AUTOMORPHISMS AND THE CANONICAL IDEAL

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ABSTRACT. The automorphism group of a curve is studied from the viewpoint of the canonical embedding and Petri's theorem. A criterion for identifying the automorphism group as an algebraic subgroup the general linear group is given. Furthermore the action of the automorphism group is extended to an action of the minimal free resolution of the canonical ring of the curve X.

1. INTRODUCTION

Let X be a non-singular complete algebraic curve defined over an algebraically closed field of characteristic $p \geq 0$. If the genus g of the curve X is $g \geq 2$ then the automorphism group $G = \mathrm{Aut}(X)$ of the curve X is finite. The theory of automorphisms of curves is an interesting object of study, see the surveys [3], [7] and the references therein.

On the other hand the theory of syzygies which originates in the work of Hilbert and Sylvester has attracted a lot of researchers and it seems that a lot of geometric information can be found in the minimal free resolution of the ring of functions of an algebraic curve. For an introduction to this fascinating area we refer to [9].

In this article we aim to put together the theory of syzygies of the canonical embedding and the theory of automorphism of curves. For this we assume that X is a non-hyperelliptic, not trigonal and a non-singular quintic of genus 6 and we also assume $p \neq 2$. These conditions are needed for Petri's theorem to hold, while the $p \neq 2$ condition is needed to ensure the faithful action of the automorphism group on the space of holomorphic differentials $H^0(X, \Omega_X)$.

More precisely, in section 2.1 we use Petri's theorem in order to give a necessary and sufficient condition for an element in $\mathrm{GL}(H^0(X, \Omega_X))$ to act as an automorphism of our curve. In section 3 we show that the automorphism group G of the curve acts on a minimal free resolution $\mathbf{F}$ of the ring S_X. Notice that an action of a group G on a graded module M gives rise to a series of linear representations $\rho_d : G \to M_d$ to all linear spaces M_d of degree d for $d \in \mathbb{Z}$. For the case of the free modules F_i of the minimal free resolution $\mathbf{F}$ we relate the actions of the group G in both F_i and in the dual F_{g-2-i} in terms of an inner automorphism of G. This information is used in order to show that the action of the group G on generators of the modules F_i sends generators of degree d to the set of generators of degree d.

In the theory of syzygies in order to provide an invariant that does not depend on the selection of the minimal resolution, the $\mathrm{Tor}_i^S(k, S_X)$ is used, which is again a graded module. The degree d-part will be denoted by $\mathrm{Tor}_i^S(k, S_X)_d$, which is a vector space of dimension $b_{i,d}$ and we prove that it is also a G-module. Moreover, the representations to the d graded space of each F_i, $\rho_{i,d} : G \to \mathrm{GL}(F_{i,d})$ can be expressed as a direct sum of the G-modules $\mathrm{Tor}_i^S(k, S_X)_d$. We conclude by showing

that the G-module structure of all F_i is determined by knowledge of the G-module structure of $H^0(X, \Omega_X)$ and the G-module structure of each $\mathrm{Tor}_i^S(k, S_X)$ for all $0 \leq i \leq g - 2$.

2. Automorphisms of curves and Petri's theorem

Consider a complete non-singular non-hyperelliptic curve of genus $g \geq 3$ over an algebraically closed field K. Let Ω_X denote the sheaf of holomorphic differentials on X.

Theorem 1 (Noether-Enriques-Petri). *There is a short exact sequence*

$$0 \to I_X \to \mathrm{Sym} H^0(X, \Omega_X) \to \bigoplus_{n=0}^{\infty} H^0(X, \Omega_X^{\otimes n}) \to 0,$$

where I_X is genereted by elements of degree 2 and 3. Also if X is not a non-singular quintic of genus 6 or X is not a trigonal curve, then I_X is generated by elements of degree 2.

For a proof of this theorem we refer to [19], [12]. The ideal I_X is called *the canonical ideal* and it is the homogeneous ideal of the embedded curve $X \to \mathbb{P}_k^{g-1}$. The automorphism group of the ambient space $\mathbb{P}^{g-1}$ is known to be $\mathrm{PGL}_g(k)$, [13, example 7.1.1 p. 151]. On the other hand every automorphism of X is known to act on $H^0(X, \Omega_X)$ giving rise to a representation

$$\rho : G \to \mathrm{GL}(H^0(X, \Omega_X)),$$

which is known to be faithful, when X is not hyperelliptic and $p \neq 2$, see [14]. The representation ρ in turn gives rise to a series of representations

$$\rho_d : G \to \mathrm{GL}(S_d),$$

where S_d is the vector space of degree d polynomials in the ring $S := k[\omega_1, \ldots, \omega_g]$.

Let $X \subset \mathbb{P}^r$ be a projective algebraic set. Is it true that every automorphism $\sigma : X \to X$ comes as the restriction of an automorphism of the ambient projective space, that is by an element of $\mathrm{PGL}_k(r)$. For instance such a criterion for complete intersections is explained in [15, sec. 2]. In the case of canonical embedded curves $X \subset \mathbb{P}^{g-1}$ it is clear that any automorphism $\sigma \in \mathrm{Aut}(X)$ acts also in $\mathbb{P}^{g-1} = \mathrm{Proj} H^0(X, \Omega_X)$. In this way we arrive to the following

Lemma 2. *Every automorphism $\sigma \in \mathrm{Aut}(X)$ corresponds to an element in $\mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$ and every element in $\mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$ gives rise to an automorphism of X.*

In the next section we will describe the elements $\sigma \in \mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$.

2.1. **Algebraic equations of automorphisms.** For now on we will assume that the canonical ideal I_X is generated by polynomials in $k[\omega_1, \ldots, \omega_g] = \mathrm{Sym}H^0(X, \Omega_X)$ of degree 2. Consider such a set of quadratic polynomials $\tilde{A}_1, \ldots, \tilde{A}_r$ generating I_X.

A polynomial of degree two $\tilde{A}_i$ can be encoded in terms of a symmetric $g \times g$ matrix $A_i = (a_{\nu,\mu})$ as follows. Set $\bar{\omega} = (\omega_1, \ldots, \omega_g)^t$. We have

$$\tilde{A}_i(\bar{\omega}) = \bar{\omega}^t A_i \bar{\omega}.$$

Write $\sigma(A_i)$ for the symmetric $g \times g$ matrix such that $\sigma(\tilde{A}_i) = \bar{\omega}^t \sigma(A)_i \bar{\omega}$.

Lemma 3. *For an element $\sigma \in \mathrm{GL}_g(k)$, $\sigma(I_X) \subset I_X$ holds if and only if for all $1 \leq i \leq r$, $\sigma(A_i) \in \mathrm{span}_k\{A_1, \ldots, A_r\}$.*

Proof. ($\Leftarrow$) If $\sigma(A_i) = \sum_j \lambda_j^i A_j$, then

$$\bar{\omega}^t \sigma^t A_i \sigma \bar{\omega} = \sum_j \lambda_j^i \bar{\omega}^t A_j \bar{\omega} \Rightarrow$$

$$(\sigma\bar{\omega})^t A_i (\sigma\bar{\omega}) = \sum_j \lambda_j^i \tilde{A}_j$$

Hence $\sigma(\tilde{A}_i) = \sigma(\bar{\omega}^t A_i \bar{\omega}) \in I_X$.
($\Rightarrow$) If $\sigma(I_X) \subseteq I_X$, there are $g_j^i(X) \in S$ such that

$$\sigma(\tilde{A}_i) = \sum_j g_j^i(X)\tilde{A}_j.$$

Since σ respects the degree of the polynomials, $\deg(g_j^i(X)) = 0$ for all i, hence $g_j^i(X) = \lambda_j^i \in k$.

$$\bar{\omega}^t \sigma(A_i)\bar{\omega} = \sigma(\bar{\omega}^t A_i \bar{\omega}) = \sigma(\tilde{A}_i) =$$

$$= \sum_j \lambda_j^i \tilde{A}_j = \sum_j \lambda_j^i \left(\bar{\omega}^t A_j \bar{\omega}\right) = \bar{\omega}^t \left(\sum_j \lambda_j^i A_j\right)\bar{\omega}$$

Since the matrices are symmetric it follows that $\sigma(A_i) \in \mathrm{span}_k\{\tilde{A}_1, \ldots, \tilde{A}_r\}$. $\quad\square$

The above lemma gives rise to a new representation

Proposition 4. *Let $\sigma = (\sigma_{ij}) \in \mathrm{GL}_g(k)$. The element σ gives rise to an automorphism of the curve with canonical ideal generated by $\tilde{A}_i, i = 1, \ldots, r$ if and only*

$$(1) \qquad\qquad \sigma\left(\tilde{A}_i\right) = \sum_{j=1}^r \lambda(\sigma)_{ji}\tilde{A}_j.$$

On the other hand σ acts on $\bar{\omega}$ by

$$\sigma(\bar{\omega}) = (\sigma_{\mu,\nu})\bar{\omega}$$

therefore eq. (1) can be written as

$$(\sigma_{\mu,\nu})^t A_i(\sigma_{\mu,\nu}) = \sum_{j=1}^r \lambda(\sigma)_{ji} A_j \qquad \text{for every } 1 \leq i \leq j.$$

Finding $g \times g$ matrices $(\sigma_{\mu,\nu})$ and elements $(\lambda_{i,j})$ so that the above equation holds will be called the *matrix automorphism problem*.

2.2. **The automorphism group as an algebraic set.** Consider a set $A_1, \ldots, A_r$ of linear independent elements in the vector space of $g \times g$ symmetric matrices. For each invertible matrix $\sigma = (\sigma_{ij}) \in \mathrm{GL}_g(k)$ we consider the r-elements $\sigma^t A_i \sigma$, $1 \leq i \leq r$. Fixing a basis for the space of $g \times g$ symmetric matrices we can write any element A_i as $g(g+1)/2 \times 1$ column matrix $\bar{A}_i$, that is

$$\bar{\cdot} : \text{Symmetric } g \times g \text{ matrices} \longrightarrow k^{\frac{g(g+1)}{2}}$$
$$A \longmapsto \bar{A}$$

We can now put together the r elements $\bar{A}_i$ as a $g(g+1)/2 \times r$ matrix $(\bar{A}_1 | \cdots | \bar{A}_r)$, which has full rank r, since $\{A_1, \ldots, A_r\}$ are assumed to be linear independent.

We have that σ is an automorphism if the $g(g+1)/2 \times 2r$-matrix

$$B(\sigma) = [\bar{A}_1, \ldots, \bar{A}_r, \overline{\sigma^t A_1 \sigma}, \ldots, \overline{\sigma^t A_r \sigma}]$$

has rank r, which means that $(r+1) \times (r+1)$-minors of $B(\sigma)$ are zero. This provides us with a description of the automorphism group as a determinantal variety.

A simpler way to obtain algebraic equations, in order to see the automorphism curve as a finite algebraic group in $\mathrm{GL}_g(k)$ is using Gauss elimination to find a $\frac{g(g+1)}{2} \times \frac{g(g+1)}{2}$ invertible matrix Q which puts the matrix $(\bar{A}_1 | \cdots | \bar{A}_r)$ in echelon form, that is

$$Q \left(\bar{A}_1 | \cdots | \bar{A}_r \right) = \begin{pmatrix} \mathbb{I}_r \\ \hline \mathbb{O}_{\left(\frac{g(g+1)}{2} - r \right) \times r} \end{pmatrix}.$$

But then for each $1 \leq i \leq r$ we have

$$\overline{\sigma^t A_i \sigma} = \sum_{j=1}^{r} \lambda_{ji} \bar{A}_i$$

if and only if the lower $(\frac{g(g+1)}{2} - r) \times r$ bottom block matrix of the matrix

$$(2) \qquad \qquad \left(\overline{\sigma^t A_1 \sigma}, \ldots, \overline{\sigma^t A_r \sigma} \right)$$

is zero, while the top $r \times r$ block matrix gives rise to the representation

$$\rho_1 : G \to \mathrm{GL}_r(k),$$

defined in proposition 4. Assuming that the lower $(\frac{g(g+1)}{2} - r) \times r$ bottom block matrix gives us $r(\frac{g(g+1)}{2} - r)$ equations where the entries $\sigma = (x_{ij})$ are seen as indeterminates. In this way we can write down elements of the automorphism group as a zero dimensional algebraic set, satisfying certain algebraic equations.

2.2.1. *An example: Fermat curve.* Consider the projective non singular curve given by equation

$$F_n : x_1^n + x_2^n + x_0^n = 0$$

This curve has genus $g = \frac{(n-2)(n-1)}{2}$. Set $x = x_1/x_0$, $y = x_2/x_0$. For $\omega = \frac{dx}{y^{n-1}} = -\frac{dy}{x^{n-1}}$ we have that the set

$$(3) \qquad \qquad x^i y^j \omega \text{ for } 0 \leq i + j \leq n - 3$$

forms a basis for holomorphic differentials, [16], [20], [21]. These g differentials are ordered lexicographically according to (i,j), that is

$$\omega_{0,0} < \omega_{0,1} < \cdots < \omega_{0,n-3} < \omega_{1,0} < \omega_{1,1} < \cdots < \omega_{1,n-4} < \cdots < \omega_{n-3,0}.$$

The case $n = 2$ is a rational curve, the case $n = 3$ is an elliptic curve, the case $n = 4$ has genus 3 and gonality 3, the case $n = 5$ has genus 6 and is quintic so the first Fermat curve which has canonical ideal generated by quadratic polynomial is the case $n = 6$ which has genus 10. The appendix is devoted to the proof of the following

Proposition 5. *The canonical ideal of the Fermat curve F_n consists of two sets of relations*

$$(4) \qquad G_1 = \{\omega_{i_1,j_1}\omega_{i_2,j_2} - \omega_{i_3,j_3}\omega_{i_4,j_4} : i_1 + i_2 = i_3 + i_4, j_1 + j_2 = j_3 + j_4\},$$

and

$$(5) \qquad G_2 = \left\{\omega_{i_1,j_1}\omega_{i_2,j_2} + \omega_{i_3,j_3}\omega_{i_4,j_4} + \omega_{i_5,j_5}\omega_{i_6,j_6} = 0 : \begin{smallmatrix} i_1+i_2=n+a, & j_1+j_2=b \\ i_3+i_4=a, & j_3+j_4=n+b \\ i_5+i_6=a, & j_5+j_6=b \end{smallmatrix} \right\}$$

where $0 \le a,b$ are selected such that $0 \le a + b \le n - 3$.

2.2.2. *Automorphisms of the Fermat curve.* The group of automorphisms of the Fermat curve in is given by [22],[17]

$$G = \begin{cases} \mathrm{PGU}(3, p^h), & \text{if } n = 1 + p^h \\ (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \rtimes S_3, & \text{otherwise} \end{cases}$$

The action of the automorphism group is given in terms of a 3×3 matrix A sending

$$x = (x_1/x_0) \mapsto \frac{\sum_{i=0}^2 a_{1,i}x_i}{\sum_{i=0}^2 a_{0,i}x_i} \qquad y = (x_2/x_0) \mapsto \frac{\sum_{i=0}^2 a_{2,i}x_i}{\sum_{i=0}^2 a_{0,i}x_i},$$

In characteristic 0, the matrix A is a monomial matrix, that is, it has only one non-zero element in each row and column and this element is an n-th root of unity. Two matrices A_1, A_2 give rise to the same automorphism if and only if they differ by an element in the group $\{\lambda \mathbb{I}_3 : \lambda \in k\}$. In any case the group G is naturally a subgroup of $\mathrm{PGL}_3(k)$. Finding the representation matrix of G as an element in $\mathrm{PGL}_{g-1}(k)$ is easy when $n \ne 1 + p^h$ and more complicated in $n = 1 + p^h$ case. We have two different embeddings of the Fermat curve F_n in projective space

$$\mathbb{P}_k^{g-1} \longleftarrow F_n \longrightarrow \mathbb{P}_k^2.$$

In both cases the automorphism group is given as restriction of the automorphism group of the ambient space.

The computation of the automorphism group in terms of the vanishing of the polynomials given in equation (2) is quite complicated. The following program gives an idea of the complexity of this approach. The automorphism group for the $n = 6$ case is described as an algebraic set described by $g^2 = 100$ variables and 756 equations.

```
1    FERMATCURVE(6, RATIONALS());
2    x_{7,8}*x_{10,10} − 2*x_{9,8}*x_{9,10} + x_{10,8}*x_{7,10},
3    ................756 equations....................
4    >x_{7,9}*x_{10,10} − 2*x_{9,9}*x_{9,10} + x_{10,9}*x_{7,10}
```

3. Syzygies

3.1. **Free resolutions.** Recall that $S = k[\omega_1, \ldots, \omega_g]$ is the polynomial ring in g variables. Let M be a graded S-module generated by the elements $m_1, \ldots, m_r$ of corresponding degrees $a_1, \ldots, a_r$. We consider the free S-module $F_0 = \bigoplus_{j=1}^r S(-a_j)$ together with the onto map

$$F_0 = \bigoplus_j S(-a_j) \xrightarrow{\pi} M.$$

Let us denote by $M_1, \ldots, M_r$ elements of F_0, such that $\pi(M_i) = m_i$, assuming also that $\deg(M_i) = \deg(m_i)$, for $1 \leq i \leq r$. In the above formula we have used the isomorphism

$$
(6) \qquad \bigoplus_{j=1}^r S(-a_j) \longrightarrow \bigoplus_{j=1}^r M_i S
$$

$$
(s_1, \ldots, s_r) \longmapsto \sum_{j=1}^s s_i M_i.
$$

We introduced the shifts in the grading so that $\deg_{S(-a_j)} s_j = \deg_M(s_j M_j)$, that is $\deg_{S(-a_j)} s_j = \deg_S(s_j) + \deg(M_j) = \deg_S(s_j) + a_j$.

The kernel of the map π is again a finite generated S-module and by continuing this operation we arrive at a free resolution of M, that is, at a sequence of free S-modules

$$
(7) \qquad 0 \longrightarrow F_g \xrightarrow{\phi_g} \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \ ,
$$

where $\operatorname{coker}(\phi_1) = F_0/\operatorname{Im}\phi_1 = F_0/\ker\pi \cong M$.

Let $\mathfrak{m}$ be the maximal ideal of S generated by $\langle \omega_1, \ldots, \omega_g \rangle$. The pair $(S, \mathfrak{m})$ behaves in many aspects like a local ring, see [11]. The graded free resolution in eq. (7) is called minimal if for each i the image of ϕ_i is contained in $\mathfrak{m}F_{i-1}$ or equivalently if the reduced maps $\bar{\phi}_i : F_i/\mathfrak{m}F_i \to F_{i-1}/\mathfrak{m}F_{i-1}$ are zero. From now on all free resolutions, we will use, are assumed to be minimal.

Each free module in the resolution can be written as

$$F_i = \bigoplus_j S(-j)^{\beta_{i,j}}.$$

The integers $\beta_{i,j}$ are called the Betti numbers of the resolution. The Betti diagram of the free resolution is given by the following table:

	0	1	$\cdots$	r
i	$\beta_{0,i}$	$\beta_{1,i+1}$	$\cdots$	$\beta_{r,i+r},$
$i+1$	$\beta_{0,i+1}$	$\beta_{1,i+2}$	$\cdots$	$\beta_{r,i+r+1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
j	$\beta_{0,j}$	$\beta_{1,j+1}$	$\cdots$	$\beta_{r,j+r}$

For example the Fermat curve for $n = 6$, that is $g = 10$ gives rise to the following Betti diagram, as we can compute using the magma algebra system, [6]:

	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	28	105	189	189	105	27	0	0
2	0	0	27	105	189	189	105	28	0
3	0	0	0	0	0	0	0	0	1

We observe that the above table is highly symmetrical and this is not a coincidence. Observe also that only the first three rows are non zero and this is due to a general result stating that the Casteluovo-Mumford regularity for the ring S/I_X is three, see [9, 9B]. Also $28 = \beta_{1,2} = \binom{g-2}{2}$. The evident symmetry in the above matrix (top to down, left to right) is part of a general theory as we will see in section 3.4. We will introduce group actions on free resolutions first.

3.2. **Extending group actions.** Let M be a finitely generated graded S-module acted on by the group G. Let $m_1, \ldots, m_r$ be minimal generators of M of corresponding degrees a_i. Every element $\sigma \in G$ sends every generator M_i of the free module F_0

$$(8) \qquad \sigma(M_i) = \sum_{\nu=1}^{r} a_{\nu,i} M_i, \text{ for some } a_{\nu,i} \in S.$$

Remark 6. We would like to point out here that unlike the theory of vector spaces, an element $x \in F_0$ might admit two different decompositions

$$x = \sum_{i=1}^{r} a_i m_i = \sum_{i=1}^{r} b_i m_i,$$

that is

$$\sum_{i=1}^{r} (a_i - b_i) m_i = 0,$$

and if $a_{i_0} - b_{i_0} \neq 0$ we cannot assume that it is invertible, so we can't express M_{i_0} as an S-linear combination of the other elements M_i, for $i_0 \neq i, 1 \leq i \leq r$. We can only deduce that $\{a_i - b_i\}_{i=1,\ldots,r}$ form a syzygy.

Therefore one might ask if the matrix expression given in eq. (8) is unique. In proposition 12 we will prove that the elements $a_{\nu,i}$ which appear as coefficients in eq. (8) are in the field k and therefore the expression is indeed unique.

Observe that we have extended the natural action of $\mathrm{Aut}(X)$ on $H^0(X, \Omega_X)$ to an action on the ring $S = \mathrm{Sym} H^0(X, \Omega_X)$, so that $\sigma(xy) = \sigma(x)\sigma(y)$. Therefore if $M = I_X$ then for all $s \in S$, $m \in I_X = M$ we have $\sigma(sm) = \sigma(s)\sigma(m)$. All the actions in the modules we will consider will have this property.

We can extend the action of G on the free S-module

$$F_0 = M_1 S \oplus \cdots \oplus M_r S$$

by the rule $\sigma(sM_i) = \sigma(s)\sigma(M_i)$. As we did in the isomorphism in eq. (6) we would like to transfer the action to the direct sum of the shifted ring $\bigoplus_{j=1}^{r} S(-a_j)$. Observe that

$$\sigma\left(\sum_{j=1}^{r} s_j M_j\right) = \sum_{j=1}^{r} \sigma(s_j) \sum_{\nu=1}^{r} a_{\nu,j}(\sigma) M_\nu = \sum_{\nu=1}^{r} \left(\sum_{j=1}^{r} a_{\nu,j}(\sigma)\sigma(s_j)\right) M_\nu,$$

where $\deg_S a_{\nu,j} + a_\nu = \deg_S m_j$. This means that under the action of $\sigma \in G$ the r-tuple $(s_1, \ldots, s_r)^t$ is sent to

$$
\begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix} \xmapsto{\sigma} \begin{pmatrix} a_{1,1}(\sigma) & a_{1,2}(\sigma) & \cdots & a_{1,r}(\sigma) \\ \vdots & \vdots & & \vdots \\ a_{r,1}(\sigma) & a_{r,2}(\sigma) & \cdots & a_{r,r}(\sigma) \end{pmatrix} \begin{pmatrix} \sigma(s_1) \\ \vdots \\ \sigma(s_r) \end{pmatrix}.
$$

If $A(\sigma) = \big(a_{i,j}(\sigma)\big)$ is the matrix corresponding to σ then for $\sigma, \tau \in G$ the following cocycle condition holds:

$$
A(\sigma\tau) = A(\sigma)A(\tau)^\sigma.
$$

If we can assume that G acts trivially on the matrix $A(\tau)$ for every $\tau \in G$ (for instance when $A(\tau)$ is a matrix with entries in k for every $\tau \in G$), then the above cocycle condition becomes a homomorphism condition.

Also if $A(\sigma)$ is a principal derivation, that is there is an $r \times r$ matrix Q, such that

$$
A(\sigma) = \sigma(Q) \cdot Q^{-1}
$$

then after a basis change of the generators we can show that the action on the coordinates is just given by

$$
(s_1, \cdots, s_r)^t \xmapsto{\sigma} (\sigma(s_1), \cdots, \sigma(s_r))^t,
$$

that is the matrix $A(\sigma)$ is the identity. We will call the action on the free resolution $\mathbf{F}$ obtained by extending the action on M the standard action.

3.3. **Group actions on free resolutions.** Assume that M is acted on by a group G and let

$$
(9) \qquad 0 \to F_n \xrightarrow{\delta_n} F_{n-1} \to \cdots F_i \xrightarrow{\delta_i} F_{i-1} \to \cdots F_1 \to M \to 0
$$

be a minimal free resolution, that is each F_i is a free graded S-module and the image of each δ_i is in $\mathfrak{m}F_{i-1}$, or equivalently δ_i maps the basis of F_i to a minimal set of generators of the image of δ_i, see [9, sec. 1B]. The image of each δ_i is the i-th module of syzygies.

Assume that each F_i is acted on by G and that the maps δ_i are G-equivariant. Then a series of representations can be defined:

$$
(10) \qquad \rho_i : G \to \mathrm{GL}(\mathrm{Tor}_i^S(k, M)).
$$

Indeed, from the resolution given in eq. (7) we can consider the complex

$$
(11) \qquad \cdots \to k \otimes F_i \xrightarrow{1_k \otimes \delta_i} k \otimes F_{i-1} \to \cdots
$$

Since the δ_i maps are zero modulo $\mathfrak{m}$ all maps in the above complex are zero. On the other hand the homology of the complex (11) is by definition $\mathrm{Tor}_i^S(k, M)$ so $\mathrm{Tor}_i^S(k, M) = k \otimes F_i$.

We will now study the action of the group G on the generators of F_i. First of all we have that

$$
F_i = \bigoplus_{\nu=1}^{r_i} \bigoplus_{\mu=1}^{b_{i,\nu}} e_{i,\nu,\mu} S \cong \bigoplus_{\nu=1}^{r_i} S(-d_{i,\nu})^{b_{i,\nu}}.
$$

In the above formula we assumed that F_i is generated by elements $e_{i,\nu,\mu}$ such that the degree of $e_{i,\nu,\mu} = d_{i,\nu}$ for all $1 \le \mu \le b_{i,\nu}$. We also assume that

$$
d_{i,1} < d_{i,2} < \cdots < d_{i,r_i}.
$$

The action of σ is respecting the degrees, so an element of minimal degree $d_{i,1}$ is sent to a linear combination of elements of minimal degree $d_{i,1}$. In this way we obtain a representation
$$\rho_{i,1} : G \to \mathrm{GL}(b_{i,1}, k).$$
In a similar way an element $e_{i,2,\mu}$ of degree $d_{i,2}$ is sent to an element of degree $d_{i,2}$ and we have that
$$\sigma(e_{i,2,\mu}) = \sum_{j_1=1}^{b_{i,2}} \lambda_{i,2,\mu,j_1} e_{i,2,j_1} + \sum_{j_2=1}^{b_{i,1}} \lambda'_{i,2,\mu,j_1} e_{i,1,j_2},$$
where all $\lambda_{i,2,\mu,j_1} \in k$ and all $\lambda'_{i,1,\mu,j_2} \in S/\mathfrak{m}^{d_{i,2}-d_{i,1}+1}$. In this case we have a representation with entries in an Artin algebra instead of a field, which has the form:
$$\rho_{i,2} : G \to \mathrm{GL}(b_{i,1} + b_{i,2}, S/\mathfrak{m}^{d_{i,2}-d_{i,1}+1}),$$
$$\sigma \mapsto \begin{pmatrix} A_1(\sigma) & A_{1,2}(\sigma) \\ 0 & A_2(\sigma) \end{pmatrix},$$
where $A_1(\sigma) \in \mathrm{GL}(b_{i,1}, k)$ and $A_2(\sigma) \in \mathfrak{m}^{d_{i,2}-di,1}\mathrm{GL}(b_{i,2}, k)$.

By induction the situation in the general setting gives rise to a series of nested representations:
$$\rho_{i,j} : G \to \mathrm{GL}(b_{i,1} + b_{i,2}, S/\mathfrak{m}^{d_{i,j}-d_{i,1}+1})$$

(12)
$$\sigma \mapsto A(\sigma) = \begin{pmatrix} A_1(\sigma) & A_{1,2}(\sigma) & \cdots & A_{1,j}(\sigma) \\ 0 & A_2(\sigma) & & A_{2,j}(\sigma) \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & A_j(\sigma) \end{pmatrix}$$

where $A_\nu(\sigma) \in \mathrm{GL}(b_{i,\nu}, k)$ and $A_{\kappa,\lambda}(\sigma)$ is an $b_{i,\kappa} \times b_{i,\lambda}$ matrix with coefficients in $\mathfrak{m}^{b_{i,\lambda}-b_{i,\kappa}}$. The representation ρ_{i,r_i} taken modulo $\mathfrak{m}$ reduces to $\mathrm{Tor}_i^S(k, M)$, seen as a $k[G]$-module.

Remark 7. Using the same construction as in the Maschke theorem, see [1, p.13] we can prove that this representation is equivalent to a block diagonal representation, if the characteristic $p \nmid |G|$. We will see in proposition 12 that under mild hypotheses this requirement can be removed.

3.4. **Gorenstein symmetry.** For the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$ we have $r = g - 1$, therefore $S(-r - 1) = S(-g)$. The S-module $S(-r - 1)$ is special, for instance since X is smooth we have by [2, th. 3.1] that the sheaf corresponding to the module $S(-g)$ is canonically isomorphic to $\wedge^{g-1}\Omega_X^1$ thus it is naturally acted by the automorphism group of X.

If **F** is a free resolution
$$\mathbf{F} : \quad 0 \longrightarrow F_r \xrightarrow{\phi_r} \cdots \xrightarrow{\phi_{i+1}} F_i \xrightarrow{\phi_i} F_{i-1} \xrightarrow{\phi_{i-1}} \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0$$

of S/I_X then we twist it by $S(-r - 1)$ in order to obtain a new free resolution $\mathrm{Hom}_S(F_i, S(-g))$ of S/I_X:
$$\mathbf{F}^* : \quad 0 \longleftarrow F_r \xleftarrow{\phi_r^*} \cdots \xleftarrow{\phi_{i+1}^*} F_i \xleftarrow{\phi_i^*} F_{i-1} \xleftarrow{\phi_{i-1}^*} \cdots \longleftarrow F_1 \xleftarrow{\phi_1^*} F_0$$

The ring $S_X := S/I_X$ is Cohen-Macauley of codimension $r - 1 = g - 2$ therefore
$$\mathrm{Ext}_S^i(S_X, S(-g)) = 0 \text{ for } i \neq g - 2.$$

This means that $\mathbf{F}$ is a free resolution of $\mathrm{Ext}_S^{g-2}(S_X, S(-g))$, which in turn is isomorphic to $S_X(1)$, see [9, prop. 9.5].

Lemma 8. *We have*
$$\mathrm{Hom}_S(S(-a), S(-b)) \cong S(-b+a).$$

Proof. There are elements e_a, e_b of degrees a, b respectively such that $e_a S \cong S(-a)$ and $e_b S \cong S(-b)$, where the S-module isomorphisms are given by sending $ef \mapsto f$. This means that $\deg_{e_a S}(e_a f) = \deg_{S(-a)} f$ and $\deg_{e_b S}(e_b f) = \deg_{S(-b)} f$. In particular $\deg_{S(-a)} 1_S = a$ or in other words $S(-a)_d = S_{-a+d}$, see also [18, prop. 2.3].

The set $\mathrm{Hom}_S(N, M)$ where N, M are graded S-modules with N finitely generated equals $\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_i(N, M)$, where $\mathrm{Hom}_i(N, M)$ consists of all morphisms of degree i, see [18, prop. 2.7].

A homomorphism ϕ satisfies $\phi(e_a f) = f \phi(e_a)$ and is described completely by $\phi(e_a) \in e_b S$. If ϕ is of degree 0 we furthermore have $\phi(e_a) = e_b F_\phi$, where the degree of F_ϕ satisfies the equation $a = b + \deg_S(F_\phi)$, that is $\deg(F_\phi) = a - b$. This proves that $\mathrm{Hom}_0(S(-a), S(-b)) \cong S_{a-b}$, thus $\mathrm{Hom}_S(S(-a), S(-b)) \cong S(-b+a)$. $\qquad \square$

Corollary 9. *For $b = g$ we have*
$$\mathrm{Hom}_S(S(-a), S(-g)) = S(-g+a)$$

Since F^* is a free resolution of $S_X(1)$ (shifted by one) we arrive at the symmetry

(13) $$\beta_{i,j} = \beta_{g-2-i, g+1-j}.$$

This symmetry can also be interpreted in terms of of Koszul cohomology, see [10, prop. 4.1]. In general the Betti table for a canonical model is given by

	0	1	$\cdots$	a	$a+1$	$\cdots$	$b-1$	b	$\cdots$	$g-3$	$g-2$
0	1	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	0
1	0	β_1	$\cdots$	β_a	β_{a+1}	$\cdots$	β_{g-3-a}	0	$\cdots$	0	0
2	0	0	$\cdots$	0	β_{g-3-a}	$\cdots$	β_{a+1}	β_a	$\cdots$	β_1	0
3	0	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	1

In the above table the i, j-entry corresponds to $\beta_{i,i+j}$. The Green conjecture states that the integer a is equal to $\mathrm{Cliff}(X) - 1$. Notice that the Green conjecture is known to fail in positive characteristic, see [5].

3.5. **Unique actions.** Let us consider two actions of the automorphisms group G on $H^0(X, \Omega_X)$, which can naturally be extended on the symmetric algebra $\mathrm{Sym} H^0(X, \Omega_X)$. We will denote the first action by $g \star v$ and the second action by $g \circ v$, where $g \in G$, $v \in \mathrm{Sym} H^0(X, \Omega_X)$.

Proposition 10. *If the curve X satisfies the conditions of faithful action of $G = \mathrm{Aut}(X)$ on $H^0(X, \Omega_X)$, that is X is not hyperelliptic and $p > 2$, [14, th. 3.2] and moreover both actions $\star, \circ$ restrict to actions on the canonical ideal I_X, then there is an automorphism $i : G \to G$, such that $g \star v = i(g) \circ v$.*

Proof. Both actions of G on $H^0(X, \Omega_X)$ introduce automorphisms of the curve X. That is since $G \star I_X = I_X$ and $G \circ I_X = I_X$, the group G is mapped into $\mathrm{Aut}(X) = G$. This means that for every element $g \in G$ there is an element $g^* \in \mathrm{Aut}(X) = G$ such that $g \star v = g^* v$, where the action on the right is the standard action of the automorphism group on holomorphic differentials. By the definition of the group

action for every $g_1, g_2 \in G$ we have $(g_1 g_2)^* v = g_1^* g_2^* v$ for all $v \in H^0(X, \omega_X)$ and the faithful action of the automorphism group provides us with $(g_1 g_2)^* = g_1^* g_2^*$, i.e. the map $i_* : g \mapsto g^*$ is a homomorphism. Similarly the map corresponding to the ∘-action, $i_\circ : g \mapsto g^\circ$ is a homomorphism and the desired homomorphism i is the composition of $i_* i_\circ^{-1}$. $\square$

So far we have introduced the map $\mathrm{Hom}_S(F_i, S(-g))$ which induces a symmetry of the free resolution $\mathbf{F}$ by sending F_i to F_{g-2-i}. Then each free module F_i of the resolution $\mathbf{F}$ is equipped by the extension of the action on holomorphic differentials, according to the construction of section 3.3. On the other hand since $S(-g)$ is a G-module we have that $F_{g-2-i} \cong \mathrm{Hom}_S(F_i, S(-g))$ is equipped by a second action namely every $\phi : F_i \to S(-g)$ is acted naturally by G in terms of $\phi \mapsto \phi^\sigma = \sigma^{-1} \phi \sigma$. How are the two actions related?

Lemma 11. *Denote by $\star$ the action of G on F_i induced by taking the $S(-g)$-dual. The standard and the $\star$-actions are connected in terms of an automorphism ψ_i of G, that is for all $v \in F_i$ $g \star v = \psi_i(g)v$.*

Proof. Assume that $i \leq g - 2 - i$. Consider the standard action of G on the free resolution $\mathbf{F}$. The module F_{g-2-i} obtains a new action $g \star v$ for $g \in G, v \in F_i$. By 3.3 this $\star$ action is transfered to an action on all F_j for $j \geq g - 2 - i$, including the final term F_{g-2} which is isomorphic to $S(-1)$. This gives us two actions on $H^0(X, \Omega_X)$ which satisfy the requirements of proposition 10. The desired result follows. $\square$

Proposition 12. *Under the faithful action requirement we have that all automorphisms $\sigma \in G$ send the direct summand $S(-j)^{\beta_{i,j}}$ of F_i to itself, that is the representation matrix in eq. (12) is block diagonal.*

Proof. Consider $F_i = \bigoplus_{\nu=1}^{r_i} M_{i,\nu} S$, where $M_{i,1_i}, \ldots, X\mathrm{ffl}_{i,r_i}$ are assumed to be minimal generators of F_i with descending degrees $a_{i,\nu} = \deg(m_{i,\nu})$, $1 \leq \nu \leq r_i$. The action of an element σ is given in terms of the matrix $A(\sigma)$ given in equation (12). The element $\phi \in \mathrm{Hom}_S(F_i, S(-g))$ is sent to

$$h : \mathrm{Hom}_S(F_i, S(-g)) \xrightarrow{\cong} F_{g-2-i}$$
$$\phi \longmapsto \big(\phi(M_{i,1}), \ldots, \phi(M_{i,r_i})\big)$$

Each $\phi(M_{i,\nu})$ can be considered as an element in $S(-g - 1 + \deg(m_{i,\nu}))$ inside F_{g-2-i}. Observe that the element $\phi \in \mathrm{Hom}_S(F_i, S(-g))$ is known if we know all $\phi(M_i, \nu)$ for $1 \leq \nu \leq r_i$. From now on we will identify such an element ϕ as a r_i-tuple $\big(\phi(M_i, \nu)\big)_{1 \leq \nu \leq r_i}$. We can consider as a basis of $\mathrm{Hom}(F_i, S(-g))$ the morphisms ϕ_μ given by

$$\phi_\mu(M_j) = \delta_{\mu,j} \cdot E,$$

where E is a basis element of degree g of the rank 1 module $S(-g) \cong S \cdot E$. This is a different basis than the basis $M_{g-2-i,\nu}$, $1 \leq n \leq r_{g-2-i}$ of F_{g-2-i} we have already introduced.

Recall that if A, B are G-modules, then there is an natural action on $\mathrm{Hom}(A, B)$, sending $\phi \in \mathrm{Hom}(A, B)$ to $^\sigma\phi$ which is the map

$$^\sigma\phi : A \ni a \mapsto \sigma\phi(\sigma^{-1} a)$$

We have also a second action on the module F_{g-2-i}. We compute $^\sigma\phi(M_{i,\nu})$ for all base elements $M_{i,\nu}$ in order to descrive $^\sigma\phi$:

$$\sigma\left(\phi(\sigma^{-1}M_{i,\nu})\right)_{1\leq\nu\leq\kappa} = \left(\sum_{\mu=1}^{r_i}\sigma\big(\alpha_{\mu,\nu}(\sigma^{-1})\big)\sigma\phi(M_{i,\mu})\right)_{1\leq\nu\leq r_i}$$

$$= \left(\sum_{\mu=1}^{r_i}\sigma\big(\alpha_{\mu,\nu}(\sigma)\big)\chi(\sigma)\phi(M_{i,\mu})\right)_{1\leq\nu\leq r_i}$$

where in the last equation we have used the fact that $\phi(M_i)$ are in rank one G-module $S(-g) \cong \wedge^{g-1}\Omega_X^1$ hence the action of $\sigma \in G$ is given by multiplication by $\chi(\sigma)$, where $\chi(\sigma)$ is an invertible element is S.

In order to simplify the notation consider i fixed, and denote $M_\nu = M_{i,\nu}$, $r = r_i$, $a_{i,j} = a_j$.

According to eq. (13) if M_j has degree a_j then the element ϕ_j has degree $g+1-a_j$. Assume that M_r has maximal degree a_r. Then, ϕ_r has minimal degree. Moreover, in order to describe $^\sigma\phi_r$ we have to consider the tuple $(^\sigma\phi_r(M_1),\ldots,^\sigma\phi_r(M_r))$. We have

$$\left(^\sigma\phi_r(M_\nu)\right)_{1\leq\nu\leq r} = \left(\sum_{\mu=1}^{r}\sigma\big(\alpha_{\mu,\nu}^{(i)}(\sigma^{-1})\big)\chi(\sigma)\phi_r(M_\mu)\right)_{1\leq\nu\leq r}$$

$$= \left(\sigma\big(\alpha_{r,\nu}^{(i)}(\sigma^{-1})\big)\chi(\sigma)E\right)_{1\leq\nu\leq r}$$

and we finally conclude that

$$^\sigma\phi_r = \sum_{\nu=1}^{r}\sigma^{-1}\big(\alpha_{r,\nu}^{(i)}(\sigma^{-1})\big)\chi(\sigma)\phi_\nu.$$

In this way every element $x \in F_{g-2-i}$ is acted on by σ in terms of the action

$$\sigma \star x = h\big(^\sigma h^{-1}(x)\big).$$

On the other hand the elements $h(\phi_r)$ are in F_{g-2-i} and by lemma 11 there is an element $\sigma' \in G$ such that

$$\sigma'h(\phi_r) = \sum_{\nu=1}^{r}\alpha_{\nu,r}^{(g-2-i)}(\sigma')h(\phi_\nu).$$

Since the element ϕ_ν has maximal degree among generators of F_i the element $h(\phi_r)$ has minimal degree. This means that all coefficients

$$\alpha_{\nu,r}^{(g-2-i)}(\sigma') = \sigma\big(\alpha_{r,\nu}^{(i)}(\sigma^{-1})\big)\chi(\sigma)$$

are zero for all ν such that $\deg m_\nu < \deg \mu_r$. Therefore all coefficients $a_{\nu,r}^{(i)}(\sigma)$ for ν such that $\deg m_\nu < \deg m_r$ are zero. This holds for all $\sigma \in G$. By considering in this way all elements $\phi_{r-1}, \phi_{r-2}, \ldots, \phi_1$, which might have greater degree than the degree of ϕ_r the result follows. $\qquad\square$

4. Representations on the free resolution

Each S-module F_i in the minimal free resolution can be seen as a series of representations of the group G. Indeed, the modules F_i are graded and there is an action of G on each graded part, given by representations

$$\rho_{i,d} : G \to \mathrm{GL}\big(F_{i,d}\big),$$

where $F_{i,d}$ is the degree d part of the S-module F_i. In equation (10) we have already considered the representation on $\mathrm{GL}(\mathrm{Tor}_i^S(k, S_X))$ which is $\rho_{i,d}$. Proposition 12 shows us that there is a decomposition

$$\mathrm{Tor}_i^S(k, S_X) = \bigoplus_{j \in \mathbb{Z}} \mathrm{Tor}_i^S(k, S_X)_j,$$

where $\mathrm{Tor}_i^S(k, S_X)_j$ is the k-vector space generated by generators of F_i that have degree j. This is a vector space of dimension $b_{i,j}$.

Denote by $\mathrm{Ind}(G)$ the set of isomorphism classes of indecomposable $k[G]$-modules. If k is of characteristic $p > 0$ and G has no-cyclic p-Sylow subgroup then the set $\mathrm{Ind}(G)$ is infinite, see [4, p.26]. Suppose that each $\mathrm{Tor}_i^S(k, S_X)_j$ admits the following decomposition in terms of $U \in \mathrm{Ind}(G)$:

$$\mathrm{Tor}_i^S(k, S_X)_j = \bigoplus_{U \in \mathrm{Ind}(G)} a_{i,j,U} U \text{ where } a_{i,j,U} \in \mathbb{Z}.$$

We obviously have that

$$b_{i,j} = \sum_{U \in \mathrm{Ind}(G)} a_{i,j,U} \dim_k U.$$

The G-structure of F_i is given by

$$\mathrm{Tor}_i^S(k, S_X) \otimes S,$$

that is the G-module structure of $F_{i,d}$ is given by

$$F_{i,d} = \bigoplus_{d \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} \mathrm{Tor}_i^S(k, S_X)_{d-j} \otimes S_j.$$

Notice that structure of S_d can be expressed in terms of the decomposition of $H^0(X, \Omega_X)$ in terms of indecomposable modules using the operations in the Grothendieck ring of G.

Appendix

In what follows we will study the canonical ideal of the Fermat curve, for $n \geq 6$, following the method developed in [8] and we will prove proposition 5.

Observe that the holomorphic differentials given in eq. (3) are in 1-1 correspondence with the elements of the set $\mathbf{A} = \{(i,j) : 0 \leq i + j \leq n - 3\} \subset \mathbb{N}^2$. First we introduce the following term order on the polynomial algebra $S := \mathrm{Sym} H^0(X, \Omega_X)$.

Definition 13. Choose any term order $\prec_t$ for the variables $\{\omega_{N,\mu} : (N, \mu) \in A\}$ and define the term order $\prec$ on the monomials of S as follows:

(14) $\qquad \omega_{N_1,\mu_1} \omega_{N_2,\mu_2} \cdots \omega_{N_d,\mu_d} \prec \omega_{N_1',\mu_1'} \omega_{N_2',\mu_2'} \cdots \omega_{N_s',\mu_s'}$ if and only if

- $d < s$ or

- $d = s$ and $\sum \mu_i > \sum \mu_i'$ or

- $d = s$ and $\sum \mu_i = \sum \mu_i'$ and $\sum N_i < \sum N_i'$

- $d = s$ and $\sum \mu_i = \sum \mu_i'$ and $\sum N_i = \sum N_i'$ and

$$\omega_{N_1,\mu_1}\omega_{N_2,\mu_2}\cdots\omega_{N_d,\mu_d} \prec_t \omega_{N_1',\mu_1'}\omega_{N_2',\mu_2'}\cdots\omega_{N_s',\mu_s'}.$$

T

Lemma 14. *The number of natural numbers $0 \le i, j$ such that $0 \le i + j \le E \in \mathbb{N}$ equals $(E+1)(E+2)/2$.*

Proof. Evaluate $\sum_{i=0}^{E}\sum_{j=0}^{E-i} 1$. $\qquad\qquad\square$

We will use the following lemma

Lemma 15. *Let J be the ideal generated by the elements G_1, G_2 and let I be the canonical ideal. Assume that the cannonical is generated by elements of degree 2. If $\dim_L (S/\mathrm{in}_\prec(J))_2 \le 3(g-1)$, then $I = J$.*

We extend the correspondence between the variables $\omega_{i,j}$ and the points of $\mathbf{A}$ to a correspondence between monomials in S of standard degree 2 and points of the Minkowski sum of $\mathbf{A}$ with itself, defined as

(15) $$\mathbf{A} + \mathbf{A} = \{(i + i', j + j') \mid (i, j), (i', j') \in \mathbf{A}\} \subseteq \mathbb{N}^2.$$

Proposition 16. *Let $\mathbf{A}$ be the set of exponents of the basis of holomorphic differentials, and let $\mathbf{A} + \mathbf{A}$ denote the Minkowski sum of $\mathbf{A}$ with itself, as defined in (15). Then*

$$(\rho, T) \in \mathbf{A} + \mathbf{A} \Leftrightarrow \exists\, \omega_{i,j}\omega_{i',j'} \in S \text{ such that } \mathrm{mdeg}(\omega_{i,j}\omega_{i',j'}) = (2, \rho, T).$$

For each $n \in \mathbb{N}$ we write $\mathbb{T}^n$ for the set of monomials of degree n in S and proceed with the characterization of monomials that do not appear as leading terms of binomials in $G_1 \subseteq J$.

Proposition 17. *Let σ be the map of sets*

$$\begin{aligned}\sigma : \mathbf{A} + \mathbf{A} &\rightarrow \mathbb{T}^2 \\ (\rho, T) &\mapsto \min_\prec\{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid (\rho, T) = (i + i', j + j')\}\end{aligned}$$

Then

$$\sigma(\mathbf{A} + \mathbf{A}) = \{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid \omega_{i,j}\cdot\omega_{i',j'} \ne \mathrm{in}_\prec(f), \ \forall\, f \in G_1\}$$

The above proposition gives a characterization of the monomials that do not appear as initial terms of elements of G_1. However, some of these monomials appear as initial terms of polynomials in G_2:

Proposition 18. *Let*

$$C = \{(\rho, b) \in \mathbf{A} + \mathbf{A} \mid \rho = n + a, 0 \le a + b \le n - 6, a, b \in \mathbb{N}\}$$

Then

$$\sigma(C) \subseteq \{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid \exists\, g \in G_2 \text{ such that } \omega_{i,j}\omega_{i',j'} = \mathrm{in}_\prec(g)\}$$

Moreover $\#C = \#\sigma(C) = (n-5)(n-4)/2$.

Proof. By the form of the equations of G_2 we have that

$$i_1 + i_2 = n + a = \rho, j_1 + j_2 = b$$

so

$$i_3 + i_4 = a = \rho - n, j_3 + j_4 = n + b, i_5 + i_6 = a = \rho - n, j_5 + j_6 = b = T$$

We should have

$$0 \leq a + b \leq n - 6$$

and by lemma 14 we have that the cardinality of C equals $(n-5)(n-4)/2$. □

We now compute that

$$3(g-1) - (\#(\mathbf{A} + \mathbf{A}) - C) = 3(g-1) - \frac{(2n-5)(2n-4)}{2} + \frac{(n-5)(n-4)}{2} = 0,$$

so by lemma 15 we have that $I = J$.

References

[1] J. L. Alperin. *Local representation theory*, volume 11 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Modular representations as an introduction to the local representation theory of finite groups.

[2] Allen Altman and Steven Kleiman. *Introduction to Grothendieck duality theory*. Lecture Notes in Mathematics, Vol. 146. Springer-Verlag, Berlin-New York, 1970.

[3] Jannis A. Antoniadis and Aristides Kontogeorgis. *Automorphisms of Curves*, pages 339–361. Springer International Publishing, Cham, 2017.

[4] David J. Benson. *Modular representation theory*, volume 1081 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. New trends and methods, Second printing of the 1984 original.

[5] Christian Bopp and Frank-Olaf Schreyer. A version of green's conjecture in positive characteristic, 2018.

[6] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system I: The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[7] A. Broughton, T. Shaska, and A. Wootton. On automorphisms of algebraic curves. In *Algebraic curves and their applications*, volume 724 of *Contemp. Math.*, pages 175–212. Amer. Math. Soc., Providence, RI, 2019.

[8] Hara Charalambous, Kostas Karagiannis, and Aristides Kontogeorgis. The relative canonical ideal of the Artin-Schreier-Kummer-Witt family of curves, 2019.

[9] David Eisenbud. *The geometry of syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry.

[10] Gavril Farkas. Progress on syzygies of algebraic curves. In *Moduli of curves*, volume 21 of *Lect. Notes Unione Mat. Ital.*, pages 107–138. Springer, Cham, 2017.

[11] Shiro Goto and Keiichi Watanabe. On graded rings. I. *J. Math. Soc. Japan*, 30(2):179–213, 1978.

[12] Mark Green and Robert Lazarsfeld. A simple proof of Petri's theorem on canonical curves. In *Geometry today (Rome, 1984)*, volume 60 of *Progr. Math.*, pages 129–142. Birkhäuser Boston, Boston, MA, 1985.

[13] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[14] Bernhard Köck and Joseph Tait. Faithfulness of actions on Riemann-Roch spaces. *Canad. J. Math.*, 67(4):848–869, 2015.

[15] Aristides Kontogeorgis. Automorphisms of Fermat-like varieties. *Manuscripta Math.*, 107(2):187–205, February 2002.

[16] Aristides I. Kontogeorgis. The group of automorphisms of the function fields of the curve $x^n + y^m + 1 = 0$. *J. Number Theory*, 72(1):110–136, 1998.

[17] Heinrich-Wolfgang Leopoldt. Über die Automorphismengruppe des Fermatkörpers. *J. Number Theory*, 56(2):256–282, 1996.

[18] Irena Peeva. *Graded syzygies*, volume 14 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011.

[19] B. Saint-Donat. On Petri's analysis of the linear system of quadrics through a canonical curve. *Math. Ann.*, 206:157–175, 1973.

[20] Christopher Towse. Weierstrass points on cyclic covers of the projective line. *Trans. Amer. Math. Soc.*, 348(8):3355–3378, 1996.

[21] Christopher Wayne Towse. *Weierstrass points on cyclic covers of the projective line*. ProQuest LLC, Ann Arbor, MI, 1993. Thesis (Ph.D.)–Brown University.

[22] Pavlos Tzermias. The group of automorphisms of the Fermat curve. *J. Number Theory*, 53(1):173–178, 1995.

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