AUTOMORPHISMS OF GENERALIZED FERMAT CURVES

RUBÉN A. HIDALGO, ARISTIDES KONTOGEORGIS, MAXIMILIANO LEYTON-ÁLVAREZ, AND PANAGIOTIS PARAMANTZOGLOU

ABSTRACT. Let K be an algebraically closed field of characteristic $p \geq 0$. A generalized Fermat curve of type (k,n), where $k,n \geq 2$ are integers (for $p \neq 0$ we also assume that k is relatively prime to p), is a non-singular irreducible projective algebraic curve $F_{k,n}$ defined over K admitting a group of automorphisms $H \cong \mathbb{Z}_k^n$ so that $F_{k,n}/H$ is the projective line with exactly $(n+1)$ cone points, each one of order k. Such a group H is called a generalized Fermat group of type (k,n). If $(n-1)(k-1) > 2$, then $F_{k,n}$ has genus $g_{n,k} > 1$ and it is known to be non-hyperelliptic. In this paper, we prove that every generalized Fermat curve of type (k,n) has a unique generalized Fermat group of type (k,n) if $(k-1)(n-1) > 2$ (for $p > 0$ we also assume that $k-1$ is not a power of p).

Generalized Fermat curves of type (k,n) can be described as a suitable fiber product of $(n-1)$ classical Fermat curves of degree k. We prove that, for $(k-1)(n-1) > 2$ (for $p > 0$ we also assume that $k-1$ is not a power of p), each automorphism of such a fiber product curve can be extended to an automorphism of the ambient projective space. In the case that $p > 0$ and $k-1$ is a power of p, we use tools from the theory of complete projective intersections in order to prove that, for k and $n+1$ relatively prime, every automorphism of the fiber product curve can also be extended to an automorphism of the ambient projective space.

In this article we also prove that the set of fixed points of the non-trivial elements of the generalized Fermat group coincide with the hyper-osculating points of the fiber product model under the assumption that the characteristic p is either zero or $p > k^{n-1}$.

1. INTRODUCTION

In this paper, K will denote an algebraically closed field of characteristic $p \geq 0$. A generalized Fermat curve of type (k,n), where $k,n \geq 2$ are integers (and for $p > 0$ we also assume that k is relatively prime to p), is a non-singular irreducible projective algebraic curve $F_{k,n}$ defined over K admitting a group of automorphisms $H \cong \mathbb{Z}_k^n$ so that $F_{k,n}/H$ is the projective line with exactly $(n+1)$ cone points, each one of order k. Such a group H is called a generalized Fermat group of type (k,n). If $(n-1)(k-1) > 2$, then $F_{k,n}$ has genus $g_{n,k} > 1$ (see Section 2) and it is known to be non-hyperelliptic [6].

The generalized Fermat curves are objects with a very interesting geometry. These curves provide us with a considerable amount of examples, and their study could eventually help us to generalize certain important results. More precisely one of our future

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objectives is to generalize the work of of Y. Ihara [10] on Braid representations of the absolute Galois groups. By Belyi theorem he considered covers of the projective line ramified above $\{0, 1, \infty\}$ and the Fermat curve and its arithmetic emerged naturally. If one tries to generalise to the more general $n+1$-ramified covers the generalised Fermat curves and their arithmetic emerged in a similar way. This will be the object of another article.

To finish this paragraph, we would like to discuss a second interesting aspect of these curves. It is known that the geometry of compact Riemann Surfaces can be described via projective algebraic curves, Fuchsian group, Schottky groups, Abelian varieties, etc. However, given one of these descriptions, explicitly obtaining the others is a difficult problem, in fact in general it is a problem that has not been solved. The majority of examples of Riemann Surfaces where we explicitly know the uniformizing Fuchsian group, and the equations of an algebraic curve which represents them, are rigid examples, in other words they are not families. The generalized Fermat Curves of the type (n, k) over $K = \mathbb{C}$ form a family of algebraic curves of complex dimensions $n-2$ in which we explicitly know, for each member of the family, a representation as an algebraic curve and the uniformizing Fuchsian group (see [5]).

In the case of arbitrary characteristics, the Generalized Fermat curves can be studied as Kummer extensions of the rational function field.

We study the full group of automorphisms of generalized Fermat curves and the uniqueness of generalized Fermat groups. Our main result is Theorem 3 which states the uniqueness of generalized Fermat groups of type (k, n) if $(k-1)(n-1) > 2$ (for $p > 0$ we also assume that $k-1$ is not a power of p).

A generalized Fermat curve of type (k, n) can be seen as a complete intersections in a projective space defined by the set of equations given in eq. (2) in Section 2. Recall that a closed subscheme Y of $\mathbb{P}^s$ is called a (strict) complete intersection, if the homogeneous ideal in $K[x_1, \ldots, x_{n+1}]$ can be generated by $\mathrm{codim}(Y, \mathbb{P}^s)$ elements. By looking at the defining equations, we may see the generalized Fermat curves as a suitable fiber product of $(n-1)$ classical Fermat curves of degree k. We prove that in such algebraic model the full group of automorphism is a subgroup of the linear group under the assumptions that $(n-1)(k-1) > 2$ (if $p > 0$ we also assume that $k-1$ is not a power of p).

In the case that $p > 0$ and $k-1$ is a power of p, then we may obtain a similar result under the assumption that $n+1$ is relatively prime to k (Theorem 9). The different behaviour in the case $k-1 = q = p^h$ is an expected phenomenon, seen also in the case of the Fermat curves $x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0$, where $q = p^h$, which have $\mathrm{PGU}_3(q^2)$ as automorphism group, see [14]. Essentially this happens since raising to a p-power is linear and the Fermat curve in this case behaves like a quadratic form.

Our strategy, in the positive characteristic case, is the following. By a degree argument we show that the group of linear automorphisms is normal in the whole automorphism group. The group of linear automorphisms is studied by finding all linear transformations which leave the defining ideal of the curve invariant. For higher dimensional varieties there is an argument proving that every automorphism is linear, based on the fact that the Picard group is free. This argument can not be used in the case of curves, since the Picard groups of curves are known to have torsion. Nevertheless we can use a derivation argument in order to settle some cases.

This paper is organized as follows. In Section 2 we describe a fiber product of generalized Fermat curves and introduce the main results of the paper. The most important is Theorem 3 which states the uniqueness of the generalized Fermat groups of type (k, n), when $(k-1)(n-1) > 2$ (and for $p > 0$ the extra assumption that $k-1$ is not a power of

p). In the fiber product model, under the same hypothesis, we obtain that the full group of automorphisms is linear. The proof of the above is provided in Section 5.

In Section 3 we restrict our study to zero characteristic or to positive characteristic $p > k^{n-1}$ and prove that the set of fixed points of the non-trivial elements of the generalized Fermat group in the fiber product model coincide with the set of hyper-osculating points of the fiber product model.

In Section 6 we provide the proof of Theorem 9, concerning the linearity of the full group of automorphisms in the case when $k-1$ is a power of $p > 0$ and k is relatively prime to p, under the extra condition that k and $n+1$ are also relatively prime.

2. Main results

We use the notation (a, b) to denote the maximum common divisor between the positive integers a and b; so $(a, b) = 1$ states that a and b are relatively prime integers.

Let K be an algebraically closed field of characteristic $p \geq 0$, let $n, k \geq 2$ be integers (if $p > 0$, then we also assume that $(k, p) = 1$).

A pair $(F_{k,n}, H)$ is called a generalized Fermat pair of the type (k, n) if $F_{k,n}$ is a generalized Fermat curve of type (k, n), defined over K, and $H \cong \mathbb{Z}_k^n$ is a generalized Fermat group of type (k, n) of $F_{k,n}$. The genus of $F_{k,n}$ is

$$g_{(k,n)} = 1 + \frac{k^{n-1}}{2}((n-1)(k-1) - 2). \tag{1}$$

In particular, $g_{(k,n)} > 1$ if and only if $(k-1)(n-1) > 2$; in this case the generalized Fermat curve is non-hyperelliptic [6]. If $K = \mathbb{C}$, then $F_{k,n}$ defines a closed Riemann surface. Riemann surfaces of this kind were studied in [5].

Two generalized Fermat pairs of same type, say $(F_{k,n}, H)$ and $(\widehat{F}_{k,n}, \widehat{H})$, are called equivalent if there is an isomorphism $\phi : F_{k,n} \to \widehat{F}_{k,n}$ so that $\phi H \phi^{-1} = \widehat{H}$.

2.1. A fiber product description.

Let us consider a generalized Fermat pair $(F_{k,n}, H)$. Let us consider a branched regular covering $\pi : F_{k,n} \to \mathbb{P}^1$, whose deck group is H. By composing by a suitable Möbius transformation (that is, an element of $\mathrm{PSL}_2(K)$) at the left of π, we may assume that the branch values of π are given by the points

$$\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2},$$

where $\lambda_i \in K - \{0, 1\}$ are pairwise different.

Let us consider the non-singular complex projective algebraic curve

$$C^k(\lambda_1, \ldots, \lambda_{n-2}) := \left\{ \begin{array}{rcl} x_0^k + x_1^k + x_2^k & = & 0 \\ \lambda_1 x_0^k + x_1^k + x_3^k & = & 0 \\ \vdots & \vdots & \vdots \\ \lambda_{n-2} x_0^k + x_1^k + x_n^k & = & 0 \end{array} \right\} \subset \mathbb{P}^n. \tag{2}$$

Remark 1 ($C^k(\lambda_1, \ldots, \lambda_{n-2})$ as a fiber product of classical Fermat curves)**.** Set $\lambda_0 = 1$ and, for each $j \in \{0, 1, \ldots, n-2\}$, let C_j be the classical Fermat curve defined by $\lambda_j x_1^k + x_2^k + x_{3+j}^k = 0$. Let us consider the rational maps $\pi_j : C_j \to \mathbb{P}^1 = K \cup \{\infty\}$ defined by $\pi_j([x_1 : x_2 : x_{3+j}]) = -(x_2/x_1)^k$. The branch values of π_j are ∞, 0 and λ_j. If we consider the fiber product of all these curves, with the given maps, we obtain a reducible projective algebraic curve with k^{n-2} irreducible components. All of these components are isomorphic to $C^k(\lambda_1,\ldots,\lambda_{n-2})$.

