ON THE GALOIS-MODULE STRUCTURE OF POLYDIFFERENTIALS OF SUBRAO CURVES, MODULAR AND INTEGRAL REPRESENTATION THEORY.

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Abstract. We study the Galois-module structure of polydifferentials on Mumford curves, defined over field in positive charactersitics. We give the complete structure for the Subrao curves using the theory of harmonic cocycles.

1. Introduction

Let X be a curve of genus $g \geq 2$ defined over an algebraically closed field K of characteristic $p > 0$. The automorphism group G of X acts on the space of n-polydifferentials $H^0(X, \Omega_X^{\otimes n})$. In characteristic zero the Galois-module structure for the $n = 1$ case is a classical result due to Hurwitz [16] and this result can be easily generalized for $n \geq 1$. In positive characteristic in general the Galois-module structure is unknown.

There are only some partial results known. Let us give a brief overview. If the cover $X \to X/G$ is unramified or if $(|G|, p) = 1$ Tamagawa [37] determined the Galois-module structure of $H^0(X, \Omega_X^{\otimes 1})$. Valentini [39] generalized this result to unramified extensions with p-groups as Galois groups. Moreover, Salvador and Bautista [25] determined the semi-simple part of the representation with respect to the Cartier operator in the case of a p-group. For the cyclic group case Valentini and Madan [40] and S. Karanikolopoulos [17] determined the structure of $H^0(X, \Omega_X^{\otimes 1})$ in terms of indecomposable modules. The same study has been done for the elementary abelian case by Calderón, Salvador and Madan [32]. Finally, N. Borne [3] developed a theory using advanced techniques from both modular representation theory and K-theory in order to compute in some cases the G-module structure of polydifferentials $H^0(X, \Omega_X^{\otimes n})$.

Let us point out that there are several applications of this module structure. For example, the second author in [22] [21] connected the G-module structure of $H^0(X, \Omega_X^{\otimes 2})$ to the tangent space to the global deformation functor of curves.

There are two main reasons why it is difficult to determine this Galois-module structure:

(1) it is in general very difficult to find an explicit basis for the space of holomorphic polydifferentials,
(2) for modular representations and non-cyclic p-groups, the indecomposable modules are unknown, see the introduction of [2].

In this paper we determine the Galois-module structure of Subrao curves (see below). We do give for this curves explicit bases for the space of holomorphic polydiffertials. We show that all indecomposable modules for G are inside the group algebra $K[G]$. We employ the celebrated theory of B. Köck[19] for ordinary curves.

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Over non-archimedean complete, discretely valued fields K, D. Mumford [28] has shown that curves whose stable reduction is split multiplicative, (*i.e.,* a union of rational curves intersecting in K-rational points with K-rational nodal tangents) are isomorphic to an analytic space of the form $\Gamma\backslash(\mathbb{P}^1 - \mathcal{L}_\Gamma)$. Here Γ is a finitely generated torsion free discrete subgroup of $\mathrm{PGL}(2, K)$ with $\mathcal{L}_\Gamma$ as set of limit points. These curves are called *Mumford curves*, and the uniformization just described provides us with a set of tools similar to those coming from the uniformization theory of Riemann surfaces. The first and second authors together with G. Cornelissen have used this technique in order to bound the automorphism groups of Mumford curves in [7]. Also the deformation theory of such curves was studied by the first author and G. Cornelissen in [4].

We study the G-module structure of holomorphic differentials. One of the tools we use is the explicit description of holomorphic differentials in terms of harmonic cocycles. More precisely, motivated by the classical theory of modular forms, P. Schneider and J. Teitelbaum [33], [38] defined a notion of modular forms on graphs (or harmonic measures as they are known in the literature) and established isomorphisms

$$\begin{tikzcd} C_{\mathrm{har}}(\Gamma, n) \arrow[r, bend left, "\mathrm{Poisson(n)}"] & H^0(X_\Gamma, \Omega_{X_\Gamma}^{\otimes n}) \arrow[l, bend left, "\mathrm{Res}(n)"] \end{tikzcd},$$

for notation and explanation see also [9]. In literature there is a graph theoretic definition of $C_{\mathrm{har}}(\Gamma, n)$. In this article we will use the equivalent, more algebraic definition, $C_{\mathrm{har}}(\Gamma, n) = H^1(\Gamma, P_{2n-2})$, see section 2.

In modular representation theory the notion of irreducible and indecomposable modules differ. In general, there are two ways to describe a G-module M in this setting.

(1) Describe the indecomposable summands of M with their multiplicities.
(2) Write M as a sum $M = \mathrm{core}(M) \oplus P_G(M)$, where $\mathrm{core}(M)$ and $P_G(M)$ are the core and the projective cover of M, respectively.

The first approach is most effective if a complete classification of the possible indecomposable G-modules is known. As mentioned before, this classification is known for cyclic groups only: let σ be a generator of cyclic p-group G, then the indecomposable summands are given by $K[G]/\langle(\sigma - 1)^k\rangle$, with $k = 0, \ldots, p-1$. Unfortunately, if we try to proceed to the next simplest group like $G = \mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$ then little seems to be known for the possible indecomposable G-modules.

The second approach is essentially the theory of Brauer characters and gives useful information for the projective summands of the group G. Unfortunately, if G is a p-group then there is only one projective indecomposable module, namely the $K[G]$-module. All other non-projective modules are hidden in the core part. This method was introduced by N. Borne [3] and N. Stalder [35]. We combine these two approaches together by embedding the spaces $H^1(X, \Omega_X^{\otimes n})$ into free $K[G]$-modules using the theory of B. Köck for weakly ramified covers [19].

Studying the Galois module structure using the harmonic cocycle approach leads to difficult group theoretic problems on how finite groups embed into the automorphism group of the free group in g-generators. For the following families of Mumford curves we have a complete result.

Let K be a non-archimedean valued and complete field of characteristic $p > 0$, and $A, B \subset \mathrm{PGL}(2, K)$ are the finite subgroups of order p generated respectively

by
$$\epsilon_A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \epsilon_B = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},$$
where $s \in K^*$. For a general choice of s, the groups A and B generate a discrete subgroup N isomorphic to the free product $A * B$. The group $\Gamma := [A, B]$ is 1) a normal subgroup of N such that $N/\Gamma \cong A \times B$ and 2) a free group of rank $(p-1)^2$. A basis of Γ is given by $[a, b]$ for $a \in A \backslash \{1\}$ and $b \in B \backslash \{1\}$. The corresponding Mumford curve X_Γ is of genus $g = \text{rank}(\Gamma) = (p-1)^2$. It admits an algebraic model
$$(y^p - y)(x^p - x) = c \quad \text{where } |c| < 1. \tag{1}$$
These families of curves were first studied by D. Subrao [36], and were studied further by several authors, e.g., [7], [8], [31]. The group of automorphisms of these Mumford curves contains $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, where $\epsilon_A \bmod \Gamma, \epsilon_B \bmod \Gamma$ are the two generators of the group G.

With this notation for the Subrao curve given in eq. (1) we have:

Theorem 1. *The structure of holomorphic differentials as a $K[A]$-module is given as*
$$H^0(X, \Omega_X) = M^{p-1} \otimes_{\mathbb{Z}} K,$$
where M is the integral representation of a cyclic group of order p with minimal degree $p-1$, see eq. (6) for an explicit description of M. The same space considered as an $A \times B$-module is indecomposable.

Notice that since holomorphic differentials ($n = 1$) have a combinatorial interpretation, the above module structure is obtained from an integral representation by extension of scalars. For polydifferentials ($n > 1$) such an integral representation is not possible.

Theorem 2. *For $n > 1$ we write $2n - 1 = q \cdot p + r$ with $0 \leq r < p$.*

(1) *As a $K[A]$-module the following decomposition holds*
$$H^1(\Gamma, P_{2n-2}) = H^0(X, \Omega_X^{\otimes n}) = K[A]^{(p-1)(2n-1)-p\lceil \frac{2n-1}{p} \rceil} \bigoplus \left(K[A]/(\epsilon_A - 1)^{p-r} \right)^p.$$
A similar result holds for the group B.

(2) *As a $K[G]$-module the following decomposition holds:*
$$H^0(X, \Omega_X^{\otimes n}) = K[G]^{2n-1-2\lceil \frac{2n-1}{p} \rceil} \bigoplus K[G]/(\epsilon_A - 1)^{p-r} \bigoplus K[G]/(\epsilon_B - 1)^{p-r}.$$

Let us now describe the structure of the article. Section 2 is concerned with a short description on the holomorphic differentials of Mumford curves as cohomology classes. In section 3 we focus on holomorphic differentials. These objects have a more combinatorial nature and their study is more geometric, see [27]. As a side result we obtain a bound for the order of an automorphism acting on them. In this section we also give a criterion for a module to be indecomposable based on the dimension of the invariant subspace. Next section is devoted to the study of polydifferentials. We recall some notions for the theory of derivations in free groups and then we spend section 5 on doing computations on computing the structure of derivations on Subrao curves. In section 6 we use the theory of projective covers in order to study the $K[A]$-structure. We show how results of S. Nakajima [29]) can be applied without the usage of the theory of Mumford curves. For the $K[A \times B]$-structure we employ both the theory of projective covers and the theory of B. Köck on the Galois-module structure of weakly ramified covers. We are not aware of a method outside the theory of harmonic cocycles for Mumford curves for proving theorem 2.2.

2. Polydifferentials on Mumford curves

We begin with giving some definitions. Let K be a field, non-archimedean valued and complete. Consider the space of polynomials $P = P_{2(n-1)} \subset K[X]$ of degree $\leq 2(n-1)$. The group $\mathrm{PGL}(2,K)$ acts on P from the right: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,K)$ and $F \in P$,

$$F^{\phi}(X) := \frac{(cX+d)^{2(n-1)}}{(ad-bc)^{n-1}} F\left(\frac{aX+b}{cX+d}\right). \tag{2}$$

Let Γ be a Schottky group, i.e., a free discrete subgroup $\Gamma \subset \mathrm{PGL}(2,K)$, and let N be a finitely generated discrete subgroup of $\mathrm{PGL}(2,K)$ containing Γ as a normal subgroup. Set $G = N/\Gamma$, since the automorphism group $\mathrm{Aut}(X_\Gamma)$ is isomorphic to the quotient of the normalizer of Γ in $\mathrm{PGL}(2,K)$ by Γ, it follows that $G \leq \mathrm{Aut}(X_\Gamma)$. In this paper we only study curves X_Γ of genus $g(X_\gamma) \geq 2$, and which have finite automorphism group.

