorphism of $V(f_1)$. We consider X as a divisor of $V(f_1)$ so $\mathrm{Aut}(X)$ can be interpreted as the decomposition group of X, in the cover

$$V(f_1) \longrightarrow V(f_1)^{\mathrm{Aut}(V(f_1))}.$$

Hence, $\mathrm{Aut}(X)$ is the identity, unless X is ramified in the above cover.

Remark 3.3. If $\deg f_1 = \deg f_2$, then there is a base f_1', f_2' generating the ideal I such that

$$\mathrm{Aut}(X) = \mathrm{Aut}(V(f_1')) \cap \mathrm{Aut}(V(f_2')),$$

if and only if the representation

$$\rho : \mathrm{Aut}(X) \longrightarrow \mathrm{PGL}(2, k)$$

can be decomposed as a direct sum of one dimensional characters.

3.2.1. Complete Intersections of Fermat hypersurfaces Denote by $f_1 = x_0^m + ... + x_r^m$, $f_2 = x_0^n + ... + x_r^n$ two Fermat polynomials, $p \nmid n, m$, and $n \neq m$. We will prove that the ideal $I = \langle f_1, f_2 \rangle$ is prime of codimension two. Then I defines a complete intersection X, which is a variety (*i.e.* reduced and irreducible).

For this we observe first that $\{f_1, f_2\}$ form a regular sequence in the polynomial ring $k[x_0, ..., x_r]$. Indeed, f_1, f_2 are irreducible so if f_1 is a zero divisor in the quotient ring $k[x_0, ..., x_r]/\langle f_2 \rangle$ then $f_1 \in \langle f_2 \rangle$, a contradiction, since the degree of f_1 in x_0 is strictly less than the degree of f_2 in x_0. Since $k[x_0, ..., x_r]$ is Cohen-Macaulay, the codimension of I is two.

Proposition 3.2. *The ideal I is prime.*

Proof: Since $\{f_1, f_2\}$ is a regular sequence in the Cohen-Macaulay ring $k[x_0, ..., x_r]$, proposition 18.13 in [4] implies that R/I is a Cohen-Macaulay ring. We will use now the following

Theorem 3.1. *Let $R = k[x_0, ..., x_r]/I$ where $I = (f_1, ..., f_s)$ is a homogeneous ideal of codimension c. Let $J \subset R$ be the ideal generated by the $c \times c$ minors of the Jacobian matrix $\mathcal{J} = (\partial f_i/\partial x_j)$, taken modulo I. Suppose also that R is Cohen-Macaulay.*

- *R is reduced if and only if J has codimension ≥ 1 in R.*
- *R is a direct product of normal domains if and only if J has codimension ≥ 2 in R.*

Notice that since R is a graded ring, if it is a direct product of normal domains, R is a domain, therefore I is prime.

Proof. [4, Th. 18.15]

Remark 3.4. In geometric terms the above theorem ensures that if the codimension of the singular locus is big enough, then I is prime. In particular, if the algebraic set corresponding to I is non singular, then $\mathrm{codim}(J) = \dim X$ and the theorem holds, provided that $\dim X \geq 2$.

The Jacobian matrix in our case is

$$\mathcal{J} = \begin{pmatrix} mx_0^{m-1} & mx_1^{m-1} & \cdots & mx_r^{m-1} \\ nx_0^{n-1} & nx_1^{n-1} & \cdots & nx_r^{n-1} \end{pmatrix},$$

and the 2×2 minors are of the form:

$$mnx_i^{m-1}x_j^{m-1}\prod(x_i - \zeta x_j),$$

where ζ runs over the $n-m$ roots of one. Since the characteristic p does not divide m, n the singular locus $X \backslash X^{\mathrm{sm}}$ of X is contained in the intersection of a finite union of lines L_ν with X, where ν runs over a finite index set. For example one such line is given by $x_i = \zeta x_j$ for all i, j, where ζ is a $n-m$ root of one. In general we will have all the combinations for different values of zeta, and also for some i, j the equation $x_i - \zeta x_j$ might be replaced by $x_i = 0$.

