GALOIS ACTION ON HOMOLOGY OF GENERALIZED FERMAT CURVES

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ABSTRACT. The fundamental group of Fermat and generalized Fermat curves is computed. These curves are Galois ramified covers of the projective line with abelian Galois groups H. We provide a unified study of the action of both cover Galois group H and the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the pro-ℓ homology of the curves in study. Also the relation to the pro-ℓ Burau representation is investigated.

1. Introduction

In [12] we have studied the actions of the braid group and of the absolute Galois group on a cyclic cover of the projective line. In that article we have exploited the fact that in a ramified cover $X \to \mathbb{P}^1$ of the projective line when the ramified points are removed, then we obtain an open cover $X^0 \to \mathbb{P}^1 - \{P_1, \ldots, P_s\} = X_s$ of the projective line minus the ramified points. By covering space theory the open curve X_0 can be described as a quotient of the universal covering space $\tilde{X}_s$ by the fundamental group of the open curve $\pi_1(X^0, x_0') < \pi_1(X_s, x_0) = F_{s-1}$, where F_{s-1} is the free group in $s-1$ generators. Also the group $\pi_1(X^0, x_0')$ can be described as a subgroup of F_{s-1} by using the Reidemeister-Screier technique, see [12, sec. 3].

Y. Ihara in [10], [9] observed that if we pass to the pro-ℓ completions of the fundamental groups of the above curves, then the the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ can be realized as a group of automorphisms of $\mathfrak{F}_{s-1}$ and it can also act on certain subgroups $\mathfrak{F}_{s-1}$, corresponding to topological covers of X_s. The fundamental group of X_s admits the presentation

$$\pi_1(X_s.x_0) = \langle x_1, \ldots, x_s | x_1 x_2 \cdots x_s = 1 \rangle.$$

In this way we can unify the study of both Braid group and the absolute Galois group on (co)homology spaces of cyclic curves. In [12, thm. 1] the fundamental group R_{ℓ^k} of the open cyclic cover Y_n of the projective line minus s-points is computed to be equal to

$$R_{\ell^k} = \langle x_1^i x_j x_1^{-i+1}, 0 \leq i \leq \ell^k - 2, 2 \leq j \leq s-1, x_1^{\ell^k - 1} x_j, 1 \leq j \leq s-1 \rangle. \tag{1}$$

In this article we continue this study by focusing on the case of certain abelian coverings of the projective line.

Theorem 1. *Let α, β be the generators of the group $H_0 = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Also for an element $x \in F_{s-1}$ we will denote by x^α (resp. x^β) the element $x^\alpha = axa^{-1}$ (resp. $x^\beta = bxb^{-1}$). The fundamental group of the open Fermat curve seen as $H_0 = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$-cover of $\mathbb{P}^1 - \{0, 1, \infty\}$ is the subgroup of the free group $F_2 = \langle a, b \rangle$ on the generators*

Date: May 14, 2019.

$$
\begin{aligned}
A_1 &= \left\{ (b^n)^{\alpha^i} : 0 \leq i \leq n-1 \right\}, && \#A_1 = n \\
A_2 &= \left\{ [b^j, a]^{\alpha^i} : 1 \leq j \leq n-1, 0 \leq i \leq n-2 \right\} && \#A_2 = (n-1)^2 \\
A_3 &= \{ a^n [a^{-1}, b^j] : 0 \leq j \leq n-1 \} && \#A_3 = n
\end{aligned}
$$

The module R_F/R'_F is generated as $\mathbb{Z}[H_0]$-module by the elements $a^n, b^n, [a,b]$. More precisely we have

$$R_F/R'_F = \mathbb{Z}[\langle \alpha \rangle] \bigoplus \mathbb{Z}[\langle \beta \rangle] \bigoplus \mathbb{Z}[H_0]/I,$$

where I is the ideal of $\mathbb{Z}[H_0]$ generated by $\sum_{i=0}^{n-1} \alpha^i, \sum_{i=0}^{n-1} \beta^i$, or equivalently

$$\mathbb{Z}[H_0]/I = I_{\langle \alpha \rangle} \otimes I_{\langle \beta \rangle}.$$

Finally, let X_F be the corresponding closed curve, and K a field which contains the n-different n-th roots of 1, then

$$H_1(X_F, \mathbb{Z}) \otimes_{\mathbb{Z}} K = \bigoplus_{\substack{i,j=1 \\ i+j \neq n}} K\chi_{i,j}, \tag{2}$$

where $\chi_{i,j}$ is the character of H_0 such that $\chi_{i,j}(\alpha^\nu, \beta^\mu) = \zeta_n^{i\nu + j\mu}$.

The generalized Fermat curves plays the role of Fermat curve in the more general setting of abelian coverings of the projective line, minus more that three points removed. Their automorphism group was recently studied by R. Hidalgo, M. Leyton-Álvarez and the authors in [7].

A generalized Fermat curve of type $(k, s-1)$, where $k, s-1 \geq 2$ are integers, is a non-singular irreducible projective algebraic curve $F_{k,s-1}$ defined over K admitting a group of automorphisms $H \cong (\mathbb{Z}/k\mathbb{Z})^{s-1}$ so that $F_{k,s-1}/H$ is the projective line with exactly s branch points, each one of order k. Such a group H is called a generalized Fermat group of type $(k, s-1)$. Let us consider a branched regular covering $\pi : F_{k,s-1} \to \mathbb{P}^1$, whose deck group is H. By composing by a suitable Möbius transformation (that is, an element of $\mathrm{PSL}_2(K)$) at the left of π, we may assume that the branch values of π are given by the points

$$\infty, 0, 1, \lambda_1, \ldots, \lambda_{s-3},$$

where $\lambda_i \in K - \{0, 1\}$ are pairwise different.

A generalized Fermat curve of type $(k, s-1)$ can be seen as a complete intersection in a projective space $\mathbb{P}^{s-1}$, defined by the following set of equations

$$C^k(\lambda_1, \ldots, \lambda_{s-3}) := \left\{ \begin{array}{rcl} x_0^k + x_1^k + x_2^k &=& 0 \\ \lambda_1 x_0^k + x_1^k + x_3^k &=& 0 \\ \vdots & \vdots & \vdots \\ \lambda_{s-3} x_0^k + x_1^k + x_{s-1}^k &=& 0 \end{array} \right\} \subset \mathbb{P}^{s-1}. \tag{3}$$

Observe that topologically the construction of generalized Fermat curves does not depend on the configuration of the ramification points. On the other hand the

Riemann surface/algebraic curve structure depends heavily on this configuration. For instance the automorphism group depends on the configuration of these points, see [7].

The genus of $F_{k,n}$ can be computed using the Riemann-Hurwitz formula:

$$g_{(k,s-1)} = 1 + \frac{k^{s-2}}{2}((s-2)(k-1)-2). \tag{4}$$

It is known [5] that generalized Fermat curves, have the orbifold uniformization $\mathbb{H}/\Gamma$ in terms of the Fuchsian group

$$\Gamma = \langle x_1, x_2, \ldots, x_s \mid x_1^k = \cdots = x_s^k = x_1 x_2 \cdots x_s = 1 \rangle. \tag{5}$$

The surface group is given [5], [15] as $F_{s-1} \cdot \langle x_1^k, \ldots, x_{s-1}^k, (x_1 \cdots x_{s-1})^k \rangle$. We will compute the genus of the generalized Fermat curves by two more different methods in eq. 11 and in section 5.1.2.

Theorem 2. *Let $R_{k,s-1}$ be the fundamental group of the generalized Fermat curve of type $(k, s-1)$. It is a free group generated by the elements*

$$\begin{aligned}
A_{s-1} &= \{(x_{s-1}^k)^{x_{1,s-2}}\} \\
A_\nu &= \{(x_\nu^k)^{x_{1,\nu-1} \cdot x_{\nu+1,s-1}}\}, \text{ for } 1 \leq \nu \leq s-2 \\
A'_\nu &= \{[x_j, x_\nu]^{x_{1,\nu-1} \cdot x_\nu^{i_\nu} \cdot x_{\nu+1,s-1}}\}, \text{ for } 1 \leq \nu \leq s-2,
\end{aligned}$$

where $x_{\ell_1,\ell_2} = x_{\ell_1}^{i_{\ell_1}} x_{\ell_1+1}^{i_{\ell_1+1}} \cdots x_{\ell_2}^{i_{\ell_2}}$, $0 \leq i_j \leq k-1$, $1 \leq j \leq s-1$. In particular, in our pro-ℓ setting we are interested in generalized Fermat curves of type $(\ell^k, s-1)$. The pro-ℓ homology of the closed curve is given by

$$H_1(F_{\ell^k,s-1}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \left(\frac{\mathfrak{R}_{k,s-1} \cap \mathfrak{R}_k}{\mathfrak{R}_k} \right)^{\mathrm{ab}} = \frac{\mathfrak{F}'_{s-1,k}}{\mathfrak{F}''_{s-1,k}},$$

where $\mathfrak{R}_{k,s-1}$ is the pro-ℓ completion of the group $R_{\ell^k,s-1}$, $\mathfrak{R}_k$ is the smallest closed subgroup containing all elements $x_i^{\ell^k}, 0 \leq i \leq s$ and $\mathfrak{R}_{k,s-1} = \mathfrak{F}_{s-1}/\mathfrak{R}_k$.

Let K be a field containing $\mathbb{Z}_\ell$ and the ℓ^k-roots of unity. We have the following decomposition:

$$H_1(F_{k,s-1}, K) = \bigoplus_{i_1,\ldots,i_s}^{\ell^k - 1} C(i_1, \ldots, i_s) \chi_{i_1,\ldots,i_s}$$

where

$$C(i_1, \ldots, i_s) = \begin{cases} s - z(i_1, \ldots, i_s) - 2 & \text{if } (i_1, \ldots, i_s) \neq (0, \ldots, 0) \\ s - z(i_1, \ldots, i_s) & \text{if } (i_1, \ldots, i_s) = (0, \ldots, 0) \end{cases}$$

Moreover

$$\mathrm{rank}_{\mathbb{Z}_\ell} H_1(F_{k,s-1}, \mathbb{Z}_\ell) = (s-1)\left(\ell^k\right)^{s-1} + 2 - s\left(\ell^k\right)^{s-2}.$$

Fix the number of ramified points s. If K_k is the function field of the generalized Fermat curve then

$$\frac{\mathfrak{F}'_{s-1,k}}{\mathfrak{F}''_{s-1,k}} = \mathrm{Gal}(K_k^{\mathrm{urab}}/K_k),$$

where K_k^{urab} *is the maximal abelian unramified extension of the function field* K_k.

We can consider the curve F_s which is a Galois cover of the projective line ramified over the set of s-points with Galois group $\mathrm{Gal}(F_s/\mathbb{P}^1) = \mathbb{Z}_\ell^{s-1}$ and can be approached as the limit case of the generalized Fermat curves $F_{k,s-1}$ as shown in the diagram on the right. In this way instead of considering a simple generalized Fermat cover we consider all of them, together.

$$\begin{tikzcd}
F_s \arrow[dd, "\mathbb{Z}_\ell^{s-1}"'] \arrow[dr, "(\ell^k\mathbb{Z}_\ell)^{s-1}"] & \\
& F_{k,s-1} \arrow[dl, "(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}"] \\
\mathbb{P}^1 &
\end{tikzcd}$$

The limit

$$\mathbb{T} := \lim_{\substack{\leftarrow \\ k}} T(\mathrm{Jac}(F_{k,s-1})) = \lim_{\substack{\leftarrow \\ k}} \frac{\mathfrak{F}'_{s-1,k}}{\mathfrak{F}''_{s-1,k}}$$

corresponds to the $\mathbb{Z}_\ell$ homology of the curve F_s and all the knowledge of the Galois module structure of all Tate modules of the curves $F_{k,s-1}$ is equivalent to the knowledge of the Galois module stucture of $\mathbb{T}$.

The situation is similar with the pro-ℓ Burau representation, defined in [12]. We also in section 5.1.5 how we can pass from the $\mathbb{Z}_\ell^{s-1}$-covers corresponding to generalized Fermat curves, to the $\mathbb{Z}_\ell$-case corresponding to the pro-ℓ Burau representation, using the ideas of [13]. Section 3 is an introduction to Ihara's ideas on the study of the absolute Galois groups as a profinite braid [9], [10] following [11]. In section 4.1 we compute the Alexander module for the generalized Fermat curves, while section 5 is devoted to the $\mathbb{Z}_\ell^{s-1}$ cover of the projective line, seen as a limit of $F_{k,s-1}$-curves and the relation to the Tate modules of them. Finally we consider the passage to the Burau representation by comparing the corresponding Crowell sequences, in terms of the viewpoint developed in [13].

1.1. **Geometric Interpretation.** We consider a Galois covering $\pi : \bar{Y} \to \mathbb{P}^1$ of the projective line ramified above the points in $S \subset \mathbb{P}^1_{\bar{\mathbb{Q}}}$, and the corresponding covering of compact Riemann surfaces. We also assume that the genus g of $\bar{Y}$ is ≥ 2. The curve $Y_0 = \bar{Y} - \pi^{-1}(S)$ is a topological covering of $X_s = \mathbb{P}^1_{\mathbb{C}} - S$, which can be described in terms of covering theory and corresponds to a subgroup R_0 of $\pi_1(X_s)$. We have seen in [12] and we will see in section 2.1 how this group can be computed by using the Schreier lemma. For an application of this method to cyclic covers of the projective line we refer to [12]. In order to pass from the open curve to the corresponding closed Riemann surface we consider the quotient by the group Γ, which is the closure of the subgroup of $\mathfrak{F}_{s-1}$ generated by the stabilizers of ramification points, that is

$$\Gamma = \langle x_1^{e_1}, \ldots, x_s^{e_s} \rangle, \tag{6}$$

where $e_1, \ldots, e_s$ are the ramification indices of the ramification points of $\pi : \bar{Y} \to \mathbb{P}^1$. The group $R = R_0/R_0 \cap \Gamma$ corresponds to the closed curve $\bar{Y}$ as a quotient of the hyperbolic plane.

