ON THE PRINCIPAL IDEAL THEOREM IN ARITHMETIC TOPOLOGY

DIMOKLIS GOUNDAROULIS AND ARISTIDES KONTOGEORGIS

Abstract. In this paper we state and prove the analogous of the principal ideal theorem of algebraic number theory for the case of 3-manifolds from the point of view of arithmetic topology.

1. Introduction

There are certain analogies between the notions of number theory and those of 3-dimensional topology, that are described by the following dictionary, named after Mazur, Kapranov and Reznikov.

- Closed, oriented, connected, smooth 3-manifolds correspond to affine schemes Spec$\mathcal{O}_K$, where K is an algebraic number field and $\mathcal{O}_K$ denotes the ring of algebraic integers of K.
- A link in M corresponds to an ideal in $\mathcal{O}_K$ and a knot in M corresponds to a prime ideal in $\mathcal{O}_K$.
- An algebraic integer $w \in \mathcal{O}_K$ is analogous to an embedded surface (possibly with boundary).
- The class group $\mathrm{Cl}(K)$ corresponds to $H_1(M, \mathbb{Z})$.
- Finite extensions of number fields L/K correspond to finite branched coverings of 3-manifolds $\pi : M \to N$. A *branched cover* M of a 3-manifold N is given by a map π such that there is a link L of N with the following property: The restriction map $\pi : M \backslash \pi^{-1}(L) \to N \backslash L$ is a topological cover.

For the necessary background in algebraic number theory the reader should look at any standard book, for example [2]. For the topological part: by the term *knot* (resp. *link*) we mean *tame knot* (resp. *tame link*). By the term *embedded surface* we mean an embedding $f : E \to M$, of a two dimensional oriented, connected, smooth manifold E. A tame knot is an embedding $f : S^1 \to M$ that can be extended to an embedding of $f : S^1 \times B(0, \epsilon) \to M$. In other words tame knots admit a tubular neighborhood embedding. We will call a manifold *tamely path connected* if for every two points P, Q of M there is a path $\gamma : [0, 1] \to M$ with $\gamma(0) = P$, $\gamma(1) = Q$ with the additional property that for a suitable small disk $B(0, \epsilon)$ the path γ can be extended to an embedding $\gamma : B(0, \epsilon) \times [0, 1] \to M$. It is not clear to the authors whether all path connected 3 manifolds are tamely path connected. In what follows we will be concerned only with tamely path connected 3 manifolds. We also consider manifolds M so that $\pi_1(M)$ is a finitely generated group.

This is just a small version of the dictionary. More precise versions can be found in [5], [6].

Date: September 27, 2007.

One of the differences between the two theories is that the group $\mathrm{Cl}(K)$ is always finite while $H_1(M,\mathbb{Z}) = \mathbb{Z}^r \oplus H_1(M,\mathbb{Z})_{tor}$ is not. Many authors proposed that the analogue of the class group for arithmetic topology should be the torsion part $H_1(M,\mathbb{Z})_{tor}$, but we think that one advantage, of taking as analogue of the class group, the whole $H_1(M,\mathbb{Z})$ is that $H_1(M,\mathbb{Z})$ is the Galois group of the *Hilbert manifold* $M^{(1)}$ over M, where the Hilbert manifold $M^{(1)}$ is the maximal unramified abelian cover of M.

Theorem 1.1 (Principal Ideal Theorem for Number Fields.)**.** *Let K be a number field and let $K^{(1)}$ be the Hilbert class field of K. Let $\mathcal{O}_K$, $\mathcal{O}_{K^{(1)}}$ be the rings of integers of K and $K^{(1)}$ respectively. Let P be a prime ideal of $\mathcal{O}_{K^{(1)}}$. We consider the prime ideal*

$$\mathcal{O}_K \rhd p = P \cap \mathcal{O}_K$$

and let

$$p\mathcal{O}_{K^{(1)}} = (PP_2 \ldots P_r)^e = \prod_{g \in \mathrm{CL}(K)} g(P)$$

be the decomposition of $p\mathcal{O}_{K^{(1)}}$ in $\mathcal{O}_{K^{(1)}}$ into prime ideals. The ideal $p\mathcal{O}_{K^{(1)}}$ is principal in $K^{(1)}$.

This theorem was conjectured by Hilbert and the proof was reduced to a purely group theoretic problem by E. Artin. The group theoretic question was resolved by Ph. Furtwangler [1]. For a modern account we refer to [2, V.12].

Aim of this paper is to state and prove a natural generalization of the principal ideal theorem for number fields in the case of knot theory. It is interesting to point out that using this generalization we are able to prove the clasical Seifert theorem (cor. 2.6) and we are able to characterize 3-manifolds that have the *Seifert property* (cor. 2.7).