In the algebraic setting,

$$\mathrm{rad}\left(\frac{J+I}{I}\right) = I(X^{\mathrm{sm}}) \supset I(\cup_i(L_i \cap X)) = \cap I_i,$$

where $I_i := I(L_i \cap X)$. Therefore,

$$\mathrm{codim}(\mathrm{rad}\left(\frac{J+I}{I}\right)) \geq \mathrm{codim}(\cap I_i) = \min_i(\mathrm{codim}(I_i)) \geq \dim X - 1.$$

Since the ring $k[x_0, ..., x_r]/I$ is Cohen-Macaulay, we have that $\mathrm{codim}\left(\frac{J+I}{I}\right) = \mathrm{depth}\left(\frac{J+I}{I}\right)$, and by [4, Corollary 17.8] $\mathrm{depth}(\mathrm{rad}\left(\frac{J+I}{I}\right)) = \mathrm{depth}\left(\frac{J+I}{I}\right)$. Hence, if $\dim X \geq 2$, X is reduced, and if $\dim X \geq 3$, then X is also irreducible. $\square$

According to proposition 2.2 every automorphism of X is linear, *i.e.* it is induced by a matrix $A = (a_{ij}) \in \mathrm{PGL}(r+1, k)$, such that for every $f \in I$, $fA \in I$.

We apply this to $f_1 = x_0^m + ... + x_r^m$, first. The polynomial

$$f_1 A = \sum_{i=0}^{r}\left(\sum_{j=0}^{r} a_{ij}x_j\right)^m$$

is an element of the ideal I, *i.e.*

$$f_1 A = g_{11}(A) f_1 + g_{12}(A) f_2,$$

for two suitable polynomials $g_{11}(A)$, $g_{12}(A)$. Since the degree of $f_1 A$ is m, we have that $g_{12}(A) = 0$ and g_{11} is a character of $\mathrm{Aut}(X)$. This proves that the group $\mathrm{Lin}(X)$, of linear automorphisms of X, is a subgroup, of the group of automorphisms of the Fermat hypersurface given by $x_0^m + ... + x_r^m = 0$.

If m is not a power of the characteristic, then the automorphism group G of the Fermat hypersurface $x_0^m + \cdots + x_r^m = 0$ is $\mathbb{Z}_m^r \rtimes \mathrm{S}_{r+1}$, and the group $\mathrm{Aut}(X)$ is the subgroup of G, consisting of elements that keep f_2 in I, and a simple calculation shows that

$$\mathrm{Aut}(X) = \mathrm{Aut}(V(f_1)) \cap \mathrm{Aut}(V(f_2)) = \mathbb{Z}_{(n,m)}^r \rtimes \mathrm{S}_{r+1}.$$

We assume now that $m-1 = q$ hence $\mathrm{Aut}(V(f_1)) = \mathrm{PGU}(r+1, q^2)$. Let A be an automorphism of X represented by the matrix (a_{ij}). If the ideal $I_1 := \langle f_2, f_2 A\rangle$ is prime then $g_{21} f_1 \in I_1$, hence either g_{21} or f_1 is in I_1, and since $\deg(f_1)$, $\deg(g_{21}) <$

n, the polynomial g_{21} is zero, equivalently A can be extended to an automorphism of $V(f_2)$.

Assume now that the automorphism A cannot be extended to an automorphism of $V(f_2)$, and the ideal I_1 is not prime. Therefore, the polynomial f_2A is not in the ideal generated by f_2, so $\{f_2, f_2A\}$ is a regular sequence.

We will use the following generalization of the division algorithm in polynomials rings

Lemma 3.2. *Let $>$ be a fixed monomial order on $\mathbb{Z}^n_{\geq 0}$, and g_1, g_2 be an ordered pair of polynomials in $k[x_0, ..., x_r]$. Every f in $k[x_0, ..., x_r]$ can be written as*

$$f = a_1 g_1 + a_2 g_2 + r,$$

where $a_i, r \in k[x_0, ..., x_r]$ and either $r = 0$, or r is a k-linear combination of monomials, none of which is divisible by any of the leading terms of g_1, g_2. Furthermore, if $a_i g_i \neq 0$, then $\mathrm{multideg(f)} \geq \mathrm{multideg(a_i g_i)}$.