This geometric situation can be expressed in terms of the short exact sequence of groups

$$1 \to R = \frac{R_0}{\Gamma \cap R_0} \cong \frac{R_0 \cdot \Gamma}{\Gamma} \to \frac{\mathfrak{F}_{s-1}}{\Gamma} \xrightarrow{\psi} \frac{\mathfrak{F}_{s-1}}{R_0 \cdot \Gamma} \to 1. \tag{7}$$

In this article we focus on the study of Fermat and generalized Fermat curves. Namelly, in sections 2.1 and 2.3 we compute the fundamental group of the corresponding curves. We also treat the classical Fermat curves $s = 3$ since this computation is elementary, while for the generalized Fermat curves $s \geq 3$ more advanced tools are needed, namely the usage of Alexander modules and the Crowell exact sequence.

2. Generalized Fermat Curves

2.1. **Fermat Curves.** These curves are ramified curves over $\mathbb{P}^1 - \{0, 1, \infty\}$ with deck group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. We have $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, x_0) = F_2 = \langle a, b\rangle$.

We will employ the Schreier lemma for describing the fundamental group of the Fermat curve, as explained in [12, sec. 3]. A transversal set T for $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ in F_2 is given by $a^i b^j$, $0 \leq i, j \leq n - 1$. We also compute:

$$\overline{a^i b^j b} = \begin{cases} a^i b^{j+1} & \text{if } j < n-1 \\ a^i & \text{if } j = n-1 \end{cases}$$

and

$$\overline{a^i b^j a} = \begin{cases} a^{i+1} b^{j} & \text{if } i < n-1 \\ b^j & \text{if } i = n-1 \end{cases}$$

Thus

$$\begin{aligned} a^i b^j b \left(\overline{a^i b^j b}\right)^{-1} &= \begin{cases} a^i b^j b b^{-j-1} a^{-i} = 1 & \text{if } j < n-1 \\ a^i b^n a^{-i} & \text{if } j = n-1 \end{cases} \\ a^i b^j a \left(\overline{a^i b^j a}\right)^{-1} &= \begin{cases} a^i b^j a b^{-j} a^{-i-1} & \text{if } i < n-1, j \neq 0 \\ 1 & \text{if } i < n-1, j = 0, \\ a^{n-1} b^j a b^{-j} & \text{if } i = n-1 \end{cases} \end{aligned}$$

Consider the generators α, β of the cyclic group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Let R_F be the fundamental group of the Fermat curve, seen as a subgroup of the free group F_2. Observe that there is a well defined action of α (resp. β) on R_F/R_F' given by conjugation, i.e.

$$x^\alpha = x^a = axa^{-1} \qquad x^\beta = x^b = bxb^{-1}$$

for all $x \in R_F/R_F'$. Notice that this is indeed an action which implies that

$$(x^\alpha)^\beta = x^{\alpha\beta} = x^{\beta\alpha} = (x^\beta)^\alpha$$

i.e. the actions of α and β commute.

Remark 3. The commutator $[a, b]$ of two elements a, b in a group is defined as $[a, b] = aba^{-1}b^{-1}$.

We can consider the conjugation action of $\mathbb{Z}/n\mathbb{Z} = \langle\alpha\rangle$ and then we have the following sets of generators, of the free group R_F:

$$\begin{aligned} A_1 &= \left\{(b^n)^{\alpha^i} : 0 \leq i \leq n-1\right\}, & \#A_1 &= n \\ A_2 &= \left\{[b^j, a]^{\alpha^i} : 1 \leq j \leq n-1, 0 \leq i \leq n-2\right\} & \#A_2 &= (n-1)^2 \\ A_3 &= \{a^n [a^{-1}, b^j] : 0 \leq j \leq n-1\} & \#A_3 &= n \end{aligned}$$

We finally arrive at $n^2 + 1$ generators as predicted by Schreier index formula since $\#A_1 + \#A_2 + \#A_3 = n + (n-1)^2 + n = n^2 + 1$.

Lemma 4. *For any two elements of a group and any positive integer j we have*
(1) $[x^j, y] = [x, y]^{x^{j-1}} \cdot [x, y]^{x^{j-2}} \cdots [x, y]^x \cdot [x, y]$
(2) $[x, y^j] = [x, y] \cdot [x, y]^y \cdot [x, y]^{y^{j-1}}$.

Proof. See [4, 0.1 p.1]. □

Fix $0 \le i \le n-2$. We will prove that the $\mathbb{Z}$-module generated by the elements $\Sigma_1 := \{[b^j, a]^{\alpha^i}, 1 \le j \le n-1\}$ is the same with the $\mathbb{Z}$-module generated by the elements $\Sigma_2 := \{[b, a]^{\alpha^i \beta^j} 1 \le j \le n-2\}$. Indeed by lemma 4.(1) we have the identities (written aditively) for $1 \le j \le n-1$ and $0 \le i \le n-2$ we have

$$[b^j, a]^{\alpha^i} = [b, a]^{(\beta^{j-1} + \beta^{j-2} + \beta + 1)\alpha^i} \tag{8}$$

This proves that the elements of the set Σ_1 are transofrmed to the elements of the set Σ_2 in terms of an invertible block matrix where each block is the invertible $(n-1) \times (n-1)$ matrix with entries in $\mathbb{Z}$:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix}$$

Therefore Σ_1 and Σ_2 generate the same $\mathbb{Z}$-module.

Notice also that

$$\begin{aligned}(a^n)^{\beta^j} &= b^j a^{n-1} b^{-j} a^{-n+1} \cdot a^{n-1} b^j a b^{-j} = [b^j, a^{n-1}] + \underbrace{a^{n-1} b^j a b^{-j}}_{\in A_3} \\ &= [b^j, a]^{\alpha^{n-2} + \alpha^{n-3} + \cdots \alpha + 1} + a^{n-1} b^j a b^{-j}.\end{aligned}$$

The above computation shows that $(a^n)^{\beta^j}$ can be written as a $\mathbb{Z}$-linear combination of elements of Σ_2 (which generate A_2) and A_3. Moreover

$$a^n [a^{-1}, b^j] = (a^n)^{\beta^j} - [b, a]^{\left(\sum_{k=0}^{j-1} \beta^k\right)\left(\sum_{\lambda=0}^{n-2} \alpha^\lambda\right)}.$$

We have shown that

Lemma 5. *The free $\mathbb{Z}$-module R_F/R_F' can be generated by the n^2+1 elements*

$$(a^n)^{\beta^i}, (b^n)^{\alpha^i}, 0 \le i \le n-1 \text{ and } [a, b]^{\alpha^i \cdot \beta^j}, 0 \le i, j \le n-2.$$

2.2. **Structure as a $\mathbb{Z}/\mathbf{n}\mathbb{Z} \times \mathbb{Z}/\mathbf{n}\mathbb{Z}$-module.** We can now consider the homology group as the rank n^2+1 free $\mathbb{Z}$-module R_F/R_F'. Since R_F is a characteristic subgroup the group $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} = \langle \alpha \rangle \times \langle \beta \rangle$ acts on R_F/R_F' by conjugation making R_F/R_F' a G-module.

For a finite group G the coaugmenation ideal J_G is defined as the quotient $J_G = \mathbb{Z}[G]/\langle \sum_{g \in G} g \rangle$.

Lemma 6. *Set $H_0 = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. The module R_F/R_F' is generated as $\mathbb{Z}[H_0]$-module by the elements $a^n, b^n, [a, b]$. More precisely we have*

$$R_F/R_F' = \mathbb{Z}[\langle \alpha \rangle] \bigoplus \mathbb{Z}[\langle \beta \rangle] \bigoplus \mathbb{Z}[H_0]/I,$$

where I is the ideal of $\mathbb{Z}[H_0]$ generated by $\sum_{i=0}^{n-1} \alpha^i, \sum_{i=0}^{n-1} \beta^i$, or equivalently

$$\mathbb{Z}[H_0]/I = J_{\langle \alpha \rangle} \otimes J_{\langle \beta \rangle}$$

Proof. This is evident from the $\mathbb{Z}$-basis given in lemma 5. Notice that the elements $[a,b]^{\alpha^i\beta^j}$ are subject to the condition given in eq. (8) which implies that for all i

$$[a,b]^{\alpha^i(1+\beta+\beta^2+\cdots\beta^{n-1})} = [a,b^n]^{\alpha^i} = 0.$$

This means that the operator $1+\beta+\cdots+\beta^{n-1}$ is zero. A similar equation to eq. (8) holds which forces $\sum_{i=0}^{n-1}\alpha^i = 0$. Therefore, in the $\mathbb{Z}[H_0]$-module generated by $[a,b]$ we have that $1+\beta+\cdots+\beta^{n-1}$ and $1+\alpha+\cdots+\alpha^{n-1}$ both annihilate $[a,b]$. We compute that

$$\mathbb{Z}[H_0]/\langle 1+\beta+\beta^2+\cdots\beta^{n-1})\rangle = \bigoplus_{i=0}^{n-1}\alpha^i J_{\langle b\rangle}.$$

The result follows. $\square$

We will now prove that in R_F/R_F' there are exacty $3n$ elements which are fixed by an element of $\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$.

The $2n$ elements are $(a^n)^{\beta^i}$ (resp. $(b^n)^{\alpha^i}$) which are fixed by $\langle\alpha\rangle$ (resp. $\langle\beta\rangle$).

The other n-elements are the elements $((ab)^n)^{\alpha^i}$ which are fixed by $\langle ab\rangle$. It is a simple computation to verify that we can write the elements $((ab)^n)^{\alpha^i}$ as follows:

$$(ab)^n = [b,a]^{\alpha^{n-1}\left(\sum_{\nu=0}^{n-2}\beta^\nu\right)+\cdots+\alpha^2(\beta+1)+\alpha}a^nb^n.$$

Hence

$$((ab)^n)^{\alpha^i} = [b,a]^{\alpha^{n-1+i}\left(\sum_{\nu=0}^{n-2}\beta^\nu\right)+\cdots+\alpha^{2+i}(\beta+1)+\alpha^i}a^n\,(b^n)^{\alpha_i}.$$

We can see that the trasformation matrix from elements $[b^j,a]^{\alpha^i}$ to elements of the form $[b,a]^{\beta^j\alpha^i}$ is invertible. This allows us to prove that the elements in the sets A_2 and A_3 can be written as linear combinations of elements of the form $[b,a]^{\alpha^i\beta^j}$ and $(a^n)^{\beta^j}$ for $1\le j\le n-1$, $0\le i\le n-2$. It is clear that the elements $(a^n)^{\beta^i},(b^n)^{\alpha^i},((ab)^n)^{\alpha^i}$ as given in the table below are fixed by the cyclic group mentioned in the third collumn. The elements γ_i are the n-elements $(b^n)^{\alpha^i}$ fixed by β, the n-elements $(a^n)^{\beta^i}$ fixed by α and the n invariant elements $((ab)^n)^{\alpha^i}$ in the module generated by commutators. In the following table we enumerate the fixed elements γ_i:

Invariant element γ_i	Index	Fixed by
$(a^n)^{\beta^i}$	$1\le i\le n$	$\langle\alpha\rangle$
$(b^n)^{\alpha^i}$	$n+1\le i\le 2n$	$\langle\beta\rangle$
$((ab)^n)^{\alpha^i}$	$2n+1\le i\le 3n$	$\langle\alpha\beta\rangle$

So far we have computed the open Fermat curve addmitting a presentetation

$$R_F = \langle a_1,b_1,\ldots,a_g,b_g,\gamma_1,\ldots,\gamma_{3n}|\gamma_1\gamma_2\cdots\gamma_{3n}\cdot[a_1,b_1][a_2,b_2]\cdots[a_g,b_g]=1\rangle,$$

where g is the genus of the closed Fermat curve which equals to $(n-1)(n-2)/2$. Every ramification point of the Fermat curve is surrounded by a path γ_i and there are $3n$ such paths. We can verify that our computation so far, by computing the genus of the closed Fermat curve. We add the $3n$ missing points and we observe

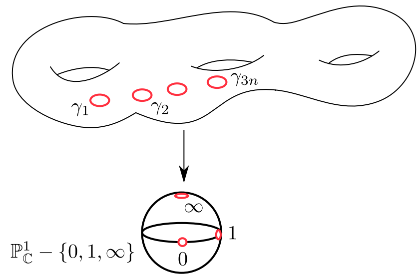

FIGURE 1. Open Fermat curve as cover of the projective line

that the rank of R_F equals $2g + 3n - 1$ so the Schreier index formula implies:

$$2g + 3n - 1 = n^2 + 1 \Rightarrow g = \frac{n^2 + 2 - 3n}{2} = \frac{(n-1)(n-2)}{2}.$$

We have that

$$H_1(X_F, \mathbb{Z}) = \frac{R_F / R_F'}{\langle \gamma_1, \ldots, \gamma_{3n} \rangle}.$$

Proposition 7. *A basis for the $\mathbb{Z}$-module $H_1(X_F, \mathbb{Z})$ consists of the set:*

$$\{[b, a]^{\alpha^i \beta^j} \mod \Gamma : 0 \leq i \leq n-2, 0 \leq j \leq n-3\},$$

where Γ is the free $\mathbb{Z}$-module generated by $\langle \gamma_1, \ldots, \gamma_{3n} \rangle$. Let K be a field that contains n different n-th roots of 1, then

$$H_1(X_F, \mathbb{Z}) \otimes_{\mathbb{Z}} K = \bigoplus_{\substack{i,j=1 \\ i+j \neq n}}^{n-1} K\chi_{i,j},$$

where $\chi_{i,j}$ is the character such that $\chi_{i,j}(\alpha^\nu, \beta^\mu) = \zeta^{i\nu + j\mu}$.

Proof. The first assertion follows by considering the action modulo the elements which are invariant by an element of G. Indeed, in order to compute the quotient we change the basis of R_F/R_F' by replacing each one of the elements $[b, a]^{\alpha^{n-1+i}\beta^{n-2}}$ by $((ab)^n)^{\alpha^i}$ for all $0 \leq i \leq n-1$, which belongs to the group $\langle \gamma_1, \ldots, \gamma_{3n} \rangle$ and is considered to be zero.