Definition 3. A map of the form

$$d : \Gamma \to P$$

is called a *derivation* if it satisfies

$$d(\gamma\gamma') = (d\gamma)^{\gamma'} + d\gamma'$$

for any $\gamma, \gamma' \in \Gamma$. These comprise a K-linear space $\mathrm{Der}(\Gamma, P)$. A *principal derivation* is a derivation of the form

$$\Gamma \ni \gamma \mapsto F^{\gamma} - F,$$

for an element $F \in P$. Principal derivations form a subspace $\mathrm{PrinDer}(\Gamma, P)$ of $\mathrm{Der}(\Gamma, P)$. The quotient is the *group cohomology*:

$$\mathrm{H}^1(\Gamma, P) = \mathrm{Der}(\Gamma, P)/\mathrm{PrinDer}(\Gamma, P).$$

The space $\mathrm{Der}(\Gamma, P)$ admits a right action of N (and hence of the group algebra $K[N]$) defined as follows: for $\phi \in N$ and $d \in \mathrm{Der}(\Gamma, P)$,

$$(d^{\phi})(\gamma) := [d(\phi\gamma\phi^{-1})]^{\phi}. \tag{3}$$

This gives rise to a right action of $G = N/\Gamma$ on the group cohomology $\mathrm{H}^1(\Gamma, P)$.

Remark 4. Notice that there is no well-defined action of N/Γ on the space $\mathrm{Der}(\Gamma, P)$, since Γ does not necessarily act trivial on derivations, i.e., if $g, \gamma \in \Gamma$ then

$$d(g\gamma g^{-1})^g = d(g)^{\gamma} - d(g) + d(\gamma).$$

3. On Holomorphic differentials

In this section we study the space of holomorphic differentials, *i.e.,* we restrict ourselves to the case $n = 1$. We derive a new upper bound for the order of an element in the automorphism group of a Mumford curve, and we derive a criterion for a modular representation to be indecomposable. Holomorphic differentials and the Jacobian of Mumford curves were studied by Y. Manin and V. Drinfeld in [27].

The space of holomorphic differentials is given by

$$\begin{aligned} \mathrm{H}^0(X, \Omega) &= \mathrm{H}^1(\Gamma, P_0) = \mathrm{H}^1(\Gamma, K) = \mathrm{Hom}(\Gamma, K) = \mathrm{Hom}(\Gamma, \mathbb{Z}) \otimes K \\ &= \mathrm{Hom}(\Gamma^{\mathrm{ab}}, \mathbb{Z}) \otimes K. \end{aligned}$$

Theorem 5. *There is a faithful representation of $G = N/\Gamma$ to $\mathrm{GL}(g, \mathbb{Z})$.*

Proof. The group N/Γ acts by conjugation on Γ^{ab} which is isomorphic to $\mathbb{Z}^g$. B. Köck in [20] proved that this action is faithful. □

Remark 6. The representation can be described by the conjugation action of N/Γ on $\Gamma/\Gamma' \cong \mathbb{Z}^g$, where $\Gamma' = [\Gamma, \Gamma]$ is the commutator of Γ. Notice that this representation is integral. The problem of describing indecomposable summands of integral representation is even more difficult than the corresponding problem for the modular theory.

Remark 7. Theorem 5 shows that holomorphic differentials on Mumford curves are, in some sense, similar to holomorphic differentials on Riemann surfaces; for a Riemann surface Y there is a faithful action of its automorphism group on $\mathrm{H}^1(Y, \mathbb{Z})$, which induces a faithful representation of a subgroup of the automorphism group on the symplectic matrices $\mathrm{Sp}(2g, \mathbb{Z})$ [12, sec. V.3 p. 269].

Corollary 8. *Every element of order p in N/Γ satisfies $p \leq g+1$.*

Proof. This is a simplified version of theorem 2.7 in [24]. □

For abelian subgroups H of the automorphism group of a general curve defined over an algebraically closed field of positive characteristic S. Nakajima [30] derived the bound

$$|H| \leq 4g + 4.$$

While for abelian subgroups of automorphism groups of Mumford curves the second author together with V. Rotger [23] proved

$$|H| \leq 3(g-1).$$

Remark 9. It is tempting here to try to find bounds for groups of automorphisms using the theory of representations for matrix groups over $\mathbb{Z}$. Let X_Γ be a Mumford curve, and let Δ be a normal abelian subgroup of $\mathrm{Aut}(X_\Gamma)$, then

$$|\mathrm{Aut}(X_\Gamma)| \leq (g+2)!|\Delta| \leq (g+2)! \cdot 3(g-1).$$

This comes from the bound of S. Friedland [14] for the order of a finite subgroup H of $\mathrm{GL}_g(\mathbb{Z})$ with a normal abelian subgroup Δ. Of course, for Mumford curves this is much worse than the bound

$$\mathrm{Aut}(X_\Gamma) \leq \min\{12(g-1), 2\sqrt{2}(\sqrt{g}+1)^2\}$$

given in [7].

3.1. **Invariants and direct factors.** Now, we develop a criterion for a modular representation to be indecomposable.

Proposition 10. *Let G be a finite cyclic group and let V be a G-module. The number of indecomposable summands of V that are G-modules equals the dimension of the space of invariants V^G.*

Proof. This is clear from the theory of Jordan normal forms; if we write the Jordan blocks of the generator of the cyclic group, then every Jordan block has an one dimensional invariant subspace. □

Remark 11. The assumption that G is cyclic is necessary. See, for example, the $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$-module given by Heller and Reiner in [42, example 1.4 p. 157.]. The space of invariants has dimension > 1 but the module is still indecomposable. However, using results of B. Köck we will prove in section 6.2 that in the cases we study the dimension of invariants count the number of indecomposable summands since all modules are submodules of projective modules.

Lemma 12. *If H is an abelian p-group acting on a K-vector space M, then $M^H \neq \{0\}$.*

Proof. Since H is a finite group, it is generated by elements $a_1, \dots a_r$. All elements a_i have order a power of p, hence all eigenvalues are 1 and the eigenspaces are non-empty by Jordan decomposition theory. If E_1 is the eigenspace of a_1 then a_i act on it since

$$a_1 v = v \Rightarrow a_i(a_1 v) = a_i v \Rightarrow a_1(a_i v) = a_i v.$$

Therefore, we can consider the non-zero eigenspace E_2 of a_2 acting on E_1. Hence, the eigenspace E_1 contains elements that remain invariant under the action of a_1, a_2. In order to obtain an invariant element we proceed inductively. □

Definition 13. Let H be a group such that for every H-module M, $M^H \neq \{0\}$. We will say that H has the non-trivial invariant property.

Proposition 14. *If a group H has the non-trivial invariant property and for a H-module V the space V^H is one dimensional, then V is indecomposable.*

Proof. Every indecomposable summand of V contributes at least one non-trivial invariant element in V^H. Therefore, if $\dim V^H = 1$, then there is only one indecomposable summand. □

3.2. **Subrao curves: proof of theorem 1.** We finish this section with computing the indecomposable summands for Subrao curves. Let $\Gamma, N, G = N/\Gamma$ as in the introduction. We consider the short exact sequence

$$1 \to [\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}] := \Gamma \to \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/p\mathbb{Z} := N \to \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to 1.$$

The group $N/\Gamma = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ acts on Γ/Γ' by conjugation. Assume that the two cyclic groups $\mathbb{Z}/p\mathbb{Z}$ are generated by x, y respectively. Then the commutators $e_{ij} = [x^i, y^j]$ form a basis of the free group Γ. Set $\langle x \rangle = A$ and $\langle y \rangle = B$. We can check

$$a[x,y]a^{-1} = [ax, y][a, y]^{-1} \tag{4}$$

$$b[x,y]b^{-1} = [x, b]^{-1}[x, by] \tag{5}$$

for every $a \in A$ and $b \in B$. Therefore, the action of A is

$$e_{ij}^x = [x \cdot x^i, y^j] - [x, y^j] = e_{i+1,j} - e_{1,j},$$

and is given by the following block diagonal form

$$\begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix}$$

where there are $p-1$ blocks M. Here M is a $(p-1) \times (p-1)$ block of the form:

$$\begin{pmatrix} -1 & -1 & -1 & \cdots & -1 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{6}$$

Notice that the matrix M has characteristic polynomial $\frac{x^p-1}{x-1} = 1 + x + \cdots x^{p-1}$ (it is the companion matrix of this polynomial), and is the prototype for an integral representation of a cyclic group of order p with minimal degree $p-1$, i.e., there are no integral representations of a cyclic group of order p in $r \times r$ matrices for $r < p-1$, see [24].

If we put the group B into play, then we observe that the representation of the abelian group $A \times B = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is indecomposable; indeed, let us compute the invariant elements of the action of A. We have

$$\left(\sum_{a \neq 1} \lambda_a e_{a,b} \right)^x = \sum_{a \neq 1} \lambda_a e_{xa,b} - \sum_{a \neq 1} \lambda_a e_{x,b} = \sum_{a \neq 1} \lambda_a e_{a,b}$$

By comparison we have:

$$\lambda_a = \lambda_{x^{-1}a} \text{ and } \lambda_x = -\sum_{a \neq 1} \lambda_a.$$

This implies that all λ_a should be constant and the space of invariants has dimension $p-1$, and is generated by the elements

$$e_j := \sum_{a \neq 1} e_{a,j}.$$

Now we compute the space of $A \times B$ invariants by using the fact $V^{A \times B} = (V^A)^B$. Recall that for $b \in B$ we have

$$e_{i,j}^b = e_{i,bj} - e_{i,b}.$$

We compute that under the action of $b \in B$

$$e_j^b = \sum_{a \neq 1} e_{a,j}^b = \sum_{a \neq 1} (e_{a,bj} - e_{a,j}) = e_{bj} - e_j.$$

Therefore, the space $V^{A \times B}$ is one dimensional and the representation is indecomposable by proposition 14.