Let H_0 be the group generated by the linear transformations $\varphi_0,\ldots,\varphi_n$, where

$$\varphi_j([x_0:\cdots:x_j:\cdots:x_n]) := [x_0:\cdots:w_k x_j:\cdots:x_n],$$

where w_k is a primitive k-th root of unity. In [5] the following facts were proved:

(1) $H_0 \cong \mathbb{Z}_k^n$.
(2) $\varphi_0\circ\varphi_1\circ\cdots\circ\varphi_n = 1$.
(3) $H_0 < \mathrm{Aut}(C^k(\lambda_1,\ldots,\lambda_{n-2}))$.
(4) The set $\mathrm{Fix}(\varphi_j)$ of fixed points of φ_j in $C^k(\lambda_1,\ldots,\lambda_{n-2})$ is given by the intersection

$$\mathrm{Fix}(\varphi_j) := \{x_j := 0\} \cap C^k(\lambda_1,\ldots,\lambda_{n-2}),$$

which is of cardinality k^{n-1}. Set $F(H_0) := \cup_{j=0}^n \mathrm{Fix}(\varphi_j)$.
(5) The map

$$\pi_0 : C^k(\lambda_1,\ldots,\lambda_{n-2}) \to \mathbb{P}^1 : [x_0:\cdots:x_j:\cdots:x_n] \mapsto -\left(\frac{x_1}{x_2}\right)^k \tag{3}$$

is a regular branched cover with deck group H_0 and whose branch values are

$$\infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2},$$

each one of order k. In other words, the pair $(C^k(\lambda_1,\ldots,\lambda_{n-2}), H_0)$ is a generalized Fermat pair of type (k,n).

Theorem 2. *The generalized Fermat pairs $(F_{k,n}, H)$ and $(C^k(\lambda_1,\ldots,\lambda_{n-2}), H_0)$ are equivalent. Moreover, the only non-trivial elements of H_0 acting with fixed points are the non-trivial powers of the generators $\varphi_0,\ldots,\varphi_n$.*

Proof. This result was obtained, for $K=\mathbb{C}$ in [5].

It can be seen that a generalized Fermat curve $F_{k,n}$ is in fact a fiber product of $n-1$ classical Fermat curves. In fact, the $n-1$ triples

$$\{\infty,0,1\}, \{\infty,0,\lambda_1\}, \ldots, \{\infty,0,\lambda_{n-2}\}$$

produce, respectively, the Fermat curves

$$C_0 : x_1^k + x_2^k + x_3^k = 0, C_1 : \lambda_1 x_1^k + x_2^k + x_4^k = 0, \ldots, C_{n-2} : x_1^k + x_2^k + x_n^k = 0.$$

If we set $\lambda_0 = 1$, then on C_j we consider the map $\pi_j : C_j \to \mathbb{P}^1$ defined by $\pi([x_1 : x_2 : x_{3+j}]) = -(x_2/x_1)^k$. The branch values of π_j are $\infty, 0, \lambda_j$. If we consider the fiber product of the above curves, using the above maps, we obtain a reducible algebraic curve admitting a group $(\mathbb{Z}_k^2)^{n-1}$ as a group of automorphisms and k^{n-2} irreducible components. All its irreducible components are isomorphic and they are generalized Fermat curves of type (k,n), each one is invariant by a subgroup isomorphic to $\mathbb{Z}_k^n$. Let $\widehat{C}$ be one of these irreducible components and let $\widehat{H}$ be its stabilizer in the above group. Then the quotient $\widehat{C}/\widehat{H} = F_{k,n}/H$. Now, the universality property of the fiber product ensures that $(F_{k,n}, H)$ and $(\widehat{C}, \widehat{H})$ are isomorphic. By the construction of the fiber product, it can be seen that in fact $(\widehat{C}, \widehat{H})$ and $(C^k(\lambda_1,\ldots,\lambda_{n-2}), H_0)$ are isomorphic. $\square$

2.2. **Automorphisms of generalized Fermat curves.** Let us consider a generalized Fermat pair $(F_{k,n}, H)$. By Theorem 2 we may assume (and this will be from now on) that

$$(F_{k,n}, H) = (C^k(\lambda_1, \ldots, \lambda_{n-2}), H_0).$$

If $n = 2$, then $F_{k,2}$ is an ordinary Fermat curve of degree k and its automorphism group was studied by P. Tzermias [19] for $p = 0$ and by H. Leopoldt [14] for $p > 0$. These results state that H_0 is the unique generalized Fermat group of type $(k, 2)$ if $k \geq 4$ (in the case $k < 4$ it is unique up to conjugation).

If $n \geq 3$, then in [5, Cor. 9] it was proved that, for $K = \mathbb{C}$, every automorphism which normalizes H_0 is linear i.e., the normalizer of H_0 is a subgroup of $\mathrm{PGL}_{n+1}(\mathbb{C})$. The arguments are still valid for any characteristic.

Again, assuming $K = \mathbb{C}$, the following uniqueness results of the generatlized Fermat groups are known. In the case that $k = 2$ (these are also called generalized Humbert curves) it was proved in [1] that for $n = 4, 5$ the generalized Fermat group of type (k, n) is unique. In [4] Y. Fuertes, G. González-Diez, the first and third author proved that for $k \geq 3$ and $n = 3$ the generalized Fermat group of type (k, n) is also unique. In the same paper it was conjectured that the uniqueness holds for $(k-1)(n-1) > 2$ (in particular, that it is normal in the whole automorphism group). Here we solve positively such a conjecture.

Theorem 3. *Let $k, n \geq 2$ be integers so that $(k-1)(n-1) > 2$. If $p > 0$, then we also assume that $k-1$ is not a power of p and that $(p, k) = 1$. Then H_0 is the only generalized Fermat group of type (k, n) of $F_{k.n}$. Moreover, $\mathrm{Aut}(F_{k,n})$ is linear and it consists of matrices such that only an element in each row and column is non-zero.*

Remark 4. If $k-1$ is a power of p, the previous theorem is, in general, false. For example, if $n = 2$ and $k = 1 + p^h$, $p > 0$, the group H_0 is not always a normal subgroup of $\mathrm{Aut}(F_{k,n}) = \mathrm{PGU}_3(p^{2h})$.

Corollary 5. *Every generalized Fermat curve of type (k, n) has a unique generalized Fermat group of same type if $(k-1)(n-1) > 2$ and, for $p > 0$, that $k-1$ is not a power of p.*

Corollary 6. *Let $k > 2$ and, for $p > 0$, let us assume that $(p, k) = 1$ and that $k-1$ is not a power of p. Then H_0 is a normal subgroup of $\mathrm{Aut}(F_{k,n})$.*

Remark 7. If $(k-1)(n-1) \leq 2$, then it is known that $\mathrm{Aut}(F_{k,n}) < \mathrm{PGL}_{n+1}(K)$. Let us now assume that $(k-1)(n-1) > 2$ and, for $p > 0$, that $k-1$ is not a power of p. Theorem 3 asserts that $\mathrm{Aut}(F_{k,n})$ coincides with the normalizer $N(H_0)$ of H_0, so by the results in [5, Cor. 9] we obtain $\mathrm{Aut}(F_{k,n}) < \mathrm{PGL}_{n+1}(K)$. In the same paper it is mentioned how to compute $\mathrm{Aut}(F_{k,n})$. This is done observing the short exact sequence:

$$1 \to H_0 \to \mathrm{Aut}(F_{k,n}) \to G_0 \to 1,$$

where G_0 is the subgroup of $\mathrm{PGL}_2(K) = \mathrm{Aut}(\mathbb{P}^1)$ which leaves invariant the set of branch points $\{0, 1, \infty, \lambda_1, \ldots, \lambda_{n-2}\}$.

In the case that $K = \mathbb{C}$, the above uniqueness results provides the following "kind of Torelli's" result.

Corollary 8. *Let* $\Gamma_1, \Gamma_2 < \mathrm{PSL}_2(\mathbb{R})$ *be Fuchsian groups acting on the upper-half plane* $\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ *so that* $\mathbb{H}^2/\Gamma_j$ *has signature* $(0; k, \overset{n+1}{\ldots}, k)$. *Let* Γ'_j *be the commutator subgroup of* Γ_j. *If* $\Gamma'_1 = \Gamma'_2$, *then* $\Gamma_1 = \Gamma_2$.

Theorem 3 states that if $(k-1)(n-1) > 2$ (and $k-1$ not a power of p in the case $p > 0$), then $\mathrm{Aut}(F_{k,n})$ is a linear group. The following states a similar result for the case that $p > 0$ and $k-1$ is a power of p under an extra condition.

Theorem 9. *Let* $p > 0$, $(p, k) = 1$ *and assume that* $k-1$ *is a power of* p. *If* $(k, n+1) = 1$, *then* $\mathrm{Aut}(F_{k,n})$ *is a subgroup of* $\mathrm{PGL}_{n+1}(K)$ *and it consists of elements* $A = (a_{ij})$ *such that*

$$A^t \Sigma_i A^q = \sum_{\mu=0}^{n-2} b_{i,\mu} \Sigma_\mu,$$

for a $(n-1) \times (n-1)$ *matrix* $(b_{i,\mu})$, *where* Σ_i *are certain* $(n+1) \times (n+1)$ *matrices, defined in eq. (13).*

3. Hyper-osculating points of $C^k(\lambda_1, \ldots, \lambda_{n-2})$

In this section we demonstrate, in characteristic zero or in characteristic $p > k^{n-1}$, that the set $F(H_0)$ of fixed points of the generalized Fermat group H_0 coincides with the set of hyper-osculating points of the curve $F_{k,n} = C^k(\lambda_1, \ldots, \lambda_{n-2})$.

We begin by explaining the theory of hyper-osculating points of curves over fields of characteristic 0 following essentially [9]. In positive characteristic a variety of new, very interesting phenomena appear. Also all definitions need appropriate modification in order to work. For the positive characteristic case we will follow the Laksov approach [12], [13], since his theory was successful in giving a version of the generalized Plücker formulas.

Essentially the results of Laksov, for the case of generalized Fermat curves, show that if we assume that the characteristic $p > k^{n-1}$, then we have exactly the same behavior as in characteristic 0.