For the second assertion let us write

$$J_{\langle \alpha \rangle} \otimes J_{\langle \beta \rangle} = \left(\bigoplus_{i=1}^{n-1} K\chi_{i,0} \right) \bigotimes \left(\bigoplus_{j=1}^{n-1} K\chi_{0,j} \right) = \bigoplus_{i,j=1}^{n-1} K\chi_{i,j}$$

We are looking for the elements which are stabilized by $\alpha\beta$, that is $\chi_{i,j}(\alpha\beta) = \zeta^{i+j} = 1$. This is the module $\oplus_{i=1}^{n-1} K\chi_{i,n-i}$, which has n-elements. The desired result follows. $\square$

Observe that the above computation agrees with $\dim_K H_1(X_F, \mathbb{Z}) \otimes_{\mathbb{Z}} K = (n-1)(n-2)$.

2.2.1. *Braid group action.* We will now consider the action of the Braid group B_3 on $H_1(X_F,\mathbb{Z})$ of the closed Fermat surface. By the faithful Artin representation we observe that the braid group in three strings is generated by the elements σ_1,σ_2, where

$$\sigma_1(a)=aba^{-1} \qquad \sigma_2(a)=a \qquad \sigma_1(b)=a \qquad \sigma_2(b)=a^{-1}b^{-1}.$$

Notice that the above two automorphism in the abelianized free group with two generators acts like the matrices

$$\bar{\sigma}_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \bar{\sigma}_2=\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix},$$

in $\mathrm{GL}(2,\mathbb{Z})$, reflecting the fact that $B_3/Z(B_3)\cong \mathrm{PSL}(2,\mathbb{Z})$. Therefore,

$$\begin{aligned}\sigma_1[a,b]&=[aba^{-1},a] &&=[b,a]^{\alpha}=-[a,b]^{\alpha}\\ \sigma_2[a,b]&=[a,a^{-1}b^{-1}]&&=[b^{-1},a^{-1}].\end{aligned}$$

and more generally

$$\sigma_1([b,a]^{\alpha^i\beta^j})=[b,a]^{\alpha^{i+1}\beta^j} \qquad \sigma_2([b,a]^{\alpha^i\beta^j})=[b^{-1},a^{-1}]^{\alpha^i\beta^j}$$

We also compute

$$\begin{aligned}\sigma_1((b^n)^{\alpha^i})&=(a^n)^{\beta^i} &\qquad \sigma_1((a^n)^{\beta^j})&=(b^n)^{\alpha^{j+1}}\\ \sigma_2((b^n)^{\alpha^i})&=((ba)^n)^{-\alpha^i} &\qquad \sigma_2((a^n)^{\beta^j})&=(a^n)^{(\beta\alpha)^{-j}}.\end{aligned}$$

2.3. **The Generalized Fermat Curve.**

2.3.1. *Application of the Schreier lemma.*

Consider the curve $X=\mathbb{P}^1-\{0,1,\infty,\lambda_1,\ldots,\lambda_{s-3}\}$ with fundamental group $F_{s-1}=\langle x_1,\ldots,x_{s-1}\rangle$ and univeral covering space $\tilde{X}$. We compute $H_1(X,\mathbb{Z})=\mathbb{Z}^{s-1}$. We have the picture on the right. We will now employ the Schreier lemma in order to compute the free subgroup $R_{k,s-1}\subset F_{s-1}$. A transversal set is given by

$$T=\left\{x_1^{i_1}x_2^{i_2}\cdots x_{s-1}^{i_{s-1}}, 0\leq i_j\leq k-1, 1\leq j\leq s-1\right\}.$$

$$\begin{tikzcd}
\tilde{X} \arrow[rr, bend left, "F'_{s-1}"] \arrow[dr, "R_{k,s-1}"] \arrow[dd, bend right, "F_{s-1}"'] & & Y \arrow[dl, "I"'] \arrow[ddll, bend left, "H_1(X{,}\mathbb{Z})"] \\
 & F_{k,s-1} \arrow[dl, "H_{k,s-1}"'] & \\
X & &
\end{tikzcd}$$

For given $1\leq \nu\leq s-1$ we have

$$\overline{x_1^{i_1}x_2^{i_2}\cdots x_{s-1}^{i_{s-1}}\cdot x_\nu}=\begin{cases}x_1^{i_1}x_2^{i_2}\cdots x_\nu^{i_\nu+1}\cdots x_{s-1}^{i_{s-1}} & \text{if } i_\nu<k-1\\ x_1^{i_1}x_2^{i_2}\cdots x_{\nu-1}^{i_{\nu-1}}x_{\nu+1}^{i_{\nu+1}}\cdots x_{s-1}^{i_{s-1}} & \text{if } i_\nu=k-1\end{cases}$$

Denote by $\bar{x}^{\bar{i}}=x_1^{i_1}x_2^{i_2}\cdots x_{s-1}^{i_{s-1}}$. We now compute

Case I For $1\leq\nu\leq s-1$:

$$\bar{x}^{\bar{i}}\cdot x_\nu\cdot\left(\overline{\bar{x}^{\bar{i}}\cdot x_\nu}\right)^{-1}=\begin{cases}\bar{x}^{\bar{i}}\cdot x_\nu\cdot x_1^{-i_1}x_2^{-i_2}\cdots x_\nu^{-i_\nu-1}\cdots x_{s-1}^{-i_{s-1}} & \text{if } i_\nu<k-1\\ \bar{x}^{\bar{i}}\cdot x_\nu\cdot x_1^{-i_1}x_2^{-i_2}\cdots x_{\nu-1}^{-i_{\nu-1}}x_{\nu+1}^{-i_{\nu+1}}\cdots x_{s-1}^{-i_{s-1}} & \text{if } i_\nu=k-1.\end{cases}$$

Case II For $\nu=s-1$:

$$\bar{x}^{\bar{i}}\cdot x_{s-1}\cdot\left(\overline{\bar{x}^{\bar{i}}\cdot x_{s-1}}\right)^{-1}=$$

$$= \begin{cases} 1 & \text{if } i_{s-1} < k-1 \\ \left(x_1^{i_1} x_2^{i_2} \cdots x_{s-2}^{i_{s-2}}\right) \cdot x_{s-1}^k \cdot \left(x_1^{i_1} x_2^{i_2} \cdots x_{s-2}^{i_{s-2}}\right)^{-1} & \text{if } i_{s-1} = k-1. \end{cases}$$

The generators of the free group $R_{k,s-1}$ are falling in the following categories:

$$\begin{aligned} A_{s-1} &= \left\{ (x_{s-1}^k)^{x_1^{i_1} \cdots x_{s-2}^{i_{s-2}}} \right\} \\ A_\nu &= \left\{ x_{1,\nu-1} \cdot x_\nu^{i_\nu} x_{\nu+1,s-1} \cdot x_\nu \cdot x_{\nu+1,s-1}^{-1} \cdot x_\nu^{-i_\nu - 1} \cdot x_{1,\nu-1}^{-1} \right\} \\ &= \left\{ [x_{\nu+1,s-1}, x_\nu]^{x_{1,\nu-1} \cdot x_\nu^{i_\nu}} \right\} \\ A'_\nu &= \left\{ x_1^{i_1} \cdots x_{\nu-1}^{i_{\nu-1}} \cdot x_\nu^{k-1} \cdot x_{\nu+1}^{i_{\nu+1}} \cdots x_{s-1}^{i_{s-1}} \cdot x_\nu \cdot x_{\nu+1,s-1}^{-1} \cdot x_{1,\nu-1}^{-1} \right\} \\ &= \left\{ \left(x_\nu^k [x_\nu^{-1}, x_{\nu+1,s-1}] \right)^{x_{1,\nu-1}} \right\}, \end{aligned} \tag{9}$$

where $x_{\ell_1,\ell_2} = x_{\ell_1}^{i_{\ell_1}} x_{\ell_1+1}^{i_{\ell_1+1}} \cdots x_{\ell_2}^{i_{\ell_2}}$. We now count the sizes of the above sets.

$$\begin{aligned} \#A_{s-1} &= k^{s-2} \\ \#A_\nu &= (k-1) \cdot k^{\nu-1} \cdot (k^{s-1-\nu} - 1), \text{ for } 1 \le \nu \le s-2 \\ \#A'_\nu &= k^{\nu-1} \cdot k^{s-1-\nu} = k^{s-2}, \text{ for } 1 \le \nu \le s-2 \end{aligned}$$

which gives in total

$$\#A_{s-1} + \sum_{\nu=1}^{s-2} \#A_\nu + \sum_{\nu=1}^{s-2} \#A'_\nu = (s-2) \cdot k^{s-1} + 1. \tag{10}$$

2.3.2. *Elements stabilized.* Observe first that the group generated by x_{s-1} stabilizes $(x_{s-1}^k)^{x_{1,s-2}}$. In this way we see that all k^{s-2} elements of A_{s-1} have non trivial stabilizer. Now we observe that

$$\begin{aligned} & x_{1,\nu-1} \cdot x_\nu^{k-1} \cdot x_{\nu+1,s-1} \cdot x_\nu \cdot x_{\nu+1,s-1}^{-1} \cdot x_{1,\nu-1}^{-1} = \\ & = [x_\nu^{k-1}, x_{\nu+1,s-1}]^{x_{1,\nu-1}} \cdot (x_\nu^k)^{x_{1,\nu-1} \cdot x_{\nu+1,s-1}}. \end{aligned}$$

Observe that $\langle x_\nu \rangle$ stabilizes the k^{s-2} elements of A'_ν of the form $(x_\nu^k)^{x_{1,\nu-1} \cdot x_{\nu+1,s-1}}$ and the element $\langle x_1 \cdots x_{s-1} \rangle$ stabilizes all elements $((x_1 \cdots x_{s-1})^k)^{x_{1,s-2}}$, which are k^{s-2}.

Invariant element γ_i	Cardinal	Fixed by
$(x_{s-1}^k)^{x_{1,s-2}}$	k^{s-2}	$\langle x_{s-1} \rangle$
$(x_\nu^k)^{x_{1,\nu-1} \cdot x_{\nu+1,s-1}}$	$(s-2)k^{s-2}$	$\langle x_\nu \rangle$, where $1 \le \nu \le s-2$
$((x_1 \cdots x_{s-1})^k)^{x_{1,s-2}}$	k^{s-2}	$\langle x_1 \cdots x_{s-1} \rangle$

In total we have sk^{s-2} fixed elements γ_i. Because of the above relations and the following computation

$$\begin{aligned} & [x_{\nu+1,s-1}, x_\nu]^{x_{1,\nu-1} \cdot x_\nu^{i_\nu}} = \\ & \qquad ([x_{\nu+1}^{i_{\nu+1}}, x_\nu]^{{x_\nu}^{i_\nu}} + [x_{\nu+2}^{i_{\nu+2}}, x_\nu]^{{x_\nu}^{i_\nu} \cdot {x_{\nu+1}}^{i_{\nu+1}}} + \ldots + [x_{s-1}^{i_{s-1}}, x_\nu]^{x_{\nu,s-2}})^{x_{1,\nu-1}}. \end{aligned}$$

Similarly to the computation of the classical Fermat curves we change to a more suitable basis. This is given by the following sets:

$$\begin{aligned} A_{s-1} &= \{(x_{s-1}^k)^{x_{1,s-2}}\} \\ A_\nu &= \{(x_\nu^k)^{x_{1,\nu-1}\cdot x_{\nu+1,s-1}}\}, \text{ for } 1 \le \nu \le s-2 \\ A'_\nu &= \{[x_j, x_\nu]^{x_{1,\nu-1}\cdot x_\nu^{i_\nu}\cdot x_{\nu+1,s-1}}\}, \text{ for } 1 \le \nu \le s-2. \end{aligned}$$

Remark 8. For the homology of the closed curve we have:

$$H_1(F_{k,s-1}, \mathbb{Z}) = \frac{R_{k,s-1}/R'_{k,s-1}}{\langle \gamma_1, \ldots, \gamma_{sk^{s-1}} \rangle}.$$

Using eq. (4) and the fact that $\text{rank} H_1(F_{k,s-1}, \mathbb{Z}) = 2g_{F_{k,s-1}}$ it is easy to verify that

$$(s-2)k^{s-1} + 1 - (s \cdot k^{s-2} - 1) = 2g_{F_{k,s-1}}. \tag{11}$$

In the above formula we have subtracted one from the number of invariant elements γ_i since $\gamma_1 \cdots \gamma_{sk^{s-2}} = 1$.

Describing the action in this case is not as straightforward as it was for the case of classical Fermat curve. We will use the theory of Alexander modules instead and postpone this computation to section 4.1.

3. On the representation of Ihara

3.1. **Pro-ℓ braid groups.** Let ℓ be a prime number and let $\mathfrak{F}_s$ denote the pro-ℓ free group with s free generators. Let $S \subset \mathbb{P}^1_{\bar{\mathbb{Q}}}$ be a set consisted of s points, $s \ge 3$, on the projective line and suppose that $P \in \mathbb{Q}$ for all $P \in S - \{\infty\}$. In this way the absolute Galois group corresponds to "pure braids". Ihara in [9] introduced the monodromy representation

$$\text{Ih}_S : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}(\mathfrak{F}_{s-1}).$$

Here the group $\mathfrak{F}_{s-1} = \pi_1^{\text{pro}-\ell}(\mathbb{P}^1_{\bar{\mathbb{Q}}} - S)$ is the pro-ℓ étale fundamental group and is known to admit a presentation

$$\mathfrak{F}_{s-1} = \langle x_1, \ldots, x_s | x_1 x_2 \cdots x_s = 1 \rangle. \tag{12}$$

Given a set $\{x_i, i \in I\}$ in a topological group we will denote by $\langle x_i, i \in I \rangle$ the topological closure of the group generated by the group elements x_i, $i \in I$. Ihara was mainly interested for the case $S = \{0, 1, \infty\}$, since by Belyi's theorem [2] the branched covers of $\mathbb{P}^1 - \{0, 1, \infty\}$ are exactly the curves defined over $\bar{\mathbb{Q}}$ but the case of $s \ge 3$ is also interesting. Using a Möbious transformation we can assume that the set S consists of the elements $0, 1, \lambda_1, \cdots, \lambda_{s-3}, \infty$.

