FRAMIZATION OF THE TEMPERLEY–LIEB ALGEBRA

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Abstract. In this paper we propose a framization of the Temperley–Lieb algebra. The framization is a procedure that can briefly be described as the adding of framing to a known knot algebra in a way that is both algebraically consistent and topologically meaningfull. Here, our framization is defined as a quotient of the Yokonuma–Hecke algebra. The main theorem provides necessary and sufficient conditions for the Markov trace defined on the Yokonuma–Hecke algebra to pass through to our framization.

Contents

1. Introduction

Since the original construction of the Jones polynomial the Temperley–Lieb algebra has become a cornerstone of a fruitful interaction between Knot theory and Representation theory. The Temperley–Lieb algebra was introduced by Temperley and Lieb [23] and was rediscovered by Jones [8] as a knot algebra [9].

A knot algebra is an algebra that is used in the construction of invariants of classical links. Our main interest lies in those knot algebras that are involved in the construction of invariants through Jones' formula [9]. More precisely, a knot algebra A is a triplet (A, π, τ), where π is an appropriate representation of the braid group in A and τ is a Markov trace function defined on A. The Temperley–Lieb algebra, the Iwahori–Hecke algebra and the BMW algebra are the most important examples of knot algebras.

The 'framization' is a mechanism designed by the second and fourth authors that consists of a generalization of a knot algebra via the addition of framing generators. In this way we obtain a new algebra which is related to framed braids and framed knots. More precisely, the framization

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procedure can roughly be described as the procedure of adding framing generators to the generating set of a knot algebra, of defining interacting relations between the framing generators and the original generators of the algebra and of applying framing on the original defining relations of the algebra. The resulting framed relations should be topologically consistent. The most difficult problem in this procedure is to apply the framization on the relations of polynomial type.

The basic example of framization is the Yokonuma–Hecke algebra, which can be regarded as a framization of the Iwahori–Hecke algebra [13, 16]. This framization of the Iwahori–Hecke algebra gives the recipe of how to apply the framization technique on the quadratic Iwahori–Hecke relation. Having in mind this example, the second and fourth authors proposed framizations of several knot algebras [17, 18]. Further, the second author constructed a unique Markov trace function, tr, on the Yokonuma–Hecke algebras $\mathrm{Y}_{d,n}(u)$ with parameters $z, x_1, \ldots, x_{d-1}$ [11]. Consequently, invariants for framed, classical and singular oriented links have been constructed [16, 15, 14] by applying the 'E–condition' on the parameters $x_1, \ldots, x_{d-1}$ so that tr re–scales and respects the negative stabilization move between framed braids [16]. The derived invariants for classical links do not coincide with the 2–variable Jones or HOMFLYPT polynomial (here denoted Homflypt) [1] except in few trivial cases, yet they are conjectured to be topologically equivalent to the Homlfypt polynomial [5].

The Temperley–Lieb algebra can be regarded as a quotient of the Iwahori–Hecke algebra, therefore it is natural to look for a quotient of the Yokonuma–Hecke algebra that can be considered as a framization of the Temperley–Lieb algebra. In this paper we propose a framization of the Temperley–Lieb algebra as a quotient of the Yokonuma–Hecke algebra over an appropriate two–sided ideal. Although such an ideal is not unique, it will become clear that our choice for the ideal that leads to the framization of the Temperley–Lieb algebra is the one that reflects the construction of a 'framed Jones Polynomial' in the most natural way.

The outline of the paper is as follows. Section 2 is dedicated to providing necessary definitions and results, including: the Iwahori–Hecke algebra, the Ocneanu trace, and the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$. In Section 3 we recall some basic tools from harmonic analysis of finite groups such as the convolution product, the product by coordinates and the Fourier transform, necessary for exploring the 'E–system'.

In Section 4, where we present two natural quotients of $\mathrm{Y}_{d,n}(u)$ that could possibly lead to a framization of the Temperley–Lieb algebra, the Yokonuma–Temperley–Lieb algebra, $\mathrm{YTL}_{d,n}(u)$ (introduced and studied in [6]) and the Complex Reflection Temperley–Lieb algebra, $\mathrm{CTL}_{d,n}(u)$. After deducing that these two quotient algebras are not suitable for our purpose, we introduce a third quotient of $\mathrm{Y}_{d,n}(u)$, the Framization of the Temperley–Lieb algebra, $\mathrm{FTL}_{d,n}(u)$. The connection between all possible quotients of $\mathrm{Y}_{d,n}(u)$ is then analyzed and we prove that the defining ideals of $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ are principal. Furthermore, we provide presentations with non–invertible generators for the quotient algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$. We conclude this section with the formula for the dimension of $\mathrm{FTL}_{d,n}(u)$ by Chlouveraki and Pouchin [2] and we provide a linear basis for the case $d = 2$, $n = 3$.

The main theorems are given in Section 5. They provide the necessary and sufficient conditions for the Markov trace tr [11] on the Yokonuma–Hecke algebra to pass through to the quotient algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ respectively. The corresponding results for the algebra $\mathrm{YTL}_{d,n}(u)$ are given in [6]. More precisely, we first find the necessary and sufficient conditions for the case of $\mathrm{FTL}_{d,3}(u)$ using tools from harmonic analysis on finite groups (Lemma 8) and then we generalize using induction on n (Theorem 5). Using the same ideas we prove the analogous theorem for $\mathrm{CTL}_{d,n}(u)$ (Theorem 6). In Section 6 we discuss the connection between the necessary and sufficient conditions such that tr passes to all three quotients algebras $\mathrm{CTL}_{d,n}(u)$, $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{YTL}_{d,n}(u)$.

Finally, in Section 7 we define invariants for framed and classical links through the quotient algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$, and we deduce that the invariants for framed links that are derived from $\mathrm{FTL}_{d,n}(u)$ provide a framed analogue of the Jones polynomial, while the invariants from the algebras $\mathrm{CTL}_{d,n}(u)$ coincide with those from $\mathrm{FTL}_{d,n}(u)$. We note that the invariants for classical links from the algebras $\mathrm{YTL}_{d,n}(u)$ recover the Jones polynomial [6].

2. Preliminaries

2.1. *Notations.* Throughout the paper by the term algebra we mean an associative unital (with unity 1) algebra over $\mathbb{C}$. Thus we can regard $\mathbb{C}$ as a subalgebra of the center of the algebra. We will also fix two positive integers, d and n.

As usual we denote $\mathbb{Z}/d\mathbb{Z}$ the group of integers modulo d, $\mathbb{Z}/d\mathbb{Z} = \{0, 1, \ldots, d-1\}$.

We denote S_n the symmetric group on n symbols. Let s_i be the elementary transposition $(i, i+1)$ and let $\langle s_i, s_j\rangle$ denote the the subgroup generated by s_i and s_j. We also denote by l the length function on S_n with respect to the s_i's.

Denote $C_d = \langle t \,|\, t^d = 1\rangle$ the cyclic group of order d. Let $t_i := (1, \ldots, 1, t, 1, \ldots, 1) \in C_d^n$, where t is in the i–th position. We then have:

$$C_d^n = \langle t_1, \ldots, t_n \,|\, t_i t_j = t_j t_i,\ t_i^d = 1\rangle.$$

Define $C_{d,n} := C_d^n \rtimes S_n$, where the action is defined by permutation on the indices of the t_i's, namely: $s_i t_j = t_{s_i(j)} s_i$. Notice that $C_{d,n}$ is isomorphic to the *complex reflection group* $G(d, 1, n)$.

Denote also B_n the braid group of type A, that is, the group generated by the elementary braidings $\sigma_1, \ldots, \sigma_{n-1}$, subject to the following relations: $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$, for $|i-j| = 1$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $|i-j| > 1$. We will also use the *d–modular framed braid group* $\mathcal{F}_{d,n} := C_d^n \rtimes B_n$, where the action of B_n on C_d^n is defined by the induced permutation on the indices of the t_i's and C is the infinite cyclic group, which surjects naturally on $\mathcal{F}_{d,n}$. We will also refer to the *framed braid group* $\mathcal{F}_n := C^n \rtimes B_n$. Of course,

$$\mathcal{F}_n \cong \mathbb{Z}^n \rtimes B_n \quad \text{and} \quad \mathcal{F}_{d,n} \cong (\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n.$$

Remark 1. We would like to point out that $C_{d,n}$ and $\mathcal{F}_{d,n}$ appear in the theory of "fields with one element". This is a theory dreamt by J. Tits in his study of algebraic groups. According to the seminal article of Kapranov and Smirnov [19], $\mathrm{GL}_n(\mathbb{F}_1) = S_n$, $\mathrm{GL}_n(\mathbb{F}_1[t]) = B_n$, $\mathrm{GL}_n(\mathbb{F}_{1^n}) = C_{d,n}$ and $\mathrm{GL}_n(\mathbb{F}_{1^n}[t]) = \mathcal{F}_{d,n}$, where $\mathrm{GL}_n(\mathbb{F}_{1^n})$ (resp. $\mathrm{GL}_n(\mathbb{F}_{1^n}[t])$) is in "some sense" the limit case $q \to 1$ of $\mathrm{GL}_n(\mathbb{F}_q)$ (resp. $\mathrm{GL}_n(\mathbb{F}_q[t])$).

2.2. *Background material.* From now on we fix a non–zero complex number u. We denote by $\mathrm{H}_n(u)$ the *Iwahori–Hecke algebra* associated to S_n, that is, the $\mathbb{C}$–algebra with linear basis $\{h_w \,|\, w \in S_n\}$ and the following rules of multiplication:

$$h_{s_i}h_w = \begin{cases} h_{s_i w} & \text{for } l(s_i w) > l(w) \\ u h_{s_i w} + (u-1)h_w & \text{for } l(s_i w) < l(w) \end{cases} \tag{2.1}$$

Set $h_i := h_{s_i}$. Then $\mathrm{H}_n(u)$ is presented by $h_1, \ldots, h_{n-1}$ subject to the following relations:

$$\begin{aligned} h_i h_j &= h_j h_i \quad \text{for all} \quad |i-j| > 1 \\ h_i h_j h_i &= h_j h_i h_j \quad \text{for all} \quad |i-j| = 1 \\ h_i^2 &= u + (u-1)h_i \end{aligned}$$

Definition 1. The *Temperley–Lieb algebra* $\mathrm{TL}_n(u)$ can be defined as the quotient of the algebra $\mathrm{H}_n(u)$ over the two–sided ideal generated by the *Steinberg elements* $h_{i,j}$:

$$h_{i,j} := \sum_{w \in \langle s_i, s_j \rangle} h_w, \quad \text{for all} \quad |i-j| = 1. \tag{2.2}$$

Equivalently, $\mathrm{TL}_n(u)$ can be presented by $h_1, \ldots, h_{n-1}$ subject to the following relations:

$$\begin{aligned} h_i h_j &= h_j h_i \quad \text{for all} \quad |i-j|>1 \\ h_i h_j h_i &= h_j h_i h_j \quad \text{for all} \quad |i-j|=1 \\ h_i^2 &= u + (u-1) h_i \\ 1 + h_i + h_j + h_i h_j + h_j h_i + h_i h_j h_i &= 0 \quad \text{for all} \quad |i-j|=1 \end{aligned}$$

Furthermore, using the transformation:

$$f_i := \frac{1}{u+1}(h_i + 1) \tag{2.3}$$

the algebra $\mathrm{TL}_n(u)$ can be presented by the non–invertible generators $f_1, \ldots, f_{n-1}$ subject to the following relations:

$$\begin{aligned} f_i^2 &= f_i \\ f_i f_j f_i &= \delta f_i, \quad \text{for all} \quad |i-j|=1 \\ f_i f_j &= f_j f_i, \quad \text{for all} \quad |i-j|>1 \end{aligned}$$

where $\delta^{-1} = 2 + u + u^{-1}$ [9].

In [7, 9] Ocneanu constructed a unique Markov trace on $\mathrm{H}_n(u)$. More precisely, we have the following theorem.