For a curve C (non-singular, projective) defined over a field K we consider the function field $K(C)$ which plays the role of the field of meromorphic functions. The points of the curve can be seen as places (equivalence classes of valuations) and a function f in $K(C)$ is called *holomorphic* at P if $v_P(f) \geq 0$, where $v_P(f)$ is the valuation of f at P. Holomorphic functions admit Taylor expansions at the completions of the valuation rings. For the general theory of functions fields over arbitrary fields we refer to [17], [8].

3.1. **Preliminaries on hyper-osculating points of curves.** Let C be a projective smooth curve of the projective space $\mathbb{P}^n$. Let us consider an s-plane $\Pi \subset \mathbb{P}^n$, $1 \leq s \leq n-1$, and let us define the multiplicity of Π in $P \in C$ as

$$\mathrm{mult}_P(\Pi \cap C) := \text{Order of contact of } \Pi \text{ and } C \text{ in } P.$$

It is known that there exists a unique s-plane, denoted by $\Pi(s, P)$, such that

$$\mathrm{mult}_P(\Pi(s,P) \cap C) \geq s+1,$$

and that there exists at most a finite number of points $P \in C$ such that

$$\mathrm{mult}_P(\Pi(s,P) \cap C) > s+1.$$

The s-plane $\Pi(s, P)$ is called the *osculating s-plane of C at P* and a point $P \in C$ is called a *hyper-osculating point* if

$$\mathrm{mult}_P(\Pi(n-1,P) \cap C) > n.$$

Remark 10. Let $\varphi \in \mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}_{n+1}(K)$. Observe that

$$\mathrm{mult}_{\varphi(P)}(\varphi(\Pi) \cap \varphi(C)) = \mathrm{mult}_P(\Pi \cap C).$$

In particular, P is a hyper-osculating point of C if and only if $\varphi(P)$ is a hyper-osculating point of $\varphi(C)$.

3.2. **Laskov's theory of osculating planes.** Let C be a smooth curve of genus g over a general field K and let D be a divisor in C. Moreover, let V be a linear system in $H^0(C, D)$ of projective dimension n. We note $\deg D$ the degree of the divisor D.

Tensor powers of the sheaf of differentials can be interpreted as

$$(\Omega^1_C)^{\otimes m} = I^m/I^{m+1},$$

where I is the ideal defining the diagonal in the product $C \times C$. Let p, q be the two projections $C \times C$ into the first and second factor respectively. Laksov defined the bundle of principal parts $P^m(D) = p_*(q^*\mathscr{O}_C(D)|C(m))$, where $C(m)$ is the subscheme of $C\times C$ defined by I^{m+1}. He then introduced a family of maps

$$v^m(D) : H^0(C, D)_C := H^0(C, D) \otimes_K \mathscr{O}_C \to P^m(D)$$

and the corresponding map $v^m : V_C := V \otimes_K \mathscr{O}_C \to P^m(D)$. Let B^m and A^m be the image and the cokernel of v^m. The Corollary 2 of [13] implies that there are integers

$$0 = G_0 < G_1 < \cdots < G_n \leq \deg D < G_{n+1} = \infty$$

such that $\mathrm{rank} B^j = (s+1)$ for $G_s \leq j < G_{s+1}$. The above sequence is called the *gap sequence* of the linear system V. If $G_m = m$ for $m = 0, 1, \ldots, n$ then the gap sequence is called *classical*.

Definition 11 (Associated Curves)**.** The surjection $V_C \to A^s$ induced by the map v^{b_s} defines a map

$$f_s : C \to \mathbb{G}(s, n)$$

to the grassmanian of s-planes in $\mathbb{P}^n$. The s-plane in $\mathbb{P}^n$ is called the *associated s-plane* to V at P, and the degree d_s of the map f_s is called the s-rank of the linear system V.

The grassmanian can be embedded in terms of the Plücker coordinates in a projective space $\mathbb{P}^N$, where $N = \binom{n+1}{s+1} - 1$. We will denote by $b_s(P)$ the ramification index and by b_s the sum of all ramification indices of the composition $C \xrightarrow{f_s} \mathbb{G}(s,n) \to \mathbb{P}^N$. The image of the later map is called the s-associated curve.

Remark 12. Geometrically d_s can be interpreted as the number of associated s-planes to V which intersect a generic $(n-s-1)$-plane of $\mathbb{P}^n$. In addition, we have that $d_s = \mathrm{rank} A^s$. See section 5 of the article [13].

Let $e_0, e_1, \ldots, e_n$ be a basis of V. Using the canonical maps $v^0 : V_C \to \mathscr{O}_C(D)$, we can prove that this basis induces a set of linearly independent functions $v_0, v_1, ..., v_n$ belonging to the local ring $\mathscr{O}_{C,P}$, $P \in C$, such that there exists a sequence of integers $h_0 < h_1 < \cdots < h_n$, where $h_i := \mathrm{Ord}_P v_i$. These integers are called the *Hermitian invariants at P*.

The s-plane associated to the sub-space of V spanned by $e_{s+1}, ..., e_n$ is the unique s-plane with maximal contact order with V at P (the order of contact is equal to $h_{s+1} - h_0$). This s-plane is called the *osculating s-plane to V at P*.

Let C be a projective smooth curve of the projective space $\mathbb{P}^n$. If $f_0 : C \to \mathbb{P}^n$ is the natural embedding defined by the inclusion $C \subset \mathbb{P}^n$, and the divisor D is the inverse image of a hyperplane Π of $\mathbb{P}^n$, we obtain that $h_0 = 0$ for all $P \in C$ and that the concepts of *osculating s-plane to V at P* and *osculating s-plane of C at P* coincide.

Additionally, given a local uniformizer z at the point p, the normal form of f_0 in P is obtained in the following manner:

$$f_0(z) := [v_0(z) : \cdots : v_n(z)].$$

When the characteristic p is small, then a lot of new phenomena appear, however for $p > \deg D$ the situation is similar as in characteristic zero:

Theorem 13 (See [13, Th. 15]). *Assume that the characteristic p of the ground field is zero or strictly grater than $\deg D$. Fix a point $P \in C$ and let $h_0, h_1, \ldots, h_n$ be the Hermite invariants of the linear system V at P. Then:*

1. *The linear system V has classical gap sequence, i.e. $G_m = m$ for $m = 0, 1, \ldots, n$.*
2. *The ramification index $b_s(P)$ of f_s at P is equal to $h_{s+1} - h_s - 1$ for $s = 0, 1, \ldots, n-1$.*
3. *The Plücker formulas take the form:*
$$d_{s+1} - 2d_s + d_{s-1} = (2g-2) - b_s \text{ for } s = 0, 1, \ldots, n-1,$$
where $d_{-1} = 0$ and $d_n = 0$.
4. *The osculating and associated s-planes to V at P coincide.*

3.3. **The hyper-osculating points of $F_{k,n}$.** Let $f_0 : F_{k,n} \to \mathbb{P}^n$ be the natural embedding defined by the inclusion $F_{k,n} \subset \mathbb{P}^n$. Let P be a point in $F(H_0)$ and let z be a local uniformizer at P. The following lemma helps us to find the normal form of f_0 around $z(P) = 0$.

Let Π be a hyperplane section of the projective space $\mathbb{P}^n$ and $D = f_0^\star(\Pi)$ the inverse image divisor of Π. Using the Bezout theorem we obtain that $\deg D = k^{n-1}$. In the rest of this section Theorem 13 will be used quite a lot, for this reason we will impose, as a general hypothesis in the entire rest of the section, that the characteristic of the ground field is zero, or stricly greater than k^{n-1}.

Lemma 14. *Let us conserve the previously defined notations. Assume that we are working over a field of zero characteristic or stricly greater than k^{n-1}. Then there exists a sequence of $n-1$ integers,*

$$1 = l_0 < 2 = l_1 < l_2 < \cdots < l_j < \cdots < l_{n-2} \leq k^{n-2},$$

such that the normal form of f_0 around $z(P) = 0$ is the following:

$$f_0(z) = [1 : z : g_0(z^k) : g_1(z^k) : \cdots : g_i(z^k) : \cdots : g_{n-2}(z^k)]$$

where the g_i admit an expansion $g_i(z) = z^{l_i} + \cdots + \cdots$.

Proof. We will begin by the case of the characteristic of the field being zero.

Using linear substitutions in the system of equations which define the curve $F_{k,n} = C^k(\lambda_1, \ldots, \lambda_{n-2})$, followed by an automorphism of $\mathbb{P}^n$, we can suppose that $F_{k,n} = C^k(\hat{\lambda}_1, \ldots, \hat{\lambda}_{n-2})$ and that $P \in \mathrm{Fix}(\varphi_1)$, These transformations do not affect the condition of being or not being a point of hyper-osculation, see Remark 10.

In order to simplify the notations, we say that $\hat{\lambda}_0 = 1$. Then the point P in homogeneous coordinates is

$$P := [1 : 0 : \rho_1 : \rho_2 : \cdots : \rho_{n-1}],$$

where $\rho_i^k = -\hat{\lambda}_{i-1}$, $0 \leq i \leq n-1$.

Let $f_0 : F_{k,n} \to \mathbb{P}^n$ be the natural embedding defined by the inclusion $F_{k,n} \subset \mathbb{P}^n$, and let us consider the following Taylor series centered in $t = 0$:

$$\sqrt[k]{1+t} = \textstyle\sum_{i=0}^{\infty} \binom{k^{-1}}{i} t^i, \; |t| < 1,$$

where

$$\binom{k^{-1}}{i} := \frac{\Gamma(k^{-1}+1)}{\Gamma(i+1)\Gamma(k^{-1}-i+1)} = \prod_{\nu=1}^{i} \frac{k^{-1}+1-\nu}{\nu} = \frac{1}{i!k^i} \prod_{\nu=1}^{i-1} (1-k\nu) \in \mathbb{Q}. \tag{4}$$

Remark 15. The binomial coefficient $\binom{n}{i}$ for $n, i \in \mathbb{N}$ has always meaning in fields of positive characteristic p, since we can always reduce it modulo p. The binomial coefficients in eq. (4) are not defined if $p \leq i$.

Remark 16. If $Mk < p$ then $\binom{k^{-1}}{i} \neq 0$ for all $i < M$. Indeed, by eq. (4) we observe that for $1 \leq \nu \leq i-1 < M$ the quantity $k\nu - 1 \not\equiv 0 \mod p$. Otherwise, $0 < \mu p = k\nu - 1 < p/M \cdot i - 1 < p$ for $\nu, \mu \in \mathbb{N}$, a contradiction.