The Ihara representation can be explained in terms of Galois theory as follows: Consider the maximal pro-ℓ extension $\mathscr{M}$ of $\mathbb{Q}(t)$ unramified outside the set S. The Galois group $\text{Gal}(\mathscr{M}/\bar{\mathbb{Q}}(t))$ is known to be the pro-ℓ free group $\mathfrak{F}_{s-1}$ of rank $s-1$. A selection of generators $x_1, \ldots, x_{s-1}$ corresponds to an isomorphism $i : \mathfrak{F}_{s-1} \to \text{Gal}(\mathscr{M}/\bar{\mathbb{Q}}(t))$, such that $i(x_\nu)$ $(1 \le \nu \le s)$ generates the inertia group of some place ξ_ν of $\mathscr{M}$ extending the place P_i of $\bar{\mathbb{Q}}(t)$, corresponding to the i-th element of the set S.

We have the following exact equence:

$$
\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(\mathscr{M}/\bar{\mathbb{Q}}(t)) \arrow[r] \arrow[d, equal] & \mathrm{Gal}(\mathscr{M}/\mathbb{Q}(t)) \arrow[r] & \mathrm{Gal}(\bar{\mathbb{Q}}(t)/\mathbb{Q}(t)) \arrow[r] \arrow[d, "\cong"] & 1 \\
& \mathfrak{F}_{s-1} & & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) &
\end{tikzcd}
\tag{13}
$$

Every element $\rho \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ gives rise to an element $\rho^* \in \mathrm{Gal}(\mathscr{M}/\mathbb{Q}(t))$, and to an automorphism $x \mapsto \rho^* x \rho^{-1} \in \tilde{P}(\mathfrak{F}_{s-1})$, where

$$\tilde{P}(\mathfrak{F}_{s-1}) := \left\{ \phi \in \mathrm{Aut}(\mathfrak{F}_{s-1}) | \phi(x_i) \sim x_i^{N(\phi)} (1 \leq i \leq s) \text{ for some } N(\phi) \in \mathbb{Z}_\ell^* \right\},$$

where $\sim$ denotes the conjugation equivalence.

Ihara [8, p.52], proved that the action of $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the topological generators of $\mathfrak{F}_{s-1}$ is given by

$$\sigma(x_i) = w_i(\sigma) x_i^{N(\sigma)} w_i(\sigma)^{-1}$$

where $N(\sigma) \in \mathbb{Z}_\ell^*$. In this way the outer Galois representation

$$\Phi_S : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \tilde{P}(\mathfrak{F}_{s-1})/\mathrm{Int}(\mathfrak{F}_{s-1})$$

is defined.

By selecting the representatives of elements $\tilde{P}(\mathfrak{F}_{s-1})$ we can define the Ihara representation

$$\mathrm{Ih}_S : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to P(\mathfrak{F}_{s-1}) \subset \mathrm{Aut}(\mathfrak{F}_{s-1}),$$

where

$$P(\mathfrak{F}_{s-1}) = \left\{ \phi \in \mathrm{Aut}(\mathfrak{F}_{s-1}) \;\middle|\; \begin{array}{l} \phi(x_i) \sim x_i^{N(\phi)}, (1 \leq i \leq s-2), \phi(x_{s-1}) \approx x_{s-1}^{N(\phi)} \\ \phi(x_s) = x_s^{N(\phi)}, \text{ for some } N(\phi) \in \mathbb{Z}_\ell^\times \end{array} \right\}, \tag{14}$$

where $\approx$ denotes conjugacy by an element of the subgroup of $\mathfrak{F}_s$ generated by the commutator $\mathfrak{F}_s'$ and $x_1, \ldots, x_{s-3}$. The composition $N \circ \mathrm{Ih}_S$ equals the cyclotomic character $\chi_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^*$. For more details on these constructions see [9, prop.3 p.55], [11, prop. 2.2.2].

3.2. **Magnus embedding.** We will explain now the Magnus embedding following [11]. This embedding is given by the map

$$\mathfrak{F}_{s-1} \to \mathbb{Z}_\ell[[u_1, u_2, \ldots, u_{s-1}]]_{\mathrm{nc}}$$

of $\mathfrak{F}_{s-1}$ into the "non-commutative" formal power series algebra ($x_i \to 1 + u_i$ for $1 \leq i \leq s-1$). Let $\mathfrak{H}$ denote the abelianization of $\mathfrak{F}_{s-1}$, and H the abelianization of F_{s-1}

$$H := \mathrm{gr}_1(F_{s-1}) = H_1(F_{s-1}, \mathbb{Z}) \qquad \mathfrak{H} =: \mathrm{gr}_1(\mathfrak{F}_{s-1}) = H_1(\mathfrak{F}_{s-1}, \mathbb{Z}_\ell) = H \otimes_{\mathbb{Z}} \mathbb{Z}_\ell.$$

The term gr_1 above has its origin on the graded Lie algebra corresponding to a (pro-ℓ) free group, see [9, p. 58] and [14]. Following [11], [16] we consider the tensor algebras

$$T(H) = \bigoplus_{n \geq 0} H^{\otimes n}, \qquad T(\mathfrak{H}) = \bigoplus_{n \geq 0} \mathfrak{H}^{\otimes n},$$

where $\mathfrak{H}^0 = \mathbb{Z}_\ell$ and $\mathfrak{H}^{\otimes n} := \mathfrak{H} \otimes_{\mathbb{Z}_\ell} \cdots \otimes_{\mathbb{Z}_\ell} \mathfrak{H}$ (n-times) (resp. $H^0 = \mathbb{Z}$, $H^{\otimes n} = H \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} H$)). If $u_0, \ldots, u_{s-1}$ is a $\mathbb{Z}_\ell$ basis of the free $\mathbb{Z}_\ell$-module $\mathfrak{H}$, then

$$T(\mathfrak{H}) = \mathbb{Z}_\ell \langle u_1, \ldots, u_{s-1} \rangle,$$

is the non-commutative polynomial algebra over $\mathbb{Z}_\ell$.

We will denote by $\widehat{T}(\mathfrak{H})$ the completion of $T(\mathfrak{H})$ with respect to the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the two sided ideal generated by $u_1,\dots,u_{s-1}$ and ℓ. This algebra is the algebra of non-commutative formal power series over $\mathbb{Z}_\ell$ with variables $u_1,\dots,u_{s-1}$:

$$\widehat{T}(\mathfrak{H}) = \prod_{n\geq 0} \mathfrak{H}^{\otimes n} = \mathbb{Z}_\ell\langle\langle u_1,\dots,u_{s-1}\rangle\rangle.$$

Let $\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]$ be the complete group algebra of $\mathfrak{F}_{s-1}$ over $\mathbb{Z}_\ell$, and let

$$\epsilon_{\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]} : \mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]] \to \mathbb{Z}_\ell$$

be the augmentation homomorphism. Denote by $I_{\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]} := \ker \epsilon_{\mathbb{Z}_\ell[[\mathfrak{F}]]}$ the augmentation ideal. The correspodence $x_i \mapsto 1+u_i$ for $1\leq i\leq s-1$ induces an isomorphism of topological $\mathbb{Z}_\ell$-algebras, the *pro-ℓ Magnus isomorphism*.

$$\Theta : \mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]] \xrightarrow{\cong} \widehat{T}(\mathfrak{H}).$$

Example 9. The map Θ sends $\mathbb{Z}_\ell[\mathbb{Z}] = \mathbb{Z}_\ell[t,t^{-1}]$ to $\mathbb{Z}_\ell[[u]]$ by mapping $\Theta(t) = 1+u$ and $\Theta(t^{-1}) = (1+u)^{-1} = \sum_{i=0}^{\infty}(-1)^i u^i$. The image $\Theta(\mathbb{Z}_\ell[t,t^{-1}])$ is not onto $\hat{T}(\mathfrak{H})$, but $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]] = \mathbb{Z}_\ell[[\mathfrak{F}_1]]$ is mapped isomorphically to $\hat{T}(\mathfrak{H})$ by Θ.

For an multiindex $I = (i_1,\dots,i_{s-1})$ we set $u_I = u_{i_1}\cdots u_{i_{s-1}}$. The coefficient of u_I in $\Theta(\alpha)$ is called the Magnus coefficient of α and it is denoted by $\mu(I,\alpha)$, that is

$$\Theta(\alpha) = \varepsilon_{\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}]]}(\alpha) + \sum_{|I|\geq 1} \mu(I,\alpha) u_I.$$

For certain properties of the Magnus embedding and a fascinating application to ℓ-adic Milnor invariants we refer to [16, chap. 8], [11, sec. 3.2].

3.3. **Milnor invariants.** Consider the group $\mathfrak{H} := \mathfrak{F}_{s-1}^{\mathrm{ab}} = \mathfrak{F}_{s-1}/[\mathfrak{F}_{s-1},\mathfrak{F}_{s-1}]$. For $f\in\mathfrak{F}_{s-1}$ denote by $[f]$ its image in $\mathfrak{H}$. We will write $\mathfrak{H}$ as an additive $\mathbb{Z}_\ell$-module, which is generated by $[u_1],\dots,[u_{s-1}]$. Notice that the following relation holds:

$$[u_1] + \cdots + [u_{s-1}] + [u_s] = 0.$$

Every automorphism $\phi \in \mathrm{Aut}(\mathfrak{F}_{s-1})$ gives rise to a linear automorphism of the free $\mathbb{Z}_\ell$-module $\mathfrak{H}$ and we will denote it by $[\phi]\in \mathrm{GL}(\mathfrak{H})$.

Lemma 10. *The elements $w_i(\sigma)\in\mathfrak{F}_{s-1}$ can be selected uniquely so that*

(1) $\mathrm{Ih}_S(\sigma)(x_i) = w_i(\sigma) x_i^{\chi_\ell(\sigma)} w_i(\sigma)^{-1}$, *where χ_ℓ is the ℓ-cyclotomic character.*
(2) *In the expression $[w_i(\sigma)] = c_1^{(i)}u_1 + \cdots + c_{s-1}^{(i)}u_{s-1}, c_j^{(i)} \in \mathbb{Z}_\ell$, we have $c_i^{(i)} = 0$.*

Proof. See [11, lemma 3.2.1] □

For a multiindex $I = (i_1,\dots,i_n)$, $1\leq i_1,\dots,i_n \leq s-1$ the ℓ-adic Milnor number for $\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is defined as the ℓ-adic Magnus coefficient of $w_i(\sigma)$, for $I' = (i_1,\dots,i_{n-1})$, that is

$$\mu(\sigma, I) := \mu(I', w_{i_n}(\sigma)),$$

see [11, eq. 3.2.2]. It is clear that the selection of $w_i(\sigma)$ describes completely the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\mathfrak{F}_{s-1}$.

3.3.1. *The commutative Magnus ring.* In this article we will consider actions of $\mathrm{Aut}(F_{s-1})$ or $\mathrm{Aut}(\mathfrak{F}_{s-1})$ on certain $\mathbb{Z}$-modules ($\mathbb{Z}_\ell$-modules) M defined as quotients of subgroups of the (pro-ℓ) free group. For example on F_{s-1}^{ab} or on $\mathfrak{F}_{s-1}^{\mathrm{ab}}$. We would like for M to be an abelian group (we also choose to write M additively) and we will entirely focus on the case $M = R/R'$, where $R < \mathfrak{F}_{s-1}$ (or $R < F_{s-1}$).

The group F_{s-1} (resp. $\mathfrak{F}_{s-1}$) acts on itself by conjugation. This action can be translated as an $T(H)$ (resp. $\hat{T}(\mathfrak{H})$) module structure on M, by setting

$$\alpha w \alpha^{-1} = \Theta(\alpha) \cdot w,$$

for $w \in F_{s-1}$ (resp. $w \in \mathfrak{F}_{s-1}$).

Lemma 11. *If $M = R/R'$ and $\mathfrak{F}'_{s-1} \subset R$ (resp. $F'_{s-1} \subset R$), then the induced conjugation action on M satisfies eq.*

$$ab \cdot m = ba \cdot m, \text{ for all } a, b \in \hat{T}(\mathfrak{H}) \text{(resp. } T(H)) \text{ and } m \in M. \tag{15}$$

Proof. For $a, b \in \mathfrak{F}_{s-1}$ and $r \in R$ we compute

$$abrb^{-1}a^{-1} = ba[a^{-1}, b^{-1}]r[a^{-1}, b^{-1}]^{-1}a^{-1}b^{-1}.$$

So a sufficient condition for eq. (15) to hold is $[R, \mathfrak{F}'_{s-1}] \subset R'$ (resp. $[R, F'_{s-1}] \subset R'$). This condition is satisfied if $\mathfrak{F}'_{s-1} \subset R$ (resp. $F_{s-1} \subset R$) then eq. (15) holds. □

Therefore, if the assumption of lemma 11 holds, instead of considering the action of the non-commutative ring $\hat{T}(\mathfrak{H})$ (resp. $T(H)$) it makes sense to consider the action of the corresponding abelianized ring.

Definition 12. Consider the commutative $\mathbb{Z}_\ell$-algebra of formal power series

$$\begin{aligned} \mathscr{A} &= \mathbb{Z}_\ell[[u_i : 1 \leq i \leq s]] / \langle (1+u_1)(1+u_2)\cdots(1+u_s) - 1 \rangle \\ &\cong \mathbb{Z}_\ell[[u_i : 1 \leq i \leq s-1]]. \end{aligned} \tag{16}$$

The algebra $\mathscr{A}$ is the symmetric algebra of $\mathfrak{H}$ over $\mathbb{Z}_\ell$, and there is a natural quotient map $\hat{T}(\mathfrak{H}) \to \mathrm{Sym}(\mathfrak{H}) = \mathscr{A}$.