Proof. [3, Thm. 3 p.63]

Since we have assumed that the ideal $I_1 = \langle f_2, f_2A\rangle$ is not prime, 3.1 implies that the codimension of the ideal J generated by the 2×2 minors of the Jacobian matrix is zero or one. The 2×2 minors of the Jacobian matrix are computed as follows

$$D_{i_1,i_2} = \begin{pmatrix} nx_{i_1}^{n-1} & \sum_{\nu=0}^r a_{\nu i_1}\left(\sum_{s=0}^r a_{s i_1} x_s\right)^{n-1} \\ nx_{i_2}^{n-1} & \sum_{\nu=0}^r a_{\nu i_2}\left(\sum_{s=0}^r a_{s i_2} x_s\right)^{n-1} \end{pmatrix}.$$

Assume first that the codimension of J is zero. Then all the 2×2-minors are zero divisors, *i.e.* there are polynomials $h_{i_1,i_2}, u_{i_1,i_2}, v_{i_1,i_2}$ such that

$$h_{i_1,i_2} D_{i_1,i_2} = u_{i_1,i_2} f_2 + v_{i_1,i_2} f_2 A. \tag{3.12}$$

By dividing h_{i_1,i_2} by f_2, f_2A, according to lemma 3.2 we can assume that

$$\deg_{x_j} h_{i_1,i_2} < deg_{x_j} f_2 A, \text{ where } j \neq i_1, i_2 \tag{3.13}$$

For a polynomial $w \in k[x_0, ..., x_r]$ we denote by $\mathrm{sp}_{i_1,i_2}(w)$, the polynomial defined by $\mathrm{sp}_{i_1,i_2}(w) = w|_{x_{i_1}=x_{i_2}=0}$. Similarly, we will denote by $\mathrm{sp}_{i_1,i_2,i_3}(w)$ the polynomial defined by $\mathrm{sp}_{i_1,i_2,i_3}(w) = w|_{x_{i_1}=x_{i_2}=x_{i_3}=0}$.

We specialize equation (3.12) for $(i_1, i_2) = (1, 2)$.

$$0 = \mathrm{sp}_{1,2}(u_{1,2})\mathrm{sp}_{1,2}(f_2) + \mathrm{sp}_{1,2}(v)\mathrm{sp}_{1,2} f_2 A.$$

The polynomial $\mathrm{sp}_{1,2}(f_2)$ is irreducible, hence either $\mathrm{sp}_{1,2}(v_{1,2})$ or $\mathrm{sp}_{1,2}(f_2A)$ are multiples of $\mathrm{sp}_{1,2}(f_2)$. But if $\mathrm{sp}_{1,2}(v_{1,2}) \in \langle \mathrm{sp}_{1,2}(f_2)\rangle$ then (3.13) implies that

$$\mathrm{multideg(u_{1,2}f_2A)} > \mathrm{multideg(h_{1,2}D_{1,2})},$$

with respect to the lexicographic order, a contradiction. Hence, $\mathrm{sp}_{1,2}(f_2A) = \mathrm{sp}_{1,2}(f_2)$, equivalently

$$f_2 A = f_2 + d_{1,2},$$

where $d_{1,2}$ is a polynomial, such that $d_{1,2}|_{x_1=x_2=0} = 0$. We arrive at the same conclusion for all 2×2 minors, so

$$f_2 A = f_2 + a,$$

where a is a polynomial such that $\mathrm{sp}_{i_1,i_2}(a) = 0$. Assume that for a fixed j_0 there are more than one elements a_{ij_0}, with $a_{ij_0} \neq 0$, say a_{i_1,j_0} and a_{i_2,j_0}. By comparing coefficients of the terms $x_{i_1}^{\mu} x_{i_2}^{n-1-\mu}$ in $f_2 A = g_{22} f_1 + g_{21} f_2$, we obtain

$$\binom{n-1}{\mu} \sum_{\nu=0}^{r} a_{\nu i}^{\mu} a_{\nu j}^{n-1-\mu} = \delta_{ij},$$

for all $\mu = 1, ..., n-1$, and this gives that $n-1$, is a power of the characteristic and A is an automorphism of f_2, a contradiction to our assumption A is not an automorphism of f_2.