Theorem 1 (Ocneanu)**.** *For any $\zeta \in \mathbb{C}^\times$ there exists a linear trace τ on $\cup_{n=1}^{\infty} \mathrm{H}_n(u)$ uniquely defined by the inductive rules:*

(1) $\tau(ab) = \tau(ba), \quad a, b \in \mathrm{H}_n(u)$
(2) $\tau(1) = 1$
(3) $\tau(a h_n) = \zeta\, \tau(a), \quad a \in \mathrm{H}_n(u)$ *(Markov property).*

The Ocneanu trace τ passes through to $\mathrm{TL}_n(u)$. Indeed, as it turned out [9], to factorize τ to the Temperley–Lieb algebra, we only need the fact that τ kills the expression of Eq. 2.2. So, in [9] it is proved that τ passes to the Temperley–Lieb algebra if and only if:

$$\zeta = -\frac{1}{u+1} \quad \text{or} \quad \zeta = -1. \tag{2.4}$$

2.3. *The Yokonuma–Hecke algebra.* The Yokonuma–Hecke algebra of type A, denoted $\mathrm{Y}_{d,n}(u)$ [25], can be defined by generators and relations [11] and can be regarded as a quotient of $\mathbb{C}\mathcal{F}_{d,n}$ over the two–sided ideal that is generated by the elements:

$$\sigma_i^2 - (u-1)e_i - (u-1)e_i\sigma_i - 1,$$

where e_i is the idempotent defined by:

$$e_i := \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_{i+1}^{d-s}, \qquad i = 1, \ldots, n-1. \tag{2.5}$$

Equivalently, one can define $\mathrm{Y}_{d,n}(u)$ as follows:

Definition 2. The *Yokonuma–Hecke algebra* $\mathrm{Y}_{d,n}(u)$ is the algebra presented by generators $g_1, \ldots, g_{n-1}, t_1, \ldots, t_n$ subject to the following relations:

$$
\begin{aligned}
g_i g_j &= g_j g_i \quad \text{for all} \quad |i-j|>1 && (2.6)\\
g_{i+1} g_i g_{i+1} &= g_i g_{i+1} g_i && (2.7)\\
t_i t_j &= t_j t_i \quad \text{for all} \quad i,j && (2.8)\\
t_i^d &= 1 \quad \text{for all} \quad i && (2.9)\\
g_i t_i &= t_{i+1} g_i && (2.10)\\
g_i t_{i+1} &= t_i g_i && (2.11)\\
g_i t_j &= t_j g_i \quad \text{for } j \neq i,\, i+1 && (2.12)\\
g_i^2 &= 1 + (u-1) e_i + (u-1) e_i g_i && (2.13)
\end{aligned}
$$

Note that for $d = 1$ the quadratic relation (2.13) becomes:

$$g_i^2 = (u-1) g_i + u$$

So the Yokonuma–Hecke $\mathrm{Y}_{1,n}(u)$ coincides with the Iwahori–Hecke algebra.

The algebra $\mathrm{Y}_{d,n}(u)$ can also be regarded as a u–deformation of the group algebra $\mathbb{C}C_{d,n}$. Indeed, if $w \in S_n$ is a reduced word in S_n with $w = s_{i_1} \ldots s_{i_k}$ then the expression $g_w = g_{s_{i_1}} \ldots g_{s_{i_r}} \in \mathrm{Y}_{d,n}(u)$ is well–defined since the generators $g_i := g_{s_i}$ satisfy the same braiding relations as the generators of S_n [20]. We have the following multiplication rule in $\mathrm{Y}_{d,n}(u)$ (see Proposition 2.4[10]):

$$g_{s_i} g_w = \begin{cases} g_{s_i w} & \text{for } l(s_i w) > l(w) \\ g_{s_i w} + (u-1) e_i g_{s_i w} + (u-1) e_i g_w & \text{for } l(s_i w) < l(w). \end{cases} \tag{2.14}$$

Note also that the generators g_{t_i} correspond to t_i and so we have that: $g_{t_i w} = g_{t_i} g_w = t_i g_w$.

The definition of the idempotents e_i can be generalized in the following way. For any indices i, j we define the following elements in $\mathrm{Y}_{d,n}(u)$:

$$e_{i,j} := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_j^{d-s}. \tag{2.15}$$

We also define, for any $0 \leq m \leq d-1$, *the shift of* e_i *by* m:

$$e_i^{(m)} := \frac{1}{d} \sum_{s=0}^{d-1} t_i^{m+s} t_{i+1}^{d-s}. \tag{2.16}$$

Notice that $e_i = e_{i,i+1} = e_i^{(0)}$. Notice also that $e_i^{(m)} = t_i^m e_i = t_{i+1}^m e_i$. Then one deduces easily that:

$$
\begin{aligned}
e_i^{(m)} e_{i+1} &= e_i e_{i+1}^{(m)} \\
t_1^a t_2^b t_3^c e_1 e_2 &= e_1^{(a+b+c)} e_2
\end{aligned} \tag{2.17}
$$

for all $0 \leq m, a, b, c \leq d-1$.

The following lemma collects some of the relations among the e_i's, the t_j's and the g_i's. These relations will be used in the paper.

Lemma 1 ([6] Lemma 1)**.** *For the idempotents e_i and for $1 \leq i,\, j \leq n-1$ the following relations hold:*

$$\begin{array}{rcl} t_j e_i & = & e_i t_j \\ e_{i+1} g_i & = & g_i e_{i,i+2} \\ e_i g_j & = & g_j e_i, \quad \text{for } j \neq i-1, i+1 \\ e_j g_i g_j & = & g_i g_j e_i \quad \text{for } |i-j| = 1 \\ e_i e_{i+1} & = & e_i e_{i,i+2} \\ e_i e_{i+1} & = & e_{i,i+2} e_{i+1}. \end{array}$$

Using the multiplication formulas (2.14), the second author proved in [11] that $\mathrm{Y}_{d,n}(u)$ has the following standard linear basis:

$$\{t_1^{a_1} \dots t_n^{a_n} g_w \,|\, a_i \in \mathbb{Z}/d\mathbb{Z},\, w \in S_n\}. \tag{2.18}$$

Notice now that using relations (2.10) and (2.11) one can write any monomial $\mathfrak{m}$ in $\mathrm{Y}_{d,n}(u)$ in the following form:

$$\mathfrak{m} = t_1^{a_1} \dots t_n^{a_n} \mathfrak{m}'$$

where $\mathfrak{m}' = g_{i_1} \dots g_{i_n}$. We say that every monomial in $\mathrm{Y}_{d,n}(u)$ has the *splitting property,*, which is in fact inherited from the framed braid group $\mathcal{F}_n$. That is, one can separate the *framing part* from the *braiding part.*

Further, we have an inductive basis of the Yokonuma–Hecke algebra, which is used in the proof of the main theorem, Theorem 5. More precisely:

Proposition 1 ([11] Proposition 8)**.** *Every element in $\mathrm{Y}_{d,n+1}(u)$ is a unique linear combination of words, each of one of the following types:*

$$\mathfrak{m}_n g_n g_{n-1} \dots g_i t_i^k \quad \text{or} \quad \mathfrak{m}_n t_{n+1}^k$$

where $0 \leq k \leq d-1$ and $\mathfrak{m}_n$ is a word in the inductive basis of $\mathrm{Y}_{d,n}(u)$.

2.4. Using the above basis, the second author proved that $\mathrm{Y}_{d,n}(u)$ supports a unique Markov trace. We have the following theorem:

Theorem 2 ([11] Theorem 12)**.** *For indeterminates z, x_1, $\dots, x_{d-1}$ there exists a unique linear Markov trace* tr*:*

$$\mathrm{tr} : \cup_{n=1}^{\infty} \mathrm{Y}_{d,n}(u) \longrightarrow \mathbb{C}[z, x_1, \dots, x_{d-1}]$$

defined inductively on n by the following rules:

$$\begin{array}{rcll} \mathrm{tr}(ab) & = & \mathrm{tr}(ba) & \\ \mathrm{tr}(1) & = & 1 & \\ \mathrm{tr}(a g_n) & = & z\,\mathrm{tr}(a) & (\textit{Markov property}) \\ \mathrm{tr}(a t_{n+1}^s) & = & x_s \mathrm{tr}(a) & (s = 1, \dots, d-1) \end{array}$$

where $a, b \in \mathrm{Y}_{d,n}(u)$.

Using the trace rules of Theorem 2 and including $x_0 := 1$, we deduce that $\mathrm{tr}(e_i)$ takes the same value for all i, and this value is denoted by E:

$$E := \mathrm{tr}(e_i) = \frac{1}{d} \sum_{s=0}^{d-1} x_s x_{d-s}.$$

Moreover, we also define *the shift by m of E*, where $0 \leq m \leq d-1$, by:

$$E^{(m)} := \mathrm{tr}(e_i^{(m)}) = \frac{1}{d} \sum_{s=0}^{d-1} x_{m+s} x_{d-s}.$$

Notice that $E = E^{(0)}$.

3. Fourier transform and the E–system

An important tool in the proof of the main theorem are some classical identities of harmonic analysis on the group of integers modulo d. More precisely, we will use identities linking the convolution product and the product by coordinates through the Fourier transform. These tools were also used in solving the so–called E–system, see Appendix [16]. Thus, in this section we shall give some notations and recall some well–known and useful facts of the Fourier transform along with some facts for the E–system.

3.1. In our setting it is convenient to see the group of integers modulo d as the group C_d. Hence, the *product by coordinates* in $\mathbb{C}C_d$ is defined by the formula

$$\left(\sum_{r=0}^{d-1} a_r t^r\right) \cdot \left(\sum_{s=0}^{d-1} b_s t^s\right) = \sum_{i=0}^{d-1} a_i b_i t^i$$

and the *convolution product* is defined by the formula:

$$\left(\sum_{r=0}^{d-1} a_r t^r\right) * \left(\sum_{s=0}^{d-1} b_s t^s\right) = \sum_{r=0}^{d-1} \left(\sum_{s=0}^{d-1} a_s b_{r-s}\right) t^r \tag{3.1}$$

In order to define the Fourier transform on C_d we need to introduce the following elements:

$$\mathbf{i}_a := \sum_{s=0}^{d-1} \chi_a(s) t^s \qquad (a \in \mathbb{Z}/d\mathbb{Z}).$$

where the χ_k's denote the characters of the group $\mathbb{Z}/d\mathbb{Z}$, namely:

$$\chi_k(m) = \cos\frac{2\pi km}{d} + i\sin\frac{2\pi km}{d} \qquad (k, m \in \mathbb{Z}/d\mathbb{Z})$$

One can verify that:

$$\mathbf{i}_a * \mathbf{i}_b = \begin{cases} d\,\mathbf{i}_a & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

On the other hand, we shall denote by δ_a the element of the canonical linear basis of $\mathbb{C}C_d$, that is, $\delta_a := t^a$. It is clear that:

$$\delta_a \cdot \delta_b = \begin{cases} \delta_a & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

The *Fourier transform* is the linear automorphism on $\mathbb{C}C_d$, defined as:

$$y := \sum_{r=0}^{d-1} a_r t^r \mapsto \widehat{y} := \sum_{s=0}^{d-1} (y * \mathbf{i}_s)(0) t^s \tag{3.2}$$

where $(y * \mathbf{i}_s)(0)$ denote the coefficient of δ_0 in the convolution $y * \mathbf{i}_s$.

The next proposition gathers the most important properties of the Fourier transform used in the paper.