4. On polydifferentials

The situation for polydifferentials is more subtle. A derivation of the free group Γ is defined by its values on a set of generators $f_1, \ldots, f_g$ of Γ. Therefore, the dimension of the space $\mathrm{Der}(\Gamma, P_{2(n-1)})$ is $(2n-1)g$. Let Γ be generated by $f_1, \ldots f_g$, and let $\{m_1, \ldots, m_{2n-1}\}$ be a basis of $P_{2(n-1)}$.

We consider the set of derivations

$$d_{i,\ell} : \Gamma \to P_{2(n-1)}, \quad 1 \leq i \leq g, 1 \leq \ell \leq 2n-1 \tag{7}$$

defined by

$$d_{i,\ell}(f_j) = m_\ell \delta_{ij}.$$

Since a derivation is uniquely determined by its values on f_i we see that the above derivations form a basis on $\mathrm{Der}(\Gamma, P_{2(n-1)})$ and the dimension equals:

$$\dim \mathrm{Der}(\Gamma, P_{2(n-1)}) = (2n-1)g.$$

Notice that for $n > 1$ if we subtract from this space the dimension $2n-1$ of principal derivations we obtain the correct dimension

$$\dim \mathrm{H}^1(\Gamma, P_{2(n-1)}) = (2n-1)(g-1) = \dim H^0(X, \Omega_X^{\otimes n})$$

for the space of holomorphic polydifferentials, as it is computed using the Riemann–Roch theorem.

An important tool for studying derivations on free groups is the theory of Fox derivatives developed by R. Fox in [13]. These are derivations

$$\frac{\partial}{\partial f_i} : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma],$$

defined by

$$\frac{\partial}{\partial f_i}(f_j) = \delta_{ij}$$

and extended to $\mathbb{Z}[\Gamma]$ by the derivation rule:

$$\frac{\partial}{\partial f_i}(f_j \cdot f_k) = (\frac{\partial}{\partial f_i}(f_j))^{f_k} + \frac{\partial}{\partial f_i}(f_k).$$

The basis for the space $\mathrm{Der}(\Gamma, P_{2(n-1)})$ given in eq. (7) can be expressed in terms of Fox derivatives:

$$d_{i,\ell} = \left\{ m_\ell \otimes \frac{\partial}{\partial f_i} \right\}_{\ell=1,\ldots,2n-1, i=1,\ldots,g}. \tag{8}$$

Next theorem guaranties that a derivation is defined if we define its values on the generators of the free group.

Theorem 15. *Let $\delta : \Gamma \to \mathbb{Z}[\Gamma]$ be a derivation such that $\delta(f_i) = h_i$. Then the derivation can be written as linear combination of the Fox derivatives:*

$$\delta(\cdot) = \sum_{i=1}^{g} h_i \frac{\partial(\cdot)}{\partial f_i}. \tag{9}$$

Proof. This is (2.2) in Fox [13]. Notice that he has a left action while we have a right action. □

We will now explain our strategy for studying the N/Γ-module structure of the space of polydifferentials in the case of a general N and Γ.

1. Give a description of the conjugation action of N on Γ. Actually we would like to describe the action of N/Γ, but this action is not well-defined unless we take out the quotient by principal derivations. However, it makes sense to fix a set of representatives $\{n_i \in N\}$ for N/Γ, $1 \leq i \leq \#N/\Gamma$. Set

$$\Gamma \ni w_{ij} = n_i f_j n_i^{-1} \quad 1 \leq i \leq \#N/\Gamma, 1 \leq j \leq g, \tag{10}$$

where w_{ij} are words in $f_1, \ldots, f_g$. A different set of representatives gives a different representation.

2. In order to find the representation matrix of the action we have to express

$$d(n_i f_j n_i^{-1})^{n_i} = d(w_{ij})^{n_i}$$

as an element of the basis derivations. For this we will need lemma 16 which allows us to compute derivations on words. Here one has to be careful because well-defined on derivations is only the action of N not the action of N/Γ. The action of N/Γ is well-defined only up to principal derivations.

Lemma 16. *Let P be a Γ-module and let $d : \Gamma \to P$ be a derivation. The derivation of a power f^k on an arbitrary element $f \in \Gamma$ is given by:*

$$d(f^k) = d(f)^{1+f+f^2+\cdots+f^{k-1}}.$$

Recall that $f_1, \ldots, f_g$ is a set of generators for Γ. Fix j and assume that $d(f_i) = 0$ if $i \neq j$. Write $\Gamma \ni w$ as a word

$$w = u_0 f_j^{p_1} u_1 f_j^{p_2} \cdots f_j^{p_r} u_r,$$

where u_i are words which do not contain the basis element f_j. Then

$$d(w) = \sum_{\nu=1}^{r} d(f_j)^{(1+f_j+f_j^2+\cdots+f_j^{\nu-1}) u_\nu f_j^{p_\nu+1} u_{\nu+1} \cdots f_j^{p_r} u_r}.$$

Proof. Both equations follow by induction. □

The element $w_{k\mu} := n_k f_\mu n_k^{-1}$ is a word in elements $f_1, \ldots, f_g$. Assume that this word can be written as

$$w_{k\mu} = u_0 f_j^{p_1} u_1 f_j^{p_2} \cdots f_j^{p^r} u_r,$$

where u_i are words which do not contain the element f_j. Using the derivation definition we compute, the action of n_k on the derivation $d_{j\ell}$

$$\begin{aligned} \left(d_{j\ell}^{n_k}\right)(f_j) &= d_{j\ell}(w_{k,\mu})^{n_k} \\ &= \left(\sum_{\nu=1}^{r} m_\ell^{(1+f_j+f_j^2+\cdots f_j^{\nu-1})u_\nu f_j^{p_\nu+1} u_{\nu+1}\cdots f_j^{p_r} u_r} \right)^{n_k} \\ &= \left(m_\ell^{\frac{\partial(w_{k\mu})}{\partial f_j}} \right)^{n_k} \end{aligned}$$

We consider the derivation $\delta : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$ using eq. (9) in theorem 15, which has the following form: (notice that in the notation of theorem 15, $h_i = \frac{\partial(w_{k\mu})}{\partial f_i} n_k$)

$$\begin{aligned} \left(d_{j\ell}^{n_k}\right) &= \sum_{\nu=1}^{g} m_\ell^{\frac{\partial(w_{k\mu})}{\partial f_\nu} n_k \frac{\partial}{\partial f_\nu}} \\ &= \sum_{\nu=1}^{g} \sum_{i=1}^{2n-1} \rho\left(\frac{\partial(w_{k\mu})}{\partial f_\nu} n_k \right)_{\ell,i} d_{\nu,i}, \end{aligned}$$

where

$$m_\ell^{\frac{\partial(w_{k\mu})}{\partial x_\nu} n_k} = \sum_{i=1}^{2n-1} \rho\left(\frac{\partial(w_{k\mu})}{\partial f_\nu} n_k \right)_{\ell,i} m_i. \tag{11}$$

The elements $\rho\left(\frac{\partial(w_{k\mu})}{\partial f_\nu} n_k\right)_{\ell,i}$ are the coefficients needed in expressing $m_\ell^{\frac{\partial(w_{k\mu})}{\partial x_\nu} n_k}$ as a linear combination of basis elements of $P_{2(n-1)}$. Equation (11) is quite complicated. We will apply this to our favored example in section 5.

3. We now explain the effect of taking the quotient. Consider the space of principal derivations $\mathrm{PrinDer}(\Gamma, P_{2(n-1)})$ inside $\mathrm{Der}(\Gamma, P_{2(n-1)})$. The space $\mathrm{Der}(\Gamma, P_{2(n-1)})$ is acted on by N but not by N/Γ. Observe also that there is the embedding:

$$P_{2(n-1)} \to \mathrm{PrinDer}(\Gamma, P_{2(n-1)}), \tag{12}$$

$$m \mapsto d_m : \big(\gamma \mapsto m^\gamma - m\big). \tag{13}$$

This action is compatible with the action of N, i.e.,

$$(d_m)^n = d_{m^n}.$$

Fix the first element f_1 in the generating set of Γ. We consider the effect of taking the quotient by $\mathrm{PrinDer}(\Gamma, P_{2(n-1)})$ by defining the normalization of a derivation d to $\tilde{d}$ where $d = \tilde{d}$ modulo $\mathrm{PrinDer}(\Gamma, P_{2(n-1)})$, and $\tilde{d}(f_1) \in K$.

Remark 17. We just remarked that there is no natural action of N/Γ on the space of derivations, only an action of N. However the vertex stabilizers $N(v)$ of the tree of the Mumford curve are subgroups of N, such that $N(v)/(\Gamma \cap N(v)) = N(v)$. Therefore, we can study the space of derivations as $N(v)$-modules. Section 5 is devoted to the study the A-module structure of the space of derivations.

Lemma 18. *Let f be a hyperbolic element in $\mathrm{PGL}(2,K)$. There is an element $\alpha \in \mathrm{PGL}(2,K)$, such that*

$$f = \alpha \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \alpha^{-1} \tag{14}$$

where μ is an element in the maximal ideal of the local field K. There is a basis $\{m_i\}$, $i = 0, \ldots, 2(n-1)$ of $P_{2(n-1)}$ such that

$$m_i^f = \mu^{i-n+1} m_i.$$

Proof. The diagonal form given in eq. (14) follows by [28, lemma 1.1]. For the second part we set $m_i = (x^i)^{\alpha^{-1}}$ and we use eq. (2). □

In order to study the quotient of derivations modulo principal derivations we will introduce a normal form and we will put derivations into this normal form. Let $f := f_1$ be the first generator of the free group Γ.