Remark 13. As we noticed already the action of $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ can be described in terms of the cocycles $w_1(\sigma), \ldots, w_{s-1}(\sigma)$. But then we can find elements

$$\varpi_1(\sigma) = \Theta(w_1(\sigma)), \ldots, \varpi_{s-1}(\sigma) = \Theta(w_{s-1}(\sigma)) \in \mathscr{A}$$

such that

$$\sigma(x_i) = \varpi_i(\sigma) \cdot x_i^{\chi_\ell(\sigma)}. \tag{17}$$

Therefore, in order to understand the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $M = \mathfrak{F}_{s-1}/\mathfrak{F}'_{s-1}$ it makes sense to consider the $\mathscr{A}$-module structure of M.

4. Alexander modules

4.1. **Definition and Crowell exact sequence.** Consider the short exact sequence in eq. (7). The group $G = \mathfrak{F}_{s-1}/\Gamma$ admits the presentation:

$$G = \langle x_1, \ldots, x_s | x_1^{e_1} = \cdots = x_s^{e_s} = x_1 \cdots x_s = 1 \rangle. \tag{18}$$

On the other hand since we assumed that $\mathfrak{F}'_{s-1} \subset R_0$, see lemma 11, the group $\mathfrak{F}_{s-1}/R_0 \cdot \Gamma$ is isomorphic to a quotient of the abelian group $\mathbb{Z}/e_1\mathbb{Z} \times \cdots \times \mathbb{Z}/e_{s-1}\mathbb{Z}$.

We will use the Crowell Exact sequence [16, sec. 9.2, sec. 9.4],

$$0 \to R^{\mathrm{ab}} = R/R' \xrightarrow{\theta_1} \mathscr{A}_\psi^{R_0,\Gamma} \xrightarrow{\theta_2} \mathscr{A}^{R_0,\Gamma} \xrightarrow{\varepsilon_{\mathscr{A}}} \mathbb{Z}_\ell \to 0, \tag{19}$$

where

$$\mathscr{A}^{R_0,\Gamma} = \mathbb{Z}_\ell[[\mathscr{F}_{s-1}/R_0 \cdot \Gamma]],$$

and $\mathscr{A}_\psi^{R_0,\Gamma}$ is the *Alexander module*, a free $\mathbb{Z}_\ell$-module

$$\mathscr{A}_\psi^{R_0,\Gamma} = \left(\bigoplus_{g \in \mathfrak{F}_{s-1}/\Gamma} \mathscr{A}^{R_0,\Gamma} dg \right) / \left\langle d(g_1 g_2) - dg_1 - \psi(g_1) dg_2 : g_1, g_2 \in \mathfrak{F}_{s-1}/\Gamma \right\rangle_{\mathscr{A}^{R_0,\Gamma}}.$$

The map $\theta_1 : R^{\mathrm{ab}} \to \mathscr{A}_\psi^{R_0,\Gamma}$ is given by

$$R^{\mathrm{ab}} \ni n \mapsto dn. \tag{20}$$

For a description of the Alexander module in terms of differentials in non-commutative algebras we refer to [13]. Notice that when the group $\mathfrak{F}_{s-1}/R_0 \cdot \Gamma$ is finite then we will write $\mathbb{Z}_\ell[\mathfrak{F}_{s-1}/R_0 \cdot \Gamma]$ instead of $\mathbb{Z}_\ell[[\mathfrak{F}_{s-1}/R_0 \cdot \Gamma]]$.

Proposition 14. *The module $\mathscr{A}_\psi^{R_0,\Gamma}$ admits the following free resolution as an $\mathscr{A}^{R_0,\Gamma}$-module:*

$$\left(\mathscr{A}^{R_0,\Gamma}\right)^{s+1} \xrightarrow{Q} \left(\mathscr{A}^{R_0,\Gamma}\right)^{s} \longrightarrow \mathscr{A}_\psi^{R_0,\Gamma} \longrightarrow 0 \tag{21}$$

where s is the number of generators of G, given in eq. (18) and $s+1$ is the number of relations. Let $\beta_1, \ldots, \beta_{s+1} \in \mathscr{A}^{R_0,\Gamma}$. The map Q is expressed in form of Fox derivatives [3, sec. 3.1],[16, chap. 8]*, as follows*

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{s+1} \end{pmatrix} \mapsto \begin{pmatrix} \psi\pi\left(\frac{\partial x_1^{e_1}}{\partial x_1}\right) & \psi\pi\left(\frac{\partial x_2^{e_2}}{\partial x_1}\right) & \cdots \psi\pi\left(\frac{\partial x_s^{e_s}}{\partial x_1}\right) & \psi\pi\left(\frac{\partial x_1\cdots x_s}{\partial x_1}\right) \\ \psi\pi\left(\frac{\partial x_1^{e_1}}{\partial x_2}\right) & \psi\pi\left(\frac{\partial x_2^{e_2}}{\partial x_2}\right) & \cdots \psi\pi\left(\frac{\partial x_s^{e_s}}{\partial x_2}\right) & \psi\pi\left(\frac{\partial x_1\cdots x_s}{\partial x_2}\right) \\ \vdots & \vdots & & \vdots \\ \psi\pi\left(\frac{\partial x_1^{e_1}}{\partial x_s}\right) & \psi\pi\left(\frac{\partial x_2^{e_2}}{\partial x_s}\right) & \cdots \psi\pi\left(\frac{\partial x_s^{e_s}}{\partial x_s}\right) & \psi\pi\left(\frac{\partial x_1\cdots x_s}{\partial x_s}\right) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{s+1} \end{pmatrix},$$

where π is the natural epimorphism $\mathfrak{F}_s \to G$ defined by the presentation given in eq. (18).

Proof. See [16, cor. 9.6]. □

If in eq. (19) $R_0 = \mathfrak{F}'_{s-1}$ and $\Gamma = \{1\}$, then $\mathscr{A}^{\mathfrak{F}'_{s-1},\{1\}} = \mathbb{Z}_\ell[[u_1, \ldots, u_{s-1}]] = \mathscr{A}$, as defined in eq. (16).

To summarize, for $H_0 = \mathfrak{F}_{s-1}/R_0 \cdot \Gamma$, the Alexander module $\mathscr{A}_\psi^{R,\Gamma}$ can be computed as a cokernel of the function Q:

$$\mathscr{A}_\psi^{R_0,\Gamma} = \mathrm{coker} Q, \qquad \left(\mathscr{A}^{R_0,\Gamma}\right)^{s+1} = \mathbb{Z}_\ell[[H_0]]^{s+1} \xrightarrow{Q} \mathbb{Z}_\ell[[H_0]]^{s} = \left(\mathscr{A}^{R_0,\Gamma}\right)^{s}. \tag{22}$$

The exponents in the above formula reflect the fact that the group G is generated by $s+1$-relations over s-free variables.

Proposition 15. *If $\Gamma = \{1\}$ in eq. (7) the Crowell exact sequence gives the Blanchfield-Lyndon exact sequence:*

$$0 \longrightarrow R^{\mathrm{ab}} \longrightarrow \left(\mathscr{A}^{R_0,\{1\}}\right)^{s-1} \xrightarrow{d_1} \mathscr{A}^{R_0,\{1\}} \xrightarrow{e} \mathbb{Z}_\ell \longrightarrow 0.$$

Proof. See [16, p.118] for the discrete case and the pro-ℓ case follows similarly. □

4.1.1. *Alexander modules for generalised Fermat curves.* It is clear that the group $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ is generated as an $\mathscr{A}$-module by the elements $[x_i, x_j]$ for $1 \leq i < j \leq s-1$.

The structure of $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ as an $\mathscr{A}$-module is expressed in terms of the Crowell exact sequence, see section 4.1, related to the short exact sequence:

$$1 \to \mathfrak{F}'_{s-1} \to \mathfrak{F}_{s-1} \xrightarrow{\psi} \mathfrak{F}^{\mathrm{ab}}_{s-1} \to 1,$$

$$0 \to (\mathfrak{F}'_{s-1})^{\mathrm{ab}} = \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} \to A_\psi \to \mathbb{Z}_\ell[[u_1, \ldots, u_{s-1}]] \to \mathbb{Z}_\ell \to 0,$$

where $A_\psi = A_\psi^{\mathfrak{F}'_{s-1},\{1\}}$ is the Alexander module and

$$\mathscr{A} = \mathscr{A}^{\mathfrak{F}'_{s-1},\{1\}} = \mathbb{Z}_\ell[[u_1, \ldots, u_{s-1}]].$$

Example 16. Assume that in eq. (7) the group $H_0 = (\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}$ and the open curve $R_0 = \mathfrak{F}'_{s-1}$ in this case. Let $\mathfrak{R}_k = \Gamma$ be the smallest closed normal subgroup of $\mathfrak{F}_{s-1}$ generated by $x_1^{\ell^k}, \ldots, x_{s-1}^{\ell^k}$. The group $G = \mathfrak{F}_{s-1,k} = \mathfrak{F}_{s-1}/\mathfrak{R}_k$ admits the presentation:

$$\mathfrak{F}_{s-1,k} = \left\langle x_1, \ldots, x_s | x_1^{\ell^k} = \cdots = x_s^{\ell^k} = x_1 \cdots x_s = 1 \right\rangle.$$

Denote the images of the elements x_i in H_0 by $\bar{x}_i$. It is clear that $\mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ is a free $\mathbb{Z}_\ell$-module of rank

$$\mathrm{rank}_{\mathbb{Z}_\ell}(\mathrm{coker} Q) = s(\ell^k)^{(s-1)} - \mathrm{rank}_{\mathbb{Z}_\ell}(Q).$$

Observe that $\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} \cong \mathbb{Z}_\ell[H_0]$ is a free $\mathbb{Z}_\ell$-module of rank $(\ell^k)^{s-1}$. By induction we can prove

$$\frac{\partial x_i^{\ell^k}}{\partial x_j} = \delta_{ij}(1 + x_i + x_i^2 + \cdots + x_i^{\ell^k - 1}) \text{ for } 1 \leq j \leq s$$

$$\frac{\partial x_1 x_2 \cdots x_s}{\partial x_j} = x_1 \cdots x_{j-1} \tag{23}$$

Set $\Sigma_i = 1 + \bar{x}_i + \cdots + \bar{x}_i^{\ell^k - 1}$. The map Q in eq. (22) is given by the matrix [16, cor. 9.6]

$$\begin{pmatrix} \Sigma_1 & 0 & \cdots & 0 & 1 \\ 0 & \Sigma_2 & \ddots & \vdots & \bar{x}_1 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \Sigma_s & \bar{x}_1\bar{x}_2\cdots\bar{x}_{s-1} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{s+1} \end{pmatrix} = \begin{pmatrix} \Sigma_1\beta_1 + \beta_{s+1} \\ \Sigma_2\beta_2 + \bar{x}_1\beta_{s+1} \\ \vdots \\ \Sigma_s\beta_s + \bar{x}_1\cdots\bar{x}_{s-1}\beta_{s+1} \end{pmatrix} \tag{24}$$

where $\beta_i \in \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ for $1 \leq i \leq s$. Observe that

$$\Sigma_i \bar{x}_i^\nu = \Sigma_i \text{ for all } 0 \leq \nu \leq \ell^k - 1.$$

Lemma 17. *For $1 \leq i \leq s-1$ the following equation holds*

$$\Sigma_i \cdot \mathbb{Z}_\ell[(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}] = \Sigma_i \cdot \mathbb{Z}_\ell \left[\bigoplus_{\substack{\nu=1 \\ \nu \neq i}}^{s-1} \mathbb{Z}/\ell^k\mathbb{Z} \right].$$

On the other hand the module $\Sigma_s\mathbb{Z}_\ell[H_0]$ contains all elements invariant under the action of the product $\bar{x}_1 \cdots \bar{x}_{s-1}$ and is a free $\mathbb{Z}_\ell$-submodule of $\mathbb{Z}_\ell[H_0]$.

Proof. Write

$$\mathbb{Z}_\ell[H_0] = \mathbb{Z}_\ell\left[\bigoplus_{\nu=1}^{s-1} \mathbb{Z}/\ell^k\mathbb{Z}\right] = \bigotimes_{\nu=1}^{s-1} \mathbb{Z}_\ell\left[\mathbb{Z}/\ell^k\mathbb{Z}\right].$$

Therefore the multiplication by Σ_i gives rise to the tensor product

$$\left(\bigotimes_{\nu=1}^{i-1} \mathbb{Z}_\ell\left[\mathbb{Z}/\ell^k\mathbb{Z}\right]\right) \bigotimes \left(\Sigma_i \mathbb{Z}_\ell\left[\mathbb{Z}/\ell^k\mathbb{Z}\right]\right) \bigotimes \left(\bigotimes_{\nu=i+1}^{s-1} \mathbb{Z}_\ell\left[\mathbb{Z}/\ell^k\mathbb{Z}\right]\right) =$$

$$\left(\bigotimes_{\nu=1}^{i-1} \mathbb{Z}_\ell\left[\mathbb{Z}/\ell^k\mathbb{Z}\right]\right) \bigotimes \left(\Sigma_i \mathbb{Z}_\ell\right) \bigotimes \left(\bigotimes_{\nu=i+1}^{s-1} \mathbb{Z}_\ell\left[\mathbb{Z}/\ell^k\mathbb{Z}\right]\right)$$

and the desired result follows.

For the case of $\Sigma_s\mathbb{Z}_\ell[H_0]$ invariance under the action of $\bar{x}_s = \bar{x}_1^{-1}\cdots\bar{x}_{s-1}^{-1}$ is clear. The rank computation follows by changing the basis of H_0 from $\bar{x}_1, \ldots, \bar{x}_{s-1}$ to the basis $\bar{x}_2, \ldots, \bar{x}_s$ and arguing as before. $\square$

The image of the map Q equals to the space generated by elements

$$\begin{pmatrix} \Sigma_1\beta_1 \\ \Sigma_2\beta_2 \\ \vdots \\ \Sigma_s\beta_s \end{pmatrix} + \begin{pmatrix} 1 \\ \bar{x}_1 \\ \vdots \\ \bar{x}_1\cdots\bar{x}_{s-1} \end{pmatrix} \beta_{s+1}.$$

For different choices of $\beta_1, \ldots, \beta_s \in \mathscr{A}^{\mathfrak{F}'_{s-1}, \mathfrak{R}_k}$ the first summand forms a free $\mathbb{Z}_\ell$-module of rank $s(\ell^k)^{s-2}$ and the second summand is a free $\mathbb{Z}_\ell$-module of rank $(\ell^k)^{s-1}$. Also their intersection is just $\mathbb{Z}_\ell$.