Proposition 2 ([22])**.** *For any y and y' in $\mathbb{C}C_d$, we have:*

(1) $\widehat{y * y'} = \widehat{y} \cdot \widehat{y'}$
(2) $\widehat{y \cdot y'} = d^{-1}\widehat{y} * \widehat{y'}$
(3) $\widehat{\delta_a} = \mathbf{i}_{-a}$

(4) $\widehat{\mathbf{i}}_a = d\delta_a$
(5) If $y = \sum_{r=0}^{d-1} a_r t^r$, *then*

$$\widehat{\widehat{y}} = d\sum_{r=0}^{d-1} a_{-r}t^r$$

Finally, we note that the elements in the group algebra $\mathbb{C}C_d$ can also be identified to the set of functions $f : \mathbb{Z}/d\mathbb{Z} \to \mathbb{C}$, where the identification is as follows:

$$(f : \mathbb{Z}/d\mathbb{Z} \to \mathbb{C}) \mapsto \sum_{k=0}^{d-1} f(k)t^k \in \mathbb{C}C_d.$$

Some times we shall use this identification, since it makes some computations more comfortable.

3.2. The E–system is a non–linear system of equations that was introduced in order to find the necessary and sufficient conditions that need to be applied on the parameters x_i of tr so that the definition of link invariants from the Yokonuma–Hecke algebra would be possible [16].

In [16, Appendix] the full set of solutions of the E–system is given. To do that, Gérardin interpreted the solutions of the E–system as certain complex functions x over $\mathbb{Z}/d\mathbb{Z}$. More precisely, the function x is defined by mapping 0 to $x_0 := 1$ and k is mapped in the parameters x_k of tr. Thus, the function x is identified as:

$$x = \sum_{k=0}^{d-1} x_k t^k$$

Notice that the coefficient of t^k in the convolution $x * x$ (respectively $x * x * x$) is:

$$E^{(k)} \quad \text{(respectively} \quad \mathrm{tr}(e_1^{(k)}e_2))$$

(see Lemma 2 [6]). Equivalently, we have

$$x * x : k \mapsto E^{(k)} \quad \text{and} \quad x * x * x : k \mapsto \mathrm{tr}(e_1^{(k)}e_2) \tag{3.3}$$

Definition 3. We say that the $(d-1)$–tuple of complex numbers $(x_1, \ldots, x_{d-1})$ satisfies the E–*condition* if $x_1, \ldots, x_{d-1}$ satisfy the following system of non–linear equations in $\mathbb{C}$, the E–*system*:

$$E^{(m)} \quad = \quad \mathrm{x}_m E \qquad (1 \leq m \leq d-1). \tag{3.4}$$

More precisely, in [16, Appendix], Gérardin achieved to solve the E–system by interpreting each equation of (3.4) as the value at m of the convolution product of the functional $x : s \mapsto x_s$ by itself in the complex algebra $\mathbb{C}C_d$ and then using some tools of harmonic analysis on finite groups. It turns out that the solutions of the E–system are parametrized by the non–empty subsets of the cyclic group $\mathbb{Z}/d\mathbb{Z}$, that is, given a non–empty subset D of $\mathbb{Z}/d\mathbb{Z}$, the corresponding solution is:

$$x_{_D} = \frac{1}{|D|}\sum_{k\in D} \chi_k \tag{3.5}$$

Remark 2. It is worth noting that the formula for the solutions of the E–system can be interpreted as a generalization of the Ramanujan's sum. Indeed, by taking the subset P of $\mathbb{Z}/d\mathbb{Z}$ consisting of the numbers coprime to d, then the solution parametrized by P is, up to the factor $|P|$, the Ramanujan's sum $c_d(k)$ (see [21]).

Notice now, that the characters of C_d are given by $t^a \mapsto \chi_k(a)$, where k runs in $\mathbb{Z}/d\mathbb{Z}$; thus, we shall denote also this character as χ_k. We note also that the solutions x_D can be seen as elements in $\mathbb{C}C_d$, namely:

$$x_D = \sum_{j=0}^{d-1} x_j t^j \tag{3.6}$$

where $x_j = \frac{1}{|D|}\sum_{m\in D}\chi_m(j)$.

We finish this section with a theorem which yields the main connection among the solutions of the E–system and the trace tr.

Theorem 3 ([13] Theorem 7). *If the trace parameters $(x_1,\ldots,x_{d-1})$ satisfy the* E*–condition, then*

$$\mathrm{tr}(\alpha e_n) = \mathrm{tr}(\alpha)\mathrm{tr}(e_n) \qquad (a \in \mathrm{Y}_{d,n}(u)).$$

4. A Framization of the Temperley–Lieb algebra

As discussed during the Introduction, the Yokonuma–Hecke algebra can be interpreted as the framization of the Iwahori–Hecke algebra, which is a knot algebra. Thus a natural question arises, the definition of a framization for the knot algebra Temperley–Lieb. Considering the fact that the Temperley–Lieb algebra can be defined as a quotient of the Iwahori–Hecke algebra, it is natural to try and define a framization of the Temperley–Lieb algebra as a quotient of the Yokonuma–Hecke algebra. Recall now that the defining ideal of the Temperley–Lieb algebra (Definition 1) is generated by the Steinberg elements which are related to the subgroups $\langle s_i, s_{i+1}\rangle$ of S_n, for all i. These subgroups can be also regarded as subgroups of $C_{d,n}$. Therefore, using the multiplication rule of Eq. 2.14 we are able to define the analogous Steinberg elements $g_{i,i+1}$ in $\mathrm{Y}_{d,n}(u)$,

$$g_{i,i+1} := \sum_{w\in\langle s_i,s_{i+1}\rangle} g_w \qquad \text{for all } i$$

In [6, Definition 5] we defined a potential candidate for the framization of the Temperley–Lieb algebra, the *Yokonuma–Temperley–Lieb algebra*, denoted $\mathrm{YTL}_n(u)$, which is the quotient of $\mathrm{Y}_{d,n}(u)$ over the 2–sided ideal generated by all the $g_{i,i+1}$'s for all i. It is not difficult to show that this ideal is in fact principal and it is generated by the element $g_{1,2}$. Moreover, the necessary and sufficient conditions for the trace tr to pass through to $\mathrm{YTL}_n(u)$ were studied [6, Theorem 5]. Unfortunately, these conditions turn out to be too strong. Namely, the trace parameters x_i must be d^{th} roots of unity, furnishing obvious, special solutions of the E–system and resulting in loss of the framing information on the level of the invariant. Thus, the framed analogues of the Jones polynomial obtained from the algebras $\mathrm{YTL}_{d,n}(u)$ turn out to be of no interest. For this reason, $\mathrm{YTL}_{d,n}(u)$ is discarded as framization of $\mathrm{TL}_n(u)$. However, using the results of [1], the classical Jones polynomial can be still recovered through these algebras by representing the Artin braid group B_n in $\mathrm{Y}_{d,n}(u)$, considering the t_i's as formal generators, and then taking the quotient over the ideal that is generated by the $g_{i,i+1}$'s [6, Corollary 2]. Finally, we note that the representation theory of this algebra has been studied extensively in [2].

Given the fact that $\mathrm{Y}_{d,n}(u)$ can be considered as a u–deformation of $\mathbb{C}C_{d,n}$ (recall the discussion in Section 2.3), it is natural to consider subgroups of $C_{d,n}$ that involve in their generating set the framing generators of the i–th and j–th strands along with $\langle s_i, s_j\rangle$. Therefore, we have considered the following subgroups of $C_{d,n}$:

$$C_{i,i+1} := \langle t_i, t_{i+1}, t_{i+2}\rangle \rtimes \langle s_i, s_{i+1}\rangle \quad \text{for all } i.$$

Further, notice that these subgroups are isomorphic to $C_{d,3}$ and thus analogous to the classical case. We define now the elements $c_{i,i+1}$ in $\mathrm{Y}_{d,n}(u)$ as follows:

$$c_{i,i+1} = \sum_{c \in C_{i,i+1}} g_c$$

We then have the following definition:

Definition 4. For $n \geq 3$, we define the algebra $\mathrm{CTL}_{d,n}(u)$ as the quotient of the algebra $\mathrm{Y}_{d,n}(u)$ by the two–sided ideal of generated by the $c_{i,i+1}$'s, for all i. We shall call $\mathrm{CTL}_{d,n}(u)$ the *Complex Reflection Temperley–Lieb algebra.*

As it will be shown in Theorem 6 the necessary and sufficient conditions such that tr passes to $\mathrm{CTL}_{d,n}(u)$ are, contrary to the case of $\mathrm{YTL}_{d,n}(u)$, too relaxed. In particular, the E–condition does not appear on the trace parameters x_i, which is necessary for the definition of link invariants through tr (see [16, Section 4] for details). Thus, in order to define link invariants from the algebras $\mathrm{CTL}_{d,n}(u)$, the E–condition must be imposed on the x_i's resulting in invariants that are naturally obtained from a different construction (see below). Therefore, both of the natural definitions for the framed analogue of the Temperley–Lieb algebra that are derived from $\mathrm{Y}_{d,n}(u)$ should be discarded as possible framizations of the Temperley–Lieb algebra.

Since the conditions such that the trace tr passes to the quotient algebras are either too strong or too relaxed, it indicates that the desired framization of the Temperley–Lieb algebra should be an intermediate algebra between $\mathrm{YTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$. Regarding the definitions of these two algebras, it is reasonable to construct the framization in an analogous way using an intermediate subgroup between $\langle s_i, s_{i+1}\rangle$ and $C_{i,i+1}$. Thus, we consider the following subgroups of $C_{d,n}$,

$$H_{i,i+1} := \langle t_i t_{i+1}^{-1}, t_{i+1} t_{i+2}^{-1}\rangle \rtimes \langle s_i, s_{i+1}\rangle \quad \text{for all } i.$$

We now introduce the following elements:

$$r_{i,i+1} := \sum_{x \in H_{i,i+1}} g_x \quad \text{for all } i.$$

Definition 5. For $n \geq 3$, the *Framization of the Temperley–Lieb algebra*, denoted $\mathrm{FTL}_{d,n}(u)$, is defined as the quotient $\mathrm{Y}_{d,n}(u)$ over the two–sided ideal generated by the elements $r_{i,i+1}$, for all i.

The reason of considering $\mathrm{FTL}_{d,n}(u)$ as a framization of the Temperley–Lieb algebra is due to the fact that through this algebra the definition of a non–trivial framization of the Jones polynomial is possible. Therefore, the necessary and sufficient conditions for the trace tr to pass to $\mathrm{FTL}_{d,n}(u)$ are determined in Theorem 6, while in Section 7 we discuss the invariants derived from this algebra.

Remark 3. Notice that when $d = 1$, the Yokonuma–Hecke algebra collapses to the Iwahori–Hecke algebra, hence it follows that $\mathrm{YTL}_{1,n}(u)$ also collapses to $\mathrm{TL}_n(u)$. Moreover, in this case the subgroups $H_{i,i+1}$ and $C_{i,i+1}$ also collapse to $\langle s_i, s_{i+1}\rangle$. Hence, $\mathrm{FTL}_{1,n}(u)$ and $\mathrm{CTL}_{1,n}(u)$ collapse to $\mathrm{TL}_n(u)$ too.