Lemma 19. *For every element λm_i, $0 \le i \le 2(n-1)$, $i \ne n-1$ and $\lambda \in K$, there is an element $a_i \in P_{2(n-1)}$ such that $(\lambda a_i)^{f_1} - \lambda a_i = \lambda m_i$.*

Proof. Consider the element $m_i^{f_1} - m_i = (\mu^{i-n+1} - 1)m_i$. The element $\mu^{i-n+1} - 1$ is invertible in the field K for $i \ne n-1$. So we set $a_i = (\mu^{i-n+1} - 1)^{-1} m_i$. □

Lemma 20. *Fix a generator f_1 of the group Γ. Every derivation $d \in \mathrm{Der}(\Gamma, P_{2(n-1)})$ is equivalent (modulo principal derivations) to a derivation $\tilde{d}$, such that $\tilde{d}(f_1)$ is in the one dimensional vector space generated by m_{n-1}.*

Proof. We evaluate $d(f_1) = m = \sum_{i=0}^{2(n-1)} \lambda_i m_i$. Consider the element,

$$a_m = \sum_{i=1}^{2(n-1)} \lambda_i a_i,$$

where the a_i are chosen as in lemma 19. Then, the derivation

$$\tilde{d} = d - d_{a_m}$$

has the desired property. □

Remark 21. On the other generators f_i $(i \ge 2)$,

$$\tilde{d}(f_i) \quad = \quad d(f_i) + a_m^{f_i} - a_m.$$

Remark 22. In this way we can consider a basis of generators $d_{i\ell} = m_\ell \otimes \frac{\partial}{\partial f_i}$ of $\mathrm{Der}(\Gamma, P_{2(n-1)})$ and we put all of them in normal form $\tilde{d}_{i\ell}$, in order to find a basis of $H^1(\Gamma, P_{2(n-1)})$. Notice that by doing so we remove all derivations of the form $d_{1\ell}$ for $\ell \ne n-1$. The number of the derivations we remove is $2n-1$, so we get the correct dimension for the cohomology.

In order to compute the action of G on $H^1(\Gamma, P_{2(n-1)})$ we can compute the action to $\tilde{d}_{ij}$ which will give as result linear combinations of non-normalized derivations and which can be normalized by eliminating all derivations of the form $d_{1,\ell}$ for $\ell \ne n-1$.

Unfortunately, this approach is very complicated to do by hand even in the simplest example $A * B$. Also, notice that if we take the quotient of two modules M_1, M_2 such that $M_1, M_1/M_2$ are both G-modules but M_2 is not a G-module, then the indecomposable summands of M_1 may decompose in the quotient modulo M_2. Indeed, think of the Jordan normal form of a cyclic p-group $G = \langle g \rangle$, acting on $M_1 := \langle e_1, \ldots, e_r \rangle$ by $ge_i = e_i + e_{i+1}$. The module M_1 is indecomposable but the quotient $M_1/\langle e_{i_0} \rangle$ decomposes to two direct summands. This means that in our case little information from $\mathrm{Der}(\Gamma, P_{2(n-1)})$ is carried to the quotient $H^1(\Gamma, P_{2(n-1)})$. A new idea is needed. This is going to be the theory of projective modules.

Remark 23. One problem we might ask is the following inverse problem: is it correct that every finite subgroup appears as a finite subgroup for some $\mathrm{Aut}(F_g)$?. This is equivalent to the question: Is every finite group realizable as a group of automorphisms of a Mumford curve? We know that every finite group is realizable even in positive characteristic as an automorphism group of some algebraic curve [26]. Of course, this curve does not need to be a Mumford curve.

5. Computations on Subrao curves continued.

In this section we restrict ourselves to the case of Γ, N as in the introduction, i.e., $\Gamma = [A, B]$ with $A, B \cong \mathbb{Z}/p\mathbb{Z}$, $A = \langle \epsilon_A \rangle$, $B = \langle \epsilon_B \rangle$. We denote by P the space P_{2n-2}. As we have seen in section 3.1 the space of invariants contains information for the number of direct summands. So we begin our study by computing $H^1(\Gamma, P)^{N/\Gamma}$. In section 5.2 we study the A-module structure of the space of derivations. The effect on taking the quotient by principal derivations is postponed until section 6.

5.1. Computing the $H^1(\Gamma, P)^{N/\Gamma}$.

Lemma 24. *Suppose that $n > 0$ and $F \in P$ satisfies $F^\gamma = F$ for every $\gamma \in \Gamma$. Then F is zero.*

Proof. First notice that Γ contains an element δ of the form $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $c \neq 0$. Since $n > 0$, it follows that F can not be a non-zero constant. Suppose that F is non-zero of minimal degree $\deg F > 0$. Since $\epsilon_A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ normalizes Γ, we have $F^{\epsilon_A \gamma \epsilon_A^{-1}} = F$, for any γ, that is $F^{\epsilon_A} = F(X+1)$ is also an invariant under Γ. In particular, $F(X+1) - F(X)$ is invariant, which is absurd, since $F(X+1) - F(X)$ has degree less than $F(X)$. □

Lemma 25. *For $n > 0$ we have*

$$H^1(N, P) = H^1(\Gamma, P)^{N/\Gamma}.$$

Proof. Consider the 5-term restriction inflation coming from the Lyndon-Hochschild-Serre spectral sequence [41, par. 6.8.3].

(15)

$$0 \to H^1(N/\Gamma, P^\Gamma) \xrightarrow{\text{inf}} H^1(N, P) \xrightarrow{\text{res}} H^1(\Gamma, P)^{N/\Gamma} \to H^2(N/\Gamma, P^\Gamma) \to H^2(N, K)$$

Observe now that for $n > 0$ we have $P^\Gamma = 0$ by lemma 25 which forces

$$H^1(N/\Gamma, P^\Gamma) = H^2(N/\Gamma, P^\Gamma) = 0.$$

The result follows. □

Remark 26. For $n = 0$ we have $P = K$ and we compute

$$H^1(N/\Gamma, P^\Gamma) \cong H^1(N, P) \cong K^2.$$

Indeed, $H^1(N, P) = H^1(A * B, K)$. Since the module K is $A * B$-trivial we have by [41, Ex. 6.2.5 p.171] that

$$H^1(A * B, K) = H^1(A, K) \times H^1(B, K) = K^2.$$

On the other hand we know that the cohomology ring $H^*(N/\Gamma, K) = \bigwedge[\eta_1, \eta_2] \otimes [\xi_1, \xi_2]$, where $\deg \eta_i = 1$, $\deg \xi_i = 2$, $\eta_i^2 = 0$. The graded part of degree 1 is just the two dimensional vector space generated by η_1, η_2 [11, sec. 3.5 p. 32]. By counting dimensions we see that the map res is zero so $H^1(\Gamma, K)^{N/\Gamma}$ is mapped injectively into a subspace of $H^2(N/\Gamma, K)$.

Notice that $H^2(N/\Gamma, K)$ is the graded part of $H^*(N/\Gamma, K)$ which is generated by $\eta_1\wedge\eta_2, \xi_1, \xi_2$ and has dimension 3 while the space $H^2(N,K) = H^2(A,K)\times H^2(B,K)$ by [41, cor. 6.2.10 p. 170] and is two dimensional. This is of course compatible with the computation of invariants done in example 3.2.

Remark 27. Given a discrete subgroup N of $\mathrm{PGL}(2,K)$ there is no canonical way of selecting a normal free subgroup. Lemma 25 implies that no matter how we select Γ, the invariant subspace of holomorphic polydifferentials of the Mumford curve X_Γ (that depends on Γ) is the same.

Proposition 28. *The dimension of $H^1(N,P)$ equals $2n-1$.*

Proof. We have the following isomorphism of groups:

$$\mathrm{Der}(A*B,P) \to \mathrm{Der}(A,P)\times \mathrm{Der}(B,P).$$

Indeed every derivation on $A*B$ restricts to derivations of subgroups A, B and every derivation that is defined on elements of A, B can be extended, by using the derivation rule to words in $A*B$. We therefore have the short exact sequence

$$0 \to \mathrm{PrinDer}(A*B,P) \to \mathrm{Der}(A,P)\times\mathrm{Der}(B,P) \to H^1(A*B,P) \to 1,$$

which allows us to compute

$$\begin{aligned}\dim_K H^1(A*B,P) &= \dim_K \mathrm{Der}(A,P)\times\mathrm{Der}(B,P) - \dim_K \mathrm{PrinDer}(A*B,P)\\ &= 2(2n-1)-(2n-1) = (2n-1).\end{aligned}$$

□

5.2. **The A-module structure of $\mathrm{Der}(\Gamma, P)$.** We compute d^δ for $d\in\mathrm{Der}(\Gamma,P)$ and $\delta\in N$. Since we will eventually consider the action of $G = N/\Gamma$ on the cohomology $\mathrm{H}^1(\Gamma,P)$, and since $G\cong A\times B$, it suffices to consider the cases $\delta\in A$ and $\delta\in B$. First, let us consider the case $\delta\in A$. We calculate using eq. (3):

$$\begin{aligned}d^\delta([\alpha,\beta]) &= [d(\delta[\alpha,\beta]\delta^{-1})]^\delta = \left[d([\delta\alpha,\beta][\delta,\beta]^{-1})\right]^\delta\\ &= \left(d[\delta\alpha,\beta])^{[\delta,\beta]^{-1}} + d([\delta,\beta]^{-1})\right)^\delta\\ &= [d([\delta\alpha,\beta])]^{\beta\delta\beta^{-1}} - [d[\delta,\beta])]^{\beta\delta\beta^{-1}}\end{aligned}$$

Consider the derivation $d^{(k)}_{[\alpha,\beta]}$ for $k = 0,\ldots,2(n-1)$ and $\alpha\in A\backslash\{1\}, \beta\in B\backslash\{1\}$ that is characterized by

$$d^{(k)}_{[\alpha,\beta]}([\alpha',\beta']) = \begin{cases}\left[(X^p-X)^i\cdot\binom{X}{j}\right]^{\beta^{-1}} & \text{if } \alpha=\alpha' \text{ and } \beta=\beta',\\ 0 & \text{otherwise,}\end{cases}$$

where i and j are determined by $k = i\cdot p + j$ and $0\le j<p$. Then

$$\left\{d^{(k)}_{[\alpha,\beta]}\right\}_{0\le k\le 2(n-1),\alpha\in A\backslash\{1\},\beta\in B\backslash\{1\}}$$

forms a K-basis of the space $\mathrm{Der}(\Gamma,P)$. For $\delta = \begin{pmatrix}1&1\\0&1\end{pmatrix}$ we have

$$\begin{aligned}\left(d^{(k)}_{[\alpha,\beta]}\right)^\delta([\alpha',\beta']) &= \left[d^{(k)}_{[\alpha,\beta]}([\delta\alpha',\beta'])\right]^{\beta'\delta\beta'^{-1}} - \left[d^{(k)}_{[\alpha,\beta]}[\delta,\beta'])\right]^{\beta'\delta\beta'^{-1}}\\ &= \begin{cases}\left[(X^p-X)^i\cdot\binom{X+1}{j}\right]^{\beta^{-1}} & \text{if } \alpha=\delta\alpha' \text{ and } \beta=\beta',\\ -\left[(X^p-X)^i\cdot\binom{X+1}{j}\right]^{\beta^{-1}} & \text{if } \alpha=\delta \text{ and } \beta=\beta',\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Hence, due to the well-known binomial relation

$$\binom{X+1}{k} = \binom{X}{k} + \binom{X}{k-1},$$

we have

$$\left(d^{(k)}_{[\alpha,\beta]}\right)^{\delta} = \begin{cases} d^{(k)}_{[\delta^{-1}\alpha,\beta]} + d^{(k-1)}_{[\delta^{-1}\alpha,\beta]} & \text{if } j>0 \text{ and } \alpha \neq \delta, \\ d^{(k)}_{[\delta^{-1}\alpha,\beta]} & \text{if } j=0 \text{ and } \alpha \neq \delta, \\ -\sum_{\alpha'\neq 1}\left(d^{(k)}_{[\alpha',\beta]} + d^{(k-1)}_{[\alpha',\beta]}\right) & \text{if } j>0 \text{ and } \alpha = \delta, \\ -\sum_{\alpha'\neq 1}\left(d^{(k)}_{[\alpha',\beta]}\right) & \text{if } j=0 \text{ and } \alpha = \delta, \end{cases}$$

where j is the number determined by $k = i\cdot p + j$ and $0 \leq j < p$.