Using this expansion, we can describe f_0 explicitly in a neighborhood of P. Let z be a local uniformizer at P, we express locally

$$f_0(z) = \left[1 : z : \sum_{i=0}^{\infty} c_{(i,1)} z^{ik} : \sum_{i=0}^{\infty} c_{(i,2)} z^{ik} : \cdots : \sum_{i=0}^{\infty} c_{(i,n-1)} z^{ik}\right],$$

where $c_{(i,j)} := \dfrac{\rho_j}{\hat{\lambda}_{j-1}^{i}} \dbinom{k^{-1}}{i}$, $1 \leq j \leq n-1$, $i \geq 0$.

We can prove by induction on j, that for each integer $1 \leq j \leq n-2$, there exists a sequence of $n-2$ integers

$$1 = l_0 < 2 = l_1 < l_2 < \cdots < l_j \leq \cdots \leq \cdots \leq l_{n-2},$$

for which there exists a change of coordinates of $\mathbb{P}^n$ (which is to say, an automorphism of $\mathbb{P}^n$) such that

$$f_0(z) = \left[1 : z : \sum_{i=1}^{\infty} d_{(i,1)} z^{ik} : \sum_{i=2}^{\infty} d_{(i,2)} z^{ik} : \sum_{i=l_2}^{\infty} d_{(i,3)} z^{ik} : \cdots : \sum_{i=l_{n-2}}^{\infty} d_{(i,n-1)} z^{ik}\right],$$

where $d_{(l_{m-1},m)} = 1$ for all $1 \leq m \leq n-2$.

By virtue of part (iii) of theorem 10 of [13] we obtain that the Hermite invariant h_n is less than or equal to $\deg D = k^{n-1}$ (It is worth mentioning that this result is valid in the case of the positive characteristic). Implying that $l_{n-2} \leq k^{n-2}$. This will prove the lemma in the case of characteristic zero.

Using the fact that $h_n \leq k^{n-1}$, and Remark 16, we can ensure that for fields of characteristic p such that $k^{n-1} < p$ the method of recurrence raised previously functions in the same way. However the sequence of integers $l_2 < l_3 < \cdots < l_{n-2}$ obtained in the case of the positive characteristic could differ from the sequence of integers obtained in the case of characteristic zero.

Let us now do some steps of the induction in order to indicate some problems that may occur over fields of positive characteristic:

$$f_0(z) = \left[1 : z : c(0,1) + c(1,1)z^k + \cdots : c(0,2) + c(1,2)z^k + \frac{\rho_1}{\hat{\lambda}_1^2}\binom{k^{-1}}{2} z^{2k} + \cdots : \ldots :\right],$$

In the first step we subtract the constant function 1 multiplied by $c(0,i)$ from all but the first two projective coordinates of $f_0(z)$ arriving at

$$f_0(z) = \left[1 : z : c(1,1)z^k + \cdots : c(1,2)z^k + \frac{\rho_2}{\hat{\lambda}_1^2}\binom{k^{-1}}{2} z^{2k} + \cdots : \ldots :\right],$$

The coefficient $c(1,1) = \frac{\rho_1}{\hat{\lambda}_0}\binom{k^{-1}}{1} \neq 0$ so we can divide the third coordinate of $f_0(z)$ by $c(1,1)$ in order to have coefficient of z^k equal to 1. Then we subtract from all but the first two coefficients the third coefficient in order to eliminate the term z^k. The coefficient of z^{2k} in the fourth coordinate equals to

$$\begin{aligned} c(1,1)c(2,2)-c(1,2)c(2,1) &= \frac{\rho_1}{\hat{\lambda}_0}\binom{k^{-1}}{1}\frac{\rho_2}{\hat{\lambda}_1^2}\binom{k^{-1}}{2}-\frac{\rho_2}{\hat{\lambda}_1}\binom{k^{-1}}{1}\frac{\rho_1}{\hat{\lambda}_0^2}\binom{k^{-1}}{2} \\ &= \frac{\rho_1\rho_2}{\hat{\lambda}_0\hat{\lambda}_1}\binom{k^{-1}}{2}\binom{k^{-1}}{1}\left(\frac{1}{\hat{\lambda}_1}-\frac{1}{\hat{\lambda}_0}\right)\neq 0, \end{aligned}$$

since

$$\binom{k^{-1}}{2}=\frac{k^{-1}(k^{-1}-1)}{2}=\frac{1-k}{2k^2}\neq 0 \text{ and } \hat{\lambda}_1\neq 1$$

We can now normalize the coefficient of z^{2k} to 1 and subtract it multiplied by the appropriate constant from the next coordinate. Doing this subtraction it can happen that the coefficients of z^{3k}, z^{4k} etc are also eliminated. So we set $2 < l_2$ the first non zero exponent in the above subtraction. We then proceed in a similar way until all coordinates are in the form requested by the lemma. □

The next theorem describes the hyper-osculating points of $F_{k,n}$ and the ramification indices.

Theorem 17. *Assume that the characteristic p of the ground field is zero or strictly grater than k^{n-1}. Let $(n-1)(k-1)>2$. Then the following holds:*

(1) *The set of hyperosculating points of $F_{k,n}$ is the set $F(H_0)$.*

(2) *If $P\in F(H_0)$, then $b_1(P)=k-2$ and $b_l(P)=k-1$ for all $2\leq l\leq n-1$.*

The following corollary is directly derived from Theorem 17 and Lemma 14.

Corollary 18. *Let z be a local chart of $F_{k,n}$ around a point P. Then the normal form of f_0 in $z(P):=0$ is:*

(1) *If $P\in F(H_0)$*

$$f_0(z)=\left[1:z:g_0(z^k):g_1(z^k):\cdots:g_i(z^k):\cdots:g_{n-1}(z^k)\right],$$

where the g_i are holomorphic functions such that $g_i(z)=z^{i+1}+\cdots+\cdots$,

(2) *If $P\notin F(H_0)$, then*

$$f_0(z)=\left[1:z:z^2+\cdots:\cdots:z^{(n-1)}+\cdots\right].$$

Proof of the Theorem 17. Let P be a point in $F(H_0)$. Using part 2 of Theorem 13 and Lemma 14, we obtain the following system of equations:

$$\left\{\begin{array}{lcl} 2+b_1(P) & = & k \\ 3+b_1(P)+b_2(P) & = & 2k \\ 4+b_1(P)+b_2(P)+b_3(P) & = & l_2k \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots & \vdots & \vdots \\ n+b_1(P)+b_2(P)+b_3(P)+\cdots+b_{n-1}(P) & = & l_{n-2}k \end{array}\right.$$

Equivalently, we obtain

$$\left\{\begin{array}{lcl} b_1(P) & = & k-2 \\ b_2(P) & = & k-1 \\ b_3(P) & = & (l_2-2)k-1 \\ \vdots & \vdots & \vdots \\ b_{n-1}(P) & = & (l_{n-2}-l_{n-3})k-1 \end{array}\right.$$

Observe that $b_l(P) \geq k-1$ for all $2 \leq l \leq n-1$. In particular, P is a hyper-osculating point.

Since the cardinality of $F(H_0)$ is equal to $(n+1)k^{n-1}$, we have the following lower bound from the total ramification indices:

$$\left\{\begin{array}{lcl} b_1 & = & \hat{b}_1 := (n+1)k^{n-1}(k-2) \\ b_l & \geq & \hat{b}_l := (n+1)k^{n-1}(k-1) \quad \text{for every } \ 2 \leq l \leq n-1 \end{array}\right.$$

Observe that in order to finish the demonstration of the theorem, it is necessary and sufficient to prove $b_l = \hat{b}_l$, for all $1 \leq l \leq n-1$. We will now prove these equalities.

Consider the following inequality

$$0 \leq b_l - \hat{b}_l \leq \sum_{l=0}^{n-1}(n-l)(b_l - \hat{b}_l),$$

where $b_0 = \hat{b}_0 = 0$. The idea is to show that the right part of the inequality is zero.

Remember that the genus of $F_{k,n}$ is given by the following formula:

$$g_{(k,n)} := \frac{k^{n-1}(((n-1)(k-1)-2)+2}{2}.$$

Via direct calculation, we obtain the following equality:

$$\sum_{l=0}^{n-1}(n-l)\hat{b}_l = n(n+1)(g_{(k,n)}-1) + (n+1)k^{n-1}.$$

Using the Plücker formulas (part 2 of Theorem 13), we obtain

$$\begin{aligned} \sum_{l=0}^{n-1}(n-l)b_l &= \sum_{l=0}^{n-1}(n-l)(2(g_{(k,n)}-1) - \Delta^2 d_l) \\ &= n(n+1)(g_{(k,n)}-1) - \sum_{l=0}^{n-1}((n-l)\Delta^2 d_l), \end{aligned}$$

where $\Delta^2 d_l = d_{l+1} - 2d_l + d_{l-1}$.

From a simple calculation it is obtained that

$$\sum_{l=0}^{n-1}(n-l)\Delta^2 d_l = d_n - (n+1)d_1 + nd_{-1}.$$

Since $d_n = d_{-1} = 0$ and $d_1 = k^{n-1}$, therefore

$$\sum_{l=0}^{n-1}(n-l)b_l = n(n+1)(g_{(k,n)}-1) + (n+1)k^{n-1},$$

which implies that $b_l = \hat{b}_l$ for all $1 \le l \le n-1$. □

4. Complete intersections and linear automorphisms

Let $\mathbb{P}^n$ be the projective space with homogeneous coordinates $[x_1 : \cdots : x_{n+1}]$. Consider the curve $F_{k,n} = C^k(\lambda_1, \dots, \lambda_{n-2})$ embedded in $\mathbb{P}^{n+1}$ as the intersection of the $n-1$ hypersurfaces $f_i := \lambda_i x_1^k + z_2^k + z_{3+i}^k = 0$ for $0 \le i \le n-2$, where $k, n \ge 2$ are integers so that, for $p > 0$, $(k,p) = 1$ (see eq. (2)).