4.1. We shall now show how the algebras defined above are related. We start noting that

$$C_{i,i+1} = H_{i,i+1} \rtimes C_d$$

Indeed, we have a homomorphism ϕ from $C_{i,i+1}$ onto C_d defined by $\phi(x) = t_i^{a+b+c}$, where C_d is regarded as the cyclic group generated by t_i and x is written uniquely as $x = t_i^a t_{i+1}^b t_{i+2}^c w$, with

$w \in \langle s_i, s_{i+1}\rangle$. Thus, $\ker \phi = H_{i,i+1}$ and $\phi\big|_{H_{i,i+1}} = \mathrm{id}_{C_d}$, then $C_{i,i+1} = H_{i,i+1} \rtimes C_d$. Therefore, given $x \in C_{i,i+1}$ we have a unique decomposition $x = t_i^k y$, where $0 \le k \le d-1$ and $y \in H_{i,i+1}$. This decomposition of the elements of $C_{i,i+1}$ together with the multiplication rule Eq. 2.14, implies $g_x = t_i^k g_y$. Then

$$c_{i,i+1} = \sum_{\substack{0\le k\le d-1\\ y\in H_{i,i+1}}} t_i^k g_y$$

hence

$$c_{i,i+1} = \left(\sum_{k=0}^{d-1} t_i^k\right) r_{i,i+1} \tag{4.1}$$

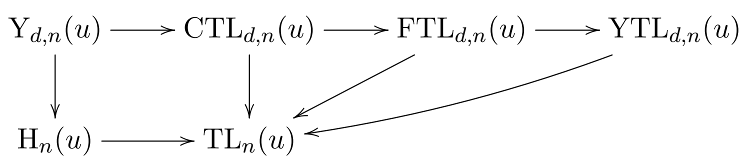
Notice that every x in $H_{i,i+1}$ can be written in the form $x = t_i^a t_{i+1}^{-a} t_{i+1}^b t_{i+2}^{-b} w = t_i^a t_{i+1}^{b-a} t_{i+2}^{-b} w$, where $w \in \langle s_i, s_{i+1}\rangle$. Therefore, by using again the multiplication rule of Eq. 2.14, we have that $g_x = t_i^a t_{i+1}^{b-a} t_{i+2}^{-b} g_w$. Then,

$$r_{i,i+1} = \sum_{\substack{a,b=0\\ w\in\langle s_i,s_{i+1}\rangle}}^{d-1} t_i^a t_{i+1}^{b-a} t_{i+2}^{-b} g_w = \left(\sum_{a,b=0}^{d-1} t_i^a t_{i+1}^{b-a} t_{i+2}^{-b}\right)\left(\sum_{w\in\langle s_i,s_{i+1}\rangle} g_w\right)$$

hence

$$r_{i,i+1} = e_i e_{i+1} g_{i,i+1} \tag{4.2}$$

Equation 4.1 implies that $\mathrm{CTL}_{d,n}(u)$ projects onto $\mathrm{FTL}_{d,n}(u)$ while Eq. 4.2 implies that $\mathrm{FTL}_{d,n}(u)$ projects onto $\mathrm{YTL}_{d,n}(u)$. Moreover, we have the following commutative diagram of epimorphisms:



where the non–horizontal arrows are defined by mapping the framing generators to 1.

4.2. It is known that the defining ideal of the Temperley–Lieb algebra is principal. We are going now to prove that the defining ideals of $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ respectively are principal ideals too. The method used in the proof is standard and we start with two technical lemmas.

Lemma 2. *The following hold in* $\mathrm{Y}_{d,n}(u)$ *for all* $i = 1,\dots,n-1$ *and* $j = 1,\dots,n$*:*

(1) $t_j = (g_1 \dots g_{n-1})^{j-1}\, t_1\, (g_1 \dots g_{n-1})^{-(j-1)}$
(2) $g_i = (g_1 \dots g_{n-1})^{i-1}\, g_1\, (g_1 \dots g_{n-1})^{-(i-1)}$

Proof. The statement (1) is true for $j = 2$. Indeed:

$$\begin{aligned}(g_1 \dots g_{n-1})\, t_1\, (g_1 \dots g_{n-1})^{-1} &= g_1 t_1 g_2 \dots g_{n-1} (g_1 \dots g_{n-1})^{-1}\\ &= t_2 (g_1 \dots g_{n-1})(g_1 \dots g_{n-1})^{-1}\\ &= t_2.\end{aligned}$$

Suppose that the statement is true for $j=k$. We will show that the statement holds for $j=k+1$. We have:

$$\begin{aligned}(g_1\dots g_{n-1})^k t_1 (g_1\dots g_{n-1})^{-k} &= (g_1\dots g_{n-1})(g_1\dots g_{n-1})^{k-1} t_1 (g_1\dots g_{n-1})^{-(k-1)}(g_1\dots g_{n-1})^{-1}\\ &= (g_1\dots g_{n-1}) t_k (g_1\dots g_{n-1})^{-1}\\ &= g_1\dots g_{k-1} g_k\, t_k\, g_{k+1}\dots g_{n-1}(g_1\dots g_{n-1})^{-1}\\ &= t_{k+1}(g_1\dots g_{n-1})(g_1\dots g_{n-1})^{-1}\\ &= t_{k+1}.\end{aligned}$$

The second statement of the Lemma has been proved in [6, (2) Lemma 5].

□

Lemma 3. *The following hold in* $\mathrm{Y}_{d,n}(u)$ *for all* $i=1,\dots,n-2$ *and* $0\le a,b,c\le d-1$*:*

$$\begin{array}{lrcl}
(1) & t_i^a t_{i+1}^b t_{i+2}^c & = & (g_1\dots g_{n-1})^{i-1}\, t_1^a t_2^b t_3^c\, (g_1\dots g_{n-1})^{-(i-1)}\\
(2) & t_i^a t_{i+1}^b t_{i+2}^c g_i & = & (g_1\dots g_{n-1})^{i-1}\, t_1^a t_2^b t_3^c g_1\, (g_1\dots g_{n-1})^{-(i-1)}\\
(3) & t_i^a t_{i+1}^b t_{i+2}^c g_{i+1} & = & (g_1\dots g_{n-1})^{i-1}\, t_1^a t_2^b t_3^c g_2\, (g_1\dots g_{n-1})^{-(i-1)}\\
(4) & t_i^a t_{i+1}^b t_{i+2}^c g_i g_{i+1} & = & (g_1\dots g_{n-1})^{i-1}\, t_1^a t_2^b t_3^c g_1 g_2\, (g_1\dots g_{n-1})^{-(i-1)}\\
(5) & t_i^a t_{i+1}^b t_{i+2}^c g_{i+1} g_i & = & (g_1\dots g_{n-1})^{i-1}\, t_1^a t_2^b t_3^c g_2 g_1\, (g_1\dots g_{n-1})^{-(i-1)}\\
(6) & t_i^a t_{i+1}^b t_{i+2}^c g_i g_{i+1} g_i & = & (g_1\dots g_{n-1})^{i-1}\, t_1^a t_2^b t_3^c g_1 g_2 g_1\, (g_1\dots g_{n-1})^{-(i-1)}
\end{array}$$

Proof. We will demonstrate the proof the statement (1) and (6). The other statements are proved in an analogous manner. For statement (1) we have from Lemma 2:

$$\begin{aligned}t_i^a t_{i+1}^b t_{i+2}^c &= (g_1\dots g_{n-1})^{i-1} t_1^a (g_1\dots g_{n-1})^{-(i-1)} (g_1\dots g_{n-1})^{i} t_1^b (g_1\dots g_{n-1})^{-i}\\ &\quad\cdot (g_1\dots g_{n-1})^{i+1} t_1^c (g_1\dots g_{n-1})^{-(i+1)}\\ &= (g_1\dots g_{n-1})^{i-1} t_1^a (g_1\dots g_{n-1}) t_1^b (g_1\dots g_{n-1})^{-1} (g_1\dots g_{n-1})^{-(i-1)}\\ &\quad\cdot (g_1\dots g_{n-1})^{i-1} (g_1\dots g_{n-1})^{2} t_1^c (g_1\dots g_{n-1})^{-2} (g_1\dots g_{n-1})^{-(i-1)}\\ &= (g_1\dots g_{n-1})^{i-1} t_1^a t_2^b t_3^c (g_1\dots g_{n-1})^{-(i-1)}.\end{aligned}$$

For statement (6) we have from Lemma 2:

$$\begin{aligned}t_i^a t_{i+1}^b t_{i+2}^c g_i g_{i+1} g_i &= (g_1\dots g_{n-1})^{i-1} t_1^a (g_1\dots g_{n-1})^{-(i-1)} (g_1\dots g_{n-1})^{i} t_1^b (g_1\dots g_{n-1})^{-i}\\ &\quad\cdot (g_1\dots g_{n-1})^{i+1} t_1^c (g_1\dots g_{n-1})^{-(i+1)} (g_1\dots g_{n-1})^{i-1} g_1 (g_1\dots g_{n-1})^{-(i-1)}\\ &\quad\cdot (g_1\dots g_{n-1})^{i} g_1 (g_1\dots g_{n-1})^{-i} (g_1\dots g_{n-1})^{i-1} g_1 (g_1\dots g_{n-1})^{-(i-1)}\\ &= (g_1\dots g_{n-1})^{i-1} t_1^a (g_1\dots g_{n-1}) t_1^b (g_1\dots g_{n-1})^{-1} (g_1\dots g_{n-1})^{-(i-1)}\\ &\quad\cdot (g_1\dots g_{n-1})^{i-1} (g_1\dots g_{n-1})^{2} t_1^c (g_1\dots g_{n-1})^{-2} (g_1\dots g_{n-1})^{-(i-1)}\\ &\quad\cdot (g_1\dots g_{n-1})^{i-1} g_1 (g_1\dots g_{n-1})^{-(i-1)} (g_1\dots g_{n-1})^{i-1} (g_1\dots g_{n-1})\\ &\quad\cdot g_1 (g_1\dots g_{n-1})^{-1} (g_1\dots g_{n-1})^{-(i-1)} (g_1\dots g_{n-1})^{i-1} g_1 (g_1\dots g_{n-1})^{-(i-1)}\\ &= (g_1\dots g_{n-1})^{i-1} t_1^a t_2^b t_3^c g_1 g_2 g_1 (g_1\dots g_{n-1})^{-(i-1)}.\end{aligned}$$

□

Theorem 4. *The defining ideal of* $\mathrm{FTL}_{d,n}(u)$ *is generated by any single element* $r_{i,i+1}$.

Proof. It is enough to prove that $r_{i,i+1} = (g_1 \dots g_{n-1})^{(i-1)} r_{1,2} (g_1 \dots g_{n-1})^{-(i-1)}$. Expanding $r_{1,2}$, we have

$$(g_1 \dots g_{n-1})^{i-1} r_{1,2} (g_1 \dots g_{n-1})^{-(i-1)} = \sum_{\substack{a+b+c=0 \\ w \in S_3}} (g_1 \dots g_{n-1})^{i-1} \, t_1^a t_2^b t_3^c \, g_w \, (g_1 \dots g_{n-1})^{-(i-1)}$$

Applying now Lemma 3 in each factor of the summation, we arrive to

$$(g_1 \dots g_{n-1})^{i-1} r_{1,2} (g_1 \dots g_{n-1})^{-(i-1)} = r_{i,i+1} \tag{4.3}$$

Therefore the proof is concluded. □

From Theorem 4 we have the following corollaries:

Corollary 1. $\mathrm{FTL}_{d,n}(u)$ *is the algebra generated by* $t_1, \dots, t_n, g_1, \dots, g_{n-1}$ *which are subject to the defining relations of* $\mathrm{Y}_{d,n}(u)$ *and the relation:*

$$r_{1,2} = 0. \tag{4.4}$$

Corollary 2. *The defining ideal of* $\mathrm{CTL}_{d,n}(u)$ *is generated by any single element* $c_{i,i+1}$. *Hence* $\mathrm{CTL}_{d,n}(u)$ *can be presented by* $t_1, \dots, t_n, g_1, \dots, g_{n-1}$ *together with the defining relations of* $\mathrm{Y}_{d,n}(u)$ *and the relation:*

$$c_{1,2} = 0. \tag{4.5}$$

Proof. As in the proof Theorem 4, it is enough to prove that $c_{i,i+1} = \gamma c_{1,2} \gamma^{-1}$ where $\gamma := (g_1 \dots g_{n-1})^{i-1}$. From Eq. 4.1, we have

$$\gamma c_{1,2} \gamma^{-1} = \left(\sum_{0 \le k \le d-1} \gamma t_1^k \gamma^{-1} \right) \gamma r_{1,2} \gamma^{-1}$$

By using now (1) of Lemma 2 and Eq. 4.3, it follows that $\gamma c_{1,2} \gamma^{-1} = c_{i,i+1}$. The rest of the statement is now clear. □

4.3. By using the analogous transformation to Eq. 2.3, we obtain presentations for $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ through non–invertible generators. More precisely, set

$$\ell_i := \frac{1}{u+1}(g_i + 1).$$

Then, by a direct computation, which is not necessary to reproduce here, we obtain the following:

Proposition 3. *The algebra* $\mathrm{FTL}_{d,n}(u)$ *can be presented with generators:*

$$\ell_1, \dots, \ell_{n-1}, t_1, \dots, t_n$$

subject to the following relations:

$$
\begin{aligned}
t_i^d = 1, \quad t_i t_j &= t_j t_i \\
\ell_i t_j &= t_j \ell_i, \quad \textit{for} \quad |i-j| > 1 \\
\ell_i \ell_j &= \ell_j \ell_i, \quad \textit{for} \quad |i-j| > 1 \\
\ell_i t_i &= t_{i+1} \ell_i + \frac{1}{u+1}(t_i - t_{i+1}) \\
\ell_i t_{i+1} &= t_i \ell_i + \frac{1}{u+1}(t_{i+1} - t_i) \\
\ell_i^2 &= \frac{(u-1)e_i + 2}{u+1}\,\ell_i, \\
\ell_i \ell_{i+1} \ell_i - \frac{(u-1)e_i + 1}{(u+1)^2}\,\ell_i &= \ell_{i+1}\ell_i\ell_{i+1} - \frac{(u-1)e_{i+1} + 1}{(u+1)^2}\,\ell_{i+1} \\
e_i e_{i+1} \ell_i \ell_{i+1} \ell_i &= \frac{u}{(u+1)^2}\, e_i e_{i+1} \ell_i
\end{aligned}
$$

Proposition 4. *The algebra* $\mathrm{CTL}_{d,n}(u)$ *can be presented with generators:*

$$\ell_1, \ldots, \ell_{n-1}, t_1, \ldots, t_n$$

subject to the following relations:

$$
\begin{aligned}
t_i^d = 1, \quad t_i t_j &= t_j t_i \\
\ell_i t_j &= t_j \ell_i, \quad \textit{for} \quad |i-j| > 1 \\
\ell_i \ell_j &= \ell_j \ell_i, \quad \textit{for} \quad |i-j| > 1 \\
\ell_i t_i &= t_{i+1} \ell_i + \frac{1}{u+1}(t_i - t_{i+1}) \\
\ell_i t_{i+1} &= t_i \ell_i + \frac{1}{u+1}(t_{i+1} - t_i) \\
\ell_i^2 &= \frac{(u-1)e_i + 2}{u+1}\,\ell_i, \\
\ell_i \ell_{i+1} \ell_i - \frac{(u-1)e_i + 1}{(u+1)^2}\,\ell_i &= \ell_{i+1}\ell_i\ell_{i+1} - \frac{(u-1)e_{i+1} + 1}{(u+1)^2}\,\ell_{i+1} \\
\sum_{k=0}^{d-1} e_i^{(k)} e_{i+1} \ell_i \ell_{i+1} \ell_i &= \sum_{k=0}^{d-1} e_i^{(k)} e_{i+1} \frac{u}{(u+1)^2}\,\ell_i
\end{aligned}
$$

Remark 4. We know that a linear basis of the Temperley–Lieb algebra can be constructed from the interpretation of the generators f_i as diagrams. In virtue of Remark 3, then it is desirable to construct a basis of $\mathrm{FTL}_{d,n}(u)$ from the presentation given in Proposition 3. Unfortunately, we do not have a diagrammatic interpretation for the generators l_i yet. Also, an explicit linear basis for both $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ is still an open problem. Indeed, the construction of such a basis for $\mathrm{FTL}_{d,n}(u)$ seems like a non–trivial task. However, in [3], by using the representation theory of Yokonuma–Hecke algebra (see [4]), Chlouveraki and Pouchin have found a formula for the dimension of $\mathrm{FTL}_{d,n}(u)$. More precisely, they proved:

$$\dim \mathrm{FTL}_{d,n}(u) = \sum_{|k_1|+|k_2|+\ldots+|k_d|=n} \left(\frac{n!}{|k_1|! \ldots |k_d|!} \right)^2 c_{|k_1|} \ldots c_{|k_d|} \tag{4.6}$$

where $n = (k_1, \ldots, k_d)$ is a d–partition of n, k_i is a usual partition whose Young diagram has at most two columns and c_k is the k–th Catalan number. We shall show an explicit basis for the case $n = 3$ and $d = 2$. To do that, we shall use the Corollary 1 to describe $\mathrm{FTL}_{d,n}(u)$. Therefore, we have that the inductive basis of $\mathrm{Y}_{2,3}(u)$ described in Proposition 1 spans $\mathrm{FTL}_{2,3}(u)$. From Eq. 4.6 it follows that the dimension of $\mathrm{FTL}_{2,3}(u)$ is 46. Thus, a linear basis of $\mathrm{FTL}_{2,3}(u)$ can be obtained by omitting 2 linearly dependent elements, from the 48 elements of the inductive basis of $\mathrm{Y}_{2,3}(u)$. Now Eq. 4.2, implies that $t_1^m e_1 e_2 g_{1,2} = 0$, for $m = 0, 1$. Therefore, it follows that the elements 1 and t_1 can be expressed, in $\mathrm{FTL}_{2,3}(u)$, as a linear combination of the standard basis of $\mathrm{Y}_{2,3}(u)$. Thus, we have a basis of $\mathrm{FTL}_{2,3}(u)$ obtained from the inductive basis $\mathrm{FTL}_{2,3}(u)$ by omitting the monomials 1 and t_1.

4.4. We finish this section with two technical lemmas which will be used in the proof of Theorems 5 and 6.

Lemma 4. *For the element $g_{1,2}$ we have in $\mathrm{Y}_{d,n}(u)$ the following:*

$$\begin{array}{lrcl}
(1) & g_1 g_{1,2} & = & [1 + (u-1)e_1]g_{1,2} \\
(2) & g_2 g_{1,2} & = & [1 + (u-1)e_2]g_{1,2} \\
(3) & g_1 g_2 g_{1,2} & = & [1 + (u-1)e_1 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2]g_{1,2} \\
(4) & g_2 g_1 g_{1,2} & = & [1 + (u-1)\, e_2 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2]g_{1,2} \\
(5) & g_1 g_2 g_1 g_{1,2} & = & [1 + (u-1)(e_1 + e_2 + e_{1,3}) + (u-1)^2 (u+2)\, e_1 e_2]g_{1,2}.
\end{array}$$

Proof. See [6, Lemma 5]. Cf. [12, Lemma 7.5]. □

Lemma 5. *For the element $r_{1,2}$ we have in $\mathrm{Y}_{d,n}(u)$:*

$$\begin{array}{lrcl}
(1) & g_1 r_{1,2} & = & [1 + (u-1)e_1]r_{1,2} \\
(2) & g_2 r_{1,2} & = & [1 + (u-1)e_2]r_{1,2} \\
(3) & g_1 g_2 r_{1,2} & = & [1 + (u-1)e_1 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2]r_{1,2} \\
(4) & g_2 g_1 r_{1,2} & = & [1 + (u-1)\, e_2 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2]r_{1,2} \\
(5) & g_1 g_2 g_1 r_{1,2} & = & [1 + (u-1)(e_1 + e_2 + e_{1,3}) + (u-1)^2 (u+2)\, e_1 e_2]r_{1,2}.
\end{array}$$

Proof. In order to prove this lemma we will make extensive use of Lemmas 4 and 1. For statement (1) we have:

$$\begin{aligned}
g_1 r_{1,2} = g_1 e_1 e_2 g_{1,2} &= e_1 e_{1,3} g_1 g_{1,2} \\
&= e_1 e_2 [1 + (u-1)e_1]g_{1,2} \\
&= [1 + (u-1)e_1]e_1 e_2 g_{1,2} \\
&= [1 + (u-1)e_1]r_{1,2}
\end{aligned}$$

In an analogous way we prove statement (2). For statement (3) we have that:

$$\begin{aligned}
g_1 g_2 r_{1,2} = g_1 g_2 e_1 e_2 g_{1,2} &= e_2 e_{1,3} g_1 g_2 g_{1,2} \\
&= e_1 e_2 [1 + (u-1)e_1 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2]g_{1,2} \\
&= [1 + (u-1)e_1 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2]e_1 e_2 g_{1,2} \\
&= [1 + (u-1)e_1 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2]r_{1,2}
\end{aligned}$$

In an analogous way we prove statement (4). Finally, we have for statement (5):

$$\begin{aligned} g_1g_2g_1r_{1,2} &= g_1g_2g_1e_1e_2g_{1,2} \\ &= e_1e_2g_1g_2g_1g_{1,2} \\ &= e_1e_2[1+(u-1)(e_1+e_2+e_{1,3})+(u-1)^2(u+2)\,e_1e_2]g_{1,2} \\ &= [1+(u-1)(e_1+e_2+e_{1,3})+(u-1)^2(u+2)\,e_1e_2]e_1e_2g_{1,2} \\ &= [1+(u-1)(e_1+e_2+e_{1,3})+(u-1)^2(u+2)\,e_1e_2]r_{1,2}. \end{aligned}$$

□

Lemma 6. *For the element $c_{1,2}$ we have in $\mathrm{Y}_{d,n}(u)$:*

$$\begin{array}{lrcl} (1) & g_1c_{1,2} & = & [1+(u-1)e_1]c_{1,2} \\ (2) & g_2c_{1,2} & = & [1+(u-1)e_2]c_{1,2} \\ (3) & g_1g_2c_{1,2} & = & [1+(u-1)e_1+(u-1)e_{1,3}+(u-1)^2e_1e_2]c_{1,2} \\ (4) & g_2g_1c_{1,2} & = & [1+(u-1)\,e_2+(u-1)e_{1,3}+(u-1)^2e_1e_2]c_{1,2} \\ (5) & g_1g_2g_1c_{1,2} & = & [1+(u-1)(e_1+e_2+e_{1,3})+(u-1)^2(u+2)\,e_1e_2]c_{1,2} \end{array}$$

Proof. The proof is completely analogous to the proof of Lemma 5. □

5. Markov traces

The main purpose of this section is to find the necessary and sufficient conditions in order that the trace tr defined on $\mathrm{Y}_{d,n}(u)$ [11] passes to the quotient algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$. Since the defining ideal of $\mathrm{FTL}_{d,n}(u)$ (respectively of $\mathrm{CTL}_{d,n}(u)$) is principal, by the linearity of tr, we have that tr passes to $\mathrm{FTL}_{d,n}(u)$ (respectively to $\mathrm{CTL}_{d,n}(u)$) if and only if we have:

$$\mathrm{tr}(\mathfrak{m}\, r_{1,2}) = 0 \quad (\text{respectively} \quad \mathrm{tr}(\mathfrak{m}\, c_{1,2}) = 0) \tag{5.1}$$

for all monomials $\mathfrak{m}$ in the inductive basis of $\mathrm{Y}_{d,n}(u)$. So, we seek necessary and sufficient conditions for (5.1) to hold. The strategy is to find such conditions first for $n = 3$ and then generalise using induction.

5.1. Recall that elements in the inductive basis of $\mathrm{Y}_{d,3}(u)$ are of the following forms:

$$t_1^a t_2^b t_3^c, \quad t_1^a g_1 t_1^b t_3^c, \quad t_1^a t_2^b g_2 g_1 t_1^c, \quad t_1^a t_2^b g_2 t_2^c, \quad t_1^a g_1 t_1^b g_2 t_2^c, \quad t_1^a g_1 t_1^b g_2 g_1 t_1^c, \tag{5.2}$$

where $0 \le a, b, c \le d-1$ (see Proposition 1). We need now to compute the trace of the elements $\mathfrak{m}\, r_{1,2}$, where $\mathfrak{m}$ runs the monomials listed in (5.2). To do these computations we will use the following lemma and proposition.