Now consider the elements

$$d^k_{ab} := d^{(k)}_{[\epsilon^a_A,\epsilon^b_B]}$$

and we order them by lexicographical order with respect to (k,a,b); that is,

$$d^0_{11}, d^1_{11}, \ldots, d^{2(n-1)}_{11}, d^0_{21}, d^1_{21}, \ldots, d^{2(n-1)}_{21}, \ldots, d^0_{(p-1),1}, d^1_{(p-1),1}, \ldots, d^{2(n-1)}_{(p-1),1}, \cdots$$

The square matrix Q of degree $(2n-1)(p-1)^2$ of the action by $\delta = \epsilon_A$ is then decomposed into $p-1$ blocks like

$$Q = \begin{pmatrix} M & & & \\ & M & & \\ & & \ddots & \\ & & & M \end{pmatrix},$$

where M is a square matrix of degree $(2n-1)(p-1)$. The matrix M is further decomposed into $p-1$ blocks

$$M = \begin{pmatrix} -N & -N & -N & \cdots & -N & -N \\ N & & & & & \\ & N & & & & \\ & & N & & & \\ & & & \ddots & & \\ & & & & N & 0 \end{pmatrix}, \tag{16}$$

where N is a square matrix of degree $2n-1$, which is of the form

$$N = \begin{pmatrix} J_p & & & & \\ & J_p & & & \\ & & \ddots & & \\ & & & J_p & \\ & & & & J_r \end{pmatrix},$$

where J_ℓ for a non-negative integer ℓ denotes the $\ell \times \ell$-Jordan block with diagonal entries equal to 1; here we put $2n-1 = q\cdot p + r$ with $0 \leq r < p$, and the number of J_p's in N is q.

In algebraic terms the above M can be described as follows: let W' be the $\mathbb{Z}[A]$-module with basis $\{w_1, \ldots, w_{p-1}\}$ and let the action be given by sending the generator ϵ_A of A to the matrix

$$\begin{pmatrix} -1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 0 \end{pmatrix}.$$

Notice that W' is an indecomposable $\mathbb{Z}[A]$ representation and the characteristic polynomial of the above matrix is $x^{p-1}+x^{p-2}+\cdots+x+1$, i.e., the p-th cyclotomic polynomial. Let $W = K \otimes_{\mathbb{Z}} W'$. We also consider the $K[A]$-module V with basis $v_1, \ldots, v_{2n-1}$ and action defined by letting the generator act in terms of the matrix N. Then, the diagonal action of $K[A]$ on $V \otimes W$ is given by the matrix M. This construction is called the *Kronecker product* in the literature.

The module V is not indecomposable and can be written as a direct sum of indecomposable modules as follows:

$$V = J_p^{\lfloor \frac{2n-1}{p} \rfloor} \oplus J_r.$$

This allows us to find the indecomposable factors of $V \otimes W$. First notice that we have a decomposition:

$$V \otimes W \quad = \quad (J_p \otimes W)^{\lfloor \frac{2n-1}{p} \rfloor} \oplus (J_r \otimes W)$$

The matrices $J_p \otimes W$ and $J_r \otimes W$ give rise to representations of dimensions $p(p-1)$ and $r(p-1)$ respectively, so they are not indecomposable.

In order to find their indecomposable summands we need to understand the space of invariants for the action of the group A on $V \otimes W$.

Lemma 29. *Consider an element*

$$d = \sum_{b=1}^{p-1} \sum_{a=1}^{p-1} \sum_{k=0}^{2n-1} \lambda_{ab}^k d_{ab}^k, \qquad \lambda_{ab}^k \in K.$$

Set $\Lambda_{ab} = {}^t(\lambda_{ab}^0, \lambda_{ab}^1, \ldots, \lambda_{ab}^{2(n-1)})$ *for* $1 \le a, b \le p-1$ *and* $\Lambda_b = {}^t(\Lambda_{1b}, \ldots, \Lambda_{(p-1)b})$. *The element d is invariant under the action of A if and only if*

$$\Lambda_{1b} = (\lambda_{1b}^0, \lambda_{1b}^1, \ldots, \lambda_{1b}^{p-2}, 0, \lambda_{1b}^p, \ldots, \lambda_{1b}^{2p-2}, 0, \lambda_{1b}^2 p, \ldots, \lambda_{1b}^{qp-2}, 0, \lambda_{1b}^{qp}, \ldots, \lambda_{1b}^{2(n-1)})^t,$$

i.e., $\lambda_{1b}^{qp-1} = 0$ *for all* $q = 1, \ldots, \left\lfloor \frac{2n-2}{p} \right\rfloor$ *and moreover* $\Lambda_{ib} = N^{i-1}\Lambda_{1,b}$.

Proof. Observe that the invariance of d is equivalent to the invariance of Λ_b under M for any b. The element Λ_b is invariant under M if and only if:

$$-N(\Lambda_{1b} + \cdots + \Lambda_{(p-1)b}) = \Lambda_{1b}$$

and

$$\Lambda_{ib} = N\Lambda_{(i-1)b} \text{ for } i = 2, \ldots, p-1.$$

These two conditions are equivalent fo

$$(1 + N + N^2 + \cdots + N^{p-1})\Lambda_{1b} = 0$$

and

$$\Lambda_{ib} = N^{i-1}\Lambda_{1b} \text{ for } i = 2, \ldots, p-1.$$

In order to calculate the matrix $1 + N + N^2 + \cdots + N^{p-1}$, we look at the Jordan block J_l. Let U_l be the matrix such that $J_l = 1 + U_l$. Note that $J_l^i = \sum_{j=0}^{i} \binom{i}{j} U_l^j$

and $\sum_{i=0}^{p-1}\binom{i}{j}=\binom{p}{j+1}$, where the latter identity is deduced from:

$$\sum_{i=0}^{p-1}(1+T)^i=\frac{(1+T)^p-1}{T}.$$

We compute

$$\begin{aligned}
1+J_l+J_l^2+\cdots+J_l^{p-1} &= \sum_{i=0}^{p-1}\sum_{j=0}^{i}\binom{i}{j}U_l^j \\
&= \sum_{j=0}^{p-1}\sum_{i=j}^{p-1}\binom{i}{j}U_l^j \\
&= \sum_{j=0}^{p-1}\left(\binom{p}{j+1}-\sum_{i=0}^{j-1}\binom{i}{j}\right)U_l^j \\
&= U_l^{p-1}-\sum_{j=0}^{p-2}\sum_{i=0}^{j-1}\binom{i}{j}U_l^j=U_l^{p-1}.
\end{aligned}$$

And the later matrix is, if $l\geq p$ of the form

$$\begin{pmatrix}
0 & \cdots & 0 & 1 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 1 & \cdots & 0\\
 & & & & \ddots & \ddots & \\
0 & \cdots & 0 & 0 & 0 & \cdots & 1\\
0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}$$

where in the first row the first $p-1$ entries are 0 or is zero if $l<p$. Of course in our case if $l\geq p$ then $l=p$. Hence $(1+N+N^2+\cdots+N^{p-1})\Lambda_{1b}$ is zero if and only if it is of the desired form. □

Lemma 30. *The equivalence classes for the indecomposable summands of cyclic groups correspond to the J_i modules for $i=1,\ldots,p$. If σ is a generator of the cyclic group then $J_r\cong K[\langle\sigma\rangle]/(\sigma-1)^r$.*

Proof. The classes are determined by the normal Jordan form of the matrix that corresponds to a generator. For the second assertion observe that the unique indecomposable representation of dimension r is $K[\langle\sigma\rangle]/(\sigma-1)^r$ [40]. □

Remark 31. Notice that in the case of $n=1$ we have the trivial action and the first condition disappears. Notice also that for fixed b the dimension of the space of invariants of the vector space generated by d_{ab}^k, $0\leq a\leq p-1$, $0\leq k\leq 2n-1$ is equal to $2n-1-\left\lfloor\frac{2n-2}{p}\right\rfloor$.

Proposition 32. *The matrix $J_p\otimes W$ gives rise to a representation of A which can be decomposed to as follows:*

$$J_p\otimes W=J_p^{p-1} \tag{17}$$

The matrix $J_r\otimes W$ is decomposed as

$$J_r\otimes W=J_p^{r-1}\oplus J_{p-r}.$$

Proof. Lemma 29 together with proposition 10 implies that the number of indecomposable summands of $J_p\otimes W$ is $p-1$, and since $\dim(J_p\otimes W)=p(p-1)$ each of the direct summands should be J_p.

The representation $J_r \otimes W$ has dimension $(p-1)r$ and is has r direct summands. We consider the short exact sequence:

$$1 \rightarrow J_{p-r} \otimes W \xrightarrow{i} J_p \otimes W \xrightarrow{j} J_r \otimes W \rightarrow 1,$$

i is easily seen to be injective, and j is surjective by counting dimensions. Notice that $J_p/J_r \cong J_{p-1}$.