Acknowledgments: The authors would like to thank Prof. E. Kehagias for usuful discussions concerning the subject of the paper.

2. The Principal Ideal Theorem for Knots

The Hilbert class field in number fields is defined to be the largest non-ramified abelian extension. Therefore we define the Hilbert manifold $M^{(1)}$ of M as the universal covering space $\widetilde{M}$ of M modulo the commutator group $[\pi_1(M), \pi_1(M)]$:

$$M^{(1)} = \widetilde{M}/[\pi_1(M), \pi_1(M)].$$

Above and in the rest of this article we will abuse the notation and we will denote by $\pi_1(M) = \pi_1(M, P)$ the first homotopy group based on a fixed point P of the manifold M. By definition $M^{(1)}$ is the largest unramified abelian cover of the manifold M. Moreover, the Galois group of the cover is:

$$G = \mathrm{Gal}(M^{(1)}/M) \cong \pi_1(M)/[\pi_1(M), \pi_1(M)] \overset{\Phi}{\cong} H_1(M,\mathbb{Z}).$$

The map Φ is the map sending a loop around a fixed point P to the homology class of the corresponding 1-cell. We have $\pi_1(M^{(1)}) = [\pi_1(M), \pi_1(M)]$. Notice that if $\pi_1(M)$ is an abelian group then $M^{(1)}$ is the universal cover of M and if $[\pi_1(M), \pi_1(M)] = \pi_1(M)$ then $M^{(1)} = M$.

Let L/K be a Galois extension of number fields and let $\mathcal{O}_L, \mathcal{O}_K$ be the corresponding rings of algebraic integers. In the case of number fields it is known that every prime ideal $p \lhd \mathcal{O}_K$ gives rise to an ideal $p\mathcal{O}_L$. This construction is not always possible in the case of 3-manifolds. Namely, if $M_1 \to M$ is a covering of 3-manifolds then an arbitrary knot does not necessarily lift to a knot in M_1. Indeed, a knot can be seen as a path $\gamma : [0,1] \to M$ so that $\gamma(0) = \gamma(1)$, and paths do lift to paths $\tilde{\gamma} : [0,1] \to M_1$, but in general $\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$. The following theorem gives a necessary and sufficient condition for liftings of maps between topological spaces.

Theorem 2.1. *Let (Y, y_0), (X, x_0) be topological spaces (arcwise connected, locally simply connected), let $p : (X', x_0') \to (X, x_0)$ be a topological covering with $p(x_0') = x_0$ and let $f : (Y, y_0) \to (X, x_0)$ be a continuous map. Then, there is a lift $\tilde{f} : Y \to X'$ of f,*

$$\begin{tikzcd} & X' \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

making the above diagram commutative if and only if

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(X', x_0')),$$

where f_*, p_* are the induced maps of fundamental groups.

Proof. [4, Chapter 5, Proposition 5]. ☐

Proposition 2.2. *Let K_1 be a knot in $M^{(1)}$. Denote by $G(K_1)$ the subgroup of G fixing K_1 and by p the covering map $M^{(1)} \to M$. Consider the link $L = p^{-1}(p(K_1)) = \bigcup_{g \in G/G(K_1)} gK_1$. Then L is the boundary of a possibly singular and possibly nonconnected surface in $M^{(1)}$.*

Proof. In number theory this theorem is proved by using the transfer map, but this method can not be applied in our case since G need not be finite. If $|H_1(M, \mathbb{Z})| < \infty$ then the classical [2, V.12] proof applies by just using the MKR dictionary, i.e. by replacing all the class groups that appear in the classical proof with the first homology groups. In the general case we will use the Theorem 2.1.

The map p when restricted to L gives a covering map $L \xrightarrow{p} \pi(K_1)$. If P is a point on $\pi(K_1)$ then $\pi^{-1}(P)$ has G elements while $\pi^{-1}(\pi(K_1))$ consists of $G/G(K_1)$ isomorphic copies of K_1, the isomorphism is given by the action of an element of G. This means that $G(K_1)$ elements of $\pi^{-1}(P)$ belong to the same copy of K_1. The situation is ilustrated in figure 1.

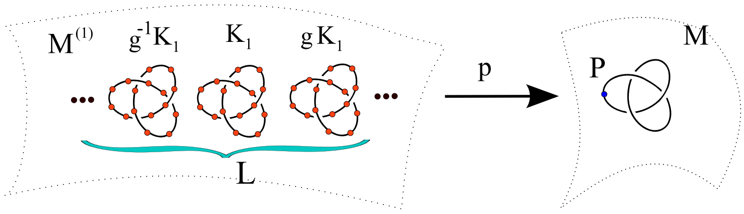

FIGURE 1. The knot K_1 and the link L.