Proposition 19. *The curve $F_{k,n}$ is a nonsingular complete intersection.*

Proof. The curve is given as the intersection of $n-1$ hypersurfaces $f_i := \lambda_i x_1^k + x_2^k + x_{3+i}^k$ for $i = 0, \dots, n-2$. We consider the matrix of ∇f_i written as rows.

$$\begin{pmatrix} kx_1^{k-1} & kx_2^{k-1} & kx_3^{k-1} & 0 & \dots & 0 \\ \lambda_1 kx_1^{k-1} & kx_2^{k-1} & 0 & kx_4^{k-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \lambda_{n-2}kx_1^{k-1} & kx_2^{k-1} & 0 & \dots & 0 & kx_{n+1}^{k-1} \end{pmatrix}. \tag{5}$$

By the defining equations of the curve we see that a point which has two variables $x_i = x_j = 0$ for $i \ne j$ and $1 \le i, j \le n+1$ has also $x_t = 0$ for $t = 1, \dots, n+1$. Therefore the above matrix has the maximal rank $n-1$ at all points of the curve.

So the defining hypersurfaces are intersecting transversally and the corresponding algebraic curve they define is non singular. □

Proposition 20. *The ideal $I_{k,n}$ defined by the $n-1$ equations defining $F_{k,n} \subset \mathbb{P}^{n+1}$ is prime.*

Proof. We will follow the method of [11, sec. 3.2.1]. Observe first that the defining equations $f_0, \dots, f_{n-2}$ form a regular sequence, and $K[x_1, \dots, x_{n+1}]$ is a Cohen-Macauley ring and the ideal $I_{k,n}$ they define is of codimension $n-1$. The ideal $I_{k,n}$ is prime by the Jacobian Criterion [3, Th. 18.15], [11, Th. 3.1] and Proposition 19. In remark [11, 3.4] we pointed out that an ideal I is prime if the the singular locus of the algebraic set defined by I has big enough codimension. □

Remark 21 (Stable Family)**.** Consider now the polynomial ring $R_1 := K[\lambda_1, \dots, \lambda_{n-2}]$ and consider the ideal J generated by $\prod_{i=1}^{n-2}\lambda_i(\lambda_i - 1) \cdot \prod_{i<j}(\lambda_i - \lambda_j)$. We consider the localization R of the polynomial ring R_1 with respect to the multiplicative set $R_1 - J$. The affine scheme $\mathrm{Spec}R$ is the space of different points $P_1, \dots, P_{n+1}$, and the family $\mathscr{X} \to \mathrm{Spec}R$ is a stable family of curves since it has non-singular fibers of genus ≥ 2.

By the results of Deligne-Mumford [2, lemma I.12] any automorphism of the generic fibre is also an automorphism of the special fibre. Special fibres have more automorphisms, when the ramified points

$$\{0, 1, \infty, \lambda_1, \ldots, \lambda_{n-2}\}$$

are in such a configuration, so that a finite automorphism group of $\mathrm{PGL}(2, K)$ permutes them.

Since $F_{k,n}$ is a projective variety, for every automorphism $\sigma \in \mathrm{Aut}(F_{k,n})$ there is a Zariski open covering of $F_{k,n}$, $(U_i)_{i\in I}$ such that the restriction of $\sigma\mid_U$ is given by $n+1$ homogeneous polynomials $g_i^{(\sigma)}$ of the same degree, i.e. if $\bar{x} = [x_1 : \cdots : x_{n+1}]$, then

$$\sigma \mid_U (\bar{x}) = [g_1^{(\sigma)}(\bar{x}) : \cdots : g_{n+1}^{(\sigma)}(\bar{x})], \tag{6}$$

see [15, prop. 6.20].

All automorphisms that come as automorphisms of the ambient projective space, i.e. they are represented on the whole curve $F_{k,j}$ as in eq. (6) with $\deg g_i = 1$ for all $1 \leq i \leq n+1$ are called linear and they form a subgroup L of $\mathrm{Aut}(F_{k,n})$.

Lemma 22. *The group L is a normal subgroup of* $\mathrm{Aut}(F_{k,n})$.

Proof. Consider a non linear automorphism $\tau \in \mathrm{Aut}(F_{k,n})$ and a linear automorphism $\sigma \in L$. Since τ is not linear there is an open $U \subset F_{k,n}$ where τ is expressed in terms of polynomials of degree $d > 1$.

Consider the element $\sigma' = \tau\sigma\tau^{-1}$. We will show that σ' is linear. Since the curve $F_{k,n}$ is connected, the open sets U and $\sigma(U)$ have non trivial intersection V. On this set V we express the automorphisms σ, τ, σ' in terms of homogenenous polynomials $g_i^{(\sigma)}, g_i^{(\tau)}, g_i^{(\sigma')}$, $1 \leq i \leq n+1$, respectively of degrees $1, d, d'$ as in eq. (6). We have $\sigma'\tau = \tau\sigma$ and this implies for $\bar{x} \in V$ the relation

$$[g_1^{(\sigma')} \circ g_1^{(\tau)}(\bar{x}) : \cdots : g_{n+1}^{(\sigma')} \circ g_{n+1}^{(\tau)}(\bar{x})] = [g_1^{(\tau)} \circ g_1^{(\sigma)}(\bar{x}) : \cdots : g_{n+1}^{(\tau)} \circ g_{n+1}^{(\sigma)}(\bar{x})].$$

Let $I_{k,n}$ be the ideal defining the curve $F_{k,n}$. For each $\bar{x} \in K^n$ there is a $\lambda_{\bar{x}} \in K$ such that

$$g_i^{(\sigma')} \circ g_i^{(\tau))}(\bar{x}) = \lambda_{\bar{x}} g_i^{(\tau)} \circ g_i^{(\sigma))}(\bar{x}) \mod I_{k,n} \text{ for all } 1 \leq i \leq n+1.$$

The left hand side has degree $d'd$ while the right hand side has degree d. So if we substitute $\mu\bar{x}$ in the above equation where $\mu^{d'} = \lambda_{\bar{x}}$ we obtain $g_i^{(\sigma')} \circ g_i^{(\tau)} = g_i^{(\tau)} \circ g_i^{(\sigma)}$ for all $1 \leq i \leq n+1$ modulo the homogenous ideal $I_{k,n}$ of the curve and this in turn is possible only if $d' = \deg g_i^{(\sigma')} = 1$, i.e. σ' is given in terms of linear polynomials.

We have proved so far that there is an open cover $(U_i)_{i\in I}$ of $F_{k,n}$ where σ' is given in terms of linear polynomials. Since every element in the defining ideal of the curve $F_{k,n}$ has degree > 1 this means that on the nonempty intersections $U_i \cap U_j$ the linear polynomials expressing σ' should not only be equal modulo the defining ideal, but equal as polynomials. This proves that σ' is given by linear polynomials on the whole space $F_{k,n}$ so $\sigma' \in L$. $\square$

4.1. **The elements of L.** In this section we describe the elements on the group L of linear automorphisms of the curve $F_{k,n}$.

All automorphisms $\sigma \in L$ are linear ones, so they are given in terms of an $(n+1) \times (n+1)$ matrix:

$$\sigma(x_i) = \sum_{\nu=1}^{n+1} a_{i,\nu} x_i. \tag{7}$$

An automorphism of $V(f_1, \ldots, f_{n-2})$ is a map σ such that if P is a point in $V(f_1, \ldots, f_{n-2})$, then $\sigma(P)$ is in $V(f_1, \ldots, f_{n-2})$. The following holds true:

$$f_i \circ \sigma = \sigma^*(f_i) \in \langle f_1, \ldots, f_{n-1} \rangle.$$

i.e.

$$f_i \circ \sigma = \sum_{\nu=1}^{n-1} g_{\nu,i} f_\nu, \tag{8}$$

for some appropriate polynomials $g_i \in K[x_1, \ldots, x_{n+1}]$. When $\sigma \in L$, so it is linear, the polynomials $g_{\nu,i}$ are just constants.

Theorem 23. *Set $Y_i = \nabla f_i$. If $\sigma \in L$, then $\sigma(Y_i)$ should be a linear combination of elements Y_i.*

Proof. By applying ∇ to eq. (8) we have for every point on the curve

$$\nabla(f_i \circ \sigma)(P) = \sum_{\nu=1}^{n-1} \big(g_{i,\nu}(P) \nabla f_\nu(P) + \nabla g_{i,\nu}(P) f_\nu(P) \big).$$

But $f_\nu(P) = 0$ so we arrive at

$$\nabla(f_i \circ \sigma)(P) = \sum_{\nu=1}^{n-1} g_{i,\nu}(P) \nabla f_\nu(P)$$

which gives rise to

$$\nabla(f_i \circ \sigma) = \sum_{\nu=1}^{n-1} g_{i,\nu} \nabla f_\nu + F,$$

where F is an element in the ideal I. The ideal I is generated by polynomials of degree k, while ∇f_i are polynomials of degree $k-1$. Therefore,

$$\nabla(f_i \circ \sigma) = \sum_{\nu=1}^{n-1} g_{i,\nu} \nabla f_\nu, \tag{9}$$

as polynomials in $K[x_1, \ldots, x_{n+1}]$. □

Now the chain rule implies that, for $\sigma \in L$,

$$\nabla(f_i \circ \sigma)(P) = \nabla(f_i)(\sigma(P)) \circ \sigma, \tag{10}$$

where σ is given by the $(n+1) \times (n+1)$ matrix $A = (a_{ij})$ given in eq. (7). We now rewrite eq. (10) and combine it with eq. (9)

$$\sigma^*(\nabla f_i) \circ \sigma = \nabla(f_i)(\sigma(P)) \circ \sigma = \nabla(f_i \circ \sigma)(P) = \sum_{\nu=1}^{n-1} g_{i,\nu} \nabla f_\nu. \tag{11}$$

Recall that $f_j = \lambda_j x_1^k + x_2^k + x_{3+j}^k$ for $1 \leq j \leq n-2$ and

$$Y_j = (k\lambda_j x_1^{k-1}, kx_2^{k-1}, 0, \ldots, 0, kx_{j+3}^{k-1}, 0, \ldots, 0),$$

where the third non zero element is at the $j+3$ position. For $1 \leq i \leq n+1$ let us write

$$\sigma^*(x_i) = \sum_{\nu=1}^{n+1} a_{i,\nu} x_\nu.$$

So

$$\sigma^*(Y_j) = k\left(\lambda_j \left(\sum_{\nu=1}^{n+1} a_{1,\nu}x_\nu\right)^{k-1}, \left(\sum_{\nu=1}^{n+1} a_{2,\nu}x_\nu\right)^{k-1}, 0\ldots 0, \left(\sum_{\nu=1}^{n+1} a_{j+3,\nu}x_\nu\right)^{k-1}, 0\ldots 0\right),$$