We now consider the functor of $A \cong \mathbb{Z}/p\mathbb{Z}$ invariants in order to obtain the long exact sequence:

$$1 \rightarrow (J_{p-r}\otimes W)^A \rightarrow (J_p \otimes W)^A \rightarrow (J_r \otimes W)^A \rightarrow H^1(A, J_{p-r}\otimes W) \rightarrow H^1(A, J_p^{p-1}) = 0,$$

the final cohomology space is zero since $J_p \cong K[A]$ is a projective A-module.

The dimension of the space of invariants corresponds to the number of indecomposable summands, and by this we compute that $H^1(A, J_{p-r} \otimes W)$ is one dimensional. Indeed, $\dim(J_{p-r} \otimes W)^A = p - r$, $\dim(J_p \otimes W)^A = p-1$ and $\dim(J_r \otimes W)^A = r$. Now if $J_{p-r} \otimes W = \oplus A_i$ is the decomposition in terms of A-modules we have

$$H^1(A, J_{p-r} \otimes W) = \oplus H^1(A, A_i).$$

Since $A_i \cong J_{\rho_i}$ for some $1 \leq \rho_i \leq p$ and $H^1(A, J_\rho) = 1$ if $1 \leq \rho \leq p-1$ and $H^1(A, J_p) = 0$ we finally obtain that all but one summands of $J_{p-r} \otimes W$ are J_p. □

Proposition 33. *The A-module structure of* $\mathrm{Der}(\Gamma, P_{2(n-1)})$ *is given by:*

$$\mathrm{Der}(\Gamma, P_{2(n-1)}) = \left(J_p^{(p-1)\lfloor \frac{2n-1}{p} \rfloor} \oplus J_p^{r-1} \oplus J_{p-r} \right)^{p-1}.$$

Proof. Observe that we have a representation of the space of derivations into $p-1$ direct summands and each one of them is given by the matrix M that corresponds to $V \otimes W$. The result follows by proposition 32: For $p \nmid 2n-1$ we have:

$$\begin{aligned} V \otimes W &\cong (J_p \otimes W)^{\lfloor \frac{2n-1}{p} \rfloor} \oplus (J_r \otimes W) \\ &= J_p^{(p-1)\lfloor \frac{2n-1}{p} \rfloor} \oplus J_p^{r-1} \oplus J_{p-r}. \end{aligned}$$

□

The $p \mid 2n-1$ case is easier:

$$V \otimes W \cong J_p^{(p-1)\frac{2n-1}{p}}.$$

6. Projective covers

In this section we focus on the computation of both the A and $A \times B = N/\Gamma = A * B/[A, B]$-module structure of $H^1(\Gamma, P_{2(n-1)})$.

Recall that a module P is called projective if for every surjective module homomorphism $N \xrightarrow{f} M$ and every module homomorphism $P \xrightarrow{g} M$ there is a homomorphism h making the diagram commutative:

(18)
$$\begin{tikzcd} & N \arrow[d, "f"] \\ P \arrow[ur, dotted, "\exists h"] \arrow[r, "g"] & M \end{tikzcd}$$

For every G-module M there is a minimal projective module $P(M)$ that fits in a short exact sequence:

$$0 \rightarrow \Omega(M) \rightarrow P(M) \xrightarrow{\phi} M \rightarrow 0.$$

The kernel of ϕ is denoted by $\Omega(M)$ and is called the loop space of M. Recall that as A-module J_r is not projective unless $r = p$. The projective cover of J_r for $1 \leq j \leq p$ is $J_p = K[A]$. If P is just a projective module that surjects to M we will denote the kernel of the epimorphism by $\tilde{\Omega}(M)$. If $M \to I$ is the embedding of a M into an injective module then we will denote the cokernel by $\tilde{\Omega}^{-1}(M)$. If I is the injective hull we will omit the tilde from the notation. For basic properties of these notions we refer to [1, sec. 1.5], also [35] contains a nice introduction tailored for the kind of problems we are interested in.

We will need the following:

Lemma 34. *The module $K[A]$ can not have a direct sum of two non-trivial $K[A]$-modules as a submodule. Moreover, the module $K[A]^n$ can not have a direct sum of $n+1$ non-trivial modules as a submodule.*

Proof. Suppose M_1 and M_2 are two non-trivial $K[A]$-modules which are direct summands of $K[A]$. Indeed, using dimensions of invariant subspaces we have $2 \leq \dim(M_1 \oplus M_2)^A \leq K[A]^A = 1$. The assertion for general n is proved similarly. $\square$

Lemma 35. *Let a' be an positive integer. If J_r is a submodule of $W = K[A]^{a'}$, then the smallest projective module that contains J_r and is a direct summand of W is $K[A]$.*

Proof. Let V be the smallest module such that $J_r \subset V$ and let V' be the module such that $W = V \oplus V'$. By the Jordan-Hölder theorem V is isomorphic to $K[A]^a$ with $1 \leq a \leq a'$.

Consider a basis $\{e_1, \ldots, e_r\}$ of J_r such that $\sigma e_i = e_i + e_{i+1}$ for $1 \leq i \leq r-1$ and $\sigma e_r = e_r$. We can complete $\{e_1, \ldots, e_r\}$ to a basis of W such that σ has the following matrix form:

$$\begin{pmatrix} X & D \\ 0 & J_r \end{pmatrix}, \tag{19}$$

where D is an $(p^a - r) \times r$ matrix and X is an $(p^a - r) \times (p^a - r)$ upper triangular matrix with 1's in the diagonal. This can be done by choosing the basis such that J_r and $K[A]^a/J_r$ are in Jordan normal form.

The existence of a basis such that σ has the decomposition given in equation (19) is equivalent to the existence of a flag of $K[A]$-modules containing J_r:

$$J_r = V_0 \subset V_1 \subset \cdots \subset V_{p^a - r} = K[A]^a.$$

When we go from V_0 to V_1 we add an extra basis element and we have to consider the following cases: the minimal polynomial of σ restricted to V_1 is either $(x-1)^{r+1}$ or $(x-1)^r$. In the first case $V_1 \cong J_{r+1}$ and in the second case $V_1 \cong J_r \oplus J_1$. When we go from V_1 to V_2 we add an extra basis element and the possibilities for the $K[A]$-module structure are $J_{r+2}, J_{r+1} \oplus J_1,\ J_r \oplus J_1 \oplus J_1$. We proceed this way inductively. In the final step we obtain the module $K[A]^a$ with minimal polynomial $x^p - 1$. Therefore, at least one of the summands in the final step should be $K[A] = J_p$ and this is a contradiction to the minimality of V unless $a = 1$. $\square$

6.1. **A-module structure of $H^1(\Gamma, P)$.** We will now give two different proofs of Theorem 2.1.

First proof of Theorem 2.1. We compute:

$$\mathrm{PrinDer}(\Gamma, P_{2n-2}) = P_{2n-2} = K[A]^{\left\lfloor \frac{2n-1}{p} \right\rfloor} \oplus J_r$$

where $2n - 1 = p \left\lfloor \frac{2n-1}{p} \right\rfloor + r$.

We have the following decompositions, see Proposition 33

$$P_{2n-2} = J_r \oplus K[A]^{\lfloor \frac{2n-1}{p} \rfloor} \tag{20}$$

$$\mathrm{Der}(\Gamma, P_{2n-2}) = \begin{cases} J_{p-r}^{p-1} \oplus K[A]^{(p-1)\lfloor \frac{(2n-1)(p-1)}{p} \rfloor} & \text{if } p \nmid 2n-1 \\ K[A]^{(p-1)^2 \frac{(2n-1)}{p}} & \text{if } p \mid 2n-1 \end{cases}. \tag{21}$$

By Lemma 35 the projective covers are

$$P(P_{2n-2}) = K[A]^{\lceil \frac{2n-1}{p} \rceil} \tag{22}$$

$$P(\mathrm{Der}(\Gamma, P_{2n-2})) = K[A]^{\lceil \frac{(2n-1)(p-1)}{p} \rceil (p-1)} \tag{23}$$

For the $p \mid 2n-1$ case we can easily see that

$$H^1(\Gamma, P_{2n-2}) \cong K[A]^{(p^2-2p)\frac{2n-1}{p}} = K[A]^{(p-2)(2n-1)}.$$

We now focus on the $p \nmid 2n-1$ case. The cokernels of the projective covers satisfy

$$\Omega(P_{2(n-1)}) = J_{p-r} \text{ and } \Omega(\mathrm{Der}(\Gamma, P_{2n-2})) = J_r^{p-1}.$$

In the following commutative diagram the first column consists of loop spaces and the second one consists of projective covers.

(24)

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & J_{p-r} \arrow[r, "i_1"] & K[A] \oplus K[A]^{\lfloor \frac{2n-1}{p} \rfloor} \arrow[r, "\pi_1"] \arrow[d, "\phi_1"'] & P_{2n-2} \arrow[r] \arrow[d, "\psi_1"] & 0 \\
0 \arrow[r] & J_r^{p-1} \arrow[r, "i_2"] & K[A]^{p-1} \oplus K[A]^{\lfloor \frac{(2n-1)(p-1)}{p} \rfloor (p-1)} \arrow[r, "\pi_2"] \arrow[d, "\phi_2"'] & \mathrm{Der}(\Gamma, P_{2n-2}) \arrow[r] \arrow[d, "\psi_2"] & 0 \\
0 \arrow[r] & \Omega(H^1(\Gamma, P_{2n-2})) \arrow[r, "i_3"] & K[A]^y \arrow[r, "\pi_3"] & H^1(\Gamma, P_{2n-2}) \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

Here y is unknown, and our aim is to find the decomposition of $H^1(\Gamma, P_{2n-2})$. Here ψ_1 is the embedding from eq. (12), and

$$\psi_2 : \mathrm{Der}(\Gamma, P_{2n-2}) \to \mathrm{Der}(\Gamma, P_{2n-2})/\mathrm{PrinDer}(\Gamma, P_{2n-2}) = H^1(\Gamma, P_{2n-2}),$$

is the natural projection map. The maps ϕ_1, ϕ_2 exist since the modules $P(P_{2n-2})$ and $P(\mathrm{Der}(\Gamma, P_{2n-2}))$ are projective, and they make the diagram commutative. For example, using the notation of eq. (18) the map π_2 is f and the map g is the composition $\psi_1 \circ \pi_1$, so the existence of ϕ_1 follows by definition.