Since the diagram

$$\begin{tikzcd} & K_1 \arrow[r] \arrow[d, "p"] & M^{(1)} \arrow[d, "p"] \\ S^1 \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & p(K_1) \arrow[r] & M \end{tikzcd}$$

commutes we have that

$$f_*(\pi_1(S^1)) \subset p_*(\pi_1(K_1)) \subset p_*(\pi_1(M^{(1)})) = p_*([\pi_1(M), \pi_1(M)]),$$

therefore the image of $f_*(\pi_1(S^1))$ under the map $\Phi : \pi_1(M) \to H_1(M, \mathbb{Z})$ is zero. By definition of $H_1(M, \mathbb{Z})$ we have that there is a topological (possibly singular) surface $\phi : E \to M$ so that

$$f(S^1) = p(K^1) = \partial\phi(E).$$

Moreover the surface E is homotopically trivial therefore theorem 2.1 implies that there is a map $\widetilde{\phi}$ making the following diagram commutative:

$$\begin{tikzcd} & M^{(1)} \arrow[d, "p"] \\ E \arrow[ur, "\widetilde{\phi}"] \arrow[r, "\phi"'] & M \end{tikzcd},$$

with the additional property $\partial\widetilde{\phi}(E) = p^{-1}(\partial\phi(E)) = L$. □

Observe that proposition 2.2 proves only that there is no topological obstruction for the link L to be the boundary of a surface. Since we have worked in terms of singular homology the boundary surface might have singularities or might consist of several components. We will use the following theorem known as "Dehn lemma" in the literature.

Theorem 2.3. *Let M be a 3-manifold and $f : D^2 \to M$ be a map such that for some neighborhood A of ∂D^2 in D^2 $f\mid_A$ is an embedding and $f^{-1}f(A) = A$. Then $f\mid_{\partial D^2}$ extends to an embedding $g : D^2 \to M$.*

Proof. [3, 4.1] □

Corollary 2.4. *Let M be a 3-manifold satisfying all the assumptions of the introduction. If a tame knot is the boundary of a topological and possibly singular surface then the knot is the boundary of an embedded surface.*

Proof. Using the embedding of a tubular neighborhood of the knot we can construct a nonsingular collar around the boundary of the topological surface and the desired result follows by theorem 2.3. □

Proposition 2.5. *Let L be a link in M that is a homologically trivial. Then there is a family of tame knots K_ϵ in M with $\epsilon > 0$, that are boundaries of embedded surfaces E_ϵ so that $\lim_{\epsilon\to 0} K_\epsilon = L$ and $E = \lim_{\epsilon\to 0} E_\epsilon$ is an embedded surface with $\partial E = L$.*

Proof. We will consider the case of a link with two components. The general case follows by induction since the group $\pi_1(M)$ has by definition countable many elements. Let $L = K_1 \cup K_2$, where K_i is given by the embedding $f_i : S^1 \to M$, a tame knot. The passage from two components to $n > 2$ components follows by induction. Notice that for the induction to work we need to know that $\pi_1(M)$ has countable many elements and this follows since $\pi_1(M)$ is finitely generated.

Select two points P_1, Q_1 on $f_1(S^1)$ and two points P_2, Q_2 on $f_2(S^1)$ so that $d(P_i, Q_i) = \epsilon$. The embedding f_i can be given as the union of two curves, namely $\gamma_i : [0,1] \to M$, $\delta_i : [0,1] \to M$, so that $\gamma_i(0) = \delta_i(1) = P_i$, $\gamma_i(1) = \delta_i(0) = Q_i$. This means that the "small" curve is the curve δ_i.

Since the manifold M is tamely path connected we can find two paths $\alpha, \beta : [0,1] \to M$ so that $\alpha(0) = P_1, \alpha(1) = Q_2$, $\beta(0) = P_2, \beta(1) = Q_1$, that are close enough so that the rectangle $\alpha(-\delta_2)\beta(-\delta_1)$ is homotopically trivial. Let $I = [0,1] \subset \mathbb{R}$. Every path in M, *i.e.* every function $f : I \to M$, defines a cycle in $H_1(M, \mathbb{Z})$. We will abuse the notation and we will denote by $f(I)$ the homology class of the path $f(I)$. We compute in $H_1(M, \mathbb{Z})$:

$$0 = f_1(S_1) + f_2(S_1) = \gamma_1(I) + \gamma_2(I) + \delta_1(I) + \delta_2(I) + 0 =$$

$$= \gamma_1(I) + \gamma_2(I) + \delta_1(I) + \delta_2(I) + \alpha(I) - \delta_2(I) + \beta(I) - \delta_1(I) =$$

$$= \gamma_1(I) + \alpha(I) + \gamma_2(I) + \beta(I).$$

This means that the tame knot $\gamma_1\alpha\gamma_2\beta$ is the boundary of a topological surface, and by Corollary 2.4 it is the boundary of an embedded surface E_ϵ.