Observe that eq. (11) implies that $\sigma^*(Y_i)$ is a linear combination of Y_i, which involves only combinations of the monomials x_i^{k-1}, while the t-th ($t = 1, 2, j+3$) coefficient of $\sigma^*(Y_i)$ involves all combinations of the terms

$$\binom{k-1}{\nu_1, \ldots, \nu_{n+1}} \left(a_{t,1}^{\nu_1} \cdots a_{t,n+1}^{\nu_{n+1}}\right) \cdot \left(x_1^{\nu_1} \cdots x_{n+1}^{\nu_{n+1}}\right) \text{ for } \nu_1 + \cdots + \nu_{n+1} = k-1.$$

For $\bar{\nu} = (\nu_1, \ldots, \nu_{n+1})$ define $\mathbf{x}^{\bar{\nu}} = x_1^{\nu_1} \cdots x_{n+1}^{\nu_{n+1}}$ and set

$$A_{t,\bar{\nu}} = a_{t,1}^{\nu_1} \cdots a_{t,n+1}^{\nu_{n+1}}.$$

Observe that if $\binom{k-1}{\nu_1,\ldots,\nu_{n+1}} \neq 0$ and $\mathbf{x}^{\bar{\nu}}$ does not appear as a term in the linear combination of Y_i, then using eq. (11) we have

$$(A_{1,\bar{\nu}}, \ldots, A_{n+1,\bar{\nu}}) \cdot A = 0.$$

But A is an invertible matrix so the above equation implies that

$$A_{t,\bar{\nu}} = 0$$

if $\mathbf{x}^{\bar{\nu}}$ does not appear as a term in the linear combination of Y_i.

Lemma 24. *The binomial coefficients* $\binom{k-1}{\nu} = 0$ *for all* $1 \leq \nu \leq k-1$ *if and only if* $k-1$ *is a power of the characteristic.*

Proof. The binomial coefficient $\binom{k-1}{\nu}$ is not divisible by the characteristic p if and only if $\nu_i \leq k_i$ for all i, where $\nu = \sum \nu_i p^i$, $k-1 = \sum k_i p^i$ are the p-adic expansions of ν and $k-1$, [3, p. 352]. The result follows. □

Lemma 25. *Let* $\sigma \in L$ *given by a* $(n+1)\times(n+1)$ *matrix* (a_{ij})*. If* $k-1$ *is not a power of the characteristic, then there is only one non-zero element in each column and row of* (a_{ij})*.*

Proof. If $k-1$ is not a power of the characteristic, then we see that the matrix $(a_{i,j})$ can have only one non zero term in each row and column. Indeed, if this was not true, then for some j we have two non-zero terms a_{j,l_1}, a_{j,l_2}. If $j \geq 3$, then we work with $\sigma^*(Y_{j-3})$ and for ν such that $\binom{k-1}{\nu} \neq 0$ we have that $a_{j,l_1}^{\nu} a_{j,l_2}^{k-1-\nu} = 0$, so the desired result follows. □

Corollary 26. *If $k-1$ is not a power of the characteristic, then every automorphism $\sigma \in L$ restricts to an automorphism of the function field $K(X)$, $X = -\frac{x_2^k}{x_1^k}$, i.e. L normalizes H_0.*

Proof. The function field of the generalized Fermat curves can be seen as Kummer extension with Galois group H of the rational function field $K(X)$, where $X = -\frac{x_2^k}{x_1^k}$ (see [5, par. 2.2] or eq. (3)). In order to prove that H is a normal subgroup of the whole automorphism group we have to show that every automorphism of the curve keeps the field $K(X)$ invariant.

Since there is only one non-zero element in each row and column of A the automorphism σ

$$\sigma^*(x_i^k) = \sum_{\nu=1}^{n+1} a_{i,\nu}^k x_\nu^k. \tag{12}$$

Therefore

$$\sigma^*(X) = -\frac{\sigma^*(x_2)^k}{\sigma^*(x_1)^k} = -\frac{\sum_{\nu=1}^{n+1} a_{2,\nu}^k x_\nu^k}{\sum_{\nu=1}^{n+1} a_{1,\nu}^k x_\nu^k}.$$

In the above equation we replace all variables x_ν for $\nu \geq 3$ using the defining equations $x_\nu^k = -\lambda_{\nu-3}x_1^k - x_2^k$ in order to arrive at an expresion involving only $X = -\frac{x_2^k}{x_1^k}$:

$$\begin{aligned}
\sigma^*(X) &= -\frac{a_{21}^k x_1^k + a_{22}^k x_2^k + \sum_{\nu=3}^{n+1} a_{2,\nu}^k \left(-\lambda_{\nu-3}x_1^k - x_2^k\right)}{a_{11}^k x_1^k + a_{12}^k x_2^k + \sum_{\nu=3}^{n+1} a_{1,\nu}^k \left(-\lambda_{\nu-3}x_1^k - x_2^k\right)} \\
&= -\frac{\left(-a_{22}^k + \sum_{\nu=3}^{n+1} a_{2,\nu}^k\right) X + \left(a_{21}^k - \sum_{\nu=3}^{n+1} \lambda_{\nu-3} a_{2,\nu}^k\right)}{\left(-a_{12}^k + \sum_{\nu=3}^{n+1} a_{1,\nu}^k\right) X + \left(a_{11}^k - \sum_{\nu=3}^{n+1} \lambda_{\nu-3} a_{1,\nu}^k\right)}.
\end{aligned}$$

□

Proposition 27. *Assume that $k-1 = p^h = q$ is a power of the characteristic. Denote by*

$$\Sigma_i = \mathrm{diag}(\lambda_i, 1, 0, \ldots, 1, 0, \ldots, 0), \tag{13}$$

with 1 in the $i+3$ position. Then a matrix $A \in \mathrm{PGL_{n+1}(K)}$ corresponding to $\sigma \in L$ should satisfy

$$A^t \Sigma_i A^q = \sum_{\mu=0}^{n-2} b_{i,\mu} \Sigma_\mu, \tag{14}$$

for a $(n-1) \times (n-1)$ matrix $(b_{i,\mu})$.

Proof. Assume that $k-1 = p^h = q$ is a power of the characteristic. Then,

$$\begin{aligned}
\sigma^*(f_i) &= \lambda_i \left(\sum_{\nu=1}^{n+1} a_{1,\nu} x_\nu\right)^{q+1} + \left(\sum_{\nu=1}^{n+1} a_{2,\nu} x_\nu\right)^{q+1} + \left(\sum_{\nu=1}^{n+1} a_{i+3,\nu} x_\nu\right)^{q+1} \\
&= \sum_{\nu,\mu=1}^{n+1} \left(\lambda_i a_{1,\nu} a_{1,\mu}^q + a_{2,\nu} a_{2,\mu}^q + a_{i+3,\nu} a_{i+3,\mu}^q\right) x_\nu x_\mu^q \\
&= \sum_{\nu,\mu=1}^{n+1} B_{\nu,\mu}^i(\sigma) x_\nu x_\mu^q.
\end{aligned}$$

Observe that by eq. (9) we have $B^i_{\nu,\mu} = 0$ for all $0 \leq i \leq n-2, 1 \leq \nu, \mu \leq n+1, n \neq \mu$.

The polynomials are in some sense "quadratic forms"

$$f_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{n+1})\Sigma_i \begin{pmatrix} x_1^q \\ x_2^q \\ \vdots \\ x_{n+1}^q \end{pmatrix}$$

so $\sigma^* f_i$ is computed as

$$\sigma^* f_i = (x_1, \ldots, x_{n+1}) A^t \Sigma_i A^q \begin{pmatrix} x_1^q \\ x_2^q \\ \vdots \\ x_{n+1}^q \end{pmatrix}$$

and the above expression should be a linear combination of f_i. The desired result follows. □

Remark 28. Matrices $A = (a_{ij})$ which satisfy eq. (14) should satisfy the following equations: For $0, \ldots, n-2$ and $1 \leq \nu, \mu \leq n+1$ we set

$$B^i_{\nu,\mu} = \lambda_i a_{1,\nu} a^q_{1,\mu} + a_{2,\nu} a^q_{2,\mu} + a_{i+3,\nu} a^q_{i+3,\mu}.$$

We have

$$B^i_{\nu,\mu} = 0 \text{ for } \nu \neq \mu.$$

Moreover the coefficients $b_{i,\mu}$ in eq. (14) satisfy the system

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_2 & \cdots & \lambda_{n-2} \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{i,1} \\ b_{i,2} \\ \vdots \\ \vdots \\ \vdots \\ b_{i,n-1} \end{pmatrix} = \begin{pmatrix} B^i_{1,1} \\ B^i_{2,2} \\ \vdots \\ \vdots \\ \vdots \\ B^i_{n+1,n+1} \end{pmatrix}$$

Which gives us that

$$b_{i,\nu} = B^i_{2+\nu,2+nu} = \lambda_i a^{q+1}_{1,2+\nu} + a^{q+1}_{2,2+\nu} + a^{q+1}_{i+3,2+\nu} \text{ for } 1 \leq \nu \leq n-1$$

plus the compatibility relations

$$\sum_{\nu=3}^{n+1} B^i_{\nu,\nu} = B^i_{2,2}$$

and

$$\sum_{\nu=3}^{n+1} \lambda_{\nu-3} B^i_{\nu,\nu} = B^i_{1,1}.$$

Solving these linear systems with $\lambda_1, \ldots, \lambda_{n-2}$ as parameters, seems a complicated problem, which is out of reach for now.

5. PROOF OF THEOREM 3

In this section, we assume $k, n \geq 2$ are integers so that $(n-1)(k-1) > 2$ and, for $p > 0$, we also assume that $(p, k) = 1$ and that $k-1$ not a power of p.

Set $F_{k,n} = C^k(\lambda_1, \ldots, \lambda_{n-2})$, where $\lambda_1, \ldots, \lambda_{n-2} \in K - \{0, 1\}$ are different.