First, we note that it is sufficient to prove that the image

$$\mathrm{Im}(\psi_1) = K[A]^{\lfloor \frac{2n-1}{p} \rfloor} \oplus J_r$$

is contained in the second summand of the compostion of $\mathrm{Der}(\Gamma, P_{2n-2})$ in eq. (21), i.e., $\phi_1(\pi_1^{-1} P_{2n-2}) \cap i_2(J_r^{p-1}) = \{0\}$. Indeed, then

$$\begin{aligned} H^1(\Gamma, P_{2n-2}) &\cong \mathrm{Der}(\Gamma, P_{2n-2})/\, \mathrm{PrinDer}(\Gamma, P_{2n-2}) \\ &\cong K[A]^{(p-1)\lfloor \frac{(2n-1)(p-1)}{p} \rfloor - 1 - \lfloor \frac{2n-1}{p} \rfloor} \oplus K[A]/J_r \oplus J_{p-r}^{p-1} \\ &\cong K[A]^{(p-1)(2n-1) - p\lceil \frac{2n-1}{p} \rceil} \oplus J_{p-r}^{p}. \end{aligned} \tag{25}$$

As ψ_1 is an injective map it follows that $\phi_1(\pi_1^{-1} P_{2n-2}) \cap i_2(J_r^{p-1}) = \{0\}$. Next we prove that the image $\mathrm{Im}(\phi_1)$ is contained in the second summand of the projective cover of $\mathrm{Der}(\Gamma, P_{2n-2})$. Suppose that $\mathrm{Im}(\phi_1) \cap K[A]^{p-1} \neq \{0\}$, then $(\mathrm{Im}(\phi_1) \cap K[A]^{p-1}) \oplus i_2(J_r^{p-1})$ is a submodule of $K[A]^{p-1}$ and, by lemma 34, this is impossible.

So $\mathrm{Im}(\phi_1) \subset K[A]^{\lfloor \frac{(2n-1)(p-1)}{p} \rfloor (p-1)}$ and $\mathrm{Im}(\psi_1)$ is inside $K[A]^{\lfloor \frac{(2n-1)(p-1)}{p} \rfloor (p-1)} \subset \mathrm{Der}(\Gamma, P_{2n-2})$. $\square$

6.1.1. *Some computations on Subrao Curves.* Our second proof of Theorem 2.1 uses the theory of Subrao curves.

Second Proof of theorem 2.1 . Recall that the Subrao curves we are studying are uniformized by $\Gamma = [\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}]$, and have the following algebraic model

$$X_c : (x^p - x)(y^p - y) = c,$$

for some $c \in K$ for which $|c| < 1$. The group $\mathbb{Z}/p\mathbb{Z}$ is a subgroup of the automorphism group and acts for instance on the curve X_c by sending the generator to $(x, y) \mapsto (x, y+1)$. We call the quotient curve Y. Note that Y is isomorphic to $\mathbb{P}^1$, and hence, that the genus g_Y is zero.

Fix a curve X_c and denote its function field by F. The extension $F/K(x)$ is a cyclic extension of the rational function field $K(x)$, We have p-places $P_i = (x - i)$ of $K(x)$, which are tamely ramified. The different is:

$$\mathrm{Diff}_{F/K(x)} = \sum_{i=0}^{p-1} 2(p-1)P_i.$$

We will study them using the results of Nakajima [29]. We have the following decomposition in terms of indecomposable modules

$$H^0(X, \Omega_X^{\otimes n}) = \bigoplus_{i=0}^{p} m_i J_i,$$

and the coefficients are given by

$$m_p = (2n-1)(g_Y - 1) + \sum_{i=1}^{p} \left\lfloor \frac{n_i - (p-1)N_i}{p} \right\rfloor,$$

where $N_i = 1$ (ordinary curves) and $n_i = 2n(p-1)$, see [22, sec. 4].

Since $g_Y = 0$ we compute:

$$\begin{aligned} m_p &= (2n-1)(g_Y - 1) + p \left\lfloor \frac{2n(p-1) - (p-1)}{p} \right\rfloor \\ &= (2n-1)(g_Y - 1) + p(2n-1) - p \left\lceil \frac{2n-1}{p} \right\rceil \\ &\overset{g_Y=0}{=} (p-1)(2n-1) - p \left\lceil \frac{2n-1}{p} \right\rceil. \end{aligned}$$

The coefficients m_j are given by the following formulas:

$$\begin{aligned} \frac{m_j}{p} &= -\left\lfloor \frac{n_i - jN_i}{p} \right\rfloor + \left\lfloor \frac{n_i - (j-1)N_i}{p} \right\rfloor \\ &= -\left\lfloor \frac{2n(p-1) - j}{p} \right\rfloor + \left\lfloor \frac{2n(p-1) - (j-1)}{p} \right\rfloor \\ &= -\left\lfloor \frac{-2n - j}{p} \right\rfloor + \left\lfloor \frac{-2n - (j-1)}{p} \right\rfloor \\ &= \left\lceil \frac{2n + j}{p} \right\rceil - \left\lceil \frac{2n + j - 1}{p} \right\rceil \end{aligned}$$

We now notice that for $0 \leq j \leq p-1$ the above expression is zero unless $p \mid 2n+j-1$.

We write $2n-1 = \left\lfloor \frac{2n-1}{p} \right\rfloor p + r$, and we see that $m_j = 0$ unless

$$j = p - r = p - (2n-1) + \left\lfloor \frac{2n-1}{p} \right\rfloor p.$$

Notice that if $p > 2n-1$ then $j = p - (2n-1)$. So we have that

$$H^1(\Gamma, P_{2n-2}) = H^0(X, \Omega_X^{\otimes n}) = K[A]^{(p-1)(2n-1)-p\lceil \frac{2n-1}{p} \rceil} \bigoplus J_{p-r}^p. \tag{26}$$

□

6.2. **Using the theory of B. Köck-study of the $A \times B$-module structure.** In this section we will employ the results of B. Köck on the projectivity of the cohomology groups of certain sheaves in the weakly ramified case. Consider a p-group G and and the cover $\pi : X \to X/G$. We first set up some notation. For every point P of X we consider the local uniformizer t at P, the stabilizer $G(P)$ of P and assign a sequence of ramification groups

$$G_i(P) = \{\sigma \in G(P) : v_P(\sigma(t) - t) \geq i + 1\}.$$

Notice that $G_0(P) = G(P)$ for p-groups, see [34, chap. IV]. Let $e_i(P)$ denote the order of $G_i(P)$. We will say that the cover $X \to X/G$ is weakly ramified if all $e_i(P)$ vanish for $i \geq 2$. Notice that Mumford curves are ordinary and that these curves only have weak ramification. We denote by Ω_X the sheaf of differentials on X and by $\Omega_X(D)$ the sheaf $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$. For a divisor $D = \sum_{P \in X} n_P P$ we denote by $D_{\mathrm{red}} = \sum_{P \in X : n_P \neq 0} P$ the associated reduced divisor. We will also denote by

$$L(D) = H^0(X, \mathcal{O}_J X(D)) = \{D + (f) > 0 : f \in F_X\} \cup \{0\},$$

where F_X is the function field of the curve X. The ramification divisor equals $R = \sum_{P \in X} \sum_{i=0}^{\infty} \big(e_i(P) - 1\big)$. Finally, Σ denotes the skyscraper sheaf defined by the short exact sequence:

$$0 \to \Omega_X^{\otimes n} \to \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big) \to \Sigma \to 0.$$

Lemma 36. *The cohomology group $H^1(X, \Omega_X^{\otimes n}) = 0$.*

Proof. There is a correspondence of sheaves between divisors and 1-dimensional $\mathcal{O}_X$-modules, $D \mapsto \mathcal{O}_X(D)$. For any differential there is a canonical divisor K, therefore there is a correspondence between Ω_X and the $\mathcal{O}(K)$.

Recall that Serre duality asserts:

$$\dim H^1(X, \mathcal{O}_X(D)) = \dim H^0(X, \Omega_X \otimes \mathcal{O}_X(D)^{-1})$$

Hence we find that

$$\dim H^1(X, \Omega_X^{\otimes n}) = \dim H^0(X, \Omega_X \otimes \Omega_X^{-\otimes n}).$$

An element of $\Omega_X \otimes \Omega_X^{-n}$ corresponds to module $\mathcal{O}_X(K - nK)$ and since

$$H^0(X, \mathcal{O}_X(K - nK)) = L(K - nK),$$

it holds that

$$\dim H^1(X, \Omega_X^{\otimes n}) = \dim L(K - nK) = 0.$$

□

Now we apply the functor of global sections to the short exact sequence above and obtain the following short exact sequence:

(27)

$$0 \to H^0(X, \Omega_X^{\otimes n}) \to H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big) \to H^0(X, \Sigma) \to H^1(X, \Omega_X^{\otimes n}) = 0.$$

Theorem 37. *The $K[G]$-module $H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big)$ is a free $K[G]$-module of rank $(2n-1)(g_Y-1+r_0)$, where r_0 denotes the cardinality of $X^G_{\mathrm{ram}} = \{P \in X/G : e(P) > 1\}$, and g_Y denotes the genus of the quotient curve $Y = X/G$.*

Proof. Since G is a p-group it suffices for the module to be free, to show that the module $H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big)$ is projective. B. Köck proved [19, Th. 2.1] that if $D = \sum_{P\in X} n_p P$ is a G-equivariant divisor, the map $\pi : X \to Y := X/G$ is weakly ramified, $n_P \equiv -1 \bmod e_P$ for all $P \in X$ and $\deg(D) \geq 2g_X - 2$, then the module $H^0(X, \mathcal{O}_X(D))$ is projective.

We have to check the conditions for the divisor $D = nK_X + (2n-1)R_{\mathrm{red}}$, where K_X is a canonical divisor on X. Notice that $K_X = \pi^* K_Y + R$ and $R = \sum_{P\in X} 2\big(e_0(P)-1\big)$, therefore

$$D = n\pi^* K_Y + \sum_{P\in X : e_0(P)>1} (2ne_0(P) - 2n + 2n - 1)\, P.$$

Therefore, the condition $n_P \equiv -1 \bmod e_0(P)$ clearly satisfied.