Choose an orientation on E_ϵ so that on $P \in \partial E_\epsilon$ one vector of the oriented basis of $T_P E_\epsilon$ is the tangent vector of the curves ∂E_ϵ and the other one is pointing inwards of E. We will denote the second vector by N_P. Moreover, we choose the same orientation on all surfaces E_ϵ in the same way, *i.e.* the induced orientation on the common curves of the boundary is the same.

We would like to take the limit surface for $\epsilon \to 0$. For this we have to distinguish the following two cases: In the first case the direction of decreasing the distance ϵ is the opposite of N_P and the limiting procedure makes the rectangle $\alpha\cdot(-\delta_2)\cdot\beta\cdot(-\delta_1)$ thinner and eventually it eliminates it. In this case the elimination of the above mentioned rectangle glues two parts of the surface E_ϵ together. The limit $\epsilon \to 0$ gives us an embedded surface E that is the boundary of our initial link L. Indeed by taking the limit the paths $\alpha(I), \beta(I)$ are identified, and this identification can be done in a smooth manner.

In the second case the direction of decreasing the distance ϵ is the same with N_P. This means that by taking the limit $\epsilon \to 0$ we don't glue two parts of the boundary of the surface E_ϵ but we make the rectangle $\alpha(-\delta_2)\beta(-\delta_1)$ thinner and after eliminating it we cut the surface in two pieces. Still the limit $\epsilon \to 0$ gives us two embedded surfaces E, E' that are the boundaries of our initial link components K_1, K_2. We can arrive at one embedded surface in the following way: We cut two disks D_1, D_2 of the interiors of E and E' and glue together them together along a tubular path T so that $\partial T = D_1 \cup D_2$. □

As a corollary of the principal ideal theorem for knots we state the following:

Corollary 2.6 (*Seifert*)**.** *Every link in a simply connected* 3 *manifold is the boundary of an embedded surface.*

Proof. Let M be simply connected. The Hilbert manifold of M coincides with M and the result follows. □

Using the "principal ideal theorem" for knots we are able to prove the following

Corollary 2.7. *In a manifold M every link is the boundary of an embeded surface if and only if $\pi_1(M) = [\pi_1(M), \pi_1(M)]$, i.e. the abelianization of $\pi_1(M)$ is trivial.*

Proof. If $\pi_1(M) = [\pi_1(M), \pi_1(M)]$ then $H_1(M, \mathbb{Z})$ is trivial and this means that the Hilbert manifold $M^{(1)}$ of M coincides with M and the result follows.

If on the other hand $\pi_1(M) \supsetneq [\pi_1(M), \pi_1(M)]$ then we can choose a point $P \in M$ and a closed path γ based on P so that $0 \neq \Phi(\gamma) \in H_1(M, \mathbb{Z})$. This means that there is no topological surface in M with boundary the path γ. $\square$

Remark: Manifolds so that $\pi_1(M)$ is a simple non abelian group satisfy the hypotheses of corollary 2.7.

References

1. Ph. Furtwangler *Beweis des Hauptidealsatzes für Klassenkörper algebraischer Zahlkörper*, Abhand. Math. Seminar Hamburg **7** (1930) 14-36.
2. G. Janusz, *Algebraic Number Fields* sec. edition. Graduate Studies in Mathematics vol 7. American Mathematical Society 1996.
3. J. Hempel 3-*Manifolds*. Ann. of Math. Studies, **No. 86**. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1976. xii+195 pp.
4. W. S. Massey, *A Basic Course in Algebraic Topology, Graduate Texts in Mathematics*, Springer-Verlag, (1991).
5. Masanori Morishita, *On Certain Analogies Between Knots and Primes*, Journal für die reine und angewandte Mathematik, 2002, 141-167.
6. Adam Sikora, *Analogies Between Group Actions on 3-Manifolds and Number Fields*, arXiv:math.GT/0107210 v2 7 Jun 2003.

National Technical University of Athens, Department of Mathematics,
E-mail address: dgound@mail.ntua.gr

Department of Mathematics, University of the Ægean, 83200 Karlovassi, Samos, Greece, http://eloris.samos.aegean.gr
E-mail address: kontogar@aegean.gr