As before, let $N(H_0) < \mathrm{Aut}(F_{k,n})$ be the normalizer of H_0 in the group $\mathrm{Aut}(F_{k,n})$.

Lemma 22 asserts that L, the group of linear automorphisms of $F_{k,n}$, is a normal subgroup of $\mathrm{Aut}(F_{k,n})$. Corollary 26 asserts that $L < N(H_0)$ and, since $H_0 < L$, that H_0 is a normal subgroup of L.

Remark 29. We may arrive to the same conclusion above using the theory of hyperosculating points under the condition $k^{n-1} < p$ or $\mathrm{char}(K) = 0$. Indeed, as a consequence of Remark 10 and Theorem 17, we have that L preserves the set of fixed points $F(H_0)$. This in particular asserts that if $\tau \in L$, then $\tau\varphi_j\tau^{-1} = \varphi_{\sigma(j)}$ for a suitable permutation σ of the set $\{0, 1, \ldots, n\}$; in particular, $\tau H_0 \tau^{-1} = H_0$. This asserts that $L < N(H_0)$.

Lemma 30. *Under the above assumptions, $N(H_0) = L$.*

Proof. As noted above (under the assumption that $k-1$ is not a power of p if $p > 0$), Corollary 26 asserts that $L < N(H_0)$. In [5] it was seen that $N(H_0) < \mathrm{PGL}_{n+1}(K)$ (in that article it was assumed that $K = \mathbb{C}$, but the general case is seen in the same way); obtaining that $N(H_0) < L$. $\square$

Lemma 31. *Under the above assumptions, H_0 is the unique generalized Fermat group of $F_{k,n}$ inside L.*

Proof. Let $H < L$ be another generalized Fermat group of type (k, n). The group H is generated by the elements φ_j^*, for $j = 0, \ldots, n$, so that the non-trivial elements of H acting with fixed points in $F_{k,n}$ are exactly the non-trivial powers of these generators and $\varphi_0^* \circ \varphi_1^* \circ \cdots \circ \varphi_n^* = 1$.

If the set of cyclic groups $\langle\varphi_0^*\rangle, \ldots, \langle\varphi_n^*\rangle$ coincides with the set of cyclic groups

$$\langle\varphi_0\rangle, \ldots, \langle\varphi_n\rangle,$$

then clearly $H_0 = H$.

So, let us assume, from now on, that the above is not the case.

Claim 1. *The set of cyclic groups $\langle\varphi_0^*\rangle, \ldots, \langle\varphi_n^*\rangle$ is not disjoint with the set of cyclic groups $\langle\varphi_0\rangle, \ldots, \langle\varphi_n\rangle$.*

Proof. Let us assume, by the contrary, that the set of cyclic groups $\langle\varphi_0^*\rangle, \ldots, \langle\varphi_n^*\rangle$ is disjoint with the set of cyclic groups $\langle\varphi_0\rangle, \ldots, \langle\varphi_n\rangle$. In this case, the group H descends under the quotient map π_0, defined in eq. (3), to a group of Möbius transformations that preserves the $n+1$ branch values ∞, 0, 1, λ_1,..., λ_{n-2}, and it is isomorphic to $\mathbb{Z}_k^t$, for some $t \geq 1$.

It is known that the finite abelian subgroups of Möbius transformations are either cyclic, isomorphic to $\mathbb{Z}_2^2$ or isomorphic to $\mathbb{Z}_p^t$, where p is the characteristic and $t \in \mathbb{N}$. The last case can not appear since $(k, p) = 1$.

Case 1. If $k \geq 3$, then $t = 1$ and $H \cap H_0 \cong \mathbb{Z}_k^{n-1}$. The cyclic group $\mathbb{Z}_k$ induced by H is generated by a Möbius transformation T that permutes the $n+1$ branch values and fixes no one. In particular, $n + 1 = rk$, for some positive integer r. It follows (see [5]) that each lifting of T (that is, the generators $\varphi_0^*, \ldots, \varphi_n^*$) is a linear transformation providing the same permutation (by conjugation action) of the generators $\varphi_0, \ldots, \varphi_n$, in r disjoint cycles of lenght k. Up to permutation of indices, we may assume that φ_0^* permutes cyclically the elements of each of the sets $\{\varphi_0, \varphi_1, \ldots, \varphi_{k-1}\}$, $\{\varphi_k, \varphi_{k+1}, \ldots, \varphi_{2k-1}\}$,..., $\{\varphi_{(r-1)k}, \varphi_{(r-1)k+1}, \ldots, \varphi_{rk-1}\}$. It follows that the maximal subgroup Q of H_0 formed by those elements that commute with φ_0^* is the one generated by the elements

$$\varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_{k-1}, \varphi_k \circ \varphi_{k+1} \circ \cdots \circ \varphi_{2k-1}, \ldots, \varphi_{(r-1)k} \circ \varphi_{(r-1)k+1} \circ \cdots \circ \varphi_{rk-1}.$$

Since the composition of all of the above elements equals the identity, $Q \cong \mathbb{Z}_k^{r-1}$.

Now, as φ_0^* must commute with each element of $H \cap H_0$, the $n-1$ generators of it must be each one invariant under conjugation by φ_0^*. As $H \cap H_0 < Q$, we must have $n \leq r$, a contradiction.

Case 2. If $k = 2$, then $t \in \{1, 2\}$. If $t = 1$, then we may proceed as in the above case to get a contradiction. If $t = 2$, then $H \cap H_0 \cong \mathbb{Z}_2^{n-2}$ and the group H induces a group of Möbius transformation isomorphic to $\mathbb{Z}_2^2$ that permutes the $n+1$ branch values and none of them is fixed by a non-trivial element. It follows that $n + 1 = 4r$, for some positive integer r.

In this case, after a permutation of the indices, we may assume that $\mathbb{Z}_2^2$ is generated by the induced elements of φ_0^* and φ_1^*. It follows that φ_i^* $(i = 0, 1)$ permutes (by conjugation action) the generators $\varphi_0, \ldots, \varphi_n$ in $2r$ disjoint cycles of lenght 2 each one. Up to a permutation of indices, we may assume that φ_0^* permutes cyclically the elements of each of the sets $\{\varphi_0, \varphi_1\}$, $\{\varphi_2, \varphi_3\}$,..., $\{\varphi_{n-1}, \varphi_n\}$. It follows that the maximal subgroup Q of H_0 formed by those elements that commute with φ_0^* is the one generated by the elements

$$\varphi_0 \circ \varphi_1, \varphi_2 \circ \varphi_3, \ldots, \varphi_{n-1} \circ \varphi_n,$$

that is, $Q \cong \mathbb{Z}_2^{2r-1}$. Since the subgroup of H_0 formed by those elements that commute with φ_0^* and with φ_1^* is a subgroup of Q, we must that that $H \cap H_0 < Q$, that is, $n-2 \leq 2r-1$. This obligates to have $r = 1$, in particular, that $n = 3$, a contradiction to the assumption that $(k-1)(n-1) > 2$. □

As a consequence of the above, the set of cyclic groups $\langle \varphi_0^* \rangle, \ldots, \langle \varphi_n^* \rangle$ is not disjoint with the set of cyclic groups $\langle \varphi_0 \rangle, \ldots, \langle \varphi_n \rangle$. We may assume, up to permutation of the indices, that $\langle \varphi_0 \rangle = \langle \varphi_0^* \rangle$. The underlying Riemann surface R of the quotient orbifold $(C(\lambda_1, \ldots, \lambda_{n-2})/\langle \varphi_0 \rangle$ is a generalized Fermat curve of type $(k, n-1)$ admiting two different generalized Fermat groups of type $(k, n-1)$; these being $H/\langle \varphi_0^* \rangle$ and the other being $H_0/\langle \varphi_0 \rangle$.

In the case that $K = \mathbb{C}$ we have the following. For $k = 2$ we have already proved the uniqueness (so normality) for $n = 4, 5$ in [1] and for $k \geq 3$, the uniqueness was obtained for $n = 3$ [4]. In this way, the above procedure asserts, by induction on n, the desired result in the zero characteristic situation.

The situation for general $p > 0$ can be done as follows. First, we know the uniqueness for $k \geq 4$ and $n = 2$ (as a consequence of the results in [19] and [14]); so again, by the induction process we are done for $k \geq 4$. The case $k = 2$ is ruled out because $1 = k - 1 = p^0$ and we are assuming that $k - 1$ is not be a power of p. In the case $k = 3$, we only need to check uniqueness for $n = 3$.

The case $(k, n) = (3, 3)$**.** In this case, our hypothesis are that $p \neq 2, 3$. Lemma 30 asserts that $H_0 \cong \mathbb{Z}_3^3$ is a normal subgroup of L and Lemma 22 asserts that L is a normal subgroup of $\mathrm{Aut}(F_{3,3})$. $W < L$ be the 3-Sylow subgroup of L containing H_0. If $W = H_0$, then the conditions of normality asserts the uniqueness. Let us now assume that $H_0 \neq W$. In this case, W/H_0 produces a 3-subgroup $G < \mathrm{PGL}_2(K)$ keeping invariant the set $\{\infty, 0, 1, \lambda_1\}$. The only possibility is to have $G \cong \mathbb{Z}_3$. Up to a transformation in $\mathrm{PGL}_2(K)$, we may assume that the generator T of G satisfies that $T(\infty) = 0$, $T(0) = 1$, $T(1) = \infty$ and $T(\lambda_1) = \lambda_1$. So, $T(x) = 1/(1 - x)$ and $\lambda_1^2 - \lambda_1 + 1 = 0$. In this case, the collection $\{\infty, 0, 1, \lambda_1\}$ is also invariant under the involutions $A(x) = \lambda_1/x$ and $B(x) = (x - \lambda_1)/(x - 1)$. The group generated by A and B is $\mathbb{Z}_2^2$. In fact, the group U generated by A and T is the alternating group $\mathscr{A}_4$ and it contains B. There are not more elements of $\mathrm{PGL}_2(K) - U$ keeping invariant the set $\{\infty, 0, 1, \lambda_1\}$; so $L/H_0 = U \cong \mathscr{A}_4$. This ensures that $|L| = 12 \times 3^3$ and also that H_0 is unique inside L (see [4, Cor. 6]). $\square$

Lemma 32. *Under the above assumptions,* $N(H_0) = \mathrm{Aut}(F_{k,n})$*, in particular, that* $\mathrm{Aut}(F_{k,n}) < \mathrm{PGL}_{n+1}(K)$*.*

Proof. Let $\tau \in \mathrm{Aut}(F_{k,n})$. Since L is a normal subgroup of the group $\mathrm{Aut}(F_{k,n})$ (see Lemma 22), then $H = \tau H_0 \tau^{-1}$ is a subgroup of L; again a generalized Fermat group of type (k, n). Since H_0 is the unique generalized Fermat group of type (k, n) inside L (see Lemma 31), we must have that $H = H_0$. $\square$

5.1. **Conclusion of the proof of Theorem 3.** Under our assumptions, H_0 is the unique generalized Fermat group of type (k, n) of $F_{k,n}$. In fact, since $L = N(H_0)$ (Lemma 30), $N(H_0) = \mathrm{Aut}(F_{k,n})$ (Lemma 32) and H_0 is the unique generalized Fermat group of type (k, n) inside L (Lemma 31), the desired uniqueness result follows.