Therefore not all minor determinants are zero divisors in $k[x_0, \cdots, x_r]/I_1$ and without restriction of generality, assume that $D_{0,1}$ is not a zero divisor. Then the polynomials $\{D_{0,1}, f_2, f_2 A\}$ form a regular sequence, and if the codimension of J is 1 then all other minors D_{i_1,i_2}, for $\{i_1, i_2\} \neq \{0, 1\}$ are zero divisors on $\langle D_{0,1}, f_2, f_2 A\rangle$, *i.e.* there are polynomials $h'_{i_1,i_2}, u'_{i_1,i_2}, v'_{i_1,i_2}, w'_{i_1,i_2}$ such that

$$h'_{i_1,i_2} D_{i_1,i_2} = w'_{i_1,i_2} D_{0,1} + u'_{i_1,i_2} f_2 + v'_{i_1,i_2} f_2 A.$$

We consider the above equation for $i_1 = 0$, and $i_2 \neq 0, 1$, and evaluate at $x_0 = x_1 = x_{i_2} = 0$. By a similar argument we obtain that

$$f_2 A = f_2 + a,$$

where $\mathrm{sp}_{0,1,i_2}(a) = 0$. Again this proves that no monomial term of the form $x_i^{\nu} x_j^{n-\nu}$ can appear as summand of a, provided $r \geq 4$, *i.e.* $\binom{n-1}{\mu} \sum_{i=0}^{r} a_{ik}^{\mu} a_{is}^{n-1-\mu} = \delta_{ks}$, and the automorphism A can be extended to an automorphism of $V(f_2)$, a contradiction.

3.3. Intersection of Fermat hypersurfaces of the same degree

The Fermat polynomials $\sum x_i^{q+1}$ over a field of characteristic p, behave like the quadratic forms $\sum x_i^2$. Indeed, we can define the bilinear form

$$\langle x, y\rangle = \sum_{i.j} x_i a_{ij} y_i^q,$$

where $x = (x_0, ..., x_r)$, $y = (y_0, ..., y_r)$ and (a_{ij}) is a nonzero matrix. Let $F : x \mapsto x^q$ be the Frobenius involution in the finite field $\mathbb{F}_{q^2}$. If $F(a_{ij}) = (a_{ij})$ then $\langle \cdot, \cdot \rangle$ is a hermitian inner product with respect to the Frobenius involution *i.e.*

$$\langle x, y\rangle = F(\langle y, x\rangle).$$

Moreover, we observe that with respect to the theory of projective duality, the quadratic and the Fermat hypersurfaces as above are the only nonsingular hypersurfaces such that the dual variety is nonsingular.[14]

Since the field of definition k is assumed to be algebraically closed, the intersection of two hypersurfaces of the form

$$\sum_{i,j} x_i a_{ij} x_j^q = 0, \text{ and } \sum_{i.j} x_i b_{ij} x_j^q$$

can be normalized, after a (not-necessary linear) change of coordinates, to the intersection of the hypersurfaces

$$\sum_{i=0}^{r} x_i^{q+1} = 0 \text{ and } \sum_{i=0}^{r} c_i x_i^{q+1} = 0$$

for suitable $c_i \in k$.

Remark 3.5. It is known that the intersection of two quadratic surfaces is an elliptic curve with infinitely many automorphisms. It seems to be interesting to study automorphism groups, of intersections of two Fermat hypersurfaces of the form:

$$\sum x_i^{q+1} \text{ and } \sum c_i x_i^{q+1}.$$

Using the Jacobian criterion 3.1 we can prove the following

Lemma 3.3. *The intersection of the Fermat hypersurfaces*

$$V(\sum x_i^{q+1}) \textit{ and } V(\sum c_i x_i^{q+1})$$

is a complete nonsingular intersection if and only if $c_i \neq c_j$ for $i \neq j$.

Let $X(c_0, ..., c_r)$ be the projective variety corresponding to the ideal

$$I = \langle \sum x_i^{q+1}, \sum c_i x_i^{q+1} \rangle,$$

$c_i \neq c_j$ for $i \neq j$.