Lemma 7. *For all $0 \le m \le d-1$, we have:*

$$\mathrm{tr}\left(e_1^{(m)} e_2 g_{1,2}\right) = (u+1)z^2 x_m + (u+2)z\, E^{(m)} + \mathrm{tr}(e_1^{(m)} e_2)$$

Proof. By direct computation we have:

$$
\begin{aligned}
\operatorname{tr}\left(e_1^{(m)} e_2 g_{1,2}\right) &= \operatorname{tr}\left(e_1^{(m)} e_2 g_1\right) + \operatorname{tr}\left(e_1^{(m)} e_2 g_2\right) + \operatorname{tr}\left(e_1^{(m)} e_2 g_1 g_2\right) \\
&+ \operatorname{tr}\left(e_1^{(m)} e_2 g_2 g_1\right) + \operatorname{tr}\left(e_1^{(m)} e_2 g_1 g_2 g_1\right) + \operatorname{tr}\left(e_1^{(m)} e_2\right) \\
&= \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} \operatorname{tr}(t_1^{m+s} t_2^{-s+k} t_3^{-k} g_1) + \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} \operatorname{tr}(t_1^{m+s} t_2^{-s+k} t_3^{-k} g_2) \\
&+ \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} \operatorname{tr}(t_1^{m+s} t_2^{-s+k} t_3^{-k} g_1 g_2) + \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} \operatorname{tr}(t_1^{m+s} t_2^{-s+k} t_3^{-k} g_2 g_1) \\
&+ \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} \operatorname{tr}(t_1^{m+s} t_2^{-s+k} t_3^{-k} g_1 g_2 g_1) + \operatorname{tr}\left(e_1^{(m)} e_2\right) \\
&= 2zE^{(m)} 2z^2 x_m + +zE^{(m)} + (u-1)zE^{(m)} + (u-1)z^2 x_m \\
&= (u+1)z^2 x_m + (u+2)zE^{(m)} + \operatorname{tr}\left(e_1^{(m)} e_2\right).
\end{aligned}
$$

□

Proposition 5. *For all* $0 \leq a, b, c \leq d-1$, *we have:*

(1) If $\mathfrak{m} = t_1^a t_2^b t_3^c$,

$$\operatorname{tr}(\mathfrak{m} r_{1,2}) = (u+1)z^2 x_{a+b+c} + (u+2)E^{(a+b+c)} z + \operatorname{tr}(e_1^{(a+b+c)} e_2)$$

(2) If $\mathfrak{m} = t_1^a g_1 t_1^b t_3^c$ *and* $\mathfrak{m} = t_1^a t_2^b g_2 t_2^c$,

$$\operatorname{tr}(\mathfrak{m} r_{1,2}) = u\left[(u+1)z^2 x_{a+b+c} + (u+2)E^{(a+b+c)} z + \operatorname{tr}(e_1^{(a+b+c)} e_2)\right]$$

(3) If $\mathfrak{m} = t_1^a t_2^b g_2 g_1 t_1^c$ *and* $\mathfrak{m} = t_1^a g_1 t_1^b g_2 t_2^c$,

$$\operatorname{tr}(\mathfrak{m} r_{1,2}) = u^2\left[(u+1)z^2 x_{a+b+c} + (u+2)E^{(a+b+c)} z + \operatorname{tr}(e_1^{(a+b+c)} e_2)\right]$$

(4) If $\mathfrak{m} = t_1^a g_1 t_1^b g_2 g_1 t_1^c$,

$$\operatorname{tr}(\mathfrak{m} r_{1,2}) = u^3\left[(u+1)z^2 x_{a+b+c} + (u+2)E^{(a+b+c)} z + \operatorname{tr}(e_1^{(a+b+c)} e_2)\right].$$

Proof. We will prove claim (1). According to Eq. 4.2 we have: $\mathfrak{m} r_{1,2} = t_1^a t_2^b t_3^c r_{1,2} = t_1^a t_2^b t_3^c e_1 e_2 g_{1,2}$. But $t_1^a t_2^b t_3^c e_1 e_2 = e_1^{(a+b+c)} e_2$, hence

$$\mathfrak{m} r_{1,2} = e_1^{(a+b+c)} e_2 g_{1,2}$$

Thus, claim (1) follows by applying Lemma 7.

The proofs for the rest claims use Lemmas 5 and 7 and follow the same argument, so we finish the proof of the proposition by proving only one more representative case. We shall prove claim (3) for $\mathfrak{m} = t_1^a g_1 t_1^b g_2 t_2^c$. This monomial can be rewritten as $t_1^a t_2^b t_3^c g_1 g_2$. Now, by using Lemma 5 on $g_1 g_2 r_{1,2}$, we obtain:

$$\mathfrak{m} r_{1,2} = t_1^a t_2^b t_3^c g_1 g_2 r_{1,2} = t_1^a t_2^b t_3^c \left[1 + (u-1)e_1 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2\right] r_{1,2}$$

then using now Eq. 4.2 and the fact the e_i's are idempotent, follows that:

$$
\begin{aligned}
\mathfrak{m} r_{1,2} &= t_1^a t_2^b t_3^c \left[e_1 e_2 + (u-1)e_1 e_2 + (u-1)e_1 e_2 + (u-1)^2 e_1 e_2\right] g_{1,2} \\
&= u^2 t_1^a t_2^b t_3^c e_1 e_2 g_{1,2}.
\end{aligned}
$$

Then, applying Eq. 2.17 we have:

$$\mathfrak{m}r_{1,2} = u^2 t_1^a t_2^b t_3^c e_1 e_2 g_{1,2} = u^2 e_1^{(a+b+c)} e_2 g_{1,2}.$$

Therefore, by using Lemma 7, we obtain the desired expression for $\mathrm{tr}(\mathfrak{m}r_{1,2})$. □

The following lemma is our main result for $n = 3$.

Lemma 8. *The trace* tr *passes to* $\mathrm{FTL}_{d,3}$ *if and only if the parameters of the trace* tr *satisfy:*

$$x_k = -z \left(\sum_{m \in \mathrm{Sup}_1} \chi(km) + (u+1) \sum_{m \in \mathrm{Sup}_2} \chi(km) \right) \quad \text{and} \quad z = -\frac{1}{|\mathrm{Sup}_1| + (u+1)|\mathrm{Sup}_2|},$$

where $\mathrm{Sup}_1 \cup \mathrm{Sup}_2$ *(disjoint union) is the support of the Fourier transform of x, and x is the complex function on $\mathbb{Z}/d\mathbb{Z}$, that maps 0 to 1 and k to the trace parameter x_k (cf. Subsection 3.2).*

Proof. Recall that the trace tr passes to $\mathrm{FTL}_{d,3}$ if and only if the Eqs. 5.1 hold, for all $\mathfrak{m}$ in the inductive basis of $\mathrm{Y}_{d,3}$. By using Proposition 5 follows that the trace tr passes to the quotient algebra $\mathrm{FTL}_{d,n}(u)$ if and only if the trace parameters$z$, $x_1, \ldots, x_{d-1}$ satisfy the following system of equations:

$$\mathbb{E}_0 = \mathbb{E}_1 = \cdots = \mathbb{E}_{d-1} = 0$$

where

$$\mathbb{E}_m := (u+1)z^2 x_m + (u+2)E^{(m)} z + \mathrm{tr}(e_1^{(m)} e_2) = 0, \quad 0 \leq m \leq d-1$$

We note now that this system of equations above is equivalent to the system:

$$\begin{aligned} \mathbb{E}_0 &= 0 \\ \mathbb{E}_m - x_m \mathbb{E}_0 &= 0 \quad \text{where} \quad 1 \leq m \leq d-1 \end{aligned} \tag{5.3}$$

We will solve this system of equations, obtaining thus the proof of the lemma.

Recall that $x_0 := 1$, $E^{(0)} = E$ and $e_i^{(0)} = e_i$, hence $\mathbb{E}_0 = (u+1)z^2 + (u+2)Ez + \mathrm{tr}(e_1 e_2)$. Then the $(d-1)$ equations $\mathbb{E}_m - x_m \mathbb{E}_0 = 0$ of Eq. 5.3 become:

$$z(u+2)\big(E^{(m)} - x_m E\big) = -\left(\mathrm{tr}(e_1^{(m)} e_2) - x_m \,\mathrm{tr}(e_1 e_2)\right), \quad 1 \leq m \leq d-1. \tag{5.4}$$

Interpreting now the above equation in the functional notation of Section 3 and having in mind Eq. 3.3, it follows that Eq. 5.4 can be rewritten as:

$$(u+2)z \left(\frac{1}{d} x * x - Ex \right) = -\left(\frac{1}{d^2} x * x * x - \mathrm{tr}(e_1 e_2) x \right).$$

Applying now the Fourier transform on the above functional equality and using Proposition 2, we obtain:

$$(u+2)z \left(\frac{\widehat{x}^2}{d} - E\widehat{x} \right) = -\left(\frac{\widehat{x}^3}{d^2} - \mathrm{tr}(e_1 e_2)\widehat{x} \right) \tag{5.5}$$

Let now $\widehat{x} = \sum_{m=0}^{d-1} y_m t^m$. Then Eq. 5.5 becomes:

$$(u+2)z \left(\frac{y_m^2}{d} - E y_m \right) = -\left(\frac{y_m^3}{d^2} - \mathrm{tr}(e_1 e_2) y_m \right)$$

Hence

$$y_m \left(\frac{y_m^2}{d^2} + (u+2)z \frac{y_m}{d} - (u+2)zE - \mathrm{tr}(e_1 e_2) \right) = 0 \tag{5.6}$$

Now, from equation $\mathbb{E}_0 = 0$, we have that $-(u+2)zE = (u+1)z^2 + \mathrm{tr}(e_1e_2)$. Replacing this expression of $-(u+2)zE$ in Eq. 5.6 we have that:

$$y_m\left(\frac{y_m^2}{d^2} + (u+2)z\frac{y_m}{d} + (u+1)z^2\right) = 0$$

or equivalently:

$$y_m\,(y_m + dz)\,(y_m + dz(u+1)) = 0 \tag{5.7}$$

Denote $\mathrm{Sup}_1 \cup \mathrm{Sup}_2$ the support of $\widehat{x}$, where

$$\mathrm{Sup}_1 := \{m \in \mathbb{Z}/d\mathbb{Z}\,;\, y_m = -dz\} \quad \text{and} \quad \mathrm{Sup}_2 := \{m \in \mathbb{Z}/d\mathbb{Z}\,;\, y_m = -dz(u+1)\}$$

hence

$$\widehat{x} = \sum_{m\in\mathrm{Sup}_1} -dzt^m + \sum_{m\in\mathrm{Sup}_2} -dz(u+1)t^m$$

Then

$$\widehat{\widehat{x}} = -dz\sum_{m\in\mathrm{Sup}_1}\widehat{\delta}_m - dz(u+1)\sum_{m\in\mathrm{Sup}_2}\widehat{\delta}_m$$

thus from argument (4) of Proposition 2 we have:

$$\widehat{\widehat{x}} = -z\left(\sum_{m\in\mathrm{Sup}_1}\mathbf{i}_{-m} + (u+1)\sum_{m\in\mathrm{Sup}_2}\mathbf{i}_{-m}\right)$$

Therefore, having in mind now (5) of Proposition 2, we deduce that:

$$x_k = -z\left(\sum_{m\in\mathrm{Sup}_1}\chi(km) + (u+1)\sum_{m\in\mathrm{Sup}_2}\chi(km)\right) \tag{5.8}$$

Having in mind that $x_0 = 1$, one can determine the values of z. Indeed, from Eq. 5.8, we have that:

$$1 = x_0 = -z(|\mathrm{Sup}_1| + (u+1)|\mathrm{Sup}_2|)$$

or equivalently:

$$z = -\frac{1}{|\mathrm{Sup}_1| + (u+1)|\mathrm{Sup}_2|}. \tag{5.9}$$

□

Keeping the same notation with the above lemma, we have:

Theorem 5. *The trace* tr *defined on* $\mathrm{Y}_{d,n}(u)$ *passes to the quotient algebra* $\mathrm{FTL}_{d,n}(u)$ *if and only if the trace parameters* $z, x_1, \ldots, x_{d-1}$ *satisfy the conditions of Lemma 8, i. e. Eqs. 5.8 and 5.9.*