Indeed, if for some $\beta_1, \ldots, \beta_{s+1} \in \mathbb{Z}_\ell[H_0]$ we have

$$\beta_{s+1}(1, \bar{x}_1, \ldots, \bar{x}_1\cdots\bar{x}_{s-1}) = (\Sigma_1\beta_1, \ldots, \Sigma_s\beta_s)$$

then by comparison of the first coordinates we see that β_{s+1} is invariant under the action of $\bar{x}_1$. So comparison of second coordinate gives us that $\bar{x}_1\beta_{s+1} = \beta_{s+1}$ is invariant under the action of $\bar{x}_2$. By continuing this way we see that β_{s+1} is invariant under the whole group H_0, that is β_{s+1} belongs to the rank one $\mathbb{Z}_\ell$-module generated by $\Sigma_1\Sigma_2\cdots\Sigma_s$. In this way we see that

Lemma 18.

$$\mathrm{Im} Q = \left(\bigoplus_{\nu=1}^{s} \Sigma_i \mathbb{Z}_\ell[H_0]\right) \bigoplus \mathbb{Z}_\ell[H_0]/\mathbb{Z}_\ell\Sigma_1\cdots\Sigma_s. \tag{25}$$

Also

$$\mathrm{rank}_{\mathbb{Z}_\ell} Q = s(\ell^k)^{s-2} + (\ell^k)^{s-1} - 1.$$

We would like to compute the cokernel of Q as $\mathbb{Z}_\ell[H_0]$-module. This computation lies within the theory of integral representation theory. This seems a very difficult problem since a complete set of representatives of the classes of indecomposable modules for groups of the form $(\mathbb{Z}/\ell^k\mathbb{Z})^t$ seems to be known onlt for $t = 1$ and $k = 1, 2$, see [18]. In this article we will not consider the problem in the integral representation setting and instead we will consider the simpler problem of determination of the H_0-action on the space $H_1(F_{k,s-1}, K)$, where K is a field which contains $\mathbb{Z}_\ell$ and the ℓ^k-roots of unity. We have the following

Lemma 19. *Consider the character $\chi_{i_1,\ldots,i_{s-1}}$ on H_0 given by*

$$\chi_{i_1,\ldots,i_{s-1}}(\bar{x}_1^{\nu_1},\ldots,\bar{x}_{s-1}^{\nu_{s-1}}) = \zeta_{\ell^k}^{\sum_{\mu=1}^{s-1}\nu_\mu i_\mu},$$

where ζ_{ℓ^k} is a primitive ℓ^k-root of unity. The decomposition of

$$\mathrm{Im}(Q)\otimes K = \bigoplus_{i_1,\ldots,i_{s-1}=0}^{\ell^k-1} c_{i_1,\ldots,i_{s-1}}\chi_{i_1,\ldots,i_{s-1}},$$

where $c_{i_1,\ldots,i_{s-1}}\in\mathbb{N}$ is the multiplicity of the corresponding character. For the $s-1$-tuple of integers $\bar{i}=(i_1,\ldots,i_{s-1})$, $0\leq i_1,\ldots,i_{s-1}\leq \ell^k-1$ let $z(\bar{i})$ be the number of $i_1,\ldots,i_{s-1},i_1+\cdots+i_{s-1}$ that are equal to zero modulo ℓ^k. We have

$$c_{i_1,\ldots,i_{s-1}} = \begin{cases} 1+z(\bar{i}) & \text{if } 0\leq z(\bar{i})\leq s-1 \\ s=z(\bar{i}) & \text{if } z(\bar{i})=s. \end{cases} \tag{26}$$

Proof. Consider the decomposition given in lemma 18. The module $K[H_0]$ contains once every possible character, therefore

$$K[H_0] = \bigoplus_{i_1,\ldots,i_{s-1}=0}^{\ell^k-1} K\chi_{i_1,\ldots,i_{s-1}}.$$

On the other hand the modules $\Sigma_i K[H_0]$ for $0\leq i\leq s-1$ are trivially acted on by elements $\bar{x}_i$. This means that

$$\Sigma_i K[H_0] = \bigoplus_{\nu_1,\ldots,\hat{\nu}_i,\ldots,\nu_{s-1}=0}^{\ell^k-1} K\chi_{\nu_1,\ldots,\nu_{i-1},0,\nu_{i+1},\ldots,\nu_{s-1}}.$$

Also the module $\Sigma_s K[H_0]$ contains elements which are invariant by elements of the group generated by $\bar{x}_1\cdots\bar{x}_{s-1}$, since $\bar{x}_s=\bar{x}_1^{-1}\cdots\bar{x}_{s-1}^{-1}$. This means that all characters which appear in the decomposition of $\Sigma_s K[H_0]$ on $\bar{x}_1^{\nu}\cdots\bar{x}_{s-1}^{\nu}$ should give 1, which is equivalent to

$$\chi_{i_1,\ldots,i_{s-1}}(\bar{x}_1^{\nu}\cdots\bar{x}_{s-1}^{\nu}) = \zeta^{\nu\sum_{\mu=1}^{s-1} i_\nu} = 1 \Rightarrow \sum_{\mu=1}^{s-1} i_\nu \equiv 0 \mod \ell^k.$$

Therefore, the decomposition into characters is given by

$$\Sigma_s K[H_0] = \bigoplus_{\substack{i_1,\ldots,i_{s-1}=0\\ i_1+\cdots+i_{s-1}=0}}^{\ell^k-1} K\chi_{i_1,\ldots,i_{s-1}}.$$

Given a character $\chi_{i_1,\ldots,i_{s-1}}$ we now count the number of times it appears. It appears on the summands $\Sigma_j K[H_0]$ for $0\leq j\leq s-1$ when $i_j=0$ and in the summand $\Sigma_s K[H_0]$ when $i_1+\cdots+i_{s-1}\equiv 0 \mod \ell^k$. Also it appears on $K[H_0]/\Sigma_1\cdots\Sigma_s$ only if $(i_1,\ldots,i_{s-1})\neq(0,\ldots,0)$. □

Lemma 20. *We have*

$$\mathrm{rank}_{\mathbb{Z}_\ell}\mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} = (s-1)(\ell^k)^{s-1} - s(\ell^k)^{s-2}+1 \tag{27}$$

and

$$\mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}\otimes K = \bigoplus_{i_1,\ldots,i_s=0}^{\ell^k-1}(s-c_{i_1,\ldots,i_s})\chi_{i_1,\ldots,i_s}. \tag{28}$$

Proof. The rank computation follows since $\mathscr{A}_{\psi}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ is the cokernel of Q, so

$$\mathrm{rank}_{\mathbb{Z}_\ell}\mathscr{A}_{\psi}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} = s(\ell^k)^{s-1} - s(\ell^k)^{s-2} - (\ell^k)^{s-1} + 1 = (s-1)(\ell^k)^{s-1} - s(\ell^k)^{s-2} + 1.$$

Similarly the decomposition in eq. (28) follows by the decomposition of $K[H_0]$ into characters. □

Proposition 21.

$$H_1(F_{k,s-1}, K) = \bigoplus_{i_1,\ldots,i_s=0}^{\ell^k-1} C(i_1,\ldots,i_s)\chi_{i_1,\ldots,i_s}$$

where

$$C(i_1,\ldots,i_s) = \begin{cases} s - z(i_1,\ldots,i_s) - 2 & \textit{if } (i_1,\ldots,i_s) \neq (0,\ldots,0) \\ s - z(i_1,\ldots,i_s) & \textit{if } (i_1,\ldots,i_s) = (0,\ldots,0) \end{cases}$$

Moreover

$$\mathrm{rank}_{\mathbb{Z}_\ell} H_1(F_{k,s-1},\mathbb{Z}_\ell) = (s-1)\left(\ell^k\right)^{s-1} + 2 - s\left(\ell^k\right)^{s-2}.$$

Proof. From the exact sequence given in eq. (19) and the rank computation given in eq. (27) in example 16 we have:

$$\begin{aligned} \mathrm{rank}\big(R_{\ell^k}/(\mathfrak{R}_k \cap R_{\ell^k})^{\mathrm{ab}}\big) &= \mathrm{rank}_{\mathbb{Z}_\ell}\mathscr{A}_{\psi}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} - \mathrm{rank}_{\mathbb{Z}_\ell}\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} + 1 \\ &= (s-2)(\ell^k)^{s-1} + 2 - s(\ell^k)^{s-2}. \end{aligned} \tag{29}$$

The above abelianization corresponds to the $\mathbb{Z}_\ell$-homology of the generalized Fermat curves of type $(k, s-1)$. The above rank coincides with genus computation given in eq. (11).

Let us write

$$H_1(F_{k,s-1}, K) = \bigoplus_{i_1,\ldots,i_{s-1}=0}^{\ell^k-1} C(i_1,\ldots,i_{s-1})\chi_{i_1,\ldots,i_{s-1}}$$

for some integers $C(i_1,\ldots,i_{s-1})$. By lemma 20 and the short exact sequence given in (19) we have

$$C(i_1,\ldots,i_{s-1}) = \begin{cases} s - c_{i_1,\ldots,i_{s-1}} - 1 & \text{if } (i_1,\ldots,i_{s-1}) \neq (0,\ldots,0) \\ s - c_{i_1,\ldots,i_{s-1}} & \text{if } (i_1,\ldots,i_{s-1}) = (0,\ldots,0) \end{cases}$$

The first assertion follows by the values of $c(i_1,\ldots,i_{s-1})$ given in eq. (26). □

Remark 22. For the case of classical Fermat curves we have $s = 3$. The character $\chi_{0,0}$ has $z(0,0) = 3$ and $C_{0,0} = 0$. Similarly the characters $\chi_{0,i}, \chi_{i,0}$ for $1 \leq i \leq \ell^k - 1$ and the character $\chi_{i,j}$ with $i + j \equiv 0 \mod \ell^k$ have $z(0,i) = z(i,0) = z(i,j) = 1$ so their contribution $C(0,i) = C(i,0) = C(i,j) = 0$. All other characters $\chi_{i,j}$ have $z(i,j) = 0$ and their contribution is $C(i,j) = 1$. In this way we arrive to the same result as in eq. (2).

Example 23. Let us now compute $\mathscr{A}_{\psi}^{R_{\ell^k},\mathfrak{R}_k}$ and R_{ℓ^k} is the the pro-ℓ completion of the group generated by

$$\{x_1^i x_j x_1^{-i-1} : 2 \leq j \leq s-1, 0 \leq i \leq \ell^k - 2\} \cup \{x_1^{\ell^k-1} x_j : 1 \leq j \leq s-1\}.$$

This group corresponds to the open cyclic cover of order ℓ^k of $\mathbb{P}^1$ ramified fully above s-points of the projective line, see [12, lemma 11]. Let $\mathfrak{R}_k = \Gamma$ be the smallest closed normal subgroup of $\mathfrak{F}_{s-1}$ generated by $x_1^{\ell^k}, \ldots, x_{s-1}^{\ell^k}$. We have the short exact sequence

$$1 \to R_{\ell^k}/R_{\ell^k} \cap \mathfrak{R}_k \to \mathfrak{F}_{s-1}/\mathfrak{R}_k \to \mathbb{Z}/\ell^k\mathbb{Z} \to 0.$$

We compute $\mathscr{A}^{R_{\ell^k},\mathfrak{R}_k} = \mathbb{Z}_\ell[\mathbb{Z}/\ell^k\mathbb{Z}]$, which is an $\mathbb{Z}_\ell$-module of rank ℓ^k. On the other hand observe that the $\mathbb{Z}_\ell$-module $\mathscr{A}_\psi^{R_{\ell^k},\mathfrak{R}_k}$ is given by exactly the same cokernel as the module $\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$. The only difference is that $\mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ is a $\mathbb{Z}_\ell[(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}]$-module while $\mathscr{A}^{R_{\ell^k},\mathfrak{R}_k}$ is a $\mathbb{Z}_\ell[\mathbb{Z}/\ell^k\mathbb{Z}]$-module.

So following exactly the same method as in example 16 we conclude that

$$\mathrm{rank}_{\mathbb{Z}_\ell}\mathscr{A}_\psi^{R_{\ell^k},\mathfrak{R}_k} = s \cdot \ell^k - s - \ell^k + 1 = (s-1)\ell^k - s + 1.$$

Also we compute the rank

$$\mathrm{rank}\big(R_{\ell^k}/(R_{\ell^k} \cap \mathfrak{R}_k)^{\mathrm{ab}}\big) = \mathrm{rank}_{\mathbb{Z}_\ell}\mathscr{A}_\psi^{R_{\ell^k},\mathfrak{R}_k} - \mathrm{rank}_{\mathbb{Z}_\ell}\mathscr{A}^{R_{\ell^k},\mathfrak{R}_k} + 1 = (s-2)(\ell^k - 1).$$

The module $\left(R_{\ell^k}/R_{\ell^k} \cap \mathfrak{R}_k\right)^{\mathrm{ab}}$ corresponds to the $\mathbb{Z}_\ell$-homology of the above curves, corresponding to R_{ℓ^k} and its rank is twice the genus of the curve, in accordance with the genus formula given in [12, eq. 21].

5. Galois modules in terms of the Magnus embedding

5.1. **The group $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ as an $\mathscr{A}$-module.** In this section we will study the $\mathscr{A}$-module structure of $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$. This is the arithmetic analogon of the Gassner representation, as Ihara points out in [10]. This consideration leads to the Galois representation of the Tate module, see section 5.1.4. Finally in section 5.1.5 we will study the passage from the Gassner representation to the Burau by seeing the generalized Fermat curve as a cover of the projective line.