The normal space of $X(c_0, ..., c_r)$ at a point $P = (x_0 : ... : x_r)$ is generated by the vectors $Y := (x_0^q : ... : x_r^q)$, $Z := (c_0 x_0^q : ... : c_r x_r^q)$. Let σ be a linear automorphism of X represented by a matrix (a_{ij}). By comparison of coefficients in (3.11) we obtain

$$\sum_i a_{ik} a_{i\nu}^q = (b_{11} + b_{12} c_k) \delta_{k\nu}$$

$$\sum_i a_{ik} a_{i\nu}^q c_i = (b_{21} + b_{22} c_k) \delta_{k\nu} \tag{3.14}$$

Denote by $e_k = (a_{ik})_{i=0,...,r}$ and $e_k' = (c_i a_{ik})_{i=0,...,r}$. By (3.14) we have

$$\langle e_k, e_\nu \rangle = (b_{11} + b_{12} c_k) \delta_{k\nu} \tag{3.15}$$

$$\langle e'_k, e_\nu \rangle = (b_{21} + b_{22}c_k)\delta_{k\nu}.$$

On the other hand $e'_k = \sum_i \lambda_i e_i$, and (3.15) implies

$$\langle e'_k, e_i \rangle = \lambda_i (b_{11} + b_{12}c_k)\delta_{ki},$$

hence $\lambda_i = \frac{b_{21}+b_{22}c_k}{b_{11}+b_{12}c_k}\delta_{ki}$ (notice that if $b_{11} + b_{12}c_k = 0$ then $b_{21} + b_{22}c_k = 0$ hence the 2×2 matrix (b_{ij}) is singular, a contradiction). The above expression for λ_i allows us to write

$$e'_k = \frac{b_{21} + b_{22}c_k}{b_{11} + b_{12}c_k} e_k,$$

therefore for all i, k

$$c_i a_{ik} = \frac{b_{21} + b_{22}c_k}{b_{11} + b_{12}c_k} a_{ik}. \tag{3.16}$$

If, for a fixed k, there are more than one (say i_1, i_2) such that $a_{i_1 k} \neq 0$, $a_{i_2 k} \neq 0$, then (3.16) implies that $c_{i_1} = c_{i_2}$, a contradiction. This proves that

$$a_{ij} = \delta_{i,\tau(j)}\mu_i,$$

where τ is an element of the symmetric group S_{r+1} and $\mu_i \in k$. Equation (3.14) implies now that

$$\sum_i a_{ik} a_{i\nu}^q = \sum_i \delta_{i,\tau(k)} \delta_{i,\tau(\nu)}^q \mu_i \mu_i^q \Rightarrow \mu_i^{q+1} = (b_{11} + b_{21}c_k).$$

We have proved so far that $H := \mathrm{Im}\rho$ is a subgroup of S_{r+1} acting on c_i by a linear fractional transformations, and that $\ker\rho = \mathbb{Z}_{q+1}^r$.

Let $(x_{ij}), (y_{ij})$ be the images of two elements $(a_{ij}), (b_{ij}) \in G$ under ρ. We write

$$a_{ij} = \mu_i \delta_{i,\sigma(j)},\, b_{ij} = \lambda_i \delta_{i,\tau(j)}.$$

Consider the product

$$c_{ij} := \sum_\nu a_{i\nu} b_{\nu j} = \sum_\nu \mu_i \lambda_\nu \delta_{i,\sigma(\nu)} \delta_{\nu,\tau(j)} = \mu_k \lambda_k \delta_{\sigma^{-1}(i)\tau(j)}.$$

Let $k := \sigma^{-1}(i) = \tau(j)$ for suitable i, j. We have

$$\mu_k^{q+1} = x_{11} + x_{12}c_k \tag{3.17}$$

and

$$\lambda_k^{q+1} = y_{11} + y_{12}c_k \tag{3.18}$$

By computing $(\mu_k\lambda_k)^{q+1}$ in two ways (by multiplying the matrices (x_{ij}) with (y_{ij}) and by multiplying equations (3.17),(3.18)) we obtain:

$$-x_{12}y_{12}c_k^2 + (x_{12}y_{11} - x_{12}y_{22})c_k + x_{12}y_{21} = 0. \tag{3.19}$$

For every k there are i, j such that $\sigma^{-1}(i) = \tau(j) = k$ and (3.19) holds for all c_k, hence the corresponding quadratic polynomial is identically zero. This implies that

$$x_{12}y_{12} = 0 \text{ and } x_{12}(y_{11} - y_{22}) = 0 \text{ and } x_{12}y_{21} = 0,$$

so $x_{12} = 0$ or ($y_{12} = y_{21} = 0$ and $y_{11} = y_{22}$) so (y_{ij}) is the identity. Therefore, $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Aut}(V(x_0^{q+1} + ... + x_r^{q+1})) = \mathrm{PGU}(r+1, q^2)$.