Proof. The proof is by induction on n. The case $n = 3$ is the lemma above. Assume now that the statement holds for all $\mathrm{FTL}_{d,k}(u)$, where $k \leq n$, that is:

$$\mathrm{tr}(a_k\, r_{1,2}) = 0$$

for all $a_k \in \mathrm{Y}_{d,k}(u)$, $k \leq n$. We will show the statement for $k = n+1$. It suffices to prove that the trace vanishes on any element of the form $a_{n+1}r_{1,2}$, where a_{n+1} belongs to the inductive basis of $\mathrm{Y}_{d,n+1}(u)$ (recall Eq. 1), given the conditions of the theorem. Namely:

$$\mathrm{tr}(a_{n+1}\, r_{1,2}) = 0.$$

Since a_{n+1} is in the inductive basis of $\mathrm{Y}_{d,n+1}(u)$, it is of one of the following forms:

$$a_{n+1} = a_n g_n \dots g_i t_i^k \quad \text{or} \quad a_{n+1} = a_n t_{n+1}^k$$

where a_n is in the inductive basis of $\mathrm{Y}_{d,n}(u)$. For the first case we have:

$$\mathrm{tr}(a_{n+1}\, r_{1,2}) = \mathrm{tr}(a_n g_n \dots g_i t_i^k\, r_{1,2}) = z\, \mathrm{tr}(a_n g_{n-1} \dots g_i t_i^k\, r_{1,2}) = z\, \mathrm{tr}(\tilde{a} r_{1,2}),$$

where $\tilde{a} := a_n g_{n-1} \dots g_i t_i^k$. Notice now that $\widetilde{a}$ is a word in $\mathrm{Y}_{d,n}(u)$ and so, by the linearity of the trace, we have that $\mathrm{tr}(\tilde{a}\, r_{1,2})$ is a linear combination of traces of the form $\mathrm{tr}(a_n\, r_{1,2})$, where a_n is in the inductive basis of $\mathrm{Y}_{d,n}(u)$. Therefore, by the induction hypothesis, we deduce that:

$$\mathrm{tr}(\widetilde{a}\, r_{1,2}) = 0$$

if and only if the conditions of the Theorem are satisfied. Therefore the statement is proved. The second case is proved similarly. Hence, the proof is concluded. □

Corollary 3. *In the case where one of the sets* Sup_1 *or* Sup_2 *is the empty set, the values of the* x_k*'s are solutions of the* E*–system. More precisely, if* Sup_1 *is the empty set, the* x_k*'s are the solutions of the* E*–system parametrized by* Sup_2 *and* $z = -1/(u+1)|\mathrm{Sup}_2|$*. If* Sup_2 *is the empty set, then* x_k*'s are the solutions of the* E*–system parametrized by* Sup_1 *and* $z = -1/|\mathrm{Sup}_1|$*.*

Proof. The proof follows from Eq. 3.5 and the expression given in theorem above for the x_k's. □

5.2. The method of finding the necessary and sufficient conditions for tr to pass to the quotient algebra $\mathrm{CTL}_{d,n}(u)$ is completely analogous to that of the previous subsection. Thus, we will need the following analogue of Proposition 5.

Proposition 6. *Define* $\mathbb{G}$*, as follows:*

$$\mathbb{G} = (u+1)z^2 \sum_{k=0}^{d-1} x_k + (u+2)z \sum_{k=0}^{d-1} E^{(k)} + \sum_{k=0}^{d-1} \mathrm{tr}(e_1^{(k)} e_2)$$

Then for all $0 \le a, b, c \le d-1$*, we have:*

(1) $\mathrm{tr}(\mathfrak{m} c_{1,2}) = \mathbb{G}$ *for* $\mathfrak{m} = t_1^a t_2^b t_3^c$
(2) $\mathrm{tr}(\mathfrak{m} c_{1,2}) = u\mathbb{G}$ *for* $\mathfrak{m} = t_1^a g_1 t_1^b t_3^c$ *and* $\mathfrak{m} = t_1^a t_2^b g_2 t_2^c$*,*
(3) $\mathrm{tr}(\mathfrak{m} c_{1,2}) = u^2\mathbb{G}$ *for* $\mathfrak{m} = t_1^a t_2^b g_2 g_1 t_1^c$ *and* $\mathfrak{m} = t_1^a g_1 t_1^b g_2 t_2^c$*,*
(4) $\mathrm{tr}(\mathfrak{m} c_{1,2}) = u^3\mathbb{G}$ *for* $\mathfrak{m} = t_1^a g_1 t_1^b g_2 g_1 t_1^c$*.*

Following now the analogous reasoning that was used to prove Theorem 5 and having in mind Eq. 5.1, Corollary 2, Lemma 6 and Proposition 6, we obtain the following theorem.

Theorem 6. *The trace* tr *passes to the quotient if and only if the parameter* z *and the* x_i*'s are related through the equation:*

$$(u+1)z^2 \sum_{k\in\mathbb{Z}/d\mathbb{Z}} x_k + (u+2)z \sum_{k\in\mathbb{Z}/d\mathbb{Z}} E^{(k)} + \sum_{k\in\mathbb{Z}} \mathrm{tr}(e_1^{(k)} e_2) = 0. \tag{5.10}$$

6. Comparison of the trace conditions

In this section we will compare the conditions that need to be applied to the trace paramaters z and x_i, $i = 1, \dots, d-1$ so that tr passes to each of the quotient algebras.

In [6] we found the necessary and sufficient conditions so that tr passes to $\mathrm{YTL}_{d,n}(u)$. Indeed, we have the following:

Theorem 7. *The trace* tr *passes to the quotient* $\mathrm{YTL}_{d,n}(u)$ *if and only if the* x_i*'s are solutions of the* E*–system and one of the two cases holds:*

(i) *For some* $0 \leq m_1 \leq d-1$ *the* x_ℓ*'s are* d^{th} *roots of unity and* $z = -\frac{1}{u+1}$ *or* $z = -1$.

(ii) *For some* $0 \leq m_1, m_2 \leq d-1$, $m_1 \neq m_2$, *the* x_ℓ*'s are expressed as:*

$$x_\ell = \frac{1}{2}\left(\exp(m_1\ell) + \exp(m_2\ell)\right) \quad (0 \leq \ell \leq d-1).$$

In this case we have $z = -\frac{1}{2}$.

The conditions for the x_i's in this case are particular solutions of the E–system. Thus, the conditions such that tr passes to $\mathrm{YTL}_{d,n}(u)$ are contained in those of Theorem 5.

Moreover, Theorem 5 can be rephrased in the following way:

Theorem 8. *The trace* tr *passes to the quotient algebra* $\mathrm{FTL}_{d,n}(u)$ *if and only if the parameter* z *and the* x_i*'s are related through the equation:*

$$(u+1)z^2 x_k + (u+2)zE^{(k)} + \mathrm{tr}(e_1^{(k)} e_2) = 0, \qquad \forall k \in \mathbb{Z}/d\mathbb{Z}$$

This implies that the conditions such that the trace passes to the quotient algebra $\mathrm{FTL}_{d,n}(u)$ are contained in those of Theorem 6. All of the above can be summarised in the following table:

	$\mathrm{Y}_{d,n}(u)$	$\twoheadrightarrow$	$\mathrm{CTL}_{d,n}(u)$	$\twoheadrightarrow$	$\mathrm{FTL}_{d,n}(u)$	$\twoheadrightarrow$	$\mathrm{YTL}_{d,n}(u)$
z x_i	free free	$\leftarrow$	Theorem 6	$\leftarrow$	Theorem 8	$\leftarrow$	Theorem 7

TABLE 1. Relations of algebras and trace conditions.

The first row includes the projections between the algebras while the second shows the inclusions of the trace conditions for each case.

7. Knot invariants

7.1. The 2–variable Jones or Homflypt polynomial, $P(\lambda, u)$, can be defined through the Ocneanu trace τ on $\mathrm{H}_n(u)$ [9]. Indeed, for any braid $\alpha \in \cup_\infty B_n$ we have:

$$P(\lambda, u)(\widehat{\alpha}) = \left(-\frac{1-\lambda u}{\sqrt{\lambda}(1-u)}\right)^{n-1} \left(\sqrt{\lambda}\right)^{\varepsilon(\alpha)} \tau(\pi(\alpha)),$$

where: $\lambda = \frac{1-u+\zeta}{u\zeta}$, π is the natural epimorphism of $\mathbb{C}B_n$ onto $\mathrm{H}_n(u)$ that sends the braid generator σ_i to h_i and $\varepsilon(\alpha)$ is the algebraic sum of the exponents of the σ_i's in α. Further, the Jones polynomial, $V(u)$, related to the algebras $\mathrm{TL}_n(u)$, can be redefined through the Homflypt polynomial, by specializing ζ to $-\frac{1}{u+1}$ [9]. This is the non–trivial value for which the Ocneanu trace τ passes to the quotient algebra $\mathrm{TL}_n(u)$. Namely:

$$V(q)(\widehat{\alpha}) = \left(-\frac{1+u}{\sqrt{u}}\right)^{n-1} \left(\sqrt{u}\right)^{\varepsilon(\alpha)} \tau(\pi(\alpha)) = P(u,u)(\widehat{\alpha}).$$

7.2. In [16] it is proved that the trace tr defined on $\mathrm{Y}_{d,n}(u)$ can be re–scaled according to the braid equivalence corresponding to isotopic framed links if and only if the x_i's furnish a solution of the E–system (recall discussion in Section 3). Let $X_D = (\mathrm{x}_1, \ldots, \mathrm{x}_{d-1})$ be a solution of the E–system parametrized by the non–empty set D of $\mathbb{Z}/d\mathbb{Z}$. We have the following definition:

Definition 6 (Definition 3 [1])**.** The trace map tr_D defined as the trace tr with the parameters x_i specialized to the values x_i, shall be called the *specialized trace* with parameter z.

By normalizing tr_D, an invariant for framed links can be obtained [16]:

$$\Gamma_D(w,u)(\widehat{\alpha}) = \left(-\frac{(1-wu)|D|}{\sqrt{w}(1-u)}\right)^{n-1} \left(\sqrt{w}\right)^{\varepsilon(\alpha)} \mathrm{tr}_D(\gamma(\alpha)), \tag{7.1}$$

where: D is a non–empty subset of $\mathbb{Z}/d\mathbb{Z}$ which parametrizes a solution of the E–system, $w = \frac{z+(1-u)E}{uz}$, $E = \frac{1}{|D|}$, γ the natural epimorphism of the framed braid group algebra $\mathbb{C}\mathcal{F}_n$ onto the algebra $\mathrm{Y}_{d,n}(u)$, and $\alpha \in \cup_\infty \mathcal{F}_n$.

Further, in [15] Juyumaya and Lambropoulou represented the classical braid group B_n in the algebra $\mathrm{Y}_{d,n}(u)$ by regarding the framing generators t_i as formal elements. This is equivalent to restricting Γ_D to classical links, seen as framed links with all framings zero. This gives rise to an invariant of classical oriented links, denoted $\Delta_D(\widehat{\alpha})$, where $\alpha \in \cup_\infty B_n$. Namely:

$$\Delta_D(w,u)(\widehat{\alpha}) = \left(-\frac{(1-wu)|D|}{\sqrt{w}(1-u)}\right)^{n-1} \left(\sqrt{w}\right)^{\varepsilon(\alpha)} \mathrm{tr}_D(\delta(\alpha)), \tag{7.2}$$

where: D, w, E as above, δ the natural homomorphism of the classical braid group algebra $\mathbb{C}B_n$ to the algebra $\mathrm{Y}_{d,n}(u)$ and $\alpha \in \cup_\infty B_n$. Further, in [14] the invariant $\Delta_D(w,u)$ was extended to an invariant for singular links.