5.1.1. *Application to Generalized Fermat curves.* Consider the the smallest closed normal subgroup $\mathfrak{R}_k$ of $\mathfrak{F}_{s-1}$ containing all $x_i^{\ell^k}$ for $1 \leq i \leq s-1$. Define also

$$\mathfrak{F}_{s-1,k} = \mathfrak{F}_{s-1}/\mathfrak{R}_k.$$

Set $\bar{\lambda} = \{0, 1, \infty, \lambda_1, \ldots, \lambda_{s-3}\}$ and let $\mathscr{M}$ be the maximum pro-ℓ extension of $K = \bar{k}(t)$ unramified outside the set of points $\bar{\lambda}$. Consider the function field of the generalized Fermat curves

$$K_k := K\left(t^{\frac{1}{\ell^k}}, (t-1)^{1/\ell^k}, (t-\lambda_1)^{1/\ell^k}, \ldots, (t-\lambda_{s-3})^{1/\ell^k}\right).$$

Let K_k^{ur} and K_k^{urab} be the maximal unramified and maximal abelian unramified extensions of K_k respectively. Also let K' be the maximum abelian unramified extension of K and K'' be the maximum abelian unramified extension of K'. By covering space theory, the fields K', K'' correspond to the groups $\mathfrak{F}'_{s-1}$ and $\mathfrak{F}''_{s-1}$, respectively. The function field K_k corresponds to the group $\mathfrak{F}'_{s-1}\mathfrak{R}_k$ and is equal to the function field of the generalized Fermat curve.

Aim of this section is the following characterization of the maximal unramified abelian extension K_k^{urab} of the function field K_k of the generalized Fermat curve. This is a generalisation of a similar construction by Ihara for the classical Fermat curves, see [9, sec. II, p. 63]

Theorem 24. *We have that* $\mathrm{Gal}(K_k^{\mathrm{urab}}/K_k) \cong \mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k}$.

Indeed, we have

$$K' = \bigcup_k K_k, \qquad K' \cap K_k^{\mathrm{ur}} = K_k$$
$$K'' = \bigcup_k K_k^{\mathrm{urab}}, \qquad K'' \cap K_k^{\mathrm{ur}} = K_k^{\mathrm{urab}}$$

The Galois correspondence is given as follows:

$$\begin{tikzcd}
 & \mathscr{M} \arrow[dl, dash] \arrow[dr, dash] & \\
K'' \arrow[d, dash] \arrow[drr, dash] & & K_k^{\mathrm{ur}} \arrow[d, dash] \\
K' \arrow[dr, dash] & & K_k^{\mathrm{urab}} \arrow[dl, dash, "\mathrm{Gal}(K_k^{\mathrm{urab}}/K_k)"] \\
 & K_k \arrow[d, dash] & \\
 & K &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{1\} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathfrak{F}''_{s-1} \arrow[d, dash] \arrow[drr, dash] & & \mathfrak{R}_k \arrow[d, dash] \\
\mathfrak{F}'_{s-1} \arrow[dr, dash] & & \mathfrak{F}''_{s-1}\mathfrak{R}_k \arrow[dl, dash] \\
 & \mathfrak{F}'_{s-1}\mathfrak{R}_k \arrow[d, dash] & \\
 & \mathfrak{F}_{s-1} &
\end{tikzcd}$$

Using standard isomorphism theorems in group theory (see also [13, sec. 1.2.1]) and the definitions we see

(30)
$$\mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} \cong \mathfrak{F}'_{s-1}/\left(\mathfrak{F}'_{s-1} \cap \mathfrak{F}''_{s-1}\mathfrak{R}_k\right) \cong \mathfrak{F}'_{s-1}\mathfrak{R}_k/\mathfrak{F}''_{s-1}\mathfrak{R}_k \cong \mathrm{Gal}(K_k^{\mathrm{urab}}/K_k)$$

is an abelian group, a free $\mathbb{Z}_\ell$-module of rank $2g$, where g is the genus of the generalized Fermat curve, $F_{\ell^k,s-1}$ so that

$$2g_{(\ell^k,s-1)} = 2 + \ell^{k(s-2)}((s-2)(\ell^k-1)-2). \tag{31}$$

Observe that according to eq. (5) we have

$$\mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} \cong H_1(F_{k,s-1}, \mathbb{Z}_\ell).$$

The last genus computation also follows from the following proposition which identifies unramified $\mathbb{Z}/\ell^k\mathbb{Z}$-extensions of a curve X with the group of ℓ^k-torsion points of the Jacobian $J(X)$.

Proposition 25. *Let Y be a complete nonsingular algebraic curve defined over a field of characteristic prime to ℓ. The étale Galois covers of Y with Galois group $\mathbb{Z}/\ell^k\mathbb{Z}$ are classified by the étale cohomology group $H^1_{\mathrm{et}}(Y, \mathbb{Z}/\ell^k\mathbb{Z})$ which is equal to the group of ℓ^k-torsion points of* $\mathrm{Pic}(Y)$.

Proof. See [6, Ex. 2.7], [19, sec. 19]. $\square$

5.1.2. *Crowell sequence for generalized Fermat curves.* Here we use the presentation $\mathfrak{F}_{s-1} = \mathfrak{F}_s/\langle x_1 \cdots x_s \rangle$. Let $H_k = (\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}$. We have the short exact sequence

$$1 \to \mathfrak{F}'_{s-1,k} = (\mathfrak{F}_{s-1}/\mathfrak{R}_k)' \to \mathfrak{F}_{s-1}/\mathfrak{R}_k \xrightarrow{\psi} H_k \to 1$$

We will use the Crowell Exact sequence [16, chap. 9]

$$0 \to (\mathfrak{F}'_{s-1,k})^{\mathrm{ab}} = \mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} \xrightarrow{\theta_1} \mathscr{A}_{\psi}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} \xrightarrow{\theta_2} \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} \xrightarrow{\varepsilon_{\mathscr{A}_k}} \mathbb{Z}_\ell \to 0, \tag{32}$$

where

$$\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} = \mathbb{Z}_\ell[H_k] = \mathbb{Z}_\ell[(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}],$$

and

$$\mathscr{A}_{\psi}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} = \mathrm{coker} Q, \qquad \mathbb{Z}_\ell[H_k]^{s+1} \xrightarrow{Q} \mathbb{Z}_\ell[H_k]^s \tag{33}$$

The Alexander module for $\mathfrak{F}_{s-1}/\mathfrak{R}_k$ was computed on example 16. Notice that $\mathscr{A}_{\psi}^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$ and the Crowell sequence know the genus of the generalised Fermat curve, see eq. (29).

5.1.3. *Representation theory on Generalized Fermat Curves.* These are representations on the free $\mathbb{Z}_\ell$-modules

$$\rho_k : G \to \mathrm{GL}(H_1(F_{k,s-1}, \mathbb{Z}_\ell)).$$

Where G is either the absolute Galois group or the braid group B_{s-1} or B_s.

Let us now combine the two Crowell sequences together.

$$\begin{tikzcd}
0 \arrow[r] & (\mathfrak{F}'_{s-1})^{\mathrm{ab}} = \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} \arrow[r] \arrow[d] & \mathscr{A}^{s-1} \arrow[r, "d_1"] \arrow[d] & \mathscr{A} \arrow[r] \arrow[d, "\omega"] & \mathbb{Z}_\ell \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (\mathfrak{F}'_{s-1,k})^{\mathrm{ab}} = \mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} \arrow[r, "\theta"] & \left(\mathscr{A}_{\psi}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}\right) \arrow[r, "d_{1,k}"] & \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} \arrow[r] & \mathbb{Z}_\ell \arrow[r] & 0
\end{tikzcd} \tag{34}$$

For an explanation of these two combined sequences in terms of the "cotangent sequence" we refer to [13].

Lemma 26. *The group $\mathfrak{R}_k$ is invariant under the action of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Proof. For every generator $x_i^{\ell^k}$ and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we have

$$\sigma(x_i^{\ell^k}) = \sigma(x_i)^{\ell^k} = \left(w_i(\sigma) x_i^{N(\sigma)} w_i(\sigma)^{-1}\right)^{\ell^k}.$$

Let a_n be a sequence of integers such that $a_n \to N(\sigma)$. We have

$$\left(w_i(\sigma) x_i^{a_n} w_i(\sigma)^{-1}\right)^{\ell^k} = \left(w_i(\sigma) x_i^{\ell^k} w_i(\sigma^{-1})\right)^{a_n}.$$

The later element is in $\mathfrak{R}_k$ since by definition is normal in $\mathfrak{F}_{s-1}$. The limit $a_n \to N(\sigma)$ is in $\mathfrak{R}_k$ since this group is by definition closed. The result follows. $\square$

It is clear that $\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} = \mathbb{Z}_\ell[H_0]$ can be considered through the vertical map ω as an $\mathscr{A}$-module and inherits an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ by ω, by writting $\alpha \in \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ as image of an element $\alpha' \in \mathscr{A}$, that is $\alpha = \omega(\alpha')$ and define

$$\sigma(\alpha) = \sigma(\omega(\alpha')) = \omega(\sigma\alpha').$$

By lemma 26 this action is well defined. On the other hand an element $\phi = \overline{[x_i, x_j]} \in \mathfrak{F}'_{s-1,k}/\mathfrak{F}''_{s-1,k} = \mathfrak{F}'_{s-1}/(\mathfrak{F}'_{s-1} \cap \mathfrak{F}_{s-1''}\mathfrak{R}_k)$ is sent to the element

$$\theta(\phi) = d[\bar{x}_i, \bar{x}_j] = -\bar{u}_j dx_i + \bar{u}_i dx_j \in \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}.$$

The module $\mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ is a $\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$-module, described by the sequence given in eq. (21) and by the matrix Q given in eq. (24) and is naturally acted on by the absolute Galois group. Observe also that the map θ sends the class of $[x_i, x_j]$ to $d[x_i, x_j] = u_i dx_j - u_j dx_i$, and this element is annihilated by the elements $\Sigma_i = \sum_{\nu=0}^{\ell^k-1} \bar{x}_i^\nu$ for $1 \leq i \leq s$. We can see this by direct computations or by observing that in $\mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$ we have

$$\Sigma_i \cdot \beta_i = \beta_{s+1}\bar{x}_1 \cdots \bar{x}_{i-1}.$$

and the image $\theta[x_i, x_j]$ has the $s+1$ coordinate $\beta_{s+1} = 0$. The above observation generalises the definition of ideal $\mathfrak{a}_n$ in eq. (8) in the article of Ihara, [9].

Therefore,

$$H_1(F_{k,s-1}, \mathbb{Z}_\ell) \cong \theta((\mathfrak{F}'_{s-1,k})^{\mathrm{ab}}) \subset \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}$$

is acted on by $\mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k}/\langle \Sigma_i : 1 \leq i \leq s \rangle$, and $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on it in terms of the action given in eq. (17). Indeed, $\mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$ is identified with the cokernel of the matrix Q, i.e. an element in $\mathscr{A}_\psi^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$ is the class of an s-tuple which is sent to

$$\sigma : (\beta_1, \ldots, \beta_s) + \mathrm{Im}(Q) \longmapsto (\sigma\beta_1, \ldots \sigma\beta_s) + \mathrm{Im}(Q).$$

This action is well defined since the space $\mathrm{Im}(Q)$ is left invariant under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Indeed, in the commutative ring $\mathscr{A}^{\mathfrak{F}_{s-1},\mathfrak{R}_k}$, the action $\sigma(\bar{x}_i) = \bar{x}_i^{N(\sigma)}$ so $\sigma(\Sigma_i) = \Sigma_i$, and invariance follows by eq. (25).

5.1.4. *On Jacobian variety of Generalized Fermat curves.* Consider the ℓ-adic Tate module $T(\mathrm{Jac}(F_{k,s-1}))$ of the Jacobian of the generalized Fermat curves $F_{k,s-1}$:

$$T(\mathrm{Jac}(F_{k,s-1})) = H_1(F_{k,s-1}, \mathbb{Z}) \otimes \mathbb{Z}_\ell = \frac{\mathfrak{F}'_{s-1,k}}{\mathfrak{F}''_{s-1,k}}.$$

Following Ihara we consider

$$\mathbb{T} := \varprojlim_k T(\mathrm{Jac}(F_{k,s-1})) = \varprojlim_k \frac{\mathfrak{F}'_{s-1,k}}{\mathfrak{F}''_{s-1,k}}, \tag{35}$$

where the inverse limit is considered with respect to the maps $T(\mathrm{Jac}(F_{k+1,s-1})) \to T(\mathrm{Jac}(F_{k,s-1}))$, which is induced by the map

$$(x_0, \ldots, x_{s-1}) \mapsto (x_0^\ell, \ldots, x_{s-1}^\ell).$$

Let $\bar{F}_{k,s-1} = F_{k,s-1} \otimes_{\mathrm{Spec}\mathbb{Q}} \mathrm{Spec}\bar{\mathbb{Q}}$. Consider also the inverse limit

$$\varprojlim_k \mathrm{Gal}(\bar{F}_{k,s-1}/\mathbb{P}^1_{\bar{\mathbb{Q}}}) = \varprojlim_k (\mathbb{Z}/\ell^k\mathbb{Z})^{s-1} = \mathbb{Z}_\ell^{s-1}.$$

Therefore

$$\varprojlim_k \mathbb{Z}_\ell[\mathrm{Gal}(\bar{F}_{k,s-1}/\mathbb{P}^1_{\bar{\mathbb{Q}}})] \cong \mathscr{A}$$

and $\mathbb{T}$ can be considered as an $\mathscr{A}$-module. Using eq. (34) we obtain

$$\frac{\mathfrak{F}'_{s-1}}{\mathfrak{F}''_{s-1}} \cong \mathbb{T}.$$

See [1, sec. 13] for the explicit isomorphism in the case of Fermat curves.