Moreover, $\mathrm{Im}\rho$ is a finite subgroup of $\mathrm{PGL}(2, k)$. The finite subgroups of $\mathrm{PGL}(2, k)$ in positive characteristic are classified in ([17]), and there are the following possibilities: $\mathrm{A}_5, \mathrm{A}_4, \mathrm{S}_4, \mathbb{Z}_n, \mathbb{Z}_p^t, \mathbb{Z}_p^t \rtimes \mathbb{Z}_n, \mathrm{PSL}(2, p^i), \mathrm{PGL}(2, p^i)$. We have proved that all automorphisms in $\mathrm{Im}\rho$ are upper triangular, hence

$$\mathrm{Im}\rho \in \{\mathbb{Z}_p^t, \mathbb{Z}_s, \mathbb{Z}_{p_1}^{t_1} \rtimes \mathbb{Z}_n\}. \tag{3.20}$$

Moreover, the fixed points of $\mathrm{Im}\rho$ are $\{\infty\}, \{0, \infty\}, \{0, \infty\}$respectively. The set of $r+1$ points $\{(1, c_0), ..., (1, c_r)\}$ acted by $\mathrm{Im}\rho$ is divided in orbits. This proves that $s \mid r+1$ or $s \mid r$, $pt = r$, $p_1t_1n \mid r+1$ or $p_1t_1n \mid r$. The proof of theorem 1.3 is now complete.

Remark 3.6. As in the the theory of intersections of quadratic forms, the set of isomorphism classes of $X(c_0, ..., c_r)$ are in one to one correspondence with elements $((1 : c_0), ..., (1 : c_r))$ of the configuration space

$$X\{2, r+1\} := \left(\mathrm{GL}(2, k)\backslash \mathrm{M}^*(2, r+1)/\mathrm{H}_{r+1}\right)/\mathrm{S}_{r+1}$$

where $\mathrm{M}^*(2, r+1)$ is the space of $2 \times r+1$ matrices, for which no 2-minor vanishes, and $\mathrm{H}_n \cong (k^*)^{r+1}$ is the subgroup of $\mathrm{GL}(r+1, k)$ consisted of diagonal matrices. If the $r+1$ points of $\mathbb{P}^1$ are in orbits of one group mentioned in (3.20), then $\mathrm{Im}\rho$ is not the identity.

Example 3.1. Automorphisms of the curves $X(c_0, c_1, c_2, c_3)$. The set of such curves corresponds to the configuration space $X\{2, 4\}$, which is isomorphic to the affine line, and the isomorphism is given in terms of the J invariant defined as

$$J := \frac{\left(D(12)^2D(34)^2 + D(13)^2D(24)^2 + D(14)^2D(23)^2\right)^3}{D(12)^2D(34)^2D(13)^2D(24)^2D(14)^2D(23)^2},$$

where $\mathrm{D}(ij) = c_j - c_i$. By theorem (1.3) the only possibilities for $\mathrm{Im}\rho$ are given in the following table:

$\mathrm{Im}\rho$	(c_0, c_1, c_2, c_3)	J	
$\mathbb{Z}_3$	$(0, c, \omega c, \omega^2 c)$	0	$\omega^2 + \omega + 1 = 0$, $\mathrm{char}k \neq 3$
$\mathbb{Z}_4$	$(c, \zeta c, \zeta^2 c, \zeta^3 c)$	$2\frac{91\zeta^2+37}{\zeta^2-1}$	$\zeta^3 + \zeta^2 + \zeta + 1 = 0$, $\mathrm{char}k \neq 2$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(c, c+1, d, d+1)$	0	$\mathrm{char}k = 2$

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