Note that for $d = 1$ the traces tr and tr_D coincide with the Ocneanu trace τ, so the invariants $\Gamma_D(u,w)$ and $\Delta_D(u,w)$ coincide with the Homflypt polynomial. Moreover, in [1] it is shown that for generic values of the parameters u, z the invariants $\Delta_D(w,u)$ do not coincide with the Homflypt polynomial except in the trivial cases $u = 1$ and $E = 1$. Yet, computational data [5] indicate that these invariants may be topologically equivalent to the Homflypt polynomial.

In [6] the invariants that are defined through the Yokonuma–Temperley–Lieb were studied. More precisely, it was shown that in order that the trace tr passes to the quotient algebra $\mathrm{YTL}_{d,n}(u)$ it is necessary that the x_i's are d^{th} roots of unity. These furnish a (trivial) solution of the E–system and in this case $E = 1$. So, by [1], the invariants we obtain from $\mathrm{YTL}_{d,n}(u)$ coincide with the Jones polynomial. This is the main reason that the algebras $\mathrm{YTL}_{d,n}(u)$ do not qualify for being the framization of the Temperley–Lieb algebra.

In this section we will define the invariants for framed and classical links that can be obtained from the algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$.

7.3. *Invariants from* $\mathrm{FTL}_{d,n}(u)$**.** As it has already been stated, the trace parameters x_i should be solutions of the E–system so that a link invariant through tr is well–defined. Moreover, the conditions of Theorem 5 include these solutions for the x_i's. In order to define a link invariant on the level of the quotient algebra $\mathrm{FTL}_{d,n}(u)$, we discard any value of the x_i's that does not comprise a solution of the E–system. Using Corollary 3 we choose a solution of the E–system and denote with D the subset of $\mathbb{Z}/d\mathbb{Z}$ that parametrizes the said solution. This leads to the following values for z:

$$z = -\frac{1}{(u+1)|D|} \quad \text{or} \quad z = -\frac{1}{|D|}$$

Further, we do not take into consideration the case where $z = -\frac{1}{|D|}$, since important topological information is lost. Indeed, the trace tr gives the same value for all even (resp. odd) powers of the g_i's, for $m \in \mathbb{Z}^{>0}$ [16]:

$$\mathrm{tr}(g_i^m) = \left(\frac{u^m-1}{u+1}\right) z + \left(\frac{u^m-1}{u+1}\right)\frac{1}{|D|} + 1 \qquad \text{if } m \text{ is even} \tag{7.3}$$

and

$$\mathrm{tr}(g_i^m) = \left(\frac{u^m-1}{u+1}\right) z + \left(\frac{u^m-1}{u+1}\right)\frac{1}{|D|} - \frac{1}{|D|} \qquad \text{if } m \text{ is odd.} \tag{7.4}$$

From the remaining case where the x_i's are solutions of the E–system and $z = -\frac{1}{(u+1)|D|}$ we deduce that $w = u$ in Eq. 7.1. We then have the following definition:

Definition 7. Let X_D be a solution of the E–system, parametrized byt the non–empty subset D of $\mathbb{Z}/d\mathbb{Z}$ and let $z = -\frac{1}{(u+1)|D|}$. We obtain from $\Gamma_D(w,u)$ the following invariant for $\alpha \in \cup_\infty \mathcal{F}_n$:

$$(i) \quad \vartheta_D(u)(\widehat{\alpha}) \;=\; \left(-\frac{(1+u)|D|}{\sqrt{u}}\right)^{n-1} (\sqrt{u})^{\varepsilon(\alpha)}\, \mathrm{tr}_D\left(\gamma(\alpha)\right) = \Gamma_D(u,u)(\widehat{\alpha}),$$

Further, from Δ_D, we obtain the following invariant for $\alpha \in \cup_\infty B_n$:

$$(ii) \quad \theta_D(u)(\widehat{\alpha}) \;=\; \left(-\frac{(1+u)|D|}{\sqrt{u}}\right)^{n-1} (\sqrt{u})^{\varepsilon(\alpha)}\, \mathrm{tr}_D\left(\delta(\alpha)\right) = \Delta_D(u,u)(\widehat{\alpha}),$$

The invariants $\vartheta_D(u)$ are analogues of the Jones polynomial in the framed category.

Remark 5. If the invariants $\Delta_D(w,u)$ on the level of the Yokonuma–Hecke algebras turn out to be topologically equivalent to the Homflypt polynomial [5] then the invariants $\theta_D(u)$ will be topologically equivalent to the Jones polynomial, and the invariants $\vartheta_D(u)$ framed analogues of the Jones polynomial.

7.4. *Invariants from* $\mathrm{CTL}_{d,n}(u)$**.** The conditions of Theorem 6 do not involve the solutions of the E–system at all, so in order to obtain a well–defined link invariant on the level of $\mathrm{CTL}_{d,n}(u)$ we must impose this condition on the x_i's. Recall that the solutions of the E–system can be expressed in the form:

$$x_{_D} = \frac{1}{|D|}\sum_{k\in D} \mathbf{i}_k \in \mathbb{C}C_d$$

where $\mathbf{i}_k = \sum_{j=0}^{d-1} \chi_k(j)t^j$, χ_k is the character that sends $m \mapsto cos\frac{2\pi km}{d} + i\, sin\frac{2\pi km}{d}$ and D is the subset of $\mathbb{Z}/d\mathbb{Z}$ that parametrizes a solution of the E–system. Let now ε be the augmentation function of the group algebra $\mathbb{C}C_d$, sending $\sum_{j=0}^{d-1} x_j t^j$ to $\sum_{j=0}^{d-1} x_j$. We have that:

$$\varepsilon(x_{_D}) = \frac{1}{|D|}\sum_{k\in D}\varepsilon(\mathbf{i}_k) = \frac{1}{|D|}\sum_{j=0}^{d-1}\sum_{k\in D}\chi_j(k) = \begin{cases} \frac{d}{|D|}, & \text{if } 0 \in D \\ 0 & \text{if } 0 \notin D \end{cases} \tag{7.5}$$

From this we deduce that:

$$\sum_{j=0}^{d-1} E^{(j)} = \varepsilon\left(\frac{x * x}{d}\right) = \frac{1}{d|D|^2}\sum_{k\in D}\varepsilon(\mathbf{i}_k * \mathbf{i}_k) = \frac{1}{|D|^2}\sum_{k\in D}\varepsilon(\mathbf{i}_k) = \begin{cases} \frac{d}{|D|^2}, & \text{if } 0 \in D \\ 0 & \text{if } 0 \notin D \end{cases} \tag{7.6}$$

and also that:

$$\sum_{j=0}^{d-1} \mathrm{tr}(e_1^{(j)}e_2) = \varepsilon\left(\frac{x * x * x}{d^2}\right) = \frac{1}{d^2|D|^3}\sum_{k\in D}\varepsilon(\mathbf{i}_k * \mathbf{i}_k * \mathbf{i}_k) = \frac{1}{|D|^3}\sum_{k\in D}\varepsilon(\mathbf{i}_k) = \begin{cases} \frac{d}{|D|^3}, & \text{if } 0 \in D \\ 0 & \text{if } 0 \notin D \end{cases} \tag{7.7}$$

Using now Eqs. 7.5, 7.6 and 7.7, Eq. 5.10 becomes for the case where $0 \in D$:

$$\frac{d}{|D|}\left((u+1)z^2 + \frac{(u+2)}{|D|}z + \frac{1}{|D|^2}\right) = 0.$$

Therefore, the trace tr passes to the quotient for the following values of z:

$$z = -\frac{1}{(u+1)|D|} \quad \text{or} \quad z = -\frac{1}{|D|}.$$

Much like the case of $\mathrm{FTL}_{d,n}(u)$, the value $z = -\frac{1}{|D|}$ is not taken into consideration, since from Eq. 5.10 we deduce that $E = \frac{1}{|D|}$ and therefore from Eqs. 7.3 and 7.4 the trace tr gives the same value for all even (resp. odd) powers of the g_i's.

We have the following definition:

Definition 8. *Let X_D be a solution of the* E*–system, parametrized byt the non–empty subset D of $\mathbb{Z}/d\mathbb{Z}$ and let $z = -\frac{1}{(u+1)|D|}$. We obtain from $\Gamma_D(w,u)$ the following invariant for $\alpha \in \cup_\infty \mathcal{F}_n$:*

$$(i) \quad \mathcal{W}_D(u)(\widehat{\alpha}) \quad = \quad \left(-\tfrac{(1+u)|D|}{\sqrt{u}}\right)^{n-1} (\sqrt{u})^{\varepsilon(\alpha)}\, \mathrm{tr}_D\left(\gamma(\alpha)\right) = \Gamma_D(u,u)(\widehat{\alpha}),$$

Further, from Δ_D, we obtain the following invariant for $\alpha \in \cup_\infty B_n$:

$$(ii) \quad W_D(u)(\widehat{\alpha}) \quad = \quad \left(-\tfrac{(1+u)|D|}{\sqrt{u}}\right)^{n-1} (\sqrt{u})^{\varepsilon(\alpha)}\, \mathrm{tr}_D\left(\delta(\alpha)\right) = \Delta_D(u,u)(\widehat{\alpha}),$$

Remark 6. It should be clear that the invariants $\mathcal{W}_D$ and W_D that are obtained from tr on the level of the quotient algebra $\mathrm{CTL}_{d,n}(u)$ coincide with the invariants ϑ_D and θ_D on the level of $\mathrm{FTL}_{d,n}(u)$. More precisely, the conditions that are applied to the trace parameters are the same for both of the quotient algebras and, consequently, so are the related invariants.

Furthermore, the solutions of the E–system (which are the necessary and sufficient conditions so that topological invariants for framed links can be defined) are included in the conditions of Theorem 5, while for the case of $\mathrm{CTL}_{d,n}(u)$ we still have to impose them. These are the main reasons that lead us to consider the quotient algebra $\mathrm{FTL}_{d,n}(u)$ as the most natural non–trivial analogue of the Temperley–Lieb algebra in the context of framed links.

The following table give a full overview of the invariants for each quotient algebra:

d, D	$\mathcal{F}_{d,n}$	B_n	u	w
$\mathrm{Y}_{d,n}(u)$	Γ_D	Δ_D	u	w
$\mathrm{YTL}_{d,n}(u)$	$\mathcal{V}_D$	V_D	u	u
$\mathrm{FTL}_{d,n}(u)$	ϑ_D	θ_D	u	u
$\mathrm{CTL}_{d,n}(u)$	ϑ_D	θ_D	u	u

$d = 1$	B_n	u	w
$\mathrm{H}_n(u)$	P	u	λ
$\mathrm{TL}_n(u)$	V	u	u

$d, \|D\| = 1$	$\mathcal{F}_{d,n}$	B_n	u	w
$\mathrm{Y}_{d,n}(u)$	Γ_D	P	u	λ
$\mathrm{YTL}_{d,n}(u)$	$\mathcal{V}_D$	V	u	u
$\mathrm{FTL}_{d,n}(u)$	$\mathcal{V}_D$	V	u	u
$\mathrm{CTL}_{d,n}(u)$	$\mathcal{V}_D$	V	u	u

$d, \|D\| > 1$	$\mathcal{F}_{d,n}$	B_n	u	w
$\mathrm{Y}_{d,n}(u)$	Γ_D	Δ_D	u	λ
$\mathrm{YTL}_{d,n}(u)$	no	no	–	–
$\mathrm{FTL}_{d,n}(u)$	ϑ_D	θ_D	u	u
$\mathrm{CTL}_{d,n}(u)$	ϑ_D	θ_D	u	u

TABLE 2. Overview of the invariants for each algebra.

7.5. *Concluding note***.** The knot invariants from the algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ still remain under investigation. If the invariants from the Yokonuma–Hecke algebras prove to be topologically equivalent to the Homflypt polynomial, then the invariants from $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ will be topologically equivalent to the Jones polynomial. If not, it would be then meaningfull to consider the corresponding 3–manifold invariants (as obtained from work of Wenzl [24]). In the case of the algebras $\mathrm{YTL}_{d,n}(u)$ the Witten invariants of 3–manifolds can be recovered, since the related knot invariants recover the Jones polynomial [6].

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