We will now compute the dimension of $H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big)$ using Riemann–Roch theorem (keep in mind that $H^1\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big) = 0$)

$$\begin{aligned} \dim_K H^0\big(X, \Omega_X^{\otimes n}\big((2n-1)R_{\mathrm{red}}\big)\big) &= n(2g_X-2) + (2n-1)|X_{\mathrm{ram}}| + 1 - g_X \\ &= (2n-1)(g_X - 1 + |X_{\mathrm{ram}}|) \\ &= |G|(2n-1)(g_Y - 1 + r_0) \end{aligned}$$

where in the last equality we have used the Riemann–Hurwitz formula [15, 7, Cor. IV 2.4]

$$g_X - 1 = |G|(g_Y - 1) + \sum_{P\in X_{\mathrm{ram}}} (e_0(P) - 1).$$

□

Remark 38. This method was applied by the second author and B. Köck in [21] for the $n = 2$ case to compute the dimension of the tangent space to the deformation functor of curves with automorphisms. Deformations of curves with automorphisms for Mumford curves were also studied by the first author and G. Cornelissen in [4].

It follows that we have the following short exact sequence of modules:

$$0 \to H^0(X, \Omega_X^{\otimes n}) \to K[G]^{(2n-1)(g_Y-1+r_0)} \to H^0(X, \Sigma) \to 0. \tag{28}$$

Since Σ is a skyscraper sheaf the space $H^0(X, \Sigma)$ is the direct sum of the stalks of Σ

$$H^0(X, \Sigma) = \bigoplus_{P\in X_{\mathrm{ram}}} \Sigma_P \cong \bigoplus_{j=1}^{r_0} \mathrm{Ind}_{G(P_j)}^G(\Sigma_{P_j}),$$

where, for a subgroup H of G, Ind_H^G denotes the induced representation of an H-module V to a G-module, i.e., $\mathrm{Ind}_H^G V = V \otimes_{K[H]} K[G]$.

6.3. **Return to Subrao curves: proof of theorem 2.2.** Recall that we are in the case $N = A * B$ and $\Gamma = [A, B]$. where $A \cong B \cong \mathbb{Z}/p\mathbb{Z}$. Set $G = N/\Gamma = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Lemma 39. *The indecomposable summands of the module $\mathrm{Ind}_{G(P_j)}^G(\Sigma_{P_j})$ are either $K[G]$ or $K[G]/\langle(\sigma-1)^\lambda\rangle$, where $\sigma = \epsilon_A$ or ϵ_B and $1 \leq \lambda \leq p-1$.*

Proof. It follows from the ramification of the function field of Subrao curves, seen as a double Artin-Schreier extension of the rational function field, that $r = 2$, i.e., only two points P_1, P_2 of $X/(A \times B)$ are ramified in the cover $X \to X/A \times B$. Another way of obtaining this result is by using the theory of Kato graphs, and by noticing that the subgroup of the normalizer of the Subrao curve giving rise to the $A \times B$ cover is just $A * B$ corresponding to a Kato graph with two ends, see [5], [18, prop. 5.6.2]. Let $G(P_1) = A$ and $G(P_2) = B$.

We will use an approach similar to [21] in order to study the Galois module structure of the stalk Σ_{P_j} as a G_{P_j}-module. Let $P = P_j$ for some j. Notice first that $nK_X = n\pi^* K_Y + nR$, so if the valuation of K_Y at $\pi(P)$ is m then the valuation of nK_X at P is $mnp + 2n(p-1)$ and the valuation of $nK_X + (2n-1)R_{\mathrm{red}}$ is $mnp + 2n(p-1) + (2n-1)$. So if t is a local uniformizer at P and s is a local uniformizer at $\pi(P)$ we have that:

$$\Sigma_P = \frac{t^{1-p(nm+2n)}k[[t]]}{t^{2n-p(nm+2n)}k[[t]]} \cong \frac{tk[[t]]}{t^{2n}k[[t]]},$$

where the $G(P)$-equivariant isomorphism is given by multiplication with an appropriate power of the invariant element s. So the space Σ_P admits the following K-vector space basis,

$$\Sigma_P = \left\langle t, t^2, \ldots, t^{2n-1} \right\rangle_K .$$

The action of $G(P)$ on Σ_p is given by the transformation $\sigma(1/t) = 1/t + 1$ for a generator σ of the cyclic group $G(P)$, or equivalently $\sigma(t) = \frac{t}{1+t}$ see [6]. Notice, that the element $t^{-p} - t^{-1} = \frac{1-t^{p-1}}{t^p}$ is invariant and so is its inverse $t^p(1-t^{p-1})^{-1}$. Here the unit $(1-t^{p-1})^{-1}$, can be seen as a polynomial modulo t^{2n}, if we expand in terms of a geometric series and truncate the terms of degree $\geq 2n$. Now we analyse the $G(P)$ module structure of Σ_P using Jordan blocks. Observe that for $0 \leq k \leq p-1$

$$\sigma \binom{1/t}{k} = \binom{1/t}{k} + \binom{1/t}{k-1},$$

where

$$\binom{1/t}{k} = \frac{1}{k!} \prod_{\nu=n-k+1}^{n} \left(\frac{1}{t} + \nu \right)$$

Note that $\binom{1/t}{k}$ is a rational function, where the denominator is a polynomial of degree k. So if we multiply it by the invariant element $t^p(1-t^{p-1})^{-1}$ we obtain a polynomial of degree $p-k$. Another K-vectorspace basis of Σ_P is given by:

$$\left(\frac{t^p}{(1-t^{p-1})} \right)^i \binom{1/t}{k}, \text{ where } 1 \leq i \leq \left\lfloor \frac{2n-1}{p} \right\rfloor + 1 \text{ and } 0 \leq k \leq p-1.$$

The above defined elements are seen as polynomials by expanding them as powerseries and truncate the powers of t greater than $2n$. Hence, for a fixed i and by allowing k to vary we obtain a Jordan block J_p. The remaining block is J_r. So the structure of Σ_P is given by

$$\Sigma_P = J_p^{\left\lfloor \frac{2n-1}{p} \right\rfloor} \bigoplus J_r.$$

Recall [10, 12.16 p.74] that if H is a subgroup of G and $g_1, \ldots, g_\ell$ is a set of coset representatives of G in H, then for an H-module M the induced module can be written as

$$\mathrm{Ind}_H^G M = \bigoplus_{\nu=1}^{\ell} g_\nu \otimes M.$$

Using the above equation for $G = A \times B$ and $H = G(P_1) = A$ (resp. $G(P_2) = B$) we have

$$\mathrm{Ind}_{G(P_j)}^G (J_p) = K[G] \text{ and } \mathrm{Ind}_{G(P_j)}^G (J_r) = K[G]/\langle (\sigma - 1)^r \rangle,$$

where $\sigma = \epsilon_A$ or ϵ_B, and both of the above $K[G]$-modules are indecomposable. □

Proposition 40. *The indecomposable summands V_i of $H^0(X, \Omega_X^{\otimes n})$ are either $K[G]$ or $K[G]/\langle(\sigma-1)^{p-r}\rangle$, where r is the remainder of the division $2n-1$ by p.*

Proof. Let V_i be a indecomposable summand of $H^0(X, \Omega_X^{\otimes n})$. Consider the injective hull of V_i. This is the smallest injective module containing V_i, and it is of the form $K[G]$, keep in mind that for group algebras of finite groups the notions of injective and projective coinside [10, th. 62.3]. Therefore we have to consider the smallest a such that $V_i \subset K[G]^a$. We have the short exact sequence:

$$0 \to V_i \to K[G]^a \to \Omega^{-1}(V_i) \to 0. \tag{29}$$

Since the algebra $K[G]$ is self injective (i.e., $K[G]$ is injective) we have

$$V_i \cong \Omega(\Omega^{-1})(V_i) \bigoplus K[G]^t,$$

see [1, exer. 1 p.12]. Since V_i is indecomposable we have either $V_i \cong K[G]$ or $V_i = \Omega(\Omega^{-1})(V_i)$. In the second case, we have the following diagram where the first raw comes from eq. (28) and the second by eq. (29):

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\Omega_X^{\otimes n}) \arrow[r] & K[G]^{(2n-1)(g_Y-1+r_0)} \arrow[r] & H^0(X,\Sigma) \arrow[r] & 0 . \\
0 \arrow[r] & V_i \arrow[r] \arrow[u] & K[G]^a \arrow[r] \arrow[u] & \Omega^{-1}(V_i) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

In this diagram the existence of the middle vertical morphism comes from the properties of the injective hull of V_i. The module $\Omega^{-1}(V_i)$ is a non-zero indecomposable non-projective factor of $H^0(X,\Sigma)$ and is $\mathrm{Ind}_{G(P_1)}^G(J_r) = K[G]/\langle(\sigma-1)^r\rangle$. It can not be $K[G]$ since $K[G]$ is projective. We compute

$$V_i = \Omega(\mathrm{Ind}_{G(P_j)}^G(J_r)) = \Omega(K[G]/\langle(\sigma-1)^r\rangle)) = K[G]/\langle(\sigma-1)^{p-r}\rangle.$$

□

Corollary 41. *The space $H^0(X, \Omega_X^{\otimes n})^G$ has dimension equal to the number of indecomposable summands.*

Proof. Notice that each indecomposable summands V_i is contained in a $K[G]$. □

Corollary 42. *If $2n-1 \equiv 0 \bmod p$ then $H^0(X, \Omega^{\otimes n})$ is projective.*

Using the sequence given in eq. (28) and the fact that only two points of Y are ramified in $X \to Y$, i.e., $r_0 = 2$, we obtain that the number of $K[G]$ summands in $H^0(X, \Omega^{\otimes n})$ is $2n-1-2\left\lceil \frac{2n-1}{p} \right\rceil$. There are two indecomposible summands in $H^0(X, \Omega_X^{\otimes n})$, V_1, V_2 such that

$$K[G]/V_1 = K[G]/(\epsilon_A - 1)^r \text{ and } K[G]/V_2 = K[G]/(\epsilon_B - 1)^r.$$

We see that

$$V_1 = K[G]/(\epsilon_A - 1)^{p-r} \text{ and } V_2 = K[G]/(\epsilon_B - 1)^{p-r}.$$

Adding all these together we obtain:

$$H^0(X, \Omega_X^{\otimes n}) = K[G]^{2n-1-2\left\lceil \frac{2n-1}{p} \right\rceil} \bigoplus K[G]/(\epsilon_A - 1)^{p-r} \bigoplus K[G]/(\epsilon_B - 1)^{p-r}.$$

The Proof of theorem 2.2 is now complete.

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