The uniqueness ensures that $\mathrm{Aut}(F_{k,n}) = N(H_0)$. In [5] we obtain that $N(H_0)$ is a subgroup of $\mathrm{PGL}_{n+1}(K)$. Now Lemma 25 provides the last part of our theorem.

6. Proof of Theorem 9

Before to provide the proof of Theorem 9 lest provide some general facts on linear automorphisms in algebraic varieties.

Proposition 33. *Consider a complete intersection* $Y \subset \mathbb{P}^s$ *of projective hypersurfaces* Y_i *of degree* d_i *for* $i = 1, \ldots, r$*. The canonical sheaf* ω_Y *is given by*

$$\omega_Y = \mathscr{O}_Y\left(\sum_{i=1}^{r} d_i - s - 1\right).$$

Proof. [6, exer. 8.4 p. 188] $\square$

The curve $F_{k,n}$ is given as complete intersection of $n-1$ hypersurfaces of degree k. Therefore, we have the following

Corollary 34. *The canonical sheaf on the curves $F_{k,n}$ is given by*

$$\omega_{F_{k,n}} = \mathscr{O}_{F_{k,n}}\big((n-1)k-n-1\big) = \mathscr{O}_{F_{k,n}}\big((n-1)(k-1)-2\big).$$

Of course this is compatible with the genus computation given in eq. (1) since the degree of $\mathscr{O}_{F_{k,n}}(1)$ is k^{n-1}.

Proposition 35. *Let $i : X \hookrightarrow \mathbb{P}^s$ be a closed projective subvariety, such that the map*

$$H^0\big(\mathbb{P}^s, \mathscr{O}_{\mathbb{P}^s}(1)\big) \xrightarrow{i^*} H^0\big(X, \mathscr{O}_X(1)\big)$$

is an isomorphism. Every automorphism of X preserving $\mathscr{O}_X(1)$ can be extended to an automorphism of the ambient projective space, i.e. it is an element in $\mathrm{PGL}_{s+1}(K)$.

Proof. [11, prop. 2.1] □

We may try to prove that every automorphism is linear in the following way. Every automorphism σ of the curve $F_{k,n}$ should preserve the canonical sheaf so it should preserve $\mathscr{O}_{F_{k,n}}\big((n-1)(k-1)-2\big)$. Does it preserve $\mathscr{O}_{F_{k,n}}(1)$? This is certainly true if $\mathrm{Pic}(F_{k,n})$ has no torsion and it is the general way how one proves linearity in higher dimensional varieties. Unfortunately curves have torsion in their Picard group.

6.1. **Proof of linearity part of Theorem 9.** Let $D = \mathscr{O}_{F_{k,n}}(1)$. For every automorphism $\sigma \in \mathrm{Aut}(F_{k,n})$ we consider the difference $T_\sigma := \sigma(D) - D$. It is a divisor of degree 0, and the divisor $\big((n-1)(k-1)-2\big)T_\sigma$ is principal. Hence T_σ is a $\big((n-1)(k-1)-2\big)$-torsion point in the Jacobian of the curve $F_{k,n}$. The automorphism is linear if and only if T_σ is zero.

Lemma 36. *The map $\sigma \mapsto T_\sigma$ is a derivation, i.e.*

$$T_{\sigma\tau} = \sigma T_\tau + T_\sigma.$$

Proof. Observe that

$$T_{\sigma\tau} = \sigma\tau(D) - D = \sigma\tau(D) - \sigma(D) + \sigma(D) - D = \sigma(T_\tau) + T_\sigma.$$

□

Lemma 37. *The torsion points T_σ are H_0-invariant.*

Proof. Using Lemma 22 we find an $\ell \in L$ such that $h\sigma = \sigma\ell$. For all linear automorphisms ℓ and in particular for $\ell \in H_0$ we have $T_\ell = 0$. We now use the derivation rules:

$$T_{h\sigma} = hT_\sigma + T_h = hT_\sigma$$

and
$$T_{\sigma\ell} = \sigma T_\ell + T_\sigma = T_\sigma.$$
The desired result follows, since $T_{h\sigma} = T_{\sigma\ell}$. □

Consider the natural map $\pi : F_{k,n} \to F_{k,n}/H_0 \cong \mathbb{P}^1$. We have two maps induced on the Jacobians, namely
$$\pi_* : \mathrm{Jac}(F_{k,n}) \to \mathrm{Jac}(F_{k,n}/H_0)$$
$$\sum n_P P \mapsto \sum n_P \pi(P),$$
and
$$\pi^* : \mathrm{Jac}(F_{k,n}/H_0) \to \mathrm{Jac}(F_{k,n})$$
$$\sum n_Q Q \mapsto \sum n_Q \sum_{P \in \pi^{-1}(Q)} e(P/Q)P,$$
where $\sum n_P P$ (resp. $\sum n_Q Q$) is a divisor of degree 0 in $F_{k,n}$ (resp. $\mathbb{P}^1$) and $e(P/Q)$ denotes the ramification index of a point P lying above Q.

Observe that the map $\pi^* \circ \pi_* : \mathrm{Jac}(F_{k,n}) \to \mathrm{Jac}(F_{k,n})$ is given by sending a point $P \in \mathrm{Jac}(F_{k,n})$ to $\sum_{h \in H_0} P$. On the other hand side $\pi_* \circ \pi^*$ is the zero map since the Jacobian of the projective line is trivial.

This means that on the H_0-invariant points P_σ, multiplication by $|H_0| = k^n$ is zero. Since T_σ is an $\big((n-1)(k-1)-2\big)$-torsion point, if $(k, n+1) = 1$, then T_σ is zero and σ is linear.

6.2. **Proof of second part of Theorem 9.** Under the extra assumption that $(k, n+1) = 1$, we have seen in Section 6.1 that $L = \mathrm{Aut}(F_{k,n})$. Now Proposition 27 states the last part of our theorem.

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References

[1] A. Carocca, V. Gonzalez-Aguilera, R. A. Hidalgo and R. E. Rodríguez. Generalized Humbert curves. *Israel Journal of Mathematics* **64** No. 1 (2008), 165–192.
[2] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.
[3] D. Eisenbud. *Commutative algebra*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.
[4] Y. Fuertes, G. González-Diez, R. A. Hidalgo, and M. Leyton-Álvarez. Automorphisms group of generalized Fermat curves of type $(k, 3)$. *J. Pure Appl. Algebra*, 217(10):1791–1806, October 2013.
[5] G. González-Diez, R. A. Hidalgo, and M. Leyton-Álvarez. Generalized Fermat curves. *J. Algebra*, 321(6):1643–1660, 15 March 2009.
[6] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
[7] R. A. Hidalgo. Homology closed Riemann surfaces. *Quarterly Journal of Math.* **63** (2012), 931–952
[8] D. Goldschmidt. Algebraic functions and projective curves Graduate Texts in Mathematics, 215 Springer-Verlag 2013
[9] P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.
[10] Y. Ihara. Profinite braid groups, galois representations and complex multiplications. *Ann. Math.*, 121(2):351–376, March 1985.

[11] A. Kontogeorgis. Automorphisms of Fermat-like varieties. *Manuscripta Math.*, 107(2):187–205, February 2002.
[12] D. Laksov. Weierstrass points on curves *Astérisque* vol 87-88, 1981 pp. 221-247.
[13] D. Laksov. Wronskians and Plücker formulas for linear systems on curves. Ann. Sci. Ec. Norm. Super. (1984) at https://eudml.org/doc/82136
[14] H.-W. Leopoldt. Über die Automorphismengruppe des Fermatkörpers. *J. Number Theory*, 56(2):256–282, 1996.
[15] J. S. Milne. Algebraic Geometry (v6.00). *Available at www.jmilne.org/math/* , 2014.
[16] I. R. Shafarevich. Basic algebraic geometry. 1 (Varieties in projective space, Translated from the 1988 Russian edition and with notes by Miles Reid) Springer-Verlag 1994
[17] H. Stichtenoth. Algebraic function fields and codes Springer Verlag 1993
[18] K. O. Stöhr and J.F. Voloch. Weierstrass points and curves over finite fields, Proc. London Math. Soc. (3) 52 (1986), 119.
[19] P. Tzermias. The group of automorphisms of the Fermat curve. *J. Number Theory*, 53(1):173–178, 1995.
[20] R. C. Valentini and M. L. Madan. A hauptsatz of L. E. Dickson and Artin-Schreier extensions. *J. Reine Angew. Math.* 318, 156–177 (1980).

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD TÉCNICA FEDERICO SANTA MARÍA, VALPARAÍSO, CHILE
E-mail address: ruben.hidalgo@mat.utfsm.cl

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ATHENS, PANEPISTIMIOUPOLIS, 15784 ATHENS, GREECE
E-mail address: kontogar@math.uoa.gr

INSTITUTO DE MATEMÁTICA Y FÍSICA, UNIVERSIDAD DE TALCA, CAMINO LIRCAY S\N, CAMPUS NORTE, TALCA, CHILE
E-mail address: leyton@inst-mat.utalca.cl

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ATHENS, PANEPISTIMIOUPOLIS, 15784 ATHENS, GREECE
E-mail address: pan_param@math.uoa.gr