The geometric interpretation of this construction is that for fixed s-number of points we can consider all generalized Fermat curves seen as $(\mathbb{Z}/\ell^k\mathbb{Z})^s$ ramified covers of the projective line, for $k \in \mathbb{N}$. In this way we obtain a curve F_s, which is a $\mathbb{Z}_\ell^{s-1}$ cover of the projective line. The Burau representation and the pro-ℓ Burau representation can be defined in terms of such an infinite Galois cover, see [12].

This construction leads to the definition of a subspace $\mathbb{T}^{\mathrm{prim}} \subset \mathbb{T}$ which is a free $\mathscr{A}$-module of rank $s-2$. Observe that the submodule of a free module is not necessarily a free module and $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ is not necessarily free. For example in the following short exact sequence

$$0 \longrightarrow (\mathfrak{F}'_{s-1})^{\mathrm{ab}} = \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} \longrightarrow \mathscr{A}^{s-1} \xrightarrow{d_1} \mathscr{A} \longrightarrow \mathbb{Z}_\ell \longrightarrow 0$$

$\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ is contained in the free $\mathscr{A}$-module $\mathscr{A}^{s-1}$, but is free itself. The $\mathscr{A}$-module $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ contains the free module of rank $s-2$ (see [17],[11, Th. 5.39])

$$\mathbb{T}^{\mathrm{prim}} := \left\{ (\lambda_j u_1 \cdots \hat{u}_j \cdots u_{s-1})_{j=1,\ldots,s-1} : \lambda_j \in \mathscr{A}, \sum_{j=1}^{s-1} \lambda_j = 0 \right\}.$$

Set $w = u_1 \cdots u_{s-1}$ a basis of $(\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1})^{\mathrm{prim}}$ is given by

$$v_1 = \left(-\frac{w}{u_1}, \frac{w}{u_2}, 0, \ldots, 0 \right)^t, \ldots, v_{s-2} = \left(0, \ldots, 0, -\frac{w}{u_{s-2}}, \frac{w}{u_{s-2}} \right)^t.$$

In the case of Fermat curves, i.e. $s = 2$ we have that $(\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1})^{\mathrm{prim}} = \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ and $\mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1}$ is a free $\mathscr{A}$-module, generated by $[x_1, x_2]$. Notice that the injective map $d : \mathfrak{F}'_{s-1}/\mathfrak{F}''_{s-1} \xrightarrow{d} \mathscr{A}^{s-1}$ is given by sending a representative

$$\begin{aligned} [x_i, x_j] \to d([x_i, x_j]) &= (1 - x_j)dx_i - (1 - x_i)dx_j \\ &= -u_j \cdot dx_i + u_i \cdot dx_j. \end{aligned}$$

Proposition 27. *Let G be either $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ or the braid group B_s. An element in $g \in G$ induces an action on both $\mathbb{T}$ and $\mathbb{T}^{\mathrm{prim}}$. In particular the subspace $\mathbb{T}^{\mathrm{prim}}$ is a free $\mathscr{A}$-module. Thus we have a cocycle map*

$$\begin{aligned} \rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) &\to \mathrm{GL}_{s-2}(\mathscr{A}) \\ \sigma &\longmapsto (a_{ij}(\sigma)) \end{aligned}$$

This cocycle can be given in terms of the matrix

$$\sigma(w_{ij} d[x_i, x_j]) = \sum_{\nu < \mu} a_{\nu,\mu}(\sigma) w_{\nu\mu} d[x_\nu, x_\mu].$$

Remark 28. In [11, sec. 5.3] this cocycle is identified as the Gassner representation and the relation with the classical definition in terms of Fox derivatives see [3, chap. 3] is studied. The Gassner cocycles when restricted to a certain subgroup $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})[1] \subset \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ give rise to a representation instead of cocycle, see [11].

5.1.5. *From generalized Fermat curves to cyclic covers of $\mathbb{P}^1$.* We will now relate the Crowell sequences for the generalized Fermat curves and cyclic covers $\bar{Y}_{\ell^k}$ of the projective line as they were defined in [12] using the results of [13]. This will provide the relation of the Gassner representation to the Burau representation. The analogon of the Burau representation was defined in [11, p.675] by reduction of the

Gassner representation. Here we also consider this reduction with respect to the curve definition of the Burau representation.

We have the following diagram of ramified coverings of curves

$$\begin{tikzcd}
F_{k,s-1} \arrow[dr, "(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}"] & & \bar{Y}_{\ell^k} \arrow[dl, "\mathbb{Z}/\ell^k\mathbb{Z}"'] \\
& \mathbb{P}^1 - \{0,1,\infty,\lambda_1,\ldots,\lambda_{s-3}\} &
\end{tikzcd}$$

The passage for the corresponding representations from $F_{k,s-1}$ to $\bar{Y}_{\ell^k}$ corresponds to the passage from the Gassner representation to the Burau representation, see [3, prop. 3.12] and [11, sec. 5].

Set $\bar{R}_{\ell^k}$ be the fundamental group of the closed curve $\bar{Y}_{\ell^k}$, which can be computed using the Reidemeister-Schreier method, see [12]:

$$\bar{R}_{\ell^k} = R_{\ell^k}/\Gamma = \left\langle (x_2 x_1^{-1})^{x_1^\nu}, \ldots, (x_{s-1}x_1^{-1})^{x_1^\nu} : 0 \leq \nu < \ell^k - 1 \right\rangle.$$

Let also C_s be the $\mathbb{Z}$ cover of the projective line ramified over s-points. Let R be its fundamental group, which by [12] equals

$$R = \left\langle \left(x_j x_1^{-1}\right)^{x_1^\nu} : \nu \in \mathbb{Z} \right\rangle$$

The fixed field of $R/\mathfrak{R}_k$ is the function field K_{ℓ^k} of the curve $\bar{Y}_{\ell^k}$, $K(C_s)$ is the function field of the curve C_s. The group R' corresponds to the maximal unramified abelian extension $K(C_s)^{\mathrm{ur}}$ of $K(C_s)$ while $\mathfrak{R}_k$ corresponds to the maximal unramified extension $K(C_s)^{\mathrm{unrab}}$. The group $R' \cdot \mathfrak{R}_k$ corresponds to the maximal abelian unramified $K_{\ell^k}^{\mathrm{unrab}}$ extension of K_{ℓ^k}. The groups $F'_{s-1} \cdot \mathfrak{R}_k$ and $F''_{s-1} \cdot \mathfrak{R}_k$ correspond to the generalized Fermat curve $F_{k,s-1}$ and the maximal unramified extension $F_{k,s-1}^{\mathrm{unrab}}$. The groups F'_{s-1}, F''_{s-1} correspond to the maximal abelian unramified extension of K_0 and the maximal abelian unramified extension of K' respectively.

$$\begin{tikzcd}
 & \mathscr{M} \arrow[dl, dash] \arrow[dr, dash] & & \\
K'' \arrow[d, dash] \arrow[dr, dash] & & K(C_s)^{\mathrm{ab}} \arrow[dl, dash] \arrow[d, dash] & K(C_s)^{\mathrm{unrab}} \arrow[dl, dash] \arrow[d, dash] \\
K' \arrow[dr, dash] & F_{k,s-1}^{\mathrm{unrab}} \arrow[r] \arrow[d, dash] & K_{\ell^k}^{\mathrm{unrab}} \arrow[d, dash, "H_1(C_{\ell^k},\mathbb{Z})"] & K(C_s) \arrow[dl, dash] \\
 & F_{k,s-1} \arrow[r] \arrow[dr, dash] & K_{\ell^k} \arrow[d, dash] & \\
 & & K_0 = k(t) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{1\} \arrow[dl, dash] \arrow[dr, dash] & & \\
F''_{s-1} \arrow[d, dash] \arrow[dr, dash] & & \mathfrak{R}_k \arrow[dl, dash] \arrow[d, dash] & R' \arrow[dl, dash] \arrow[d, dash] \\
F'_{s-1} \arrow[dr, dash] & F''_{s-1} \cdot \mathfrak{R}_k \arrow[r, hook] \arrow[d, dash] & R' \cdot \mathfrak{R}_k \arrow[d, dash, "H_1(C_{\ell^k},\mathbb{Z})"] & R \arrow[dl, dash, bend left] \\
 & F'_{s-1} \cdot \mathfrak{R}_k \arrow[r, hook] \arrow[dr, dash] & R \cdot \mathfrak{R}_k \arrow[d, dash] & \\
 & & F_{s-1} &
\end{tikzcd}$$

As in the case of generalized Fermat curves we can form the limit

$$\mathbb{T}_R := \varprojlim_k (\mathrm{Jac}(\bar{Y}_{\ell^k})) = \varprojlim_k (R_{\ell^k}/\mathfrak{R}_k)^{\mathrm{ab}} = R^{\mathrm{ab}} = H^1(\bar{Y}_{\ell^k}, \mathbb{Z}_\ell).$$

We now compare the Crowell sequences for the cyclic covers and the Fermat covers, following [13]

$$\begin{tikzcd}
 & & 0 & 0 & & \\
0 \arrow[r] & R^{\mathrm{ab}} = \mathbb{T}_R \arrow[r] & \mathscr{A}_\psi^{R,\mathfrak{R}_k} \arrow[r] \arrow[u] & \mathscr{A}^{R,\mathfrak{R}_k} = \mathbb{Z}_\ell[[\mathbb{Z}/\ell^k\mathbb{Z}]] \arrow[r] \arrow[u] & \mathbb{Z}_\ell \arrow[r] & 0 \\
0 \arrow[r] & (\mathfrak{F}'_{s-1})^{\mathrm{ab}} = \mathbb{T} \arrow[r] \arrow[u, "\phi_1"] & \mathscr{A}_\psi^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} \arrow[r] \arrow[u, "\phi_2"] & \mathscr{A}^{\mathfrak{F}'_{s-1},\mathfrak{R}_k} = \mathbb{Z}_\ell[[(\mathbb{Z}/\ell^k\mathbb{Z})^{s-1}]] \arrow[r] \arrow[u, "\phi_3"] & \mathbb{Z}_\ell \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The map $\phi_1 : \mathbb{T} \to \mathbb{T}_R$ on Tate modules is given by the first vertical map. The action module structure is given by the commutating diagram

$$\begin{tikzcd}
\mathscr{A} \times \mathbb{T} \arrow[r] \arrow[d] & \mathbb{T} \arrow[d, "\phi_1"] \\
\mathscr{A}^{R,\mathfrak{R}_k} \times \mathbb{T}_R \arrow[r] & \mathbb{T}_R
\end{tikzcd}$$

where the horizontal maps are the module actions and the first vertical map sends $(a,t) \mapsto (\phi_3(a), \phi_1(t))$. The map ϕ_3 is the reduction identifying the variables $x_1, x_2, \ldots, x_{s-1}$. Let G be as in proposition 27. In particular from the reduction $\mathbb{T} \to \mathbb{T}_R$ we obtain the diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[dr] & \mathrm{GL}_{s-2}(\mathscr{A}) \arrow[d] \\
 & \mathrm{GL}_{s-2}(\mathscr{A}^{R,\mathfrak{R}_k})
\end{tikzcd}$$

corresponding to the free parts of $\mathbb{T}$ and $\mathbb{T}_R$ respectively.

References

[1] Greg W. Anderson. The hyperadelic gamma function. *Invent. Math.*, 95(1):63–131, 1989.

[2] G. V. Belyĭ. Galois extensions of a maximal cyclotomic field. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(2):267–276, 479, 1979.

[3] Joan S. Birman. *Braids, links, and mapping class groups*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 82.

[4] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. *Analytic pro-p groups*, volume 61 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1999.

[5] Gabino González-Diez, Rubén A. Hidalgo, and Maximiliano Leyton. Generalized Fermat curves. *J. Algebra*, 321(6):1643–1660, 15 March 2009.

[6] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[7] Rubén A. Hidalgo, Aristides Kontogeorgis, Maximiliano Leyton-Álvarez, and Panagiotis Paramantzoglou. Automorphisms of generalized Fermat curves. *J. Pure Appl. Algebra*, 221(9):2312–2337, 2017.

[8] Yasutaka Ihara. Some remarks on the number of rational points of algebraic curves over finite fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 28(3):721–724 (1982), 1981.

[9] Yasutaka Ihara. Profinite braid groups, Galois representations and complex multiplications. *Ann. of Math. (2)*, 123(1):43–106, 1986.

[10] Yasutaka Ihara. Arithmetic analogues of braid groups and Galois representations. In *Braids (Santa Cruz, CA, 1986)*, volume 78 of *Contemp. Math.*, pages 245–257. Amer. Math. Soc., Providence, RI, 1988.

[11] Hisatoshi Kodani, Masanori Morishita, and Yuji Terashima. Arithmetic Topology in Ihara Theory. *Publ. Res. Inst. Math. Sci.*, 53(4):629–688, 2017.

[12] Aristides Kontogeorgis and Panagiotis Paramatzoglou. Group actions on cyclic covers of the projective line, 2018.

[13] Aristides Kontogeorgis and Panagiotis Paramatzoglou. A non-commutative differential module approach to Alexander modules, 2019.

[14] Michel Lazard. Sur les groupes nilpotents et les anneaux de Lie. *Ann. Sci. Ecole Norm. Sup. (3)*, 71:101–190, 1954.

[15] C. Maclachlan. Abelian groups of automorphisms of compact Riemann surfaces. *Proc. London Math. Soc. (3)*, 15:699–712, 1965.

[16] M Morishita. *Knots and Primes: An Introduction to Arithmetic Topology*. SpringerLink : Bücher. Springer-Verlag London Limited, 2011.

[17] T Oda. Note on meta-abelian quotients of pro-l free groups. *preprint*, 1985.

[18] Irving Reiner. Integral representations of cyclic p-groups. In *Topics in algebra (Proc. 18th Summer Res. Inst., Austral. Math. Soc., Austral. Nat. Univ., Canberra, 1978)*, volume 697 of *Lecture Notes in Math.*, pages 70–87. Springer, Berlin, 1978.

[19] Jean-Pierre Serre. Sur la topologie des variétés algébriques en caractéristique p. In *Symposium internacional de topología algebraica International symposium on algebraic topology*, pages 24–53